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SHORT COMMUNICATIONS

Measurability Properties of Mazurkiewicz Sets

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We consider the family of Mazurkiewicz subsets of the Euclidean plane from the measure-theoretical point of view. In particular, it is shown that all Mazurkiewicz sets are negligible and there exists a Mazurkiewicz set which is absolutely negligible. On the other hand, it is proved that, assuming **CH**, there exists a Mazurkiewicz set which is not absolutely negligible.

Keywords: Mazurkiewicz set, Translation invariant measure, Extension of a measure, Negligible set, Absolutely negligible set.

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In 1914, S. Mazurkiewicz [7] presented a transfinite construction of a subset A of the Euclidean plane $\mathbf{R}^2$, having the following extraordinary property: every straight line in $\mathbf{R}^2$ meets A in exactly two points.

Motivated by this result, a set $Z \subset \mathbf{R}^2$ is called a Mazurkiewicz subset of $\mathbf{R}^2$ if $\mathrm{card}(Z \cap l) = 2$ for every straight line l lying in $\mathbf{R}^2$.

The descriptive structure of Mazurkiewicz sets can be rather complicated. For instance, these three results are well known:

(1) if a Mazurkiewicz set is analytic in $\mathbf{R}^2$, then it is also Borel in $\mathbf{R}^2$;

(2) there exists a Mazurkiewicz set which is of Lebesgue measure zero and of first Baire category;

(3) there exists a Mazurkiewicz set which is Lebesgue nonmeasurable and does not possess the Baire property.

Relatively recently it was established that in the Constructible Universe of Gödel there exists a Mazurkiewicz set which is co-analytic (see [9], [10]), but it is still unknown whether a Mazurkiewicz set can be Borel in $\mathbf{R}^2$.

In this short communication, we consider measurability properties of Mazurkiewicz sets with respect to the class $\mathcal{M}(\mathbf{R}^2)$ of all nonzero σ-finite translation invariant measures on $\mathbf{R}^2$. Let us introduce some notions in terms of this class.

A set $X \subset \mathbf{R}^2$ is called negligible with respect to the class $\mathcal{M}(\mathbf{R}^2)$ if the following two conditions are satisfied:

(a) there exists at least one measure $\nu \in \mathcal{M}(\mathbf{R}^2)$ such that $X \in \mathrm{dom}(\nu)$;

(b) for every measure $\mu \in \mathcal{M}(\mathbf{R}^2)$, the relation $X \in \mathrm{dom}(\mu)$ implies the equality $\mu(X) = 0$.

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A set $Y \subset \mathbf{R}^2$ is called absolutely negligible with respect to $\mathcal{M}(\mathbf{R}^2)$ if, for every measure $\mu \in \mathcal{M}(\mathbf{R}^2)$, there exists a measure $\mu' \in \mathcal{M}(\mathbf{R}^2)$ extending μ and such that $Y \in \mathrm{dom}(\mu')$ and $\mu'(Y) = 0$.

Let Z be a subset of $\mathbf{R}^2$ and let l be a straight line in $\mathbf{R}^2$. We say that Z is finite in direction l if each straight line in $\mathbf{R}^2$ parallel to l has finitely many common points with Z.

Lemma 1: *If $Z \subset \mathbf{R}^2$ is finite in some direction l, then Z is negligible with respect to the class $\mathcal{M}(\mathbf{R}^2)$. In particular, every Mazurkiewicz set is negligible with respect to the same class of measures.*

Remark 1: It is not difficult to demonstrate that there exists a measure ν on $\mathbf{R}^2$ which extends the standard two-dimensional Lebesgue measure λ_2 on $\mathbf{R}^2$, is invariant under the group of all isometric transformations of $\mathbf{R}^2$ and contains in its domain the family of all Mazurkiewicz sets.

In connection with Lemma 1, it is natural to ask whether every Mazurkiewicz set is absolutely negligible with respect to the class $\mathcal{M}(\mathbf{R}^2)$. We shall see below that, under an additional set-theoretical assumption, there exist Mazurkiewicz subsets of $\mathbf{R}^2$ which are not absolutely negligible with respect to $\mathcal{M}(\mathbf{R}^2)$.

Nevertheless, among Mazurkiewicz sets there are absolutely negligible ones.

Let $n \geq 1$ be a natural number and let $\mathbf{R}^n$ denote the Euclidean space of dimension n. This space can also be treated as a vector space over the field $\mathbf{Q}$ of all rational numbers. Any basis of the latter space is usually called a Hamel basis.

Lemma 2: *Every Hamel basis of the space $\mathbf{R}^n$ is an absolutely negligible subset of $\mathbf{R}^n$.*

For a proof of Lemma 2, see [2]. Notice that in [2] only the case of $n = 1$ is considered, but the same argument works for any natural number $n \geq 1$.

Lemma 3: *There exists a Mazurkiewicz subset of $\mathbf{R}^2$ which is a Hamel basis of $\mathbf{R}^2$.*

The above two lemmas imply the following statement.

Theorem 1: *There exists a Mazurkiewicz subset X of $\mathbf{R}^2$ which is absolutely negligible with respect to the class $\mathcal{M}(\mathbf{R}^2)$. So, for any measure $\mu \in \mathcal{M}(\mathbf{R}^2)$, there exists a measure $\mu' \in \mathcal{M}(\mathbf{R}^2)$ extending μ and such that $X \in \mathrm{dom}(\mu')$ and $\mu'(X) = 0$.*

Lemma 4: *Assume the Continuum Hypothesis (**CH**) and let G be a countable non-collinear subgroup of $\mathbf{R}^2$. Then there are two subsets T and Y of $\mathbf{R}^2$ satisfying these four conditions:*

(1) T is a Mazurkiewicz set and $Y \subset G + T$;

(2) $\mathrm{card}(Y) = \mathbf{c}$, where $\mathbf{c}$ denotes the cardinality of the continuum;

(3) Y is λ_2-thick, i.e., $Y \cap B \neq \emptyset$ for each Borel set $B \subset \mathbf{R}^2$ with $\lambda_2(B) > 0$;

(4) Y is almost translation invariant, i.e., for any vector $e \in \mathbf{R}^2$, the inequality $\mathrm{card}((e + Y)\triangle Y) < \mathbf{c}$ holds true.

By using the last lemma, the following statement can be derived.

Theorem 2: *The Mazurkiewicz set T of Lemma 4 is not absolutely negligible with respect to the class $\mathcal{M}(\mathbf{R}^2)$. Moreover, there exists a translation invariant* *measure* ν *on* $\mathbf{R}^2$ *extending the Lebesgue measure* λ_2 *and having the property that, for every translation invariant measure* ν' *on* $\mathbf{R}^2$ *extending* ν*, the set* T *is not* ν'*-measurable.*

Remark 2 : According to Theorem 2, under **CH**, there exists a Mazurkiewicz set which is not absolutely negligible with respect to the class $\mathcal{M}(\mathbf{R}^2)$. So, the natural question arises whether it is possible to establish within **ZFC** theory the existence of Mazurkiewicz sets which are not absolutely negligible with respect to the same class of measures. This question still remains open.

Remark 3 : In classical point set theory, there are many other interesting examples of subsets of Euclidean spaces, which possess extraordinary (pathological or paradoxical) properties. In this context, let us mention Vitali sets, Bernstein sets, Luzin sets, Sierpiński sets, etc. (see, for instance, [1], [5], [6], [8], [11], [12]). All such pathological sets can be considered from the point of view of their measurability with respect to various classes of measures. In particular, certain measurability properties of Vitali sets and of Bernstein sets are discussed in [3] and [4].

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