Bulletin of TICMI
Vol. 26, No. 2, 2022, 67–80

Research of nonlinear dynamic systems describing the process of territorial stability of the state

Temur Chilachava[a*], George Pochkhua [a]

[a]*Sokhumi State University, 61 Politkovskaya St., 0186 Tbilisi, Georgia*
(*Received September 1, 2022; Revised November 3, 2022; Accepted November 17, 2022*)

The article proposes a new nonlinear mathematical model that describes the process of possibility (instability of the state) or impossibility (stability of the state) of separating the region from the state. The model is described by the Cauchy problem for a nonlinear two-dimensional dynamic system. The model assumes that only two categories of citizens live in a particular region of a state: the first category, which is a supporter of the center (unionists) and opposes the secession of the region; the second - supporters of the separation of the region (separatists), i.e. its separation from the center, in order to form a new independent state. The general mathematical model implies the presence of both federal and external sides, which influence the separatists and unionists, respectively, by various factors, in order to change their opinion (will). Natural conditions are proposed under which secession of the region is considered possible (for example, the presence of a majority or qualified majority of citizens of the region who support separatism). In the case of variable coefficients of the model, under some restrictions, in quadratures, an exact solution to the Cauchy problem was found for a two-dimensional dynamic system, and in the case of constant coefficients, conditions were found under which secession of the region is impossible. In the particular case, the absence of external stakeholders for the region, in the case of constant coefficients and opposite values of demographic factors of the sides, the problem is actually reduced to a predator-victim model and is described by a nonlinear two-dimensional dynamic system of the Lotka-Volterra type. At the same time, conditions were found for the coefficients of attracting opponents to allies, demographic factors and baseline conditions under which it is impossible to separate the region.

The article also proposes a new nonlinear mathematical model that describes in a certain politically conflicting region of the state the presence of three groups of the population with different political priorities. One part of the population (unionists) is politically oriented towards preserving the region as part of the state, the second part of the region supports the idea of separatism, separation of the region from the state in order to form a new independent state (separatists), and the third part of the region supports the idea of irredentism of the region, that is, separation in order to join another, possibly bordering state (irredentists). Weak (a simple majority of the region's population) and strong (a qualified majority of the region's population) conditions are offered, which in a legal sense may not have direct consequences, but may determine the aspirations of the majority of the region's population. The model is described by a nonlinear three-dimensional dynamic system with variable coefficients. With some constraints on model parameters, accurate analytical solutions are found. Additional conditions were also found under which: the region remains within the state; possible secession of the region; the irredentism of the region is possible, that is, its accession to another state.

Keywords: Mathematical models, Dynamic systems, Unionists, Separatists, Irredentist, Conflict, Secession.

AMS Subject Classification: 93A30, 00A71, 97M10, 97M70.

*Corresponding author. Email: temo_chilachava@yahoo.com

ISSN: 1512-0082 print

Introduction

Mathematical modeling of physical processes has a long history. Mathematical modeling of physical processes involves the model adequacy, which is validated by Newton's non-relative five laws of classical mechanics: mass conservation law; law of conservation of impulse; the law of conservation the momentum of impulse; the first law of thermodynamics, i.e. energy conservation law; the second law of thermodynamics, i.e. entropy conservation law [1-4].

The study of a number of social processes, such as information warfare, assimilation of languages (peoples), globalization, settlement of political conflicts, secession of regions, territorial integrity of states, etc. is of great interest.

From our point of view, the only scientific approach to an adequate quantitative and qualitative description of these problems is the mathematical modeling of processes, i.e. the creation of mathematical models describing these current problems.

We created a new direction of mathematical modeling, i.e. "Mathematical Modeling of Information Warfare" [5,6]. In these models two antagonistic sides waging with each other information warfare and also the third peacekeeping side trying to extinguish information warfare reconsidered. Conditions on model parameters at which the third side will be able to force the conflicted sides to completion of information warfare are found.

We have previously proposed original mathematical models: linguistic globalization, which establishes, within the framework of the model, the possibility of globalization in English [7]; two level of assimilation of languages (peoples) by more common languages [8].

We also proposed mathematical models for the settlement of political (not military confrontation) conflicts through the economic cooperation of parts of the populations of the sides with the participation of international organizations and relevant investment funds [9, 10].

Over the past few decades, due to the desire of some political players to redistribute the world map, issues related to the self-determination of nations and the possibility of creating independent states have become relevant.

As you know, according to the principles of international law, there are two fundamental principles: the inviolability of the borders of states recognized by the UN and the right of peoples to self-determination until the creation of an independent state. The two principles actually contradict each other and their interpretation is often subjective.

It seems interesting to us, from a scientific point of view, to describe by a mathematical model the dynamics of two parts of the population of a region of an independent state that pursue opposite political goals. Moreover, the first part of the population supports the territorial integrity of the existing and UN-recognized state (unionists), and the second part advocates the separation (secession) of this region from the state and the creation of a new independent state (separatists, secessionists).

In some conflicting regions of the world, three groups of the population live together with different political priorities. One part of the population (unionists) is politically oriented towards preserving the region as part of the state, the second part of the region supports the idea of separatism, separating the region from the state in order to form a new independent state (separatists), and the third part of the region supports the idea of irredentism of the region, that is, the separation of the region in order to join another, possibly bordering state (irredentists). Therefore, the description of this socio-political process by the corresponding adequate mathematical model is of particular interest.

1. Research of a two-dimensional dynamic system describing the process of possible secession of the region

Consider a new mathematical model describing the process of the possibility (instability of the state) or the impossibility (stability of the state) of separating (secession) the region from the state, which is described by the following nonlinear two-dimensional dynamic system

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha_1(t)u(t) + (\beta_1(t) - \beta_2(t))u(t)v(t) - \gamma_1(t)u(t) + \gamma_2(t)v(t) \\ \dfrac{dv(t)}{dt} = \alpha_2(t)v(t) + (\beta_2(t) - \beta_1(t))u(t)v(t) + \gamma_3(t)u(t) - \gamma_4(t)v(t) \end{cases} \tag{1.1}$$

with the initial conditions

$$u(0) = u_0, \ v(0) = v_0, \tag{1.2}$$

where
$u(t)$ is the number of supporters of the center (unionists) in the region at time t,
$v(t)$ is the number of opponents of the center (separatists) at time t,
$\alpha_1(t), \alpha_2(t)$ are demographic factors of the corresponding parts of the population of the region,
$\beta_1(t), \beta_2(t)$ are ratios of attraction of opponents into allies,
$\gamma_1(t), \gamma_3(t)$ are coefficients of influence external for the region and the state of the side on unionists, for the purpose of their attraction on the side of separatists,
$\gamma_2(t), \gamma_4(t)$ are the coefficient of influence of the federal side (the central government of the state) on the separatists, in order to attract them to the unionist side.

In the model, it is more logical (non-triviality of the model) to assume that at the initial moment of time unionists exceed separatists ($u_0 > v_0$).

We proposed two (weak and strong requirements) conditions under which secession of the region is possible, which implies the fulfillment of inequalities

$$\frac{v(t)}{u(t) + v(t)} > 0.5, \ t > t_* \tag{1.3}$$

or

$$\frac{v(t)}{u(t) + v(t)} > \frac{2}{3}, \ t > t_{**}. \tag{1.4}$$

A weak condition (1.3) implies that more than half of the population of the region supports the idea of separatism, and a strong condition (1.4) - more than two thirds of the population of the region (a qualified majority) supports the idea of secession of the region and the creation of a new independent state.

Consider the first special case problem (1.1), (1.2)

$$\alpha_1(t) - \gamma_1(t) + \gamma_3(t) = \alpha_2(t) + \gamma_2(t) - \gamma_4(t) \equiv b(t). \tag{1.5}$$

Then, taking into account (1.5), it is possible to obtain the first integral of a nonlinear dynamic system (1.1), (1.2)

$$u(t) + v(t) = (u_0 + v_0)e^{\int_0^t b(\tau)d\tau}. \tag{1.6}$$

We define the function $v(t)$ from (1.6) and substitute it into the first equation of system (1.1)

$$\begin{aligned} \dot{u}(t) = [\alpha_1(t) - \gamma_1(t) - \gamma_2(t) + (\beta_1(t) - \beta_2(t))(u_0 + v_0)e^{\int_0^t b(\tau)d\tau}]u(t) \\ -(\beta(t) - \beta(t))u^2(t) + \gamma_2(t)(u_0 + v_0)e^{\int_0^t b(\tau)d\tau}. \end{aligned} \tag{1.7}$$

The solution of the Riccati equation (1.7) is sought in the following form

$$u(t) = w(t) + p(t)e^{\int_0^t b(\tau)d\tau} + q(t). \tag{1.8}$$

Then, according to (1.6), (1.7) we get

$$\begin{aligned} \dot{w} + \dot{p}(t)e^{\int_0^t b(\tau)d\tau} + p(t)b(t)e^{\int_0^t b(\tau)d\tau} + \dot{q}(t) \\ = [\alpha_1(t) - \gamma_1(t) - \gamma_2(t) + (\beta_1(t) - \beta_2(t))(u_0 + v_0)e^{\int_0^t b(\tau)d\tau}] \\ \times[w(t) + p(t)e^{\int_0^t b(\tau)d\tau} + q(t)] \\ -(\beta_1(t) - \beta_2(t))[w(t) + p(t)e^{\int_0^t b(\tau)d\tau} + q(t)]^2 + \gamma_2(t)(u_0 + v_0)e^{\int_0^t b(\tau)d\tau}, \end{aligned} \tag{1.9}$$

where the function $w(t)$ is a solution to the Bernoulli equation

$$\begin{aligned} \dot{w} = [\alpha_1(t) - \gamma_1(t) - \gamma_2(t) + (\beta_1(t) - \beta_2(t))(u_0 + v_0)e^{\int_0^t b(\tau)d\tau} \\ -2(\beta_1(t) - \beta_2(t))(p(t)e^{\int_0^t b(\tau)d\tau} + q(t))]w(t) - (\beta_1(t) - \beta_2(t))w^2 \end{aligned} \tag{1.10}$$

and functions $p(t), q(t)$ must meet the following conditions

$$p(t) = p = u_0 + v_0 = const, \tag{1.11}$$

$$\begin{cases} q(t) = -\dfrac{\gamma_3(t)}{\beta_1(t) - \beta_2(t)} \\ \dot{q}(t) = (\alpha_1(t) - \gamma_1(t) - \gamma_2(t))q(t) - (\beta_1(t) - \beta_2(t))q^2(t) \end{cases}. \tag{1.12}$$

From system (1.12) we get the following differential equation

$$\dot{q}(t) = (\alpha_1(t) - \gamma_1(t) - \gamma_2(t) + \gamma_3(t))q(t)$$

$$= (b(t) - \gamma_2(t))q(t) = (\alpha_2(t) - \gamma_4(t))q(t)$$

the solution of which has the form

$$q(t) = q(0)e^{\int_0^t (\alpha_2(\tau)-\gamma_4(\tau))d\tau}. \tag{1.13}$$

Substituting (1.11), (1.13) in (1.8), we get

$$u(t) = w(t) + (u_0 + v_0)e^{\int_0^t b(\tau)d\tau} + q(0)e^{\int_0^t (\alpha_2(\tau)-\gamma_4(\tau))d\tau}. \tag{1.14}$$

Let us introduce the notation

$$\begin{aligned} c(t) \equiv [\alpha_1(t) - \gamma_1(t) - \gamma_2(t) + (\beta_1(t) - \beta_2(t))(u_0 + v_0)e^{\int_0^t b(\tau)d\tau} \\ -2(\beta_1(t) - \beta_2(t))e^{\int_0^t (\alpha_2(\tau)-\gamma_4(\tau))d\tau}]. \end{aligned} \tag{1.15}$$

Then for the function $w(t)$ we get the following Cauchy problem

$$\dot{w} = c(t)w(t) - (\beta_1(t) - \beta(t))w^2(t), \tag{1.16}$$

$$w_0 = -v_0 - q(0),\ q(0) = -\frac{\gamma_3(0)}{\beta_1(0) - \beta_2(0)}$$

the solution of which has the form

$$w(t) = e^{\int_0^t c(\tau)d\tau}[\frac{1}{w_0} + \int_0^t (\beta_1(t) - \beta_2(t))e^{\int_0^t c(\tau)d\tau}dt]^{-1}. \tag{1.17}$$

Substituting (1.17) in (1.14) and (1.6) we finally obtain an exact solution to problem (1.1), (1.2) under assumption (1.5)

$$u(t) = w(t) + (u_0 + v_0)e^{\int_0^t b(\tau)d\tau} - \frac{\gamma_3(0)}{\beta_1(0) - \beta_2(0)}e^{\int_0^t (\alpha_2(\tau)-\gamma_4(\tau))d\tau}, \tag{1.18}$$

$$v(t) = (u_0 + v_0)e^{\int_0^t b(\tau)d\tau} - u(t) = -w(t) + \frac{\gamma_3(0)}{\beta_1(0) - \beta_2(0)}e^{\int_0^t (\alpha_2(\tau)-\gamma_4(\tau))d\tau}.$$

Consider the second special case of problem (1.1), (1.2), i.e. all the coefficients of system (1.1) are constant

$$\begin{cases} \dot{u} = \alpha_1 u + (\beta_1 - \beta_2)uv - \gamma_1 u + \gamma_2 v \\ \dot{v} = \alpha_2 v + (\beta_2 - \beta_1)uv + \gamma_3 u - \gamma_4 v \end{cases}, \tag{1.19}$$

$$\gamma_1 > 0,\ \gamma_2 > 0,\ \gamma_3 > 0,\ \gamma_4 > 0.$$

Suppose that the coefficients also satisfy the following equalities

$$a \equiv \alpha_1 - \gamma_1 + \gamma_3 = \alpha_2 + \gamma_2 - \gamma_4. \tag{1.20}$$

Then, taking into account (1.20), it is possible to obtain the first integral of a nonlinear dynamic system (1.19), (1.2)

$$u(t) + v(t) = (u_0 + v_0)e^{at}. \tag{1.21}$$

We define the function $v(t)$ from (1.21) and substitute it into the first equation of system (1.19)

$$\dot{u} + [\gamma_1 + \gamma_2 - \alpha_1 + (\beta_2 - \beta_1)(u_0 + v_0)e^{at}]u + (\beta_1 - \beta_2)u^2 = \gamma_2(u_0 + v_0)e^{at}. \tag{1.22}$$

The solution of the Riccati equation (1.22) is sought in the following form

$$u(t) = w(t) + (u_0 + v_0)e^{at} - \frac{\gamma_3}{\beta_1 - \beta_2}, \tag{1.23}$$

$$w(t) = e^{\int_0^t s(\tau)d\tau}[\frac{1}{w_0} + (\beta_1 - \beta_2)\int_0^t e^{\int_0^\tau s(\mu)d\mu}d\tau]^{-1},$$

$$S(t) \equiv \alpha_1 - \gamma_1 - \gamma_2 + (\beta_1 - \beta_2)(u_0 + v_0)e^{at} - 2(\beta_1 - \beta_2)p(e^{at} + q),$$

$$\begin{cases} p = u_0 + v_0 \\ q = \dfrac{\alpha_1 - \gamma_1 - \gamma_2}{\beta_1 - \beta_2} = \dfrac{-\gamma_3}{\beta_1 - \beta_2} \end{cases},$$

$$w_0 = \frac{\gamma_3}{\beta_1 - \beta_2} - v_0.$$

The solution (1.23) is valid under the following additional conditions for constant coefficients of the system of equations (1.19)

$$\gamma_1 + \gamma_2 - \gamma_3 = \alpha_1,\ \alpha_2 = \gamma_4. \tag{1.24}$$

Thus, the exact solution of problem (1.19), (1.2), under the conditions (1.20), (1.24), has the form

$$\begin{cases} u(t) = e^{\int_0^t s(\tau)d\tau}\left[\frac{1}{w_0} + (\beta_1 - \beta_2)\int_0^t e^{\int_0^\tau s(\mu)d\mu} d\tau\right]^{-1} \\ \qquad + (u_0 + v_0)e^{at} - \frac{\gamma_3}{\beta_1 - \beta_2}, \\ v(t) = \frac{\gamma_3}{\beta_1 - \beta_2} - e^{\int_0^t s(\tau)d\tau}[\frac{1}{w_0} + (\beta_1 - \beta_2)\int_0^t e^{\int_0^\tau s(\mu)d\mu} d\tau]^{-1}. \end{cases} \tag{1.25}$$

Consider the third special case of problem (1.1), (1.2), when there is no influence of forces external to the region (outside the state, as well as the federal center), and unionists and separatists only among themselves decide on the choice of the path of political development of the region.

In this case, in the system of equations (1.1), it is necessary to assume

$$\gamma_1(t) \equiv 0,\ \gamma_3(t) \equiv 0,\ \gamma_2(t) \equiv 0,\ \gamma_4(t) \equiv 0. \tag{1.26}$$

If (1.26) is executed, the nonlinear system of differential equations (1.1) will be rewritten as follows:

$$\begin{cases} \dfrac{du(t)}{dt} = \alpha_1(t)u(t) + (\beta_1(t) - \beta_2(t))u(t)v(t) \\ \dfrac{dv(t)}{dt} = \alpha_2(t)v(t) + (\beta_2(t) - \beta_1(t))u(t)v(t) \end{cases}. \tag{1.27}$$

Now let us consider a special case of constancy of all coefficients of the model

$$\begin{gathered} \alpha_1(t) = \alpha_1 = const,\ \alpha_2(t) = \alpha_2 = const, \\ \beta_1(t) = \beta_1 = const,\ \beta_2(t) = \beta_2 = const. \end{gathered} \tag{1.28}$$

In case (1.28), the system of equations (1.27) and initial conditions (1.2) takes the form

$$\begin{cases} \frac{du(t)}{dt} = \alpha_1 u(t) + (\beta_1 - \beta_2)u(t)v(t) \\ \frac{dv(t)}{dt} = \alpha_2 v(t) + (\beta_2 - \beta_1)u(t)v(t) \end{cases}. \tag{1.29}$$

$$u(0) = u_0,\ v(0) = v_0.$$

Consider several particular cases of the Cauchy problem (1.29).

1. $\alpha_2 = 0, \alpha_1 = 0, \beta_2 = \beta_1$.

The demographic factors of the sides are zero, and the coefficients of attracting opponents to the allies are equal among themselves. In this case, the exact solution of system (1.29) has the form

$$u(t) = u_0,\ v(t) = v_0. \tag{1.30}$$

From (1.3), (1.4), (1.30) it is easy to determine the conditions for the possibility of secession of the region in a weak

$$v_0 > u_0$$

and strong condition

$$v_0 > 2u_0$$

which is impossible due to the assumption (logic) of non-triviality of the model.

2. $\alpha_2 \neq 0, \alpha_1 \neq 0, \beta_2 = \beta_1$.

The demographic factors of the sides are unequal to zero, and the coefficients of attracting opponents to the allies are equal. In this case, the exact solution of system (1.29) has the form

$$u(t) = u_0 e^{\alpha_1 t}, \; v(t) = v_0 e^{\alpha_2 t}. \tag{1.31}$$

Then according to (1.3), (1.31) the weak condition of secession possibility has the form

$$e^{(\alpha_2 - \alpha_1)t} > \frac{u_0}{v_0} > 1, \tag{1.32}$$

a strong condition

$$e^{(\alpha_2 - \alpha_1)t} > 2\frac{u_0}{v_0} > 2. \tag{1.33}$$

From (1.32), (1.33) it follows that when the demographic factor of separatists is less than the demographic factor of unionists, inequalities (1.32), (1.33) do not have a solution and secession of the region is impossible. If the demographic factor of separatists is greater than the demographic factor of unionists, then the solutions to inequality (1.32), (1.33) have the form

$$t > t_* = \frac{\ln \frac{u_0}{v_0}}{\alpha_2 - \alpha_1}, \; t > t_{**} = \frac{\ln 2\frac{u_0}{v_0}}{\alpha_2 - \alpha_1}. \tag{1.34}$$

According to (1.34), secession of the region is possible with a weak requirement starting from the moment of time t_*, and with a strong requirement after the moment of time t_{**}.

3. $\alpha_2 > 0, \alpha_1 < 0, \beta_2 < \beta_1$.

4. $\alpha_2 < 0, \alpha_1 > 0, \beta_2 > \beta_1$.

The third and fourth cases suggest the opposite of signs as demographic factors and differences in the coefficients of attraction of opponents into allies. Moreover, in the third and fourth cases, we have the classic predator-victim model (Lotka-Volterra system of equations), in the third case, the role of predators is played by unionists, and separatists are the victim and vice versa in the fourth case, i.e. separatists are predators, and unionists are the victim.

It is easy to obtain the first integral of the system of equations (1.29)

$$\alpha_2 \ln \frac{u}{u_0} - (\beta_1 - \beta_2)(u(t) - u_0) = \alpha_1 \ln \frac{v}{v_0} + (\beta_1 - \beta_2)(v(t) - v_0), \tag{1.35}$$

which is a closed integral curve in the first quarter of the phase plane $(O, v(t), u(t))$ of the solutions of the system of equations (1.29). Analysis (1.35) in the third and fourth cases leads to the following inequalities for the desired functions $v(t), u(t)$

$$v_{min} \leqslant v(t) \leqslant v_{max}, \tag{1.36}$$

where v_{min}, v_{max} is the smallest and largest positive roots of the next transcendent equation

$$\alpha_2 \ln \frac{a}{u_0} - (\beta_1 - \beta_2)(a - u_0) = \alpha_1 \ln \frac{v}{v_0} + (\beta_1 - \beta_2)(v(t) - v_0), \tag{1.37}$$

$$a = \frac{\alpha_2}{\beta_1 - \beta_2} > 0,$$

$$u_{min} \leqslant u(t) \leqslant u_{max}, \tag{1.38}$$

where u_{min}, u_{max} the smallest and largest positive roots of the next transcendent equation

$$\alpha_2 \ln \frac{u}{u_0} - (\beta_1 - \beta_2)(u(t) - u_0) = \alpha_1 \ln \frac{b}{v_0} + (\beta_1 - \beta_2)(b - v_0), \tag{1.39}$$

$$b = \frac{-\alpha_1}{\beta_1 - \beta_2} > 0.$$

Thus, on a closed integral curve (1.35) completely located in the first quarter of the phase plane (O,v(t),u(t))of the solutions of the system of equations (1.29), it may be minimum time point $M(v(t_1), u(t_1))$, for which weak $(v(t_1) > u(t_1))$ or strong $(v(t_1) > 2u(t_1))$ conditions of region secession possibility are fulfilled. If such a point lies on an integral curve, the secession of the region is possible and impossible otherwise.

2. Research of a three-dimensional dynamic system describing the process of rivalry between three groups of citizens of the region with different political priorities

Consider a new mathematical model describing the process of confronting citizens of the state region with various political priorities (unionists, secessionists, irredentists), which is described by the following nonlinear three-dimensional dynamic system

$$
\begin{cases}
\frac{du(t)}{dt} = \alpha_1(t)u(t) + (\beta_1(t) - \beta_2(t))u(t)v(t) + (\beta_1(t) - \beta_3(t))u(t)w(t) \\
\qquad\qquad -\gamma_5(t)u(t) - \gamma_3(t)u(t) + \gamma_1(t)v(t) + \gamma_2(t)w(t) \\
\frac{dv(t)}{dt} = \alpha_2(t)v(t) + (\beta_2(t) - \beta_1(t))u(t)v(t) + (\beta_2(t) - \beta_3(t))v(t)w(t) \\
\qquad\qquad +\gamma_3(t)u(t) - \gamma_1(t)v(t) - \gamma_6(t)v(t) + \gamma_4(t)w(t) \\
\frac{dw(t)}{dt} = \alpha_3(t)w(t) + (\beta_3(t) - \beta_1(t))u(t)w(t) + (\beta_3(t) - \beta_2(t))v(t)w(t) \\
\qquad\qquad +\gamma_5(t)u(t) + \gamma_6(t)v(t) - \gamma_2(t)w(t) - \gamma_4(t)w(t)
\end{cases} \tag{2.1}
$$

with the initial conditions

$$
u(0) = u_0,\ v(0) = v_0,\ w(0) = w_0, \tag{2.2}
$$

where
$u(t)$ is the number of supporters of the center (unionists) in the region at time t,
$v(t)$ is the number of supporters of separation from the center in order to create a new independent state (separatists) at the moment t,
$w(t)$ is the number of supporters of separation from the center in order to join another state (irredentists) at a time t,
$\alpha_1(t), \alpha_2(t), \alpha_3(t)$ are demographic factors of the corresponding parts of the population of the region,
$\beta_1(t), \beta_2(t), \beta_3(t)$ are factors of influence on opponents, in order to attract them to their side (unionism, separatism, irredentism),
$\gamma_1(t), \gamma_2(t)$ are factors of influence of the federal side (the central government of the state) on separatists and irredentists, respectively, in order to attract them to the unionist side,
$\gamma_3(t), \gamma_4(t)$ are factors of influence of external (contributing to separatism) and internal (de facto government) forces on unionists and irredentists, respectively, in order to attract them to the separatist side,
$\gamma_5(t), \gamma_6(t)$ are factors of influence of external interested forces (other state) on unionists and separatists, respectively, in order to attract them to irredentism (reunification with another state),
$[0, T]$ is time interval, model review.

In the model, it is more logical (adequacy of the model) to assume that at the initial moment of time unionists outnumber the total number of separatists and irredentists

$$
u_0 > v_0 + w_0. \tag{2.3}
$$

We will consider the weak and strong conditions under which separation of the region is possible, with the aim of creating an independent state (separation of the region) or joining another state (irredentism), which implies the fulfillment of the following inequalities

$$
\frac{v(t)}{u(t) + v(t) + w(t)} > \frac{1}{2},\ t > t_1, \tag{2.4}
$$

$$\frac{v(t)}{u(t)+v(t)+w(t)} > \frac{2}{3},\ t > t_2, \tag{2.5}$$

$$\frac{w(t)}{u(t)+v(t)+w(t)} > \frac{1}{2},\ t > t_3, \tag{2.6}$$

$$\frac{w(t)}{u(t)+v(t)+w(t)} > \frac{2}{3},\ t > t_4, \tag{2.7}$$

Weak conditions (2.4), (2.6) imply that more than half of the population of the region supports separatism or irredentism, respectively, and strong conditions (2.5), (2.7) - more than two thirds of the population of the region (a qualified majority) supports the idea of separating the region and creating a new independent state or reunification with another state.

If none of the inequalities (2.4)-(2.7), taking into account (2.3), then the separation and irredentism of the region is impossible and the conflict region remains part of the previous state.

Consider a special case where there is no influence of forces external to the region (outside the state, as well as the federal center), and unionists, separatists and irredentists only decide among themselves on the choice of the path of political development of the region.

In this case, in the system of equations (1), it is necessary to assume

$$\gamma_i(t) \equiv 0, i = \overline{1,6}. \tag{2.8}$$

Suppose also that the demographic factors of the three parts of the population of the region are zero

$$\alpha_j(t) \equiv 0, j = \overline{1,3}. \tag{2.9}$$

The nonlinear system of differential equations (2.1) (nonlinear three-dimensional dynamic system), taking

$$\begin{cases} \frac{du(t)}{dt} = (\beta_1(t)-\beta_2(t))u(t)v(t) + (\beta_1(t)-\beta_3(t))u(t)w(t) \\ \frac{dv(t)}{dt} = (\beta_2(t)-\beta_1(t))u(t)v(t) + (\beta_2(t)-\beta_3(t))v(t)w(t) \\ \frac{dw(t)}{dt} = (\beta_3(t)-\beta_1(t))u(t)w(t) + (\beta_3(t)-\beta_2(t))v(t)w(t) \end{cases} \tag{2.10}$$

From (2.10), (2.2), we get the first integral of a three-dimensional dynamic system

$$u(t) + v(t) + w(t) = u_0 + v_0 + w_0 \equiv p. \tag{2.11}$$

Consider a special case

$$\beta_1(t) \equiv \beta_2(t) \neq \beta_3(t),\ t \in [0,T]. \tag{2.12}$$

Considering (2.12), the second first integral (2.10), (2.2) has the following form:

$$v(t) = qu(t), \ q = \frac{v_0}{u_0}. \tag{2.13}$$

The first two integrals (2.11), (2.13) of the dynamic system (2.10), (2.2), allow us to find its exact analytical solution

$$\begin{cases} u(t) = \dfrac{pu_0 e^{p\int_0^t (\beta_1(\tau)-\beta_3(\tau))d\tau}}{w_0 + (u_0+v_0)e^{p\int_0^t (\beta_1(\tau)-\beta_3(\tau))d\tau}} \\ v(t) = qu(t) \\ w(t) = p-(q+1)u(t) \end{cases}. \tag{2.14}$$

Consider the second special case

$$\beta_1(t) \equiv \beta_3(t) \neq \beta_2(t), \ t \in [0,T]. \tag{2.15}$$

Considering (2.15), the second first integral (2.10), (2.2) has the following form:

$$v(t) = q_1 w(t), \ q_1 = \frac{w_0}{u_0}. \tag{2.16}$$

The first two integrals (2.11), (2.16) of the dynamic system (2.10), (2.2), allow us to find its exact analytical solution

$$\begin{cases} u(t) = \dfrac{pu_0 e^{p\int_0^t (\beta_1(\tau)-\beta_2(\tau))d\tau}}{v_0 + (u_0+w_0)e^{p\int_0^t (\beta_1(\tau)-\beta_2(\tau))d\tau}} \\ v(t) = q_1 w(t) \\ v(t) = p-(q_1+1)u(t) \end{cases}. \tag{2.17}$$

Analysis of the obtained exact analytical solution of the Cauchy problem (2.10), (2.2) for a nonlinear three-dimensional dynamic system, under the natural assumption (2.3) (unionists prevail in the region at the initial moment of time) shows that:

1. In case of execution of the inequality system

$$\begin{cases} \beta_1(t) \geqslant \beta_2(t) \\ \beta_1(t) \geqslant \beta_3(t) \end{cases}, \ t \in [0,T] \tag{2.18}$$

separation or irredentism of the region is impossible and the region in the legal sense remains within the former state.

2. In case of execution of the system

$$\begin{cases} \beta_1(t) \equiv \beta_2(t) \\ \beta_3(t) > \beta_1(t) \end{cases}, \ t \in [0,T] \tag{2.19}$$

according to (2.14), regional irredentism is possible (fulfillment of condition (2.6) or (2.7)), where in time t_3 or t_4 is determined from the integral relations

$$\int_0^{t_3} (\beta_3(\tau) - \beta_1(\tau))d(\tau) = \frac{\ln \frac{u_0+v_0}{w_0}}{p},$$
$$\int_0^{t_4} (\beta_3(\tau) - \beta_1(\tau))d(\tau) = \frac{\ln \frac{2(u_0+v_0)}{w_0}}{p}. \quad (2.20)$$

3. In case of execution of the system

$$\begin{cases} \beta_1(t) \equiv \beta_3(t) \\ \beta_2(t) > \beta_1(t) \end{cases}, \ t \in [0, T] \quad (2.21)$$

according to (2.17), it is possible to separate the region (condition (2.4) or (2.5)), wherein time t_1 or t_2 is determined from the integral relations

$$\int_0^{t_1} (\beta_2(\tau) - \beta_1(\tau))d(\tau) = \frac{\ln \frac{u_0+w_0}{v_0}}{p},$$
$$\int_0^{t_2} (\beta_2(\tau) - \beta_1(\tau))d(\tau) = \frac{\ln \frac{2(u_0+w_0)}{v_0}}{p}. \quad (2.22)$$

3. Conclusion

We would like to note that the proposed mathematical model (2.1), (2.2) is common and with variable coefficients of a dynamic system can well describe many conflict regions existing in the world. At the same time, specific conflicts have their own specific sides, characterized by the historical past, the character and mentality of the politically opposing sides (peoples), the geopolitical location and economic potential of the region, the interest of the bordering states, etc., which can be taken into account by the variable parameters of the model.

Naturally, with variable coefficients of the mathematical model (2.1), (2.2), its analytical solution is impossible, so it is necessary to use computer modeling, using tested computer computing programs.

References

[1] A. Golubyatnikov, T. Chilachava. *Estimates of the motion of detonation waves in a gravitating gas*, Fluid Dynamics, **19**, 2 (1984), 292-296

[2] T. Chilachava. *Problem of a strong detonation in a uniformly compressing gravitating gas*, Moscow State University, Bulletin, Series Mathematics and Mechanics, **1** (1985), 78-83

[3] A. Golubyatnikov, T. Chilachava. *Propagation of a detonation wave in a gravitating sphere with subsequent dispersion into a vacuum*, Fluid Dynamics, **21**, 4 (1986), 673-677

[4] T. Chilachava. *A central explosion in an inhomogeneous sphere in equilibrium in its own gravitational field*, Fluid Dynamics, **23**, 3 (1984), 472-477

[5] T. Chilachava, N. Kereselidze. *Optimizing Problem of Mathematical Model of Preventive Information Warfare, Information and Computer Technologies*, Theory and Practice: Proceedings of the International Scientific Conference ICTMC-2010 Devoted to the 80th Anniversary of I.V. Prangishvili, (2012), 525-529

[6] T. Chilachava, L. Karalashvili, N. Kereselidze. *Integrated Models of Non-Permanent Information Warfare*, International Journal on Advanced Science, Engineering, Information and Technology (IJASEIT), **10**, 6 (2020), 2222-2230

[7] T. Chilachava. *Research of The Dynamic System Describing Globalization Process*, Springer Proceedings in Mathematics & Statistics, Mathematics, Informatics and their Applications in Natural Sciences and Engineering, **276** (2019), 67-78

[8] T. Chilachava, G. Pochkhua. *Research of a three-dimensional dynamic system describing the process of assimilation*, Lecture Notes of TICMI, **22** (2021), 63-72

[9] T. Chilachava, G. Pochkhua. *Mathematical and Computer Models of Settlements of Political Conflicts and Problems of Optimization of Resources*, International Journal of Modeling and Optimization, **4** (2020), 132-138

[10] T. Chilachava, G. Pochkhua. *Conflict resolution models and resource minimization problems*, Applications of Mathematics and Informatics in Natural Sciences and Engineering. Springer Proceedings in Mathematics & Statistics, **334** (2020), 47-59