Bulletin of TICMI
Vol. 20, No. 2, 2016, 37–43

Asymptotic Center by a Sequence of Mappings

M.R. Haddadi*

Faculty of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
(*Received March 12, 2016; Revised November 28, 2016; Accepted Accepted 12, 2016*)

Main purpose of this paper is to generalize the concept of asymptotic center and give new extensions of some fixed point theorems. For this, we first prove some results by the asymptotic center definition. Next, we will introduce a new extension by sequences of functions, and we will prove existence theorems with it.

Keywords: K-Lipschitzian, Asymptotic center, Fixed points, Normal cone.

AMS Subject Classification: 46A32, 46M05, 41A17.

1. Introduction and preliminaries

In 1969, Ky Fan [4] proved that for any continuous function f from a compact convex subset C of a normed linear space X into X, there exists $x \in C$ such that $\|f(x)-x\| = dist(f(x), C)$. Since then, there have appeared several generalizations, extensions and applications of this theorem. Indeed,Reich [8] has shown that even if K is a non-empty approximately p-compact convex subset of a locally convex Hausdorff topological vector space E with a relatively compact image $f(K)$, then the same conclusion holds. Later, Segal and Singh [9] have extended this result to convex valued continuous multifunctions. Even though a best approximation theorem guarantees the existence of an approximate solution, it is contemplated to find an approximate solution which is optimal. In this direction, Srinivasan and Veeramani [10] have proved the general forms of existence theorems for best proximity pairs, and Kim and Lee [6] prove two general existence theorems of best proximity pairs in a recent paper.

Many of the generalizing topics in this paper are from Bose and Laskar [2], Downing and Kirk [3], Goebel and Kirk [5], and Lan and Webb [7].

Let X be a Banach space. Then a function $\delta_X : [0, 2] \to [0, 1]$ is said to be the modulus of convexity of X if

$$\delta_X(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \varepsilon\}.$$

Also the characteristic of convexity or the coefficient of convexity of the Banach space X is the number

$$\epsilon_0(X) = \sup\{\varepsilon \in [0, 2] : \delta_X(\varepsilon) = 0\}.$$

*Corresponding author. Email: haddadi@abru.ac.ir

ISSN: 1512-0082 print

Lemma 1.1: *[1] Let C be a weakly compact convex subset of a Banach space and $f : C \to (-\infty, \infty]$ a proper lower semicontinuous convex function. Then there exists $x_0 \in Dom(f)$ such that $f(x_0) = \inf\{f(x) : x \in C\}$.*

2. Main results

Definition 2.1: Let C be a nonempty subset of a Banach space X and let $\{f_n\}$ be a bounded sequence of continuous map on C. Consider the functional $r_a(., \{f_n\}) : C \to R^+$ defined by

$$r_a(x, \{f_n\}) = \limsup_{n\to\infty} \|f_n(x) - x\|.$$

The infimum of $r_a(x, \{f_n\})$ over C is denoted by $r_a(C, \{f_n\})$. A point $z \in C$ is said to be an asymptotic center of the sequence $\{f_n\}$ with respect to C if

$$r_a(z, \{f_n\}) = r_a(C, \{f_n\}).$$

The set of all asymptotic centers of $\{f_n\}$ with respect to C is denoted by $\mathcal{Z}_a(C, \{f_n\})$. On the other hand

$$\mathcal{Z}_a(C, \{f_n\}) = \{x : r_a(x, \{f_n\}) = r_a(C, \{f_n\})\}$$

This set may be empty, a singleton, or certain infinitely many points. In fact, if

$\lim_{n\to\infty} f_n(x) = x$, then

$$x \in \mathcal{Z}_a(C, \{f_n\}).$$

Several useful results of asymptotic center concept are discussed in the following. We now discuss the existence of asymptotic center of bounded sequences. We first establish a preliminary result:

Proposition 2.2: *Let C be a nonempty subset of X and let $\{f_n\}$ be a sequence of K-Lipschitzian maps such that $f_n : C \to X$. Then $r_a(., \{f_n\})$ is $(K + 1)$-Lipschitzian map.*

Proof: Suppose $\{f_n\}$ is a sequence of K-Lipschitzian maps. For $x, y \in X$ we have

$$\|x - f_n(x)\| \le \|x - y\| + \|y - f_n(y)\| + \|f_n(x) - f_n(y)\|.$$

Therefore

$$r_a(x, \{f_n\}) \le \|x - y\| + r_a(y, \{f_n\}) + \lim_{n\to\infty} \|f_n(x) - f_n(y)\|.$$

Thus

$$r_a(x, \{f_n\}) - r_a(y, \{f_n\}) \le \|x - y\| + K\|x - y\|.$$

Similarly by replacing roles of x and y we have

$$|r_a(x,\{f_n\}) - r_a(y,\{f_n\})| \leq (K+1)\|x-y\|.$$

□

We now discuss the existence and uniqueness of asymptotic center.

Proposition 2.3: *Let C be a nonempty weakly compact convex subset of Banach space X and let $\{f_n\}$ be a sequence of K-Lipschitzian maps such that $f_n : C \to X$. Then $\mathcal{Z}_a(C,\{f_n\})$ is nonempty.*

Proof: Since C is compact and $r_a(.,\{f_n\})$ is continuous, by Lemma 1.1 there exists $x_0 \in C$ such that $r_a(x_0,\{f_n\}) = r_a(C,\{f_n\})$ i.e. $x_0 \in \mathcal{Z}_a(C,\{f_n\})$. □

Let X be a normed linear space. We remember that a subset P of X is called a cone if
(i) P is closed, non-empty and $P \neq \{0\}$,
(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,
(iii) $P \cap -P = \{0\}$.
For a given cone $P \subseteq X$, we can define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$. The cone P is called normal if there is a number $M > 0$ such that for all $x, y \in X$, $0 \leq x \leq y$ implies $\|x\| \leq M\|y\|$.
The least positive number satisfying the above is called the normal constant of P.

Lemma 2.4: *Let C be a nonempty convex subset of Banach space X which is ordered by a normal cone P and let $\{f_n\}$ be a sequence of convex maps such that $f_n : C \to X$. Then $r_a(.,\{f_n\})$ is convex.*

Proof: We want to show that

$$r_a(\alpha x + (1-\alpha)y, \{f_n\}) \leq \alpha r_a(x,\{f_n\}) + (1-\alpha) r_a(y,\{f_n\})$$

for all $x, y \in X$ and $\alpha \in (0,1)$. Since $\{f_n\}$ is convex and X is an ordered Banach space with $\leq_p$ and normed constant $k = 1$, we have

$$f_n(\alpha x + (1-\alpha)y) - \alpha x + (1-\alpha)y \leq_p \alpha(f_n(x) - x) + (1-\alpha)(f_n(y) - y).$$

Thus

$$\|f_n(\alpha x + (1-\alpha)y) - \alpha x + (1-\alpha)y\| \leq \alpha\|f_n(x) - x\| + (1-\alpha)\|f_n(y) - y\|.$$

Hence

$$\begin{aligned}&\limsup_{n\to\infty} \|f_n(\alpha x + (1-\alpha)y) - \alpha x + (1-\alpha)y\| \\ &\qquad \leq \alpha \limsup_{n\to\infty} \|f_n(x) - x\| + (1-\alpha) \limsup_{n\to\infty} \|f_n(y) - y\|.\end{aligned}$$

□

Theorem 2.5 : *Let C be a nonempty convex compact subset of the Banach space X which is ordered by a normal cone P and $\{f_n\}$ a sequence of K-Lipschitzian convex maps such that $f_n : C \to X$. Then $\mathcal{Z}_a(C, \{f_n\})$ is a nonempty convex set.*

Proof : By Proposition 2.3 $\mathcal{Z}_a(C, \{f_n\})$ is nonempty. Suppose $x, y \in \mathcal{Z}_a(C, \{f_n\})$ and so

$$r_a(x, \{f_n\}) = r_a(y, \{f_n\}) = r_a(C, \{f_n\}).$$

By Lemma 2.4 $r_a(., \{f_n\})$ is convex, therefore for $t \in [0, 1]$ we have

$$r_a((1-t)x + ty, \{f_n\}) \leq (1-t)r_a(x, \{f_n\}) + tr_a(y, \{f_n\}) = r_a(C, \{f_n\}).$$

i.e $(1-t)x + ty \in \mathcal{Z}_a(C, \{f_n\})$. □

Theorem 2.6 : *Let C be a nonempty weakly compact convex subset of uniformly convex Banach space X which is ordered by a normal cone P and let $\{f_n\}$ be a sequence of K-Lipschitzian maps such that $f_n : C \to X$ where $K < 1$. Then $\mathcal{Z}_a(C, \{f_n\})$ is unique.*

Proof : Suppose C is an arbitrary bounded subset of X. Since $\{f_n\}$ are continuous and convex functions and $r_a(x, \{f_n\}) \to \infty$ as $\|x\| \to \infty$, by Lemma 1.1 $\mathcal{Z}_a(C, \{f_n\}) \neq \emptyset$. Suppose $\mathcal{Z}_a(C, \{f_n\})$ is not singleton. We claim that

$$(1-K)diam(\mathcal{Z}_a(C, \{f_n\})) \leq \epsilon_0(X)r_a(C, \{f_n\}).$$

Set $d = diam(\mathcal{Z}_a(C, \{f_n\}))$ that $d > 0$. Let $0 < r < d$ and $x, y \in \mathcal{Z}_a(C, \{f_n\})$ with $\|x - y\| \geq d - r$. By the convexity of $\mathcal{Z}_a(C, \{f_n\})$, $\frac{x+y}{2} \in \mathcal{Z}_a(C, \{f_n\})$. Also from the property of modulus of convexity for every $n \in \mathbb{N}$ we have

$$\|f_n(\frac{x+y}{2}) - \frac{x+y}{2}\| \leq \|\frac{f_n x - x}{2} + \frac{f_n y - y}{2}\|$$

$$\leq r_a(C, \{f_n\})[1 - \delta_X(\frac{\|f_n x - x - (f_n y - y)\|}{r_a(C, \{f_n\})})].$$

Therefore

$$\begin{aligned} r_a(C, \{f_n\}) &= \limsup_{n\to\infty} \|f_n(\frac{x+y}{2}) - \frac{x+y}{2}\| \\ &\leq r_a(C, \{f_n\}) \limsup_{n\to\infty}[1 - \delta_X(\frac{\|f_n x - x - (f_n y - y)\|}{r_a(C, \{f_n\})})] \end{aligned}$$

and thus

$$\liminf_{n\to\infty} \delta_X(\frac{\|f_n x - x - (f_n y - y)\|}{r_a(C, \{f_n\})}) \leq 0.$$

By definition of $\epsilon_0(X)$ and lim inf, there exists $n_0 \in \mathbb{N}$ such that

$$(1-K)(d-r) \leq (1-K)\|x - y\| \leq \|f_{n_0}x - x - (f_{n_0}y - y)\| \leq \epsilon_0(X)r_a(C, \{f_n\}).$$

Since $r > 0$ is arbitrary, we proved that the claim. By uniformly convexity of X, $\epsilon_0(X) = 0$ and so $diam(\mathcal{Z}_a(C, \{f_n\})) = 0$ that is contradiction. Therefore $\mathcal{Z}_a(C, \{f_n\})$ is singleton. $\square$

Theorem 2.7: *Let C be a nonempty closed convex subset of a uniformly convex Banach space. Then every bounded sequence $\{f_n\}$ in X has a unique asymptotic with respect to C, i.e., $\mathcal{Z}_a(C, \{f_n\}) = \{z\}$ and*

$$\limsup_{n\to\infty} \|f_n(z) - z\| < \limsup_{n\to\infty} \|f_n(x) - x\| \quad for\ x \neq z.$$

Proof: The result follows from Theorem 3.6. $\square$

Theorem 2.8: *Let C be a nonempty closed convex bounded subset of a uniformly convex Banach space X and let $\{f_n\}$ be a sequence of bounded maps such that $f_n : C \to X$ with $\mathcal{Z}_a(C, \{f_n\}) = \{z\}$ and $r_a(C, \{f_n\}) = r$. For $t \in (0, 1)$, let $g_n(w) = (1 - t)w + tf_n(w)$, $n \in N$, $for\ all\ w \in C$. Then $\mathcal{Z}_a(C, \{g_n\}) = \{z\}$ and $r_a(C, \{g_n\}) = tr$.*

Proof: Suppose, for contradiction, that $\mathcal{Z}_a(C, \{g_n\}) = v \neq z$. Since

$$\|g_n(z) - z\| = t\|f_n(z) - z\| \qquad for\ all\ n \in N,$$

it follows that

$$r_a(C, \{g_n\}) = \inf\{\limsup_{n\to\infty} \|g_n(w) - w\| : w \in C\} \leq tr.$$

Let $r_a(C, \{g_n\}) = r^{'}$. Since the asymptotic center v of $\{g_n\}$ is unique, we have

$$r^{'} = \limsup_{n\to\infty} \|g_n(v) - v\| \leq t \limsup_{n\to\infty} \|f_n(v) - v\| < tr.$$

For each $n \in N$, we have

$$\begin{aligned}
\|f_n(v) - v\| &= \|v - (1 - t)v - tf_n(v) + (1 - t)v - (1 - t)f_n(v)\| \\
&\leq \|v - [(1 - t)v + tf_n(v)]\| + (1 - t)\|f_n(v) - v\| \\
&= \|g_n(v) - v\| + (1 - t)\|f_n(v) - v\|,
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty} \|f_n(v) - v\| \leq r^{'} + (1 - t)r < r$$

contradicting $r_a(C, \{f_n\}) = r$. Thus, $\mathcal{Z}_a(C, \{g_n\}) = \{z\}$, we have $r_a(C, \{g_n\}) = tr$. $\square$

Let C be a nonempty subset of a Banach space X. We remember that for $x \in C$ the inward set of x relative to C is the set

$$I_C(x) = \{(1 - t)x + ty : y \in C, t \geq 0\},$$

and $T : C \to X$ is said to be a inward mapping if $Tx \in I_C(x)$ for all $x \in C$.

Theorem 2.9 : *Let C be a nonempty subset of uniformly convex Banach space X which is ordered by a normal cone P and let $\{f_n\}$ be a sequence of K-Lipschitzian bounded maps which are uniformly convergent to $f : C \to X$ and $K < 1$. If z is the asymptotic of $\{f_n\}$ with respect to C, then it is also asymptotic with respect to $I_C(z)$.*

Proof: Suppose that v is the asymptotic of $\{f_n\}$ with respect to $I_C(z)$. Suppose that $v \neq z$. For $v \neq z$ and $C \subseteq I_C(z)$, we have $v \in I_C(z)\backslash C$ and $f(v) < f(z)$ by the uniqueness of the asymptotic center and the continuity of $\{f_n\}$, there exists $w \in I_C(z)\backslash C$ such that $f(w) < f(z)$. Hence $w = (1 - t)z + ty$ for some $y \in C$ and $t > 1$. Since $f(.)$ is a convex function, $r_a(y, \{f_n\}) \leq f(t^{-1}w + (1 - t^{-1})z) = t^{-1}f(w) + (1 - t^{-1})f(z) < f(z)$, a contradiction. Hence $v = z$. □

Theorem 2.10 : *Let C be a nonempty weakly compact convex subset of a uniformly convex Banach space X which is ordered by a normal cone P and let $\{f_n\}$ be a sequence of K-Lipschitzian maps which are uniformly convergent to $f : C \to X$ and $K < 1$. If $T : C \to X$ is a inward nonexpansive mapping such that $T(\mathcal{Z}_a(I_C(z), f)) \subseteq \mathcal{Z}_a(I_C(z), f)$, then T has a fixed point.*

Proof: Let $z \in \mathcal{Z}_a(C, \{f_n\})$. Because $Tz \in I_C(z)$ and by Theorem 2.9 z is the asymptotic center of $\{f_n\}$ with respect to $I_C(z)$, i.e. $z, Tz \in \mathcal{Z}_a(I_C(z), \{f_n\})$ we conclude that from Theorem 2.6 $Tz = z$. □

Let C be a nonempty subset of Banach space X, $T : C \to X$. Then $x \in X$ is said to be foxed point T if $T(x) = x$, and we denote the set of all fixed points of T by $F(T)$. In the following we give new results in the fixed point.

Theorem 2.11 : *Let C be a nonempty subset of Banach space X, $T : C \to X$ nonexpansive and $f_n : C \to X$ such that $\mathcal{Z}_a(C, \{f_n\})$ is weakly compact and star-shaped. Also assume $T(\mathcal{Z}_a(C, \{f_n\})) \subseteq \mathcal{Z}_a(C, \{f_n\})$, $T(\partial C) \subseteq C$ and $I - T$ demiclosed on $\mathcal{Z}_a(C, \{f_n\})$. Then $F(T) \cap \mathcal{Z}_a(C, \{f_n\}) \neq \emptyset$.*

Proof: Let u be the star- of $\mathcal{Z}_a(C, \{f_n\})$ and let $\{a_n\}$ be a sequence in (0, 1) such that $a_n \to 1$. Define $T_n : \mathcal{Z}_a(C, \{f_n\}) \to \mathcal{Z}_a(C, \{f_n\})$ by

$$T_n x = (1 - a_n)u + a_n Tx.$$

For each $n \geq 1$, T_n is a contraction, so there exists exactly one fixed point x_n of T_n. Now since

$$\lim \|Tx_n - x_n\| \leq \lim_{n\to\infty} \|T_n x_n - Tx_n\|,$$

$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Since $\mathcal{Z}_a(C, \{f_n\})$ is weakly compact there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z \in \mathcal{Z}_a(C, \{f_n\})$. Since $I - T$ is demiclosed on $\mathcal{Z}_a(C, \{f_n\})$ and $x_{n_i} - Tx_{n_i} \to 0$, it follows that $z \in F(T)$. Therefore $F(T) \cap \mathcal{Z}_a(C, \{f_n\}) \neq \emptyset$. □

Corollary 2.12: *Let C be a nonempty subset of the Banach space X, $T : X \to X$ nonexpansive and $f_n : C \to X$ such that $\mathcal{Z}_a(C, \{f_n\})$ is compact and convex. If $T(\mathcal{Z}_a(C, \{f_n\})) \subseteq \mathcal{Z}_a(C, \{f_n\})$, then $F(T) \cap \mathcal{Z}_a(C, \{f_n\}) \neq \emptyset$.*

References

[1] V. Barbu and Th. Precupanu, *Convexity and optimization in Banach spaces*, Editura Academiei, Bucharest; Sijthof and Noordhof International Publishers, Alphen aan den Rijn, 1978

[2] S.C. Bose and S.K. Laskar, *Fixed point theorems for certain class of mappings*, J. Math. Phys. Sci., **19**(1985), 503-509

[3] D. Downing and W. A. Kirk, *Fixed point theorems for set-valued map- pings in metric and Banach spaces*, Math. Japonica, **22** (1977), 99-112

[4] Ky Fan, *Extensions of Two Fixed Point Theorems of F.E. Browder*, Math. Z., **112** (1969), 234-240

[5] K. Goebel and W.A. Kirk, *Topics in metric fixed point theory*, Cambridge University Press, Cambridge, 1990

[6] W.K. Kim and K.H. Lee, *Existence of best proximity pairs and equilibrium pairs*, J. Math. Anal. Appl., **316**, 2 (2006), 433-446

[7] K.Q. Lan and J.R.L. Webb, *A-properness and fixed point theorems for dissipative type maps,* Abstract Appl. Anal., **4**, 2 (1999), 83-100

[8] S. Reich, *Approximate selections, best approximations, fixed points, and invariant sets*, J. Math. Anal. Appl., **62**, 1 (1978), 104-113

[9] V.M. Sehgal and S.P. *Singh, A generalization to multifunctions of Fan's best approximation theorem*, Proc. Amer. Math. Soc., **102**, 3 (1988), 534-537

[10] P.S. Srinivasan and P. Veeramani, *On best proximity pair theorems and fixed-point theorems*, Abstr. Appl. Anal., **2003**, 1 (2003), 33-47