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Stability Parameters for Holomorphic Triples

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We describe the chamber structure of the set of stability parameters for holomorphic triples on a connected smooth projective curve over the field of complex numbers.

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1. Introduction

Quiver representations are fundamental objects of algebra [10]. Some of the basic results have even inspired progress in the realm of persistence modules and the analysis of big data [9]. Other applications include moduli spaces of dynamical systems ([6], [4]). Quiver representations have been generalized to quiver bundles [1]. Prominent examples are holomorphic triples or, more generally, holomorphic chains over connected smooth projective curves. One of the main motivations to study them is their role in understanding the geometry and topology of the moduli space of Higgs bundles. A special feature of these objects is that their notion of semistability depends on several parameters, and this makes it a fundamental task to look at the variation of their moduli spaces with the stability parameter. Examples for this may be found in [5], [2], and [7]. However, in these papers, only some of the stability parameters were moved whereas half of them were simply set to be one. The author has made major progress in understanding the nature of the "neglected" stability parameters in [3] and [13]. In this note, we will illustrate the main result of [13] for holomorphic triples. More precisely, we will give a self-contained proof in the special case at hand and determine the chamber structure of the space of stability parameters. To my knowledge, such chamber structures for all stability parameters occuring haven't appeared in the literature, so far.

Notation and conventions

We will work over the field $\mathbb{C}$ of complex numbers and fix a connected smooth projective curve X over $\mathbb{C}$. For notation and basic facts concerning vector bundles and sheaves on algebraic curves, we refer to the first part of [8].

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2. Boundedness

2.1. *Background*

A *holomorphic triple* on X is a datum (E_1, E_2, φ) in which E_1 and E_2 are algebraic vector bundles over X and $\varphi\colon E_1 \longrightarrow E_2$ is a homomorphism.[1] The *type* of (E_1, E_2, φ) is $\underline{t}(E_1, E_2, \varphi) := (\mathrm{rk}(E_1), \mathrm{rk}(E_2), \deg(E_1), \deg(E_2))$. A *subtriple* is a pair (F_1, F_2) in which F_i is a subbundle of E_i, $i = 1, 2$, and $\varphi(F_1) \subset F_2$.[2]

For an element $\underline{\pi} = (\kappa_1, \kappa_2, \chi_1, \chi_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$ and a pair of $(\mathcal{F}_1, \mathcal{F}_2)$ of coherent $\mathcal{O}_X$-modules, we define the $\underline{\pi}$*-rank*,

$$\mathrm{rk}_{\underline{\pi}}(\mathcal{F}_1, \mathcal{F}_2) := \kappa_1 \cdot \mathrm{rk}(\mathcal{F}_1) + \kappa_2 \cdot \mathrm{rk}(\mathcal{F}_2),$$

and the $\underline{\pi}$*-degree*,

$$\deg_{\underline{\pi}}(\mathcal{F}_1, \mathcal{F}_2) := \kappa_1 \cdot \deg(\mathcal{F}_1) + \kappa_2 \cdot \deg(\mathcal{F}_2) + \chi_1 \cdot \mathrm{rk}(\mathcal{F}_1) + \chi_2 \cdot \mathrm{rk}(\mathcal{F}_2).$$

If $\mathrm{rk}_{\underline{\kappa}}(\mathcal{F}_1, \mathcal{F}_2) > 0$, we also define the $\underline{\pi}$*-slope*

$$\mu_{\underline{\pi}}(\mathcal{F}_1, \mathcal{F}_2) := \frac{\deg_{\underline{\pi}}(\mathcal{F}_1, \mathcal{F}_2)}{\mathrm{rk}_{\underline{\pi}}(\mathcal{F}_1, \mathcal{F}_2)}.$$

A holomorphic triple is $\underline{\pi}$*-(semi)stable*, if the inequality

$$\mu_{\underline{\pi}}(F_1, F_2)(\leq)\mu_{\underline{\pi}}(E_1, E_2)$$

holds for every subtriple $(0, 0) \neq (F_1, F_2) \neq (E_1, E_2)$.

Remark 1 : We may view E_1 and E_2 as locally free $\mathcal{O}_X$-modules ([8], Proposition 1.8.1). The above condition is equivalent to the seemingly more restrictive condition that $\mu_{\underline{\pi}}(\mathcal{F}_1, \mathcal{F}_2)(\leq)\mu_{\underline{\pi}}(E_1, E_2)$ holds for all pairs $(0, 0) \neq (\mathcal{F}_1, \mathcal{F}_2) \neq (E_1, E_2)$ with $\mathcal{F}_i$ a coherent submodule of E_i, $i = 1, 2$, and $\varphi(\mathcal{F}_1) \subset \mathcal{F}_2$.

Remark 2 : Fix a type $\underline{t} = (r_1, r_2, d_1, d_2) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \times \mathbb{Z} \times \mathbb{Z}$ and a stability parameter $\underline{\pi}$. Then, according to [11] and [12], there exists a moduli space $\mathcal{M}(\underline{t}, \underline{\pi})$ for $\underline{\pi}$-semistable holomorphic triples (E_1, E_2, φ), satisfying $\underline{t}(E_1, E_2, \varphi) = \underline{t}$. The closed points of $\mathcal{M}(\underline{t}, \underline{\pi})$ correspond to S-equivalence classes of $\underline{\pi}$-semistable holomorphic triples of type $\underline{t}$, and there is an open subset $\mathcal{M}^{\mathrm{s}}(\underline{t}, \underline{\pi}) \subset \mathcal{M}(\underline{t}, \underline{\pi})$ which parameterizes isomorphy classes of $\underline{\pi}$-stable holomorphic triples of type $\underline{t}$.

2.2. *The main result of [13] for holomorphic triples*

Theorem 2.1 : *Fix a type $\underline{t} = (r_1, r_2, d_1, d_2)$. Then, there exists a constant C, such that, for every stability parameter $\underline{\pi} \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$, for every $\underline{\pi}$-semistable*

[1] A curve X as fixed gives rise to a connected compact Riemann surface X^{an}. According to the GAGA principles, the category of algebraic vector bundles on X with homomorphisms is equivalent to the category of holomorphic vector bundles on X^{an} with homomorphisms. This explains why the word "holomorphic" appears in the denomination of the objects that we are going to study.
[2] A subtriple induces the holomorphic triple $(F_1, F_2, \varphi_{|F_1} : F_1 \longrightarrow F_2)$.

holomorphic triple (E_1, E_2, φ) with $\underline{t}(E_1, E_2, \varphi) = \underline{t}$, and every non-trivial vector bundle F on X which is isomorphic to a subbundle of E_1 or E_2, the estimate

$$\mu(F) = \frac{\deg(F)}{\mathrm{rk}(F)} \leq C$$

holds true.

The first fundamental implication of this result is the existence of an algebraic variety S, or, rather a scheme of finite type over $\mathbb{C}$, algebraic vector bundles $E_{S,1}$, $E_{S,2}$ on $S \times X$, and a homomorphism $\varphi_S \colon E_{S,1} \longrightarrow E_{S,2}$, such that, for every stability parameter $\underline{\pi} \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$, for every $\underline{\pi}$-semistable holomorphic triple (E_1, E_2, φ) with $\underline{t}(E_1, E_2, \varphi) = \underline{t}$, there is a closed point $s \in S$, such that the restriction of $(E_{S,1}, E_{S,2}, \varphi_S)$ to $\{s\} \times X$, which is identified with X via the projection onto the second factor, is isomorphic to (E_1, E_2, φ).

Remark 3: By [8], Proposition 5.1.1, the boundedness property we have just described is not true for the family of all holomorphic triples of type $\underline{t}$.

In the next section, we will see that there are only finitely many distinct notions of semistability as $\underline{\pi}$ varies over $\mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R}$ and, so, only finitely many distinct moduli spaces. We can also say something about the way these moduli spaces are related (see Proposition 3.1).

Proof: A holomorphic triple (E_1, E_2, φ) leads to the dual holomorphic triple $(E_2^\vee, E_1^\vee, \varphi^\vee)$. Here, (E_1, E_2, φ) is semistable with respect to $(\kappa_1, \kappa_2, \chi_1, \chi_2)$ if and only if $(E_2^\vee, E_1^\vee, \varphi^\vee)$ is semistable with respect to $(\kappa_2, \kappa_1, -\chi_2, -\chi_1)$ ([3], Remark 7 (iii)).[1] So, we may assume without loss of generality that $r_1 \leq r_2$. Define

$$N := \Big\{ (\nu_1, \nu_2, \eta_1, \eta_2) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R} \times \mathbb{R} \,\Big|\, \nu_1 \cdot d_1 + \nu_2 \cdot d_2 + \eta_1 \cdot r_1 + \eta_2 \cdot r_2 = 0 \Big\} \quad (1)$$

and

$$P := \Big\{ (\nu_1, \nu_2, \eta_1, \eta_2) \in N \,\Big|\, \max\{ |\nu_1|, |\nu_2| \} = 1 \Big\}. \quad (2)$$

As explained in [3], Remark 7 (iv) and Page 174, we may assume that $\underline{\pi} \in P$.

We will derive the bound for stability parameters of the form $\underline{\pi} = (\kappa, 1, \chi_1, \chi_2) \in P$. The case of parameters of the form $\underline{\pi} = (1, \kappa, \chi_1, \chi_2) \in P$ is left as an exercise.

There is the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E_2 \arrow[d, equal] \\ E_1 \arrow[r, "\varphi"] & E_2 \end{tikzcd}.$$

By semistability,

$$d_2 + \chi_2 \cdot r_2 \leq 0, \quad (3)$$

[1] Observe different labelings of the vertices.

or, equivalently,[2] $\kappa \cdot d_1 + \chi_1 \cdot r_1 \geq 0$. This gives

$$\frac{\chi_1}{\kappa} \geq -\mu_1 \overset{d_1>0}{\geq} -d_1, \quad \mu_1 := \frac{d_1}{r_1}. \tag{4}$$

For $K := \ker(\varphi)$ and a subbundle $F \subset K$, there is the commutative diagram

$$\begin{array}{ccc} F & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow \\ E_1 & \xrightarrow{\varphi} & E_2 \end{array}.$$

Semistability implies $\kappa \cdot \deg(F) + \chi_1 \cdot \mathrm{rk}(F) \leq 0$, i.e.,

$$\mu(F) \leq -\frac{\chi_1}{\kappa} \overset{(4)}{\leq} \mu_1 \tag{5}$$

and

$$\deg(F) \leq \mathrm{rk}(F) \cdot \mu_1 \overset{\mu_1>0}{\leq} r_1 \cdot \mu_1 = d_1. \tag{6}$$

Now, let $F \subset E_2$ be any subbundle. We first consider the case that φ is generically surjective. By the assumption that $r_1 \leq r_2$, this may occur if and only if $r_1 = r_2$ and φ is generically an isomorphism. So, there is a unique subbundle $G \subset E_1$ with $\varphi(G) \subset F$ and $\mathrm{rk}(G) = \mathrm{rk}(F)$. One verifies that

$$\deg(F) - \deg(G) = \deg(F/G) \leq \deg(E_2/E_1) = d_2 - d_1.$$

Semistability applied to the commutative diagram

$$\begin{array}{ccc} G & \longrightarrow & F \\ \big\downarrow & & \big\downarrow \\ E_1 & \xrightarrow{\varphi} & E_2 \end{array}$$

yields

$$\begin{aligned}(\kappa + 1) \cdot \deg(F) + \kappa \cdot (d_1 - d_2) \leq \kappa \cdot \deg(G) + \deg(F) &\leq -\mathrm{rk}(F) \cdot (\chi_1 + \chi_2) \\ &= \frac{\mathrm{rk}(F)}{r_2} \cdot (\kappa \cdot d_1 + d_2) \leq \kappa \cdot d_1 + d_2\end{aligned}$$

and, thus,

$$\deg(F) \leq d_2. \tag{7}$$

[2] $\pi \in N$

If φ is not generically surjective, we look at the commutative diagram

$$\begin{array}{ccc} 0 & \longrightarrow & F \\ \big\downarrow & & \big\downarrow \\ E_1 & \xrightarrow{\varphi} & E_2 \end{array},$$

so that $\deg(F) + \chi_2 \cdot \mathrm{rk}(F) \leq 0$ and

$$\chi_2 \leq -\mu(F). \tag{8}$$

The $\mathcal{O}_X$-module $Q := E_2/\mathrm{im}(\varphi)$ has positive rank and may have torsion, and there is the commutative diagram

$$\begin{array}{ccc} E_1 & \longrightarrow & \mathrm{im}(\varphi) \\ \big\| & & \big\downarrow \\ E_1 & \xrightarrow{\varphi} & E_2 \\ \big\downarrow & & \big\downarrow \\ 0 & \longrightarrow & Q \end{array}$$

with exact columns. We derive[1]

$$\deg(Q) - \mu(F) \cdot \mathrm{rk}(Q) \overset{(8)}{\geq} \deg(Q) + \chi_2 \cdot \mathrm{rk}(Q) \geq 0, \tag{9}$$

so that

$$\mu(F) \leq \mu(Q) = \frac{d_2 - \deg\big(\mathrm{im}(\varphi)\big)}{\mathrm{rk}(Q)} = \frac{d_2 - \big(d_1 - \deg(K)\big)}{\mathrm{rk}(Q)} \overset{(6)}{\leq} \frac{d_2}{\mathrm{rk}(Q)} \overset{d_2>0}{\leq} d_2. \tag{10}$$

Any subbundle $F \subset E_1$ may be written as an extension

$$0 \longrightarrow F \cap K \longrightarrow F \longrightarrow \varphi(F) \longrightarrow 0.$$

With (5), (7), and (10), we find

$$\deg(F) = \deg(F \cap K) + \deg\big(\varphi(F)\big) \leq d_1 + r_2 \cdot d_2.$$

This concludes the argument. $\square$

3. The chamber decomposition

3.1. *The general construction*

We fix a type $\underline{t} = (r_1, r_2, d_1, d_2)$ as before und look at the space P of stability parameters introduced in (2). Suppose that $\underline{\pi}_1$ and $\underline{\pi}_2$ are two elements in P which

[1] Compare Remark **1**.

induce distinct notions of semistability on the family of holomorphic triples of type $\underline{t}$. Then, after possibly exchanging the roles of $\underline{\pi}_1$ and $\underline{\pi}_2$, we may assume that there is a holomorphic triple (E_1, E_2, φ) which is $\underline{\pi}_1$-semistable but not $\underline{\pi}_2$-semistable. In this situation, there is a subtriple (F_1, F_2) of (E_1, E_2, φ) with $\deg_{\underline{\pi}_1}(F_1, F_2) \leq 0$ and $\deg_{\underline{\pi}_2}(F_1, F_2) > 0$. Set

$$\underline{\pi}(\lambda) := \big(\kappa_1(\lambda), \kappa_2(\lambda), \chi_1(\lambda), \chi_2(\lambda)\big) := (1-\lambda)\cdot \underline{\pi}_1 + \lambda \cdot \underline{\pi}_2, \quad \lambda \in [0,1].$$

So, there exists a value $\lambda_0 \in [0,1]$ with

$$\kappa_1(\lambda_0)\cdot \deg(F_1) + \kappa_2(\lambda_0)\cdot \deg(F_2) + \chi_1(\lambda_0)\cdot \mathrm{rk}(F_1) + \chi_2(\lambda_0)\cdot \mathrm{rk}(F_2) = 0. \quad (11)$$

If $\underline{\pi}_1$ and $\underline{\pi}_2$ define different notions of stability, there are a holomorphic triple (E_1, E_2, φ) which is, say, $\underline{\pi}_1$-stable but not $\underline{\pi}_2$-stable and a subtriple (F_1, F_2) of (E_1, E_2, φ) with $\deg_{\underline{\pi}_1}(F_1, F_2) < 0$ and $\deg_{\underline{\pi}_2}(F_1, F_2) \geq 0$. Again, we arrive at an equation as (11).

A *test type* is a tuple $\underline{u} = (s_1, s_2, e_1, e_2)$ with $0 \leq s_1 \leq r_1$, $0 \leq s_2 \leq r_2$, $0 < s_1 + s_2 < r_1 + r_2$, and $e_1, e_2 \in \mathbb{Z}$. A test type $\underline{u}$ determines a *wall*

$$W(\underline{u}) := \Big\{ (\nu_1, \nu_2, \eta_1, \eta_2) \in P \,\Big|\, \nu_1 \cdot e_1 + \nu_2 \cdot e_2 + \eta_1 \cdot r_1 + \eta_2 \cdot r_2 = 0 \Big\}.$$

These walls induce a decomposition

$$P := \bigsqcup_{i \in I} \mathcal{C}_i \quad (12)$$

into locally closed subsets $\mathcal{C}_i$, $i \in I$, called *chambers*. By construction, the chamber decomposition (12) has the following property.

Proposition 3.1: i) *Let $i_0 \in I$, $\underline{\pi}_1, \underline{\pi}_2 \in \mathcal{C}_{i_0}$, and (E_1, E_2, φ) be a holomorphic triple of type $\underline{t}$. Then, (E_1, E_2, φ) is $\underline{\pi}_1$-(semi)stable if and only if it is $\underline{\pi}_2$-(semi)stable.*

ii) *Assume $i_0 \in I$, $\underline{\pi}_1 \in \mathcal{C}_{i_0}$, $\underline{\pi}_2 \in \overline{\mathcal{C}}_{i_0}$,*[1] *and that (E_1, E_2, φ) is a holomorphic triple of type $\underline{t}$. If (E_1, E_2, φ) is $\underline{\pi}_2$-stable, then it is also $\underline{\pi}_1$-stable. If (E_1, E_2, φ) is $\underline{\pi}_1$-semistable, then it is also $\underline{\pi}_2$-semistable. In particular, there is a morphism*

$$\mathcal{M}(\underline{t}, \underline{\pi}_1) \longrightarrow \mathcal{M}(\underline{t}, \underline{\pi}_2).$$

An application of Theorem 2.1 shows that there is a **finite** collection of test types $\underline{u}_1, \ldots, \underline{u}_\ell$, such that the induced **finite** chamber decomposition $P = \bigsqcup_{j=1}^{m} \mathcal{K}_j$ has the same properties as the one in (12) (see [3], Proposition 21). In the next section, we will explain this observation for holomorphic triples with $r_1 = r_2 = 2$. For a finite chamber decomposition, we have only finitely many distinct moduli spaces and the last statement in Proposition 3.1 gives us a first important indication how the distinct moduli spaces interact. A concrete example for holomorphic triples and stability parameters with $\kappa_1 = 1 = \kappa_2$ is contained in [5].

[1]The symbol $\overline{\cdot}$ indicates the closure.

3.2. *Explicit examples*

We will now look at a type $\underline{t} = (2, 2, d_1, d_2)$ with $d_1 \leq d_2$, so that holomorphic triples (E_1, E_2, φ) of type $\underline{t}$ in which φ is injective are not a priori excluded. Since, for a stability parameter $\underline{\pi} \in P$, χ_2 is determined by κ_1, κ_2, and χ_1, and $\kappa_1 = 1$ or $\kappa_2 = 1$, we may think of P as the union of

$$P_1 = (0, 1] \times \{1\} \times \mathbb{R} \quad \text{and} \quad P_2 := \{1\} \times (0, 1] \times \mathbb{R}.$$

We will discuss the chamber decomposition of P_1. The one of P_2 is analogous.

Remark 1: i) Given a stability parameter $\underline{\pi} = (\kappa, 1, \chi_1, \chi_2) \in P_1$ and a $\underline{\pi}$-semistable holomorphic triple $(E_1, E_2, 0)$ of type $\underline{t}$, we have

$$\kappa \cdot d_1 + 2 \cdot \chi_1 = 0 \quad \text{and} \quad d_2 + 2 \cdot \chi_2 = 0.$$

Parameters satisfying these equations correspond to the case when equality holds in (4). A triple $(E_1, E_2, 0)$ will be semistable for such a parameter if and only if E_1 and E_2 are semistable vector bundles. We may neglect the case $\varphi = 0$ in the following discussions. Note that the above equations describe the wall that is associated with the test type $(2, 0, d_1, 0)$ or $(0, 2, 0, d_2)$.

ii) The last property stated in Proposition 3.1 shows that, for a holomorphic triple (E_1, E_2, φ) of type $(2, 2, d_1, d_2)$ which is semistable with respect to a parameter $\underline{\pi}$ which lies in a chamber which is adjacent to the line defined by equality in (4), the bundles E_1 and E_2 are semistable.

Remark 2: Let (E_1, E_2, φ) be a holomorphic triple and (F_1, F_2) a subtriple. There is an induced homomorphism $\overline{\varphi}\colon Q_1 \longrightarrow Q_2$, $Q_i := E_i/F_i$, $i = 1, 2$. If (E_1, E_2, φ) is $\underline{\pi}$-semistable and

$$\kappa \cdot \deg(F_1) + 1 \cdot \deg(F_2) + \chi_1 \cdot \mathrm{rk}(F_1) + \chi_2 \cdot \mathrm{rk}(F_2) = 0, \quad \underline{\pi} = (\kappa, 1, \chi_1, \chi_2),$$

$(F_1, F_2, \varphi_{|F_1})$, $(Q_1, Q_2, \overline{\varphi})$, and $(F_1 \oplus Q_1, F_2 \oplus Q_2, \varphi_{|F_1} \oplus \overline{\varphi})$ are $\underline{\pi}$-semistable as well.

Remark 3: Let (E_1, E_2, φ) be a holomorphic triple of type $(2, 2, d_1, d_2)$ in which $K = \ker(\varphi)$ has rank one. By (5), $\deg(K) \leq d_1/2$. Setting $Q := E_2/\mathrm{im}(\varphi)$, Inequality (3), the second inequality in (9) and the first inequality in (5) show

$$\frac{d_2}{2} \leq -\chi_2 \leq \deg(Q) = d_2 - \deg\bigl(\mathrm{im}(\varphi)\bigr) = d_2 - d_1 + \deg(K) \leq d_2 - d_1 - \frac{\chi_1}{\kappa}.$$

This yields

$$\chi_1 \leq \kappa \cdot \left(\frac{d_2}{2} - d_1 \right). \tag{13}$$

3.2.1. *Test types with $s_1 = 1$, $s_2 = 0$*

Of course, these are relevant only for holomorphic triples of type $\underline{t}$ in which φ is not generically surjective. A test type $(1, 0, e_1, 0)$ defines the line

$$\kappa \cdot e_1 + \chi_1 = 0 \quad \text{or} \quad \chi_1 = -e_1 \cdot \kappa.$$

By Inequalities (5) and (13), we need to look only at the cases where

$$d_1 - \frac{d_2}{2} \leq e_1 \leq \frac{d_1}{2}.$$

3.2.2. Test types with $s_1 = 1$, $s_2 = 2$

By Remark **2**, the walls defined by test types of the form $(1, 2, e_1, e_2)$ are included in the walls defined by test types of the form $(1, 0, f_1, f_2)$ and vice versa.

3.2.3. Test types with $s_1 = 0$, $s_2 = 1$

For $e_2 \in \mathbb{Z}$, the test type $(0, 1, 0, e_2)$ defines the line

$$0 = e_2 + \chi_2 = e_2 - \chi_1 - \kappa \cdot \frac{d_1}{2} - \frac{d_2}{2}.$$

In view of Remark **3** and $0 < \kappa \leq 1$, we may limit ourselves to test types with

$$\frac{d_2}{2} \leq e_2 < d_2 - \frac{d_1}{2}.$$

Because of Inequality (13), we have to draw the lines defined by these test types only to the point where they hit the line defined by the equation $2 \cdot \chi_1 = \kappa \cdot (d_2 - 2 \cdot d_1)$.

3.2.4. Test types with $s_1 = 2$, $s_2 = 1$

The walls obtained from test types with $s_1 = 2$ and $s_2 = 1$ are included in the walls obtained from test types with $s_1 = 0$ and $s_2 = 1$ and vice versa.

3.2.5. Test types with $s_1 = 1$, $s_2 = 1$

Suppose that $\underline{\pi} = (\kappa, 1, \chi_1, \chi_2) \in P_1$, that (E_1, E_2, φ) is $\underline{\pi}$-semistable and has type $\underline{t}$, and that (L_1, L_2) is a subtriple with $\mathrm{rk}(L_1) = 1 = \mathrm{rk}(L_2)$, such that

$$\kappa \cdot \deg(L_1) + \deg(L_2) + \chi_1 + \chi_2 = 0. \tag{14}$$

If $L_1 \subset \ker(\varphi)$, then (14) is equivalent to

$$\kappa \cdot \deg(L_1) + \chi_1 = 0 \quad \text{and} \quad \deg(L_2) + \chi_2 = 0.$$

The walls corresponding to these cases have already been treated. So, we may assume for the following that $\deg(L_1) \not\subset \ker(\varphi)$, so that $\varphi_{|L_1}\colon L_1 \longrightarrow L_2$ is injective. In particular, $\deg(L_1) \leq \deg(L_2)$. By the same token, we may assume that $\overline{\varphi}\colon E_1/L_1 \longrightarrow E_2/L_2$ is non-zero. By Remark **2**, we are reduced to the case that $(E_1, E_2, \varphi) = (L_1, L_2, \chi) \oplus (M_1, M_2, \psi)$ with line bundles L_1, L_2, M_1, M_2 and non-zero homomorphisms $\chi\colon L_1 \longrightarrow L_2$ and $\psi\colon M_1 \longrightarrow M_2$. We may assume without loss of generality that $\deg(M_2) \leq \deg(L_2)$. Again, by Remark **2**, (L_1, L_2, χ) is itself a $\underline{\pi}$-semistable holomorphic triple. This amounts to the inequality

$$0 \geq \deg(L_2) + \chi_2 = \deg(L_2) - \frac{1}{2} \cdot (\kappa \cdot d_1 + d_2) - \chi_1,$$

i.e.,

$$\chi_1 \geq \deg(L_2) - \frac{1}{2} \cdot (\kappa \cdot d_1 + d_2). \tag{15}$$

Since $\deg(M_2) \leq \deg(L_2)$, (M_1, M_2, ψ) will then be $\underline{\pi}$-semistable, too. Equation (14) the implies that $(L_1, L_2, \chi) \oplus (M_1, M_2, \psi)$ is $\underline{\pi}$-semistable. In the special case that equality holds in (15), the holomorphic triple

$$(L_1, 0, 0) \oplus (0, L_2, 0) \oplus (M_1, M_2, \psi)$$

will be $\underline{\pi}$-semistable, as well. Note that L_1 is the kernel in this holomorphic triple. By the discussion of the case $s_1 = 1$, $s_2 = 0$,

$$d_1 - \frac{d_2}{2} \leq \deg(L_1) \leq \frac{d_1}{2}. \tag{16}$$

Finally, the wall we are speaking about here is determined by the equation

$$\kappa \cdot \left(\deg(L_1) - \frac{d_1}{2}\right) = \frac{d_2}{2} - \deg(L_2). \tag{17}$$

Remark 4: By this equation, a semistable holomorphic triple $(L_1, L_2, \chi) \oplus (M_1, M_2, \psi)$ with $\deg(L_1) = d_1/2$ can only exist, if $\deg(L_2) = d_2/2$. This case may occur only if d_1 and d_2 are both even. Suppose d_1 and d_2 are both even and $(L_1, L_2, \chi) \oplus (M_1, M_2, \psi)$ is such that $\deg(L_1) = d_1/2 = \deg(M_1)$, $\deg(L_2) = d_2/2 = \deg(M_2)$. Then, Inequality (15) becomes $\chi_1/\kappa \geq -d_1/2$, i.e., it agrees with Inequality (5). This means that such a holomorphic triple will be semistable for every stability in P_1 for which semistable holomorphic triples of type $(2, 2, d_1, d_2)$ may exist. We will omit this case in the following discussion.

By (16), the positivity of κ, and (10), it follows that

$$\frac{d_2}{2} < \deg(L_2) \leq d_2. \tag{18}$$

Altogether, (16) and (18) show that there are only finitely many test types to consider. The following constraints reduce the number of test types further:

- $\deg(L_2) - \deg(L_1) = d_2 - \deg(M_2) - d_1 + \deg(M_1) \leq d_2 - d_1$,
- $\deg(L_1) - \dfrac{d_1}{2} \leq \dfrac{d_2}{2} - \deg(L_2)$.

The last inequality follows from (17), because we need $\kappa \leq 1$.

3.2.6. Example

In the picture, the scale for the x-axis is twice the one for the y-axis. The chambers in light grey are unbounded. By Proposition 3.1, the two-dimensional unbounded chambers parameterize extensions

$$0 \longrightarrow (L_1, L_2, \chi) \longrightarrow (E_1, E_2, \varphi) \longrightarrow (M_1, M_2, \psi) \longrightarrow 0$$

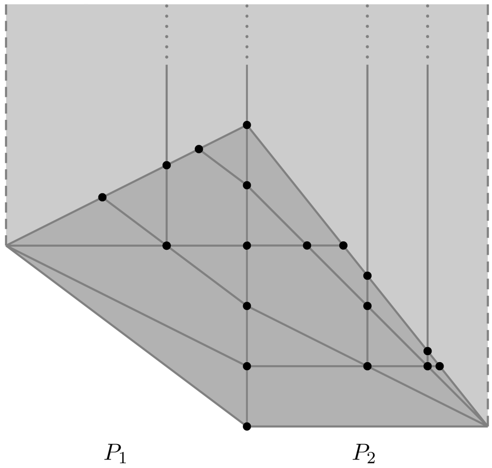

Figure 1. The chamber decomposition for $d_1 = 3$, $d_2 = 8$.

and the one dimensional unbounded chambers direct sums as seen in the last section.

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