Bulletin of TICMI
Vol. 24, No. 1, 2020, 3–11

On some Properties of Euler Integrals

Grigori Giorgadze[a,b*] and Vagner Jikia [a]

[a] *Iv. Javakhishvili Tbilisi State University, 2 University St., 0186, Tbilisi, Georgia*;
[b] *I. Vekua Institute of Applied Mathematics of Tbilisi State University*

(*Received December 18, 2019; Revised February 4, 2020; Accepted February 13, 2020*)

It is shown that Euler integrals of the first and second kind are expressed by the Dirac delta function in the domain of their singularity. Analytical extension of Euler integrals are considered as distributions on main functional space and some calculations in spirit of generalized functions in complex domain are given.

Keywords: Special functions, generalized functions, singular integrals

AMS Subject Classification: 33C45, 30G99, 32A55

1. Introduction and motivation

The main object of study in this paper is the investigation of Betta-function [1]

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \quad \Re\alpha>0, \Re\beta>0.$$

in the singular points. The following relation is known

$$B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \quad \Re\alpha>0,\ \Re\beta>0, \tag{1}$$

where

$$\Gamma(z)=\int_0^\infty t^{z-1}\exp(-t)dt, \quad \Re z>0. \tag{2}$$

The integral

$$\int_0^\infty t^{z-1}\exp(-t)dt,$$

is an analytic function of z when $\Re z>0$ and is called an Euler integral of the second kind. The many-valued function t^{z-1} is made precise by the identity $t^{z-1}=e^{(z-1)\lg t}$, where $\lg t$ is purely real. If $\Re z\le 0$ then the integral does not converge because of the singularity of the integrand at $t=0$.

*Corresponding author. Email: gia.giorgadze@tsu.ge

ISSN: 1512-0082 print

For explicitness we remark here that the Gamma-function in the expression (2) by the definition is the following infinite product

$$\Gamma(z) = \frac{1}{z}\prod_{1}^{\infty}\left(\left(1+\frac{1}{n}\right)^{z}\left(1+\frac{z}{n}\right)^{-1}\right) \tag{3}$$

and therefore represents an analytic function with the exception of points of the complex plane: $z_k = -k\,(k = 0,\,1,\,2,\,\ldots)$. The points z_k are simple poles for (3) with residues $\frac{(-1)^k}{k!}$.

It is possible to extend analyticly the right hand-side of (2) on the domain $\Re z \leq 0$ in the following manner (see [14]). Let D be a contour which starts from a point ρ on the real axis, encircles the origin once contour-clockwise and returns to ρ. Consider the contour integral

$$\int_D (-t)^{z-1}\exp(-t)dt,$$

when $\Re z > 0$ and z is not an integer. The many-valued function $(-t)^{z-1}$ is to be made definite as above and when t is on the negative part of the real axis, so that, on D, $-\pi \leq \arg(-t) \leq \pi$. The integrand is not analytic inside D, but the path of integration may be deformed (without affecting the value of the integral) into the path of integration which starts from ρ, proceeds along the real axis to δ, describes a circle of radius δ counter-clockwise round the origin and returns to ρ along the real axis.

Let $-t = \delta e^{i\theta}$ on the circle. Then for all positive ρ we have the identity

$$\int_D (-t)^{z-1}\exp(-t)dt = -2i\sin\pi z\int_0^{\rho} t^{z-1}\exp(-t)dt,$$

when $\delta \to 0$. Make $\rho \to \infty$ and let C be the limit of contour D. Then

$$\int_D (-t)^{z-1}\exp(-t)dt = -2i\sin\pi z\int_0^{\infty} t^{z-1}\exp(-t)dt$$

and therefore for all values of z except of the integer values of $z = 0, \pm 1, \pm 2, ...$, we have

$$\Gamma(z) = -\frac{1}{2i\sin\pi z}\int_C (-t)^{z-1}\exp(-t)dt. \tag{4}$$

The Gamma-function satisfies the following functional equation (see [2], pp.544-553)

$$\Gamma\left(z+1\right) = z\Gamma\left(z\right)$$

and does not satisfy any differential equation with rational coefficients.

Similarly, it is possible to replace the first kind Euler integral

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt$$

by a contour integral (Pochhammer's contour) which converges for all values of α, β and we obtain the analytic function as a contour integral

$$B(\alpha, \beta) = \int_C t^{\alpha-1}(1-t)^{\beta-1}dt, \tag{5}$$

where C here is Pochhamer's contour. Therefore, identity (1) is true for all values α, β with the exception of $\alpha = -n$ or $\beta = -n$, where $n = 0, 1, 2, ...$ and the integral (5) needs additional investigation in these points.

Below we study the Beta-function in the neighborhood of the exceptional points and give explicit formulas for the values of the function in these points.

2. Main results.

In this section we give an interpretation of Beta-functions in singular points as a generalized function (see [11]), in particular, the following theorems are valid.

Theorem 2.1: *The Beta-function $B\left(a + i\gamma,\, -a - i\gamma\right)$, which is defined by the Euler integral of the first kind*

$$B\left(a + i\gamma,\, -a - i\gamma\right) = \int\limits_0^1 t^{a+i\gamma-1}(1-t)^{-a-i\gamma-1}dt, \quad \gamma,\, a \in \mathbb{R}, \tag{6}$$

can be expressed in terms of the Dirac delta function as follows:

$$B\left(a + i\gamma,\, -a - i\gamma\right) = 2\pi \exp\left(-ia\frac{d}{d\gamma}\right)\delta\left(\gamma\right), \quad \gamma,\, a \in \mathbb{R}. \tag{7}$$

Proof: Using the following standard substitutions

$$\left(\frac{t}{1-t}\right)^{a+i\gamma} = \exp\left(a + i\gamma\right)\xi, \quad \xi = \ln\frac{t}{1-t}, \quad d\xi = \frac{dt}{t(1-t)}. \tag{8}$$

From formula (6) one obtains:

$$B\left(a + i\gamma,\, -a - i\gamma\right) = F\left[\exp\left(a\xi\right)\right]\left(\gamma\right), \quad \gamma,\, a \in \mathbb{R}, \tag{9}$$

where $F\left[\exp\left(a\xi\right)\right]\left(\gamma\right) \equiv f\left(\gamma - ia\right))$ denotes the Fourier transform of the function $\exp\left(a\xi\right)$, which has the following integral representation:

$$f\left(\gamma - ia\right) = \int\limits_{-\infty}^{+\infty} \exp i\left(\gamma - ia\right)\xi d\xi. \tag{10}$$

Calculation of the integral (10) gives (see e.,g., [10], p. 213):

$$f\left(\gamma - ia\right) = 2\pi u\delta\left(\gamma\right), \quad u = \exp\left(-ia\frac{d}{d\gamma}\right). \tag{11}$$

By formulas (9) and (11) one gets:

$$f\left(\gamma - ia\right) \equiv B\left(a + i\gamma,\, -a - i\gamma\right) = 2\pi \exp\left(-ia\frac{d}{d\gamma}\right)\delta\left(\gamma\right), \quad \gamma,\, a \in \mathbb{R}. \tag{12}$$

(12) implies the proof. □

Theorem 2.2: *The function, which is defined by the identities (6), (7), has the following analytic representation:*

$$B\left(a + iz,\, -a - iz\right) = -\frac{1}{i}\frac{1}{z - ia}, \quad z = x + iy, \quad z \neq ia, \quad a \in \mathbb{R}. \tag{13}$$

Proof: Using (12) and taking into account the next formula (see e., g., [10], p. 44):

$$\int\limits_{-\infty}^{+\infty} \varphi\left(\gamma\right)\delta^{(n)}\left(\gamma\right)d\gamma = (-1)^n \varphi^{(n)}\left(0\right),$$

one can show that

$$\int\limits_{-\infty}^{+\infty} \varphi\left(\gamma\right) f\left(\gamma - ia\right)d\gamma = 2\pi\varphi\left(ia\right), \quad a \in \mathbb{R} \tag{14}$$

for any function $\varphi\left(\gamma\right)$ of the space of the main functions (see e., g., [11], pp. 15 and 27).

Because u in (11) is a unitary operator (see e., g., [12], p. 54), by virtue of the following relations (see e., g., [10], p. 22)

$$\left(f\left(\gamma - ia\right), \varphi\right) = 2\pi\left(u\delta, \varphi\right) = 2\pi\left(\delta, \bar{u}\varphi\right) = 2\pi\left(\delta, \varphi\left(\gamma + ia\right)\right),$$

one can write (see e., g., [10], p. 200):

$$\int\limits_{-\infty}^{+\infty} \varphi\left(\gamma\right) B\left(a + i\gamma,\, -a - i\gamma\right) d\gamma = 2\pi \int\limits_{ia-\infty}^{ia+\infty} \varphi\left(\eta\right)\delta\left(\eta - ia\right) d\eta = 2\pi\varphi\left(ia\right). \tag{15}$$

Taking into account the following conditions of completenes:

$$\left|\gamma\right\rangle d\gamma \left\langle\gamma\right| = 1, \quad \left|\eta\right\rangle d\eta \left\langle\eta\right| = 1, \quad \left|\eta\right\rangle = \left|u\gamma\right\rangle$$

from expressions (15) one obtains:

$$B\left(a + i\eta,\, -a - i\eta\right) = 2\pi\delta\left(\eta - ia\right), \quad \eta = \gamma + ia, \quad a \in \mathbb{R}. \tag{16}$$

It is known that the Dirac delta function has the following analytical representation (see e., g., [10], pp. 200 and 201):

$$B\left(a+iz,\,-a-iz\right)=2\pi\delta\left(z-ia\right)=-\frac{1}{i}\frac{1}{z-ia},\quad z=x+iy,\quad z\neq ia,\quad a\in\mathbb{R}. \quad (17)$$

Equalities (17) show that the theorem under consideration is true. □

Corollary 2.3: *Replace a by $-a$, from the expressions (9) one obtains:*

$$B\left(-a+i\gamma,\,a-i\gamma\right)=2\pi\exp\left(ia\frac{d}{d\gamma}\right)\delta\left(\gamma\right),\quad \gamma,\,a\in\mathbb{R}. \quad (18)$$

Indeed, from analytical representation (13) we obtain:

$$B\left(-a+iz,\,a-iz\right)=2\pi\delta\left(z+ia\right)=-\frac{1}{i}\frac{1}{z+ia},\quad z=x+iy,\quad z\neq -ia,\quad a\in\mathbb{R}.$$

Proposition 2.4: *For non-negative integers m and n we have*

$$B\left(i\gamma-m,\,-i\gamma-n\right)=2\pi\exp\left(im\frac{d}{d\gamma}\right)\left(1+\exp\left(-i\frac{d}{d\gamma}\right)\right)^{m+n}\delta\left(\gamma\right) \quad (19)$$

and

$$B\left(iz-m,\,-iz-n\right)=-\frac{1}{i}\sum_{k=0}^{m+n}\frac{A_{m+n}^{k}}{z-i\left(k-m\right)}. \quad (20)$$

Proof: First we prove the identity (19). Due to the substitutions (8) the representation

$$B\left(i\gamma-m,\,-i\gamma-n\right)=\int_{0}^{1}t^{i\gamma-m-1}(1-t)^{i\gamma-n-1}dt$$

can be rewritten as follows:

$$B\left(i\gamma-m,\,-i\gamma-n\right)=\int_{-\infty}^{+\infty}\exp\left(i\gamma\xi\right)\exp\left(-m\xi\right)\left(1+\exp\left(\xi\right)\right)^{n+m}d\xi= \quad (21)$$

$$\sum_{k=0}^{m+n}A_{m+n}^{k}\int_{-\infty}^{+\infty}\exp\left(i\gamma\xi\right)\exp\left(\left(k-m\right)\xi\right)d\xi,$$

where A_{m+n}^{k} is the binomial coefficient of the decomposition.

Similarly to formula (10) the integral

$$\int_{-\infty}^{+\infty} \exp\left(i\gamma\xi\right) \exp\left((k-m)\xi\right) d\xi$$

from series (21) can be represented as follows:

$$\int_{-\infty}^{+\infty} \exp\left(i\gamma\xi\right) \exp\left((k-m)\xi\right) d\xi = 2\pi \exp\left(-i\left(k-m\right)\frac{d}{d\gamma}\right)\delta\left(\gamma\right). \tag{22}$$

Obviously, due to formulas (21) and (22) one can write:

$$B\left(i\gamma-m,\, -i\gamma-n\right) = 2\pi \sum_{k=0}^{m+n} A_{m+n}^{k} \exp\left(-i\left(k-m\right)\frac{d}{d\gamma}\right)\delta\left(\gamma\right). \tag{23}$$

Summation of the series (23) gives the expression (19).
For the proof of expression (20) we use formulas (12) and (23). We have:

$$B\left(i\gamma-m,\, -i\gamma-n\right) = \sum_{k=0}^{m+n} A_{m+n}^{k} B\left(k-m+i\gamma,\, -\left(k-m\right)-i\gamma\right). \tag{24}$$

It is clear that the function (24) has the following representation:

$$B\left(iz-m,\, -iz-n\right) = \sum_{k=m+1}^{m+n} A_{m+n}^{k} B\left(k-m+iz,\, -\left(k-m\right)-iz\right). \tag{25}$$

Using formula (12) for representations (25) we obtain:

$$B\left(k-m+iz,\, -(k-m)-iz\right) = -\frac{1}{i}\frac{1}{z-i(k-m)}. \tag{26}$$

Substitution of (26) in (25) yields (20). □

3. Some generalizations.

Recently in [3]-[4] the authors introduced the following extension of Gamma and Beta-functions in the following manner(see also [5]-[9]):

$$\Gamma_p(z) = \int_0^{\infty} t^{z-1} \exp\left(-t-p/t\right) dt, \quad \Re z > 0, \quad \Re p > 0 \tag{27}$$

and for $\Re\alpha > 0, \Re\beta > 0, \Re p > 0$

$$B(\alpha,\, \beta; p) = \int\limits_0^1 t^{\alpha-1}(1-t)^{\beta-1} \exp\left(-p/t\,(1-t)\right) dt. \tag{28}$$

In the framework of the previous section we have the following results.

Theorem 3.1 : *For the function $B\,(-a+i\gamma,\, a-i\gamma; p\,)$, defined by the expression*

$$B\,(-a+i\gamma,\, a-i\gamma; p\,) = \int\limits_0^1 t^{-a+i\gamma-1}(1-t)^{a-i\gamma-1} \exp\left(-p \times t/(1-t)\right) dt, \tag{29}$$

when $a \geq 0, \quad p \geq 0$ we have

$$B\,(-a+i\gamma,\, a-i\gamma; p\,) = 2\pi \exp\left(ia\frac{d}{d\gamma}\right) \times \exp\left(-p\exp\left(-i\frac{d}{d\gamma}\right)\right) \delta\,(\gamma) \tag{30}$$

and

$$B\,(-a+i\gamma,\, a-i\gamma; p\,) = \int\limits_0^\infty u^{z-1} \exp\left(-pu\right) du, \quad z = -a+i\gamma. \tag{31}$$

Proof : For the proof of (30) rewrite this expression as follows:

$$\begin{aligned} B\,(-a+i\gamma,\, a-i\gamma; p\,) &= \lim_{\varepsilon\to 0} \int\limits_{\varepsilon}^{1-\varepsilon} t^{-a+i\gamma-1}(1-t)^{a-i\gamma-1} dt \\ &\times \exp\left(-p \times t/(1-t)\right), \quad a \geq 0, \quad p \geq 0. \end{aligned} \tag{32}$$

After the decomposition of the exponential function in the integrand (32), by virtue of the consequence of the theorem 1* (see e., g., [13], p. 669), one can write:

$$B\,(-a+i\gamma,\, a-i\gamma; p\,) = \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} B\,(k-a+i\gamma,\; -(k-a)-i\gamma\,). \tag{33}$$

By the formulas (33) and (12) one obtains:

$$B\,(-a+i\gamma,\, a-i\gamma; p\,) = 2\pi \exp\left(ia\frac{d}{d\gamma}\right) \times \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} \exp\left(-ik\frac{d}{d\gamma}\right) \delta\,(\gamma)\,. \tag{34}$$

Summation of the series (34) gives (30).

Using the substitution $t = \dfrac{u}{1+u}$ in the representation (29), we obtain the expression (31). □

Corollary 3.2: *In the particular case when $p = 1$, the expression (31) reduces to* *the Euler integral of the second kind*

$$G(z) = \int_0^\infty u^{z-1} \exp(-u)\, du, \quad z = -a + i\gamma, \quad a \geq 0, \tag{35}$$

which is also integrable in the domain $\Re z \leq 0$ in terms of the Dirac function.

Indeed, by formulas (35) and (31) one can write:

$$G(-a + i\gamma) \equiv B(-a + i\gamma,\, a - i\gamma; 1), \quad a \geq 0, \tag{36}$$

where, according to expressions (36) and (30)

$$G(-a + i\gamma) = 2\pi \exp\left(ia\frac{d}{d\gamma}\right) \times \exp\left(-\exp\left(-i\frac{d}{d\gamma}\right)\right) \delta(\gamma), \quad a \geq 0.$$

By the equalities (6), (29) and (31) we obtain the following expression:

$$B(-a + i\gamma,\, a - i\gamma) = B(-a + i\gamma,\, a - i\gamma; 0) = \int_0^\infty u^{-a+i\gamma-1} du, \quad a \geq 0. \tag{37}$$

In addition, according to formulas (18) and (37) one can write:

$$\int_0^\infty u^{-a+i\gamma-1} = 2\pi \exp\left(ia\frac{d}{d\gamma}\right) \delta(\gamma)\, du, \quad a \geq 0.$$

Acknowledgements.

This work was partially supported by Shota Rustaveli National Science Foundation grant FR17_354.

References

[1] V. Ya. Arsenin, Methods of Mathematical Physics and Special Functions. Nauka, Moscow (Russian), 1984.

[2] M. Lavrentiev and B. Shabat, Methods of the theory of functions of a complex variable. Nauka, Moscow (in Russian), 1973.

[3] M. Aslam Chaudhry and S. M. Zubair, Generalized incomplete gamma functions with applications, J. Comput. Appl. Math. 55, 99-124, 1994.

[4] M. Aslam Chaudhry, A. Qadir, M. Rafique, S. M. Zubair, Extension of Eulr's beta Function, Journal of Computational and Applied Mathematics, 78, 19-32, 1997.

[5] E. Özergin, M. Ali Özarslan, A. Altin, Extension of gamma, beta and hypergeometric functions, Journal of Computational and Applied Mathematics, 235, 4601-4610, 2011.

[6] R. K. Parmar, A new Generalization of Gamma, Beta, Hypergeometric and Confluent Hypergeometric Functions, Le Matematiche Vol. LXVIII. Fasc. II, pp. 33–52, 2013

[7] K.Rakesh, Parmar and Purnima Chopra, Generalization of Incomplete Extended Beta Function and Beta Distribution, International Journal of Engineering Research and Development, Volume 2, Issue 4, pp. 58-62, 2012.

[8] I. Lomidze, N. Makhaldiani, Some Properties of the Generalized Euler Beta Function. Memoirs on Differential Equations and Mathematical Physics. Vol. 56, pp. 57-71, 2012.

[9] V. Sh. Jikia. The Euler Beta Function of imaginary parameters and it's connection with the Dirac Function. Reports of Enlarged Sessions of the Seminar of I. Vekua Institute of Applied Mathematics. Tbilisi University Press, V. 30, p. 35, 2016.
[10] I. M. Gelfand and G. E. Shilov. Generalized Functions. Properties and Operations. Volume 1. *(Russian)*, Moscow, 1959.
[11] H. Bremmerman. Distributions, Complex Variables and Fourier Transforms. Addison-Wesley Publishing Company, Reading, Massachusetts, 1965.
[12] J. P. Elliot and P. G. Dawber. Symmetry in Physics. Volume 1, *(Russian)*, Moscow, 1979.
[13] G. M. Fihtenhol'tz. Course of Differential and Integral Calculus, vol. II, *(Russian)*, Moscow, 1969.
[14] A.Kratzer and W. Franz, Transzendente Functionen, Leipzig, 1960.