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Some New Results in the Theory of L^p-Dissipativity

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In the present paper we survey some recent results obtained with Vladimir Maz'ya. They concern the L^p-dissipativity of systems of partial differential operators.

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1. Introduction

In a series of joint papers with Vladimir Maz'ya - starting from [2, 3] - we have considered the problem of characterizing the L^p-dissipativity of partial differential operators.

Our results have been fitted in the frame of the theory of semi-bounded operators in the monograph [5]. After the monograph was published, we have obtained some new results ([4, 6, 7]). The aim of the present paper is to survey them, in particular, the new ones concerning systems of partial differential operators.

In Section 2 we recall our main result concerning scalar second order operators, which was obtained in our first paper [2]. Section 3 is devoted to the elasticity system. The topic of Section 4 is the L^p-dissipativity for systems of partial differential operators of the first order. We shall see that these imply also some new criteria for systems of partial differential operators of the second order.

2. The main result for scalar second order operators

Let Ω be an open set in $\mathbb{R}^n$. By $C_0(\Omega)$ $(C_0^1(\Omega))$ we denote the space of complex valued continuous $(C^1(\Omega)$ functions having compact support in Ω.

In what follows, $\mathscr{A}$ is a $n \times n$ matrix function with complex valued entries $a^{hk} \in (C_0(\Omega))^*$, $\mathscr{A}^t$ is its transposed matrix and $\mathscr{A}^*$ is its adjoint matrix, i.e. $\mathscr{A}^* = \overline{\mathscr{A}}^t$.

Let $\mathbf{b} = (b_1, \dots, b_n)$ and $\mathbf{c} = (c_1, \dots, c_n)$ stand for complex valued vectors with $b_j, c_j \in (C_0(\Omega))^*$. By a we mean a complex valued scalar distribution in $(C_0^1(\Omega))^*$.

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We denote by $\mathscr{L}(u,v)$ the sesquilinear form

$$\mathscr{L}(u,v)=\int_\Omega(\langle\mathscr{A}\,\nabla u,\nabla v\rangle-\langle\mathbf{b}\nabla u,v\rangle+\langle u,\overline{\mathbf{c}}\nabla v\rangle-a\langle u,v\rangle)$$

defined on $C_0^1(\Omega)\times C_0^1(\Omega)$.

The integrals appearing in this definition have to be understood in a proper way. The entries a^{hk} being measures, the meaning of the first term is

$$\int_\Omega\langle\mathscr{A}\,\nabla u,\nabla v\rangle=\int_\Omega\partial_k u\,\partial_h\overline{v}\,da^{hk}\,.$$

Similar meanings have the terms involving $\mathbf{b}$ and $\mathbf{c}$. Finally, the last term is the action of the distribution $a\in(C_0^1(\Omega))^*$ on the functions $u\,\overline{v}$ belonging to $C_0^1(\Omega)$.

The form $\mathscr{L}$ is related to the operator

$$Au=\operatorname{div}(\mathscr{A}\,\nabla u)+\mathbf{b}\nabla u+\operatorname{div}(\mathbf{c}u)+au. \tag{1}$$

where div denotes the divergence operator. The operator A acts from $C_0^1(\Omega)$ to $(C_0^1(\Omega))^*$ through the relation

$$\mathscr{L}(u,v)=-\int_\Omega\langle Au,v\rangle$$

for any $u,v\in C_0^1(\Omega)$.

If $p\in(1,\infty)$, p' denotes its conjugate exponent $p/(p-1)$.

Let $1<p<\infty$. We say that the form $\mathscr{L}$ is L^p-*dissipative* if for all $u\in C_0^1(\Omega)$

$$\mathbb{R}\mathrm{e}\,\mathscr{L}(u,|u|^{p-2}u)\geqslant 0 \qquad \text{if } p\geqslant 2; \tag{2}$$

$$\mathbb{R}\mathrm{e}\,\mathscr{L}(|u|^{p'-2}u,u)\geqslant 0 \qquad \text{if } 1<p<2 \tag{3}$$

(we use here that $|u|^{q-2}u\in C_0^1(\Omega)$ for $q\geqslant 2$ and $u\in C_0^1(\Omega)$).

An algebraic necessary and sufficient condition for the L^p-dissipativity of the form $\mathscr{L}(u,v)$ was found in [2]

It concerns operator (1) without lower order terms:

$$Au=\operatorname{div}(\mathscr{A}\,\nabla u)$$

with the coefficients $a^{hk}\in(C_0(\Omega))^*$.

This result was new even for smooth coefficients, when it implies a criterion for the L^p-contractivity of the corresponding semigroup.

Theorem 2.1: *Let the matrix* $\mathbb{I}\mathrm{m}\,\mathscr{A}$ *be symmetric, i.e.* $\mathbb{I}\mathrm{m}\,\mathscr{A}^t=\mathbb{I}\mathrm{m}\,\mathscr{A}$. *The form*

$$\mathscr{L}(u,v)=\int_\Omega\langle\mathscr{A}\,\nabla u,\nabla v\rangle$$

is L^p-dissipative if and only if

$$|p-2|\,|\langle \mathbb{I}\mathrm{m}\,\mathscr{A}\,\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}\mathrm{e}\,\mathscr{A}\,\xi,\xi\rangle \tag{4}$$

for any $\xi \in \mathbb{R}^n$, where $|\cdot|$ denotes the total variation.

It is clear that condition (4) has to be understood in the sense of measures.

It is possible to prove that condition (4) holds if and only if

$$\frac{4}{p\,p'}\langle \mathbb{R}\mathrm{e}\,\mathscr{A}\,\xi,\xi\rangle + \langle \mathbb{R}\mathrm{e}\,\mathscr{A}\,\eta,\eta\rangle - 2(1-2/p)\langle \mathbb{I}\mathrm{m}\,\mathscr{A}\,\xi,\eta\rangle \geqslant 0 \tag{5}$$

for any $\xi,\eta \in \mathbb{R}^n$.

The class of partial differential operators of the second order whose principal part is such that the form (5) is not merely non-negative, but strictly positive, which could be called *p-strongly elliptic*, was very recently considered by some Authors (see [1, 8–10]).

Let us assume that either A has lower order terms or they are absent and $\mathbb{I}\mathrm{m}\,\mathscr{A}$ is not symmetric. One could prove that (4) is still a necessary condition for A to be L^p-dissipative. However, in general, it is not sufficient.

3. The L^p-dissipativity for elasticity system

In this section we consider the classical operator of linear elasticity

$$Eu = \Delta u + (1-2\nu)^{-1}\nabla\,\mathrm{div}\,u \tag{6}$$

where ν is the Poisson ratio. We assume that either $\nu > 1$ or $\nu < 1/2$. It is well known that E is strongly elliptic if and only if one of these inequalities holds.

The form $\mathscr{L}$ related to the operator (6) is

$$\mathscr{L}(u,v) = -\int_\Omega (\langle \nabla u, \nabla v\rangle + (1-2\nu)^{-1}\,\mathrm{div}\,u\ \mathrm{div}\,v)\,dx\,. \tag{7}$$

Following the approach described in the previous section, we say that the form $\mathscr{L}$ is L^p-dissipative if

$$-\int_\Omega (\langle \nabla u, \nabla(|u|^{p-2}u)\rangle + (1-2\nu)^{-1}\,\mathrm{div}\,u\ \mathrm{div}(|u|^{p-2}u))\,dx \leqslant 0 \qquad \text{if } p \geqslant 2,$$

$$-\int_\Omega (\langle \nabla u, \nabla(|u|^{p'-2}u)\rangle + (1-2\nu)^{-1}\,\mathrm{div}\,u\ \mathrm{div}(|u|^{p'-2}u))\,dx \leqslant 0 \qquad \text{if } p < 2,$$

for all $u \in (C_0^1(\Omega))^n$.

We now give a necessary and sufficient condition for the L^p-dissipativity of the form (7) in the case $n = 2$.

The next result provides a necessary and sufficient condition, which turns to be useful.

Lemma 3.1: *Let Ω be a domain of $\mathbb{R}^2$. The form* (7) *is L^p-dissipative if and only if*

$$\int_\Omega [C_p |\nabla |v||^2 - \sum_{j=1}^{2} |\nabla v_j|^2 + \gamma\, C_p\, |v|^{-2} |v_h \partial_h |v||^2 - \gamma\, |\operatorname{div} v|^2]\, dx \leqslant 0$$

for any $v \in (C_0^1(\Omega))^2$, where

$$C_p = (1 - 2/p)^2, \qquad \gamma = (1 - 2\nu)^{-1}.$$

We have also a necessary algebraic condition

Lemma 3.2: *Let Ω be a domain of $\mathbb{R}^2$. If the form* (7) *is L^p-dissipative, we have*

$$C_p[|\xi|^2 + \gamma\, \langle \xi, \omega\rangle^2] \langle \lambda, \omega\rangle^2 - |\xi|^2 |\lambda|^2 - \gamma\, \langle \xi, \lambda\rangle^2 \leqslant 0$$

for any ξ, λ, $\omega \in \mathbb{R}^2$, $|\omega| = 1$ (the constants C_p and γ being given by (3.1)*).*

We note that Lemma 3.1 can be proved in any number of variables while a proof of Lemma 3.2 is known only for $n = 2$. Combining these two results, one can prove

Theorem 3.3: *Let Ω be a domain of $\mathbb{R}^2$. The form* (7) *is L^p-dissipative if and only if*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 \leqslant \frac{2(\nu - 1)(2\nu - 1)}{(3 - 4\nu)^2}\,. \tag{8}$$

We mention that this result was proved for the first time in [3]. In [4] a simpler proof was given.

As far as $n = 3$ is concerned, condition (8) is still necessary, even in the case of a non-constant Poisson ratio. In fact, as proved in [4], we have

Theorem 3.4: *Let Ω be a bounded domain in $\mathbb{R}^3$ whose boundary is in the class C^2. Suppose $\nu = \nu(x)$ is a continuos function defined in Ω such that*

$$\inf_{x \in \Omega} |2\nu(x) - 1| > 0.$$

If the form (7) *is L^p-dissipative in Ω, then*

$$\left(\frac{1}{2} - \frac{1}{p}\right)^2 \leqslant \inf_{x \in \Omega} \frac{2(\nu(x) - 1)(2\nu(x) - 1)}{(3 - 4\nu(x))^2}\,.$$

It is not know if condition (8) is sufficient for the L^p-dissipativity of the three-dimensional elasticity (see also Problem 43 in [11]). The next result (see [4]) provides a more strict sufficient condition.

Theorem 3.5 : *Let Ω be a domain in $\mathbb{R}^3$. If*

$$(1-2/p)^2 \leqslant \begin{cases} \dfrac{1-2\nu}{2(1-\nu)} & \text{if } \nu < 1/2 \\[2ex] \dfrac{2(1-\nu)}{1-2\nu} & \text{if } \nu > 1. \end{cases}$$

the form (7) *is L^p-dissipative.*

4. The L^p-dissipativity of systems of partial differential operators

In previous papers we have considered the L^p-dissipativity for classes of systems of the second order and obtained related necessary and sufficient conditions. These results are described in detail in the monograph [5].

The kind of criteria for systems of the first order, which we have obtained in [6], is quite different.

They concern the partial differential operator

$$Eu = \mathscr{B}^h\, \partial_h u + \partial_h(\mathscr{C}^h\, u) + \mathscr{D}\, u, \tag{9}$$

where $\mathscr{B}^h$ and $\mathscr{C}^h$ $(h = 1, \dots, n)$ are $m \times m$ matrices with complex-valued entries $b^h_{ij}, c^h_{ij} \in (C_0(\Omega))^*$ $(1 \leqslant i, j \leqslant m)$ and $\mathscr{D}$ is a matrix whose elements d_{ij} are complex-valued distributions in $(C^1_0(\Omega))^*$.

Let us denote by $\mathscr{L}(u, v)$ the related sesquilinear form given by

$$\mathscr{L}(u, v) = \int_\Omega \langle \mathscr{B}^h\, \partial_h u, v\rangle - \langle \mathscr{C}^h\, u, \partial_h v\rangle + \langle \mathscr{D}\, u, v\rangle\, . \tag{10}$$

It is defined in $(C^1_0(\Omega))^m \times (C^1_0(\Omega))^m$.

As in the scalar case, we say that the form $\mathscr{L}$ is L^p-dissipative if (2)-(3) hold for all $u \in (C^1_0(\Omega))^m$.

We start by considering the operator

$$Eu = \mathscr{B}^h\, \partial_h u + \mathscr{D}\, u$$

where the entries of the matrices $\mathscr{B}^h, \mathscr{D}$ are locally integrable functions and also $\partial_h\, \mathscr{B}^h$ (where the derivatives are in the sense of distributions) is a matrix with locally integrable entries. The next Theorem provides necessary and sufficient conditions for the L^p-dissipativity of the form $\mathscr{L}$ related to this operator. The conditions are different according if $p = 2$ or not (see formulas (11) and (12) below).

Theorem 4.1 : *The form*

$$\mathscr{L}(u, v) = \int_\Omega \langle \mathscr{B}^h\, \partial_h u, v\rangle + \langle \mathscr{D}\, u, v\rangle$$

is L^p − dissipative if and only if the following conditions are satisfied:

(1)

$$\mathscr{B}^h(x) = b_h(x)\, I, \qquad \textit{if } p \neq 2, \tag{11}$$

$$\mathscr{B}^h(x) = (\mathscr{B}^h)^*(x), \qquad \textit{if } p = 2, \tag{12}$$

for almost any $x \in \Omega$ *and* $h = 1, \dots, n$. *Here* b_h *are real locally integrable functions* $(1 \leqslant h \leqslant n)$.

(2)

$$\mathbb{R}\mathrm{e}\langle (p^{-1}\partial_h \mathscr{B}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle \geqslant 0 \tag{13}$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ *and for almost any* $x \in \Omega$.

As far as the more general operator (9) is concerned, we have the following result, under the assumption that $\mathscr{B}^h$, $\mathscr{C}^h$, $\mathscr{D}$, $\partial_h \mathscr{B}^h$ and $\partial_h \mathscr{C}^h$ are matrices with complex locally integrable entries.

Theorem 4.2: *The form* (10) *is* L^p*-dissipative if and only if the following conditions are satisfied*

(1)

$$\mathscr{B}^h(x) + \mathscr{C}^h(x) = b_h(x)\, I, \qquad \textit{if } p \neq 2,$$
$$\mathscr{B}^h(x) + \mathscr{C}^h(x) = (\mathscr{B}^h)^*(x) + (\mathscr{C}^h)^*(x), \qquad \textit{if } p = 2,$$

for almost any $x \in \Omega$ *and* $h = 1, \dots, n$. *Here* b_h *are real locally integrable functions* $(1 \leqslant h \leqslant n)$.

(2)

$$\mathbb{R}\mathrm{e}\langle (p^{-1}\partial_h \mathscr{B}^h(x) - p'^{-1}\partial_h \mathscr{C}^h(x) - \mathscr{D}(x))\zeta, \zeta\rangle \geqslant 0$$

for any $\zeta \in \mathbb{C}^m$, $|\zeta| = 1$ *and for almost any* $x \in \Omega$.

The last results imply some sufficient criteria for the L^p-dissipativity of systems of the second order.

Specifically, let us consider the class of systems of partial differential equations of the form

$$Eu = \partial_h(\mathscr{A}^h(x)\partial_h u) + \mathscr{B}^h(x)\partial_h u + \mathscr{D}(x)u, \tag{14}$$

where $\mathscr{A}^h$, $\mathscr{B}^h$ and $\mathscr{D}$ are $m \times m$ matrices with complex locally integrable entries.

In [3] we have proved that if the operator (14) has no lower order terms, we have the following necessary and sufficient algebraic conditions:

Theorem 4.3: *The operator*

$$\partial_h(\mathscr{A}^h(x)\partial_h u)$$

is L^p-dissipative if and only if

$$\begin{gathered}\mathbb{R}\mathrm{e}\langle \mathscr{A}^h(x)\lambda,\lambda\rangle - (1-2/p)^2\,\mathbb{R}\mathrm{e}\langle \mathscr{A}^h(x)\omega,\omega\rangle(\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle)^2 \\ -(1-2/p)\,\mathbb{R}\mathrm{e}(\langle \mathscr{A}^h(x)\omega,\lambda\rangle - \langle \mathscr{A}^h(x)\lambda,\omega\rangle)\,\mathbb{R}\mathrm{e}\langle\lambda,\omega\rangle \geqslant 0\end{gathered} \tag{15}$$

for almost every $x\in\Omega$ and for every $\lambda,\omega\in\mathbb{C}^m$, $|\omega|=1$, $h=1,\dots,n$.

Combining this result with Theorem 4.1 we get

Theorem 4.4: *Let E be the operator* (14)*, where $\mathscr{A}^h$ are $m\times m$ matrices with complex locally integrable entries and the matrices $\mathscr{B}^h(x)$, $\mathscr{D}(x)$ satisfy the hypothesis of Theorem 4.1. If* (15) *holds for almost every $x\in\Omega$ and for every $\lambda,\omega\in\mathbb{C}^m$, $|\omega|=1$, $h=1,\dots,n$, and if conditions* (11)-(12) *and* (13) *are satisfied, the operator E is L^p-dissipative.*

For scalar operators something more can be said. Consider the operator (14) when $m=1$

$$\partial_h(a^h(x)\partial_h u) + b^h(x)\partial_h u + d(x)u$$

(a^h, b^h and d being scalar functions). In this case such an operator can be written in the form

$$Eu = \operatorname{div}(\mathscr{A}(x)\nabla u) + \mathscr{B}(x)\nabla u + d(x)\,u \tag{16}$$

where $\mathscr{A}=\{c_{hk}\}$, $c_{hh}=a^h$, $c_{hk}=0$ if $h\neq k$ and $\mathscr{B}=\{b^h\}$. One can show that (15) is equivalent to

$$\frac{4}{pp'}\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi,\xi\rangle + \langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\eta,\eta\rangle - 2(1-2/p)\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\xi,\eta\rangle \geqslant 0 \tag{17}$$

for almost any $x\in\Omega$ and for any $\xi,\eta\in\mathbb{R}^n$ (see [5, Remark 4.21, p.115]). Condition (17) is in turn equivalent to the inequality:

$$|p-2|\,|\langle \mathbb{I}\mathrm{m}\,\mathscr{A}(x)\xi,\xi\rangle| \leqslant 2\sqrt{p-1}\,\langle \mathbb{R}\mathrm{e}\,\mathscr{A}(x)\xi,\xi\rangle \tag{18}$$

for almost any $x\in\Omega$ and for any $\xi\in\mathbb{R}^n$ (see [5, Remark 2.8, p.42]). In view of the results of section 2 we have

Theorem 4.5: *Let E be the scalar operator* (16) *where $\mathscr{A}$ is a diagonal matrix. If inequality* (18) *and conditions* (11)-(12) *and* (13) *are satisfied, the operator E is L^p-dissipative.*

More generally, consider the scalar operator (16) with a matrix $\mathscr{A}=\{a_{hk}\}$ not necessarily diagonal. The following result holds true.

Theorem 4.6: *Let the matrix $\mathbb{I}\mathrm{m}\,\mathscr{A}$ be symmetric. If inequality* (18) *and conditions* (11)-(12) *and* (13) *are satisfied, the operator* (16) *is L^p-dissipative.*

We mention that in paper [7] we give necessary and, separately, sufficient conditions for the L^p- dissipativity of the "complex oblique derivative" operator. They are of a different kind and they involve the norm in a suitable space of multipliers of the imaginary part of the coefficients of the operator.

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