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Method of Fundamental Solutions for Transmission Problems

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We investigate three-dimensional basic and mixed transmission problems for Helmholtz equation in piece wise homogeneous media by the potential method. We show uniqueness and existence of solutions in the L_p based Sobolev and Bessel potential spaces and develop the fundamental solutions method for transmission problems.

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1. Preliminary material

The basic interior and exterior boundary value problems for the Helmholtz equation by different methods are studied in scientific literature in various function spaces for smooth and non-smooth domains (see [43], [37], [10], [11], [12], [31], [1]).

Applying the potential method and the theory of pseudodifferential operators, here we study three-dimensional transmission problems for Helmholtz equation, in particular, we consider two types of problems: basic transmission problem and mixed transmission problem.

First we establish the uniqueness and existence of solutions and derive the corresponding estimates, and afterwards we develop the method of fundamental solutions for the problems under consideration.

The Method of Fundamental Solutions (MFS) was first proposed by the well known Georgian mathematician Victor Kupradze in the 1960s (see the pioneering works in this direction by V. Kupradze and M. Alexidze, [23], [24], [25]). The main idea of the MFS is to distribute the singularity poles $\{y^{(k)}\}_{k=1}^{\infty}$ of the fundamental solution $\Gamma(x-y)$ of a differential operator outside the domain under consideration, construct the set of functions $\{\Gamma(x-y^{(k)})\}_{k=1}^{\infty}$, prove its density properties and linear independency in appropriate function spaces, and then ap-

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proximate the sought for solution by a linear combination of the fundamental solutions $\sum_{k=1}^{N} C_k \,\Gamma(x-y^{(k)})$ with unknown coefficients C_k, which are to be determined by satisfying the corresponding boundary conditions. Starting from the 1970s, the MFS gradually became a useful technique and is used to solve a large variety of physical and engineering problems (see [5], [23], [24], [25], [26], [30], [3], [22], [42], [34], [39], [15], [8], [9], [20], [29], [21] and the references therein). It should be mentioned that until now it has not been worked out how to apply the MFS to crack type problems, since the different approaches related to MFS described in the scientific literature are not applicable to interior crack type problems.

Note that interior crack type problems can be reformulated as mixed type transmission problems introducing an artificial interface boundary surface containing the crack faces. Consequently, the MFS for mixed type transmission problems, which is described in this paper, is applicable also to the interior crack type problems.

Let $\Omega^+ = \Omega_1 \subset \mathbb{R}^3$ be a bounded domain with a smooth boundary S. Further, let $\Omega^- = \Omega_2 := \mathbb{R}^3 \setminus \overline{\Omega^+}$, $\partial\Omega^\pm = S$, $\overline{\Omega^\pm} = \Omega^\pm \cup S$. Throughout the paper $n(x)$ stands for the outward unit normal vector at the point $x \in S$. The symbols $\{\cdot\}^\pm$ denote one sided limiting values (traces) on $S = \partial\Omega^\pm$ from $\Omega^\pm$. Without loss of generality we assume that the origin is located in the domain Ω_1.

Further, let us consider the following regular dissection of the interface S: $S = \overline{S}_T \cup \overline{S}_C$, where $S_T \cap S_C = \varnothing$. Here S_T is the so called "transmission part", while S_C represents the "interface crack part".

Throughout the paper, for simplicity we assume that the surfaces S and the curve $\ell = \overline{S}_T \cap \overline{S}_C$ are C^∞-smooth if not otherwise stated.

By $C^{k,\alpha}$ with nonnegative integer k and $0 < \alpha \leqslant 1$, we denote the space of functions whose k-th order partial derivatives are Hölder continuous functions with exponent α. By L_p, $L_{p,loc}$, $L_{p,comp}$, W_p^r, $W_{p,loc}^r$, $W_{p,comp}^r$, H_p^s, and $B_{p,q}^s$ (with $r \ge 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$) we denote the well-known Lebesgue, Sobolev-Slobodetskii, Bessel potential, and Besov function spaces, respectively (see, e.g., [40], [41], [28], [2]). Recall that $H_2^r = W_2^r = B_{2,2}^r$, $H_2^s = B_{2,2}^s$, $W_p^t = B_{p,p}^t$, and $H_p^k = W_p^k$, for any $r \ge 0$, for any $s \in \mathbb{R}$, for any positive and non-integer t, and for any non-negative integer k. In our analysis we essentially employ also the spaces:

$$\widetilde{H}_p^s(\mathcal{M}) := \{f \,:\, f \in H_p^s(\mathcal{M}_0),\ \mathrm{supp}\, f \subset \overline{\mathcal{M}}\},$$

$$\widetilde{B}_{p,q}^s(\mathcal{M}) := \{f \,:\, f \in B_{p,q}^s(\mathcal{M}_0),\ \mathrm{supp}\, f \subset \overline{\mathcal{M}}\},$$

$$H_p^s(\mathcal{M}) := \{r_{\mathcal{M}} f \,:\, f \in H_p^s(\mathcal{M}_0)\},$$

$$B_{p,q}^s(\mathcal{M}) := \{r_{\mathcal{M}} f \,:\, f \in B_{p,q}^s(\mathcal{M}_0)\,\},$$

where $\mathcal{M}_0$ is a closed manifold without boundary and $\mathcal{M}$ is an open proper submanifold of $\mathcal{M}_0$ with nonempty smooth boundary $\partial\mathcal{M} \neq \varnothing$; $r_{\mathcal{M}}$ is the restriction operator onto $\mathcal{M}$.

Remark 1 : Let a function f be defined on an open proper submanifold $\mathcal{M}$ of a closed manifold $\mathcal{M}_0$ without boundary. Let $f \in B_{p,q}^s(\mathcal{M})$ and $\widetilde{f}$ be the extension of f by zero to $\mathcal{M}_0 \setminus \mathcal{M}$. If the extension preserves the space, i.e., if $\widetilde{f} \in \widetilde{B}_{p,q}^s(\mathcal{M})$, then we write $f \in \widetilde{B}_{p,q}^s(\mathcal{M})$ instead of $f \in r_{\mathcal{M}}\widetilde{B}_{p,q}^s(\mathcal{M})$ when it does not lead to misunderstanding.

Consider the Helmholtz equation

$$L(\partial,\omega)\,u(x) := (\Delta+\varkappa^2)\,u(x) = 0, \qquad x\in\Omega^{\pm}, \tag{1.1}$$

where $\Delta = \partial_1^2+\partial_2^2+\partial_3^2$ is the Laplace operator, $\partial_j = \partial/\partial x_j$, $\partial := (\partial_1,\partial_2,\partial_3)$, $\varkappa \in \mathbb{R}$ is the so called frequency parameter.

We say that u belongs to the Sommerfeld class of radiating functions in an unbounded domain Ω^- and write $u\in\mathcal{S}(\Omega^-)$ if for sufficiently large $|x|$ the relation

$$\frac{\partial u(x)}{\partial r} - i\,\varkappa\,u(x) = o(r^{-1}), \quad r=|x|\,, \tag{1.2}$$

holds uniformly in all directions $x/|x|$ (see [43], [37], [10], [11]).

Denote by $\Gamma(x-y,\omega)$ the fundamental solution that corresponds to outgoing waves and satisfies the Sommerfeld radiation condition,

$$\Gamma(x-y,\varkappa) = -\frac{1}{4\pi}\,\frac{e^{i\,\varkappa\,|x-y|}}{|x-y|}\,. \tag{1.3}$$

Introduce the single and double layer potentials associated with the fundamental solution (1.3):

$$V(g)(x) \equiv V_{S}(g)(x) = \int\limits_S \Gamma(x-y,\varkappa)\,g(y)\,dS, \qquad x\in\mathbb{R}^3\setminus S, \tag{1.4}$$

$$W(g)(x) \equiv W_{S}(g)(y) = \int\limits_S \big[\partial_{n(y)}\Gamma(x-y,\varkappa)\big]\,g(y)\,dS, \qquad x\in\mathbb{R}^3\setminus S, \tag{1.5}$$

where f and g are densities of the potentials and $\partial_n := \frac{\partial}{\partial n}$ denotes the normal derivative.

It is well known that these potentials have the following properties (see, e.g., [32], [43], [33], [10], [11], [17], [12], [31], [6], [13], [1]).

Theorem 1.1 : *Let $1<p<\infty$ and $s\in\mathbb{R}$. The operators*

$$\begin{aligned} &V : B^s_{p,p}(S)\to H_p^{s+1+\frac1p}(\Omega^+), \qquad && V : B^s_{p,p}(S)\to H_{p,\,loc}^{s+1+\frac1p}(\Omega^-)\cap\mathcal{S}(\Omega^-)\,,\\ &W : B^s_{p,p}(S)\to H_p^{s+\frac1p}(\Omega^+), && W : B^s_{p,p}(S)\to H_{p,\,loc}^{s+\frac1p}(\Omega^-)\cap\mathcal{S}(\Omega^-)\,, \end{aligned}$$

are continuous.

If $g\in B_{p,p}^{-\frac1p}(S)$, $h\in B_{p,p}^{1-\frac1p}(S)$, then $V,W\in C^\infty(\Omega^\pm)$ and

$$L(\partial,\omega)V\big(g\big)\,(x)=0, \quad L(\partial,\omega)W\big(g\big)\,(x)=0, \qquad in \;\; \Omega^{\pm}, \tag{1.6}$$

$$\big\{V(g)(x)\big\}^+ = \big\{V(g)(x)\big\}^- = \mathcal{H}\,g(x) \qquad on \quad S, \tag{1.7}$$

$$\big\{\partial_{n(x)}\,V(g)(x)\big\}^{\pm} = \big[\,\mp 2^{-1}\,I + \widetilde{\mathcal{K}}\,\big]\,g(x), \qquad on \quad S, \tag{1.8}$$

$$\big\{W(h)\,(x)\big\}^{\pm} = \big[\,\pm\, 2^{-1} I + \mathcal{K}\,\big]\, h(x) \qquad on \qquad S, \tag{1.9}$$

$$\big\{\partial_n\, W(h)(x)\big\}^{+} = \big\{\partial_n\, W(h)(x)\big\}^{-} \equiv \mathcal{L}\, h(x) \qquad on \qquad S, \tag{1.10}$$

where I stands for the identity operator, while $\widetilde{\mathcal{K}}$, $\mathcal{K}$, and $\mathcal{H}$ are boundary integral operators

$$\widetilde{\mathcal{K}}\, g(x) := \int\limits_S \big[\partial_{n(x)}\Gamma(x-y,\omega)\big]\, g(y)\, dS, \qquad x \in S, \tag{1.11}$$

$$\mathcal{K}\, g(x) := \int\limits_S \big[\partial_{n(y)}\Gamma(x-y,\omega)\big]\, g(y)\, dS, \qquad x \in S, \tag{1.12}$$

$$\mathcal{H}\, g(x) := \int\limits_S \Gamma(x-y,\omega)\, g(y)\, dS, \qquad x \in S. \tag{1.13}$$

Moreover, the following mappings are bounded

$$\begin{aligned} &\mathcal{H} \;:\; B^s_{p,p}(S) \to B^{s+1}_{p,p}(S)\,, \qquad && \widetilde{\mathcal{K}} \;:\; B^s_{p,p}(S) \to B^s_{p,p}(S)\,,\\ &\mathcal{K} \;:\; B^s_{p,p}(S) \to B^s_{p,p}(S)\,, && \mathcal{L} \;:\; B^{s+1}_{p,p}(S) \to B^s_{p,p}(S)\,. \end{aligned}$$

For $S \in C^{1,\alpha}$, $0 < \alpha \leqslant 1$, the operators $\mathcal{H}$, $\widetilde{\mathcal{K}}$, and $\mathcal{K}$ are weakly singular operators, while $\mathcal{L}$ is a singular integro-differential operator.

If $S \in C^{k+1,\alpha}$ with $k \geqslant 1$ and $0 < \beta < \alpha \leqslant 1$. Then the following operators are continuous

$$\begin{aligned} &V \;:\; C^{k,\beta}(S) \longrightarrow C^{k+1,\beta}(\overline{\Omega^{\pm}}), \qquad && W \;:\; C^{k,\beta}(S) \longrightarrow C^{k,\beta}(\overline{\Omega^{\pm}}),\\ &\mathcal{H} \;:\; C^{k,\beta}(S) \longrightarrow C^{k+1,\beta}(S)\,, && \widetilde{\mathcal{K}}, \mathcal{K} \;:\; C^{k,\beta}(S) \longrightarrow C^{k,\beta}(S)\,,\\ &\mathcal{L} \;:\; C^{k,\beta}(S) \longrightarrow C^{k-1,\beta}(S)\,. && \end{aligned}$$

Remark 2 : The following operator equalities hold true in appropriate function spaces:

$$\mathcal{K}\,\mathcal{H} = \mathcal{H}\,\widetilde{\mathcal{K}}, \qquad\qquad \mathcal{L}\,\mathcal{K} = \widetilde{\mathcal{K}}\,\mathcal{L}, \tag{1.14}$$

$$\mathcal{H}\,\mathcal{L} = -4^{-1}\, I_6 + \mathcal{K}^{\,2}, \quad \mathcal{L}\,\mathcal{H} = -4^{-1}\, I_6 + \widetilde{\mathcal{K}}^{\,2}. \tag{1.15}$$

2. Basic transmission problem

The basic transmission problem (BT) is formulated as follows: Find complex valued functions

$$u_1 \in H^1_p(\Omega_1), \qquad u_2 \in H^1_{p,\,loc}(\Omega_2) \cap \mathcal{S}(\Omega_2), \quad 1 < p < \infty, \tag{2.1}$$

satisfying the Helmoltz equations in the corresponding domains,

$$(\Delta + \varkappa_1^2)\, u_1(x) = 0, \quad x \in \Omega_1, \quad \varkappa_1^2 = \varrho_1\, \omega^2, \tag{2.2}$$

$$(\Delta + \varkappa_2^2)\, u_2(x) = 0, \quad x \in \Omega_2, \quad \varkappa_2^2 = \varrho_2\, \omega^2, \tag{2.3}$$

and the transmission conditions on S:

$$\{u_1(x)\}^+ - \{u_2(x)\}^- = f(x), \qquad x \in S, \tag{2.4}$$

$$\{\partial_{n(x)} u_1(x)\}^+ - \{\partial_{n(x)} u_2(x)\}^- = F(x), \qquad x \in S, \tag{2.5}$$

where ϱ_j, $j = 1, 2$, are positive constants, $\omega \in \mathbb{R}$ is a frequency parameter, and

$$f \in B_{p,p}^{1-\frac{1}{p}}(S), \qquad F \in B_{p,p}^{-\frac{1}{p}}(S). \tag{2.6}$$

Here equations (2.2) and (2.3) are understood in the distributional sense, the Dirichlet type condition (2.4) is understood in the usual trace sense, while the Neumann type condition (2.5) is understood in the generalized trace sense defined by the corresponding Green's formulas (see, e.g. [31, Ch. 4]).

Applying the celebrated Rellich-Vekua lemma one can prove the following uniqueness theorem (cf. [43], [10], [10]).

Theorem 2.1: *The homogeneous basic transmission problem* (2.1)-(2.5) *(with $f = F = 0$) possesses only the trivial solution for $p = 2$.*

To prove the existence results we look for a solution pair (u_1, u_2) in the form of surface potentials:

$$u_1(x) = V_1(g_1)(x), \quad x \in \Omega_1, \tag{2.7}$$

$$u_2(x) = W_2(g_2)(x) + a\, V_2(g_2)(x), \quad x \in \Omega_2, \tag{2.8}$$

where V_j and W_j denote the single and double layer potentials on S constructed by the fundamental solution $\Gamma(x-y, \varkappa_j)$, $j = 1, 2$, and $g_1 \in B_{p,p}^{-\frac{1}{p}}(S)$ and $g_2 \in B_{p,p}^{1-\frac{1}{p}}(S)$ are unknown densities. Here a is a complex constant,

$$a = a_1 + i\, a_2, \quad a_j \in \mathbb{R},\ j = 1, 2, \quad a_2 \neq 0. \tag{2.9}$$

The contact conditions (2.4)-(2.5) lead to the following system of pseudodifferential equations with respect to g_1 and g_2:

$$\mathcal{H}_1 g_1 - \big(-2^{-1} I + \mathcal{K}_2 + a\, \mathcal{H}_2 \big)\, g_2 = f \quad \text{on} \quad S, \tag{2.10}$$

$$\big(-2^{-1} I + \widetilde{\mathcal{K}}_1 \big) g_1 - \big[\mathcal{L}_2 + a\, \big(2^{-1} I + \widetilde{\mathcal{K}}_2 \big) \big]\, g_2 = F \quad \text{on} \quad S, \tag{2.11}$$

where the pseudodifferential operators $\mathcal{H}_j$, $\mathcal{K}_j$, $\widetilde{\mathcal{K}}_j$, and $\mathcal{L}_j$ are generated by the layer potentials V_j and W_j and their normal derivatives (see (1.10)-(1.13)).

The operators

$$\mathcal{D}_2 := -2^{-1}I + \mathcal{K}_2 + a\,\mathcal{H}_2 \;:\; B^s_{p,p}(S) \to B^s_{p,p}(S), \tag{2.12}$$

$$\mathcal{N}_2 := \mathcal{L}_2 + a\left(2^{-1}I + \widetilde{\mathcal{K}}_2\right) \;:\; B^{s+1}_{p,p}(S) \to B^s_{p,p}(S) \tag{2.13}$$

are invertible operators for arbitrary $p \in (1,\infty)$ and $s \in \mathbb{R}$ (see [4], [10], [36], [27], [18], [19]). Therefore from (2.10) we can define g_2,

$$g_2 = \mathcal{D}_2^{-1}\mathcal{H}_1 g_1 - \mathcal{D}_2^{-1} f \quad \text{on} \quad S, \tag{2.14}$$

and substitute it into equation (2.11) to obtain

$$\big[\big(-2^{-1}I + \widetilde{\mathcal{K}}_1\big) - \mathcal{N}_2\mathcal{D}_2^{-1}\mathcal{H}_1\big] g_1 = F - \mathcal{N}_2\mathcal{D}_2^{-1} f \quad \text{on} \quad S, \tag{2.15}$$

where $\mathcal{D}_2^{-1}$ is the operator inverse to (2.12).

Evidently, the systems (2.10)-(2.11) and (2.14)-(2.15) are equivalent.

Denote

$$\mathcal{P} := \big(-2^{-1}I + \widetilde{\mathcal{K}}_1\big) - \mathcal{N}_2\mathcal{D}_2^{-1}\mathcal{H}_1. \tag{2.16}$$

Let us show that the operator

$$\mathcal{P} \;:\; B^s_{p,p}(S) \to B^s_{p,p}(S), \;\; 1 < p < \infty, \quad s \in \mathbb{R}, \tag{2.17}$$

is invertible.

It can easily be verified that the principal homogeneous symbol $\mathfrak{S}(\mathcal{P};x,\xi)$, $x \in S$, $\xi \in \mathbb{R}^2 \setminus \{0\}$, of the operator $\mathcal{P}$ is elliptic. Indeed, taking into account that

$$\mathfrak{S}(\mathcal{H}_j;x,\xi) = -\frac{1}{2|\xi|}, \;\; \mathfrak{S}(\mathcal{L}_j;x,\xi) = \frac{|\xi|}{2}, \;\; \mathfrak{S}(\widetilde{\mathcal{K}}_j;x,\xi) = 0, \;\; j = 1,2, \tag{2.18}$$

$$\mathfrak{S}(\mathcal{D}_2^{-1};x,\xi) = -2, \quad \mathfrak{S}(\mathcal{N}_2\mathcal{H}_1;x,\xi) = \mathfrak{S}(\mathcal{N}_2\mathcal{H}_2;x,\xi) = -\frac{1}{4}, \tag{2.19}$$

we get

$$\mathfrak{S}(\mathcal{P};x,\xi) = -1. \tag{2.20}$$

This implies that the operator (2.17) is Fredholm with zero index. Now we show that the null-space of the operator (2.17) is trivial. Indeed, let first $p = 2$, $s = -\frac{1}{2}$, and let $\widetilde{g}_1 \in B^{-\frac{1}{2}}_{2,2}(S) = H_2^{-\frac{1}{2}}(S)$ be a solution of the homogeneous equation $\mathcal{P}\widetilde{g}_1 = 0$ on S. Then the functions $\widetilde{g}_1$ and $\widetilde{g}_2 = \mathcal{D}_2^{-1}\mathcal{H}_1\widetilde{g}_1$ will be solutions to the homogeneous system (2.10)-(2.11) with $f = F = 0$. It then follows that the functions $\widetilde{u}_1$ and $\widetilde{u}_2$ defined by the equalities

$$\widetilde{u}_1(x) = V_1(\widetilde{g}_1)(x), \; x \in \Omega_1, \qquad \widetilde{u}_2(x) = W_2(\widetilde{g}_2)(x) + a\,V_2(\widetilde{g}_2)(x), \; x \in \Omega_2,$$

belong to the spaces $H_2^1(\Omega_1)$ and $H_{2,loc}^1(\Omega_2)\cap\mathcal{S}(\Omega_2)$ respectively and solve the homogeneous basic transmission problem. Consequently, due to the uniqueness Theorem 2.1, we deduce:

$$\widetilde{u}_1(x)=0,\ x\in\Omega_1,\qquad \widetilde{u}_2(x)=0,\ x\in\Omega_2.$$

From these relations the equalities $\widetilde{g}_1=0$ and $\widetilde{g}_2=0$ follow immediately, implying that the null-space of the operator (2.17) is trivial for $p=2,\ s=-\frac{1}{2}$. Thus the operator (2.17) is invertible for these particular values of the parameters. Due to the general theory of pseudodifferential equations on manifolds without boundary then it follows that the operator (2.17) is invertible for an arbitrary $1<p<\infty$ and $s\in\mathbb{R}$.

In particular, the operator

$$\mathcal{P}\ :\ B_{p,p}^{-\frac{1}{p}}(S)\to B_{p,p}^{-\frac{1}{p}}(S),\ \ 1<p<\infty, \tag{2.21}$$

is invertible and equation (2.15) is uniquely solvable for an arbitrary right hand side function. Therefore the solution pair (g_1,g_2) of system (2.10)-(2.11) is representable as follows:

$$g_1=\mathcal{P}^{-1}F-\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1}f\in B_{p,p}^{-\frac{1}{p}}(S), \tag{2.22}$$

$$g_2=\mathcal{D}_2^{-1}\mathcal{H}_1\mathcal{P}^{-1}F-\mathcal{D}_2^{-1}\mathcal{H}_1\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1}f-\mathcal{D}_2^{-1}f\in B_{p,p}^{1-\frac{1}{p}}(S). \tag{2.23}$$

Remark 1 : With the help of the imbedding theorems it can be shown that if

$$S\in C^{2,\alpha},\quad f\in C^{1,\beta}(S),\quad F\in C^{0,\beta}(S),\quad 0<\beta<\alpha\le 1, \tag{2.24}$$

then for the functions g_1 and g_2 defined by (2.18) and (2.19) the following inclusions are true:

$$g_1\in C^{0,\beta}(S),\quad g_2\in C^{1,\beta}(S), \tag{2.25}$$

implying the following inclusions in view of formulas (2.7)-(2.8) and mapping properties of potentials and boundary integral operators described in Theorem 1.1:

$$u_1\in C^{\infty}(\Omega_1)\cap C^{1,\beta}(\overline{\Omega}_1),\quad u_2\in C^{\infty}(\Omega_2)\cap C^{1,\beta}(\overline{\Omega}_2)\cap\mathcal{S}(\Omega_2). \tag{2.26}$$

The above results lead to the following existence theorem.

Theorem 2.2 : *Let conditions* (2.6) *be satisfied. Then the basic transmission problem* (2.1)-(2.5) *is uniquely solvable and solutions can be represented in the form* (2.7)-(2.8)*, where the densities g_1 and g_2 are given by formulas* (2.22)-(2.23)*. Moreover we have the following estimates*

$$\|u_1\|_{H_p^1(\Omega_1)}\le C_1\left(\|f\|_{B_{p,p}^{1-\frac{1}{p}}(S)}+\|F\|_{B_{p,p}^{-\frac{1}{p}}(S)}\right), \tag{2.27}$$

$$\|u_2\|_{H_p^1(\Omega_2\cap B(R))}\le C_2(R)\left(\|f\|_{B_{p,p}^{1-\frac{1}{p}}(S)}+\|F\|_{B_{p,p}^{-\frac{1}{p}}(S)}\right), \tag{2.28}$$

where C_1 is some positive constant, $B(R)$ is a ball centered at the origin and radius R, such that $\overline{\Omega}_1 \subset B(R)$, and $C_2(R)$ is a positive constant which depends on R.

Proof: Existence of a solution for $p \in (1,\infty)$ follows from the invertibility of the operator (2.21), formulas (2.22)-(2.23), and representations (2.7)-(2.8). It remains to prove uniqueness of solutions for arbitrary $p > 1$. Let $(u_1, u_2) \in H^1_p(\Omega_1) \times \big(H^1_{p,loc}(\Omega_2) \cap \mathcal{S}(\Omega_2)\big)$ be a solution to the homogeneous basic transmission problem. It can be shown that any solution of the homogeneous Helmholtz equation in Ω_1 from the space $H^1_p(\Omega_1)$ is uniquely representable in the form of a single layer potential (2.7) with $g_1 = [T]^{-1}(\{\partial_n u_1\}^+ + a\{u_1\}^+) \in B^{-1/p}_{p_1p}(S)$, where

$$T = -\frac{1}{2} I + \widetilde{\mathcal{K}}_1 + a\varkappa_1 : B^{-1/p}_{p_1p}(S) \to B^{-1/p}_{p_1p}(S)$$

ia invertible operator. Similarly, any solution of the homogeneous Helmholtz equation in Ω_2 from the class $H^1_{p,loc}(\Omega_2) \cap \mathcal{S}(\Omega_2)$ is uniquely representable by the linear combination of layer potentials (2.8) with $g_2 = \mathcal{D}_2^{-1}\{u_2\}^+ \in B^{1-\frac{1}{p}}_{p,p}(S)$ (cf. [43], [18], [19]) . Since the homogeneous system (2.10)-(2.11) in the space $B^{-\frac{1}{p}}_{p,p}(S) \times B^{1-\frac{1}{p}}_{p,p}(S)$ has only the trivial solution, we conclude that $g_1 = g_2 = 0$ on S implying $u_1 = 0$ in Ω_1 and $u_2 = 0$ in Ω_2.

The norm estimates (2.27)-(2.28) follow from the properties of the layer potentials described in Theorem 1.1 and invertibility of the boundary operators involved in formulas (2.22)-(2.23). □

3. Mixed transmission problem

The mixed transmission problem (MT) is formulated as follows: Find complex valued functions

$$u_1 \in H^1_p(\Omega_1), \qquad u_2 \in H^1_{p,\,loc}(\Omega_2) \cap \mathcal{S}(\Omega_2), \quad 1 < p < \infty, \tag{3.1}$$

satisfying the Helmoltz equations in the corresponding domains,

$$(\Delta + \varkappa_1^2)\, u_1(x) = 0, \quad x \in \Omega_1, \quad \varkappa_1^2 = \varrho_1\, \omega^2, \tag{3.2}$$

$$(\Delta + \varkappa_2^2)\, u_2(x) = 0, \quad x \in \Omega_2, \quad \varkappa_2^2 = \varrho_2\, \omega^2, \tag{3.3}$$

and the mixed transmission conditions on the transmission part S_T and crack part S_C:

$$\{u_1(x)\}^+ - \{u_2(x)\}^- = f_1(x), \qquad x \in S_T, \tag{3.4}$$

$$\{\partial_{n(x)} u_1(x)\}^+ - \{\partial_{n(x)} u_2(x)\}^- = F_1(x), \qquad x \in S_T, \tag{3.5}$$

$$\{\partial_{n(x)} u_1(x)\}^+ = F^{(+)}(x), \qquad x \in S_C, \tag{3.6}$$

$$\{\partial_{n(x)} u_2(x)\}^- = F^{(-)}(x), \qquad x \in S_C, \tag{3.7}$$

where ϱ_j, $j = 1, 2$, and ω are as in the formulation of the basic transmission problem (BT), and

$$f_1 \in B_{p,p}^{1-\frac{1}{p}}(S_T), \quad F_1 \in B_{p,p}^{-\frac{1}{p}}(S_T), \quad F^{(\pm)} \in B_{p,p}^{-\frac{1}{p}}(S_C). \tag{3.8}$$

In addition we require that the following compatibility condition is satisfied:

$$F := \begin{cases} F_1 & \text{on } S_T, \\ F^{(+)} - F^{(-)} & \text{on } S_C, \end{cases} \qquad F \in B_{p,p}^{-\frac{1}{p}}(S). \tag{3.9}$$

It is evident that the crack type conditions (3.6)-(3.7) are equivalent to the following two conditions:

$$\{\partial_{n(x)} u_1(x)\}^+ - \{\partial_{n(x)} u_2(x)\}^- = F^{(+)}(x) - F^{(-)}(x), \qquad x \in S_C, \tag{3.10}$$

$$\{\partial_{n(x)} u_1(x)\}^+ + \{\partial_{n(x)} u_2(x)\}^- = F^{(+)}(x) + F^{(-)}(x), \qquad x \in S_C, \tag{3.11}$$

Denote by $\widetilde{f}_1$ some fixed extension of the function f_1 from S_T onto the whole of S preserving the smoothness:

$$\widetilde{f}_1 \in B_{p,p}^{1-\frac{1}{p}}(S), \qquad r_{_{S_T}} \widetilde{f}_1 = f_1 \quad \text{on } S_T. \tag{3.12}$$

Then an arbitrary extension f of the function f_1 from S_T onto the whole of S preserving the smoothness has the form $f = \widetilde{f}_1 + g$ where

$$g \in \widetilde{B}_{p,p}^{1-\frac{1}{p}}(S_C). \tag{3.13}$$

Now we can reformulate equivalently the mixed transmission conditions (3.4)-(3.7) as follows:

$$\{u_1(x)\}^+ - \{u_2(x)\}^- = f_1(x), \qquad x \in S_T, \tag{3.14}$$

$$\{\partial_{n(x)} u_1(x)\}^+ - \{\partial_{n(x)} u_2(x)\}^- = F(x), \qquad x \in S, \tag{3.15}$$

$$\{\partial_{n(x)} u_1(x)\}^+ + \{\partial_{n(x)} u_2(x)\}^- = F^{(+)}(x) + F^{(-)}(x), \qquad x \in S_C, \tag{3.16}$$

where F is defined by (3.9).

Applying again the Rellich-Vekua lemma one can prove the following uniqueness theorem (cf. [43], [10], [10]).

Theorem 3.1: *The homogeneous mixed transmission problem* (3.1)-(3.7) *possesses only the trivial solution for p=2.*

Motivated by the results obtained in the previous section, we look for solution pair of the mixed transmission problem (MT) in the form of layer protentials

$$u_1(x) = V_1(g_1)(x), \quad x \in \Omega_1, \tag{3.17}$$

$$u_2(x) = W_2(g_2)(x) + a\, V_2(g_2)(x), \quad x \in \Omega_2, \tag{3.18}$$

where

$$g_1 = \mathcal{P}^{-1}F - \mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1}(\widetilde{f}_1 + g), \tag{3.19}$$

$$g_2 = \mathcal{D}_2^{-1}\mathcal{H}_1\mathcal{P}^{-1}F - \mathcal{D}_2^{-1}\mathcal{H}_1\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1}(\widetilde{f}_1 + g) - \mathcal{D}_2^{-1}(\widetilde{f}_1 + g) \tag{3.20}$$

with unknown function g satisfying the inclusion (3.13).

It can be verified easily that the conditions (3.2)-(3.3), (3.14), and (3.15) are satisfied automatically, while the condition (3.16) leads to the following pseudodifferential equation with respect to the unknown function g:

$$\big(-2^{-1}I + \widetilde{\mathcal{K}}_1\big)g_1 + \big[\mathcal{L}_2 + a\left(2^{-1}I + \widetilde{\mathcal{K}}_2\right)\big]\, g_2 = F^{(+)} + F^{(-)} \quad \text{on} \ \ S_C,$$

which can be rewritten as

$$r_{S_C}\Big\{-\big(-2^{-1}I + \widetilde{\mathcal{K}}_1\big)\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1} \\ - \big[\mathcal{L}_2 + a\left(2^{-1}I + \widetilde{\mathcal{K}}_2\right)\big]\mathcal{D}_2^{-1}\big(\mathcal{H}_1\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1} + I\big)\Big\}g = \Psi \quad \text{on} \ \ S_C, \tag{3.21}$$

where

$$\Psi = F^{(+)} + F^{(-)} - r_{S_C}\Big\{\big(-2^{-1}I + \widetilde{\mathcal{K}}_1\big)\big(\mathcal{P}^{-1}F - \mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1}\big)\widetilde{f}_1 \\ + \big[\mathcal{L}_2 + a\left(2^{-1}I + \widetilde{\mathcal{K}}_2\right)\big]\mathcal{D}_2^{-1}\big[\mathcal{H}_1\mathcal{P}^{-1}F - \big(\mathcal{H}_1\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1} + I\big)\widetilde{f}_1\big]\Big\} \in B_{p,p}^{-\frac{1}{p}}(S_C). \tag{3.22}$$

Let us introduce the notation

$$\mathcal{Q} := -\big(-2^{-1}I + \widetilde{\mathcal{K}}_1\big)\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1} \\ - \big[\mathcal{L}_2 + a\left(2^{-1}I + \widetilde{\mathcal{K}}_2\right)\big]\mathcal{D}_2^{-1}\big(\mathcal{H}_1\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1} + I\big) \tag{3.23}$$

and rewrite equation (3.21) as

$$r_{S_C}\,\mathcal{Q}\,g = \Psi \quad \text{on} \ \ S_C. \tag{3.24}$$

The operator $\mathcal{Q}$ has the following mapping properties:

$$r_{S_C}\mathcal{Q} \,:\, \widetilde{H}_p^s(S_C) \to H_p^{s-1}(S_C), \tag{3.25}$$

$$:\, \widetilde{B}_{p,p}^s(S_C) \to B_{p,p}^{s-1}(S_C), \tag{3.26}$$

for $s \in \mathbb{R}$ and $1 < p < \infty$.

To establish Fredholm properties of the operators (3.25) and (3.26) first we calculate the principal homogeneous symbol. With the help of relations (2.18)-(2.20), we find:

$$\mathfrak{S}(\mathcal{Q}; x, \xi) = 2\,|\xi|, \quad x \in S, \quad \xi \in \mathbb{R}^2 \setminus \{0\}. \tag{3.27}$$

Therefore by Theorem 5.1 in Appendix A, we conclude that if the following inequalities hold

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2}\,, \tag{3.28}$$

then the operators (3.25) and (3.26) are Fredholm with zero index.

Now let us show that these operators possess the trivial null-spaces for the parameters $p = 2$ and $s = \frac{1}{2}$ satisfying the inequalities (3.28). Let $\widetilde{g} \in \widetilde{H}_2^{\frac{1}{2}}(S_C) = \widetilde{B}_{2,2}^{\frac{1}{2}}(S_C)$ be a solution of the homogeneous equation $r_{S_C}\,\mathcal{Q}\,\widetilde{g} = 0$ on S_C, and consider the functions

$$v_1(x) = V_1(h_1)(x), \quad x \in \Omega_1, \tag{3.29}$$

$$v_2(x) = W_2(h_2)(x) + a\,V_2(h_2)(x), \quad x \in \Omega_2, \tag{3.30}$$

where

$$h_1 = -\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1}\,\widetilde{g}, \tag{3.31}$$

$$h_2 = -\mathcal{D}_2^{-1}\big(\mathcal{H}_1\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1} + I\big)\,\widetilde{g}. \tag{3.32}$$

It is easy to check that the pair $(v_1, v_2) \in H_2^1(\Omega_1) \times \big(H_{2,loc}^1(\Omega_2) \cap \mathcal{S}(\Omega_2)\big)$ solves the homogeneous mixed transmission problem and due to the uniqueness Theorem 3.1 we have $v_1 = 0$ in Ω_1 and $v_2 = 0$ in Ω_2. These relations imply $h_1 = 0$ and $h_2 = 0$ on S, whence the equality $\widetilde{g} = 0$ on S follows due to the invertibility of the operator

$$\mathcal{P}^{-1}\mathcal{N}_2\mathcal{D}_2^{-1} \,:\, H_2^{\frac{1}{2}}(S) \to H_2^{-\frac{1}{2}}(S),$$

which in turn is a consequence of the invertibility of the operators (2.12), (2.13), and (2.17). Thus the null-spaces of the operators (3.25) and (3.26) are trivial for the particular values $p = 2$ and $s = \frac{1}{2}$ and consequently they are invertible. Due to Theorem 5.1 then we deduce that the operators (3.25) and (3.26) are invertible for arbitrary p and s satisfying the inequalities (3.28).

These results lead to the following existence theorem.

Theorem 3.2: *Let conditions* (3.8)-(3.9) *hold and* $\frac{4}{3} < p < 4$. *Then the mixed transmission problem is uniquely solvable and the solution pair* $(u_1, u_2) \in H_p^1(\Omega_1) \times \big(H_{p,\,loc}^1(\Omega_2) \cap \mathcal{S}(\Omega_2)\big)$ *is representable in the form* (3.17)-(3.20), *where* $g \in \widetilde{B}_{p,p}^{1-\frac{1}{2}}(S_C)$ *is defined by the uniquely solvable pseudodifferential equation* (3.24).

Moreover we have the following estimates

$$\|u_1\|_{H^1_p(\Omega_1)} \le C_1 \Big(\|f_1\|_{B^{1-\frac{1}{p}}_{p,p}(S_T)} + \|F\|_{B^{-\frac{1}{p}}_{p,p}(S)} + \|F^{(+)} + F^{(-)}\|_{B^{-\frac{1}{p}}_{p,p}(S_C)} \Big), \tag{3.33}$$

$$\|u_2\|_{H^1_p(\Omega_2 \cap B(R))} \le C_2(R) \Big(\|f_1\|_{B^{1-\frac{1}{p}}_{p,p}(S_T)} + \|F\|_{B^{-\frac{1}{p}}_{p,p}(S)} + \|F^{(+)} + F^{(-)}\|_{B^{-\frac{1}{p}}_{p,p}(S_C)} \Big), \tag{3.34}$$

where C_1 is some positive constant, $B(R)$ is a ball centered at the origin and radius R, such that $\overline{\Omega}_1 \subset B(R)$, and $C_2(R)$ is a positive constant which depends on R.

Proof: It is quite similar to the proof of Theorem 2.2. □

Remark 1: Note that if in the formulation of the mixed transmission problem $\varkappa_1 = \varkappa_2$ and the conditions (3.4)-(3.5) are homogeneous, i.e., $f_1 = F_1 = 0$, then we obtain the interior crack problem with S_C treated as a crack surface. Consequently, the theoretical results obtained in Subsection 3 for the mixed transmission problem as well as the results related to the FSM obtained in the next section can be applied also to the interior crack type problems.

4. Method of fundamental solutions for the mixed transmission problem (MT)

Now we develop the Method of Fundamental Solutions (MFS) for transmission problems.

Let us introduce two artificial Lipschitz surfaces S_i and S_e, where $S_i \subset \Omega_1 = \Omega^+$, i.e., S_i is located inside the surface $S = \partial\Omega_1 = \partial\Omega_2$, while $S_e \subset \Omega_2 = \Omega^-$ and S is located inside the surface S_e. Denote by Ω_i the bounded domain surrounded by the surface S_i, and by Ω_e the domain exterior to S_e. Evidently $\overline{\Omega}_i \subset \Omega_1$ and $\overline{\Omega}_e \subset \Omega_2$.

Further, let $\{y^{(k)}\}_{k=1}^{\infty} \subset S_i$ be a dense set of points in S_i and let $\{z^{(j)}\}_{j=1}^{\infty} \subset S_e$ be a dense set of points in S_e.

We use the notation introduced in the previous sections and construct the following sets of functions:

$$\gamma^{(1)}_j(x) := \Gamma(x - z^{(j)}, \varkappa_1), \quad j = 1, 2, 3, \cdots \tag{4.1}$$

$$\gamma^{(2)}_k(x) := \Gamma(x - y^{(k)}, \varkappa_2), \quad k = 1, 2, 3, \cdots \tag{4.2}$$

$$\varphi^{(2)}_k(x) := \big[(\partial_{n(y)} + a)\Gamma(x - y, \varkappa_2)\big]_{y=y^{(k)}}, \quad k = 1, 2, 3, \cdots \tag{4.3}$$

Motivation of our further analysis is the following. If one looks for approximate solution pair $\big(u^{(N)}_1, u^{(M)}_2\big)$ of the exact solution (u_1, u_2) of the mixed transmission problem (3.1), (3.2), (3.3), (3.14), (3.15), and (3.16) in the form of linear combinations:

$$u_1^{(N)}(x) = \sum_{j=1}^{N} A_j\, \gamma_j^{(1)}(x), \quad x \in \Omega_1, \tag{4.4}$$

$$u_2^{(M)}(x) = \sum_{k=1}^{M} B_k\, \varphi_k^{(2)}(x), \quad x \in \Omega_2, \tag{4.5}$$

in accordance with Theorem 3.2 we need to approximate the functions F on S, $F^{(+)} + F^{(-)}$ on S_C, and f_1 on S_T by choosing the unknown constants A_j and B_k appropriately,

$$\sum_{j=1}^{N} A_j\, \partial_{n(x)}\gamma_j^{(1)}(x) - \sum_{k=1}^{M} B_k\, \partial_{n(x)}\varphi_k^{(2)}(x) \approx F \quad \text{on} \;\; S, \tag{4.6}$$

$$\sum_{j=1}^{N} A_j\, \partial_{n(x)}\gamma_j^{(1)}(x) + \sum_{k=1}^{M} B_k\, \partial_{n(x)}\varphi_k^{(2)}(x) \approx F^{(+)} + F^{(-)} \quad \text{on} \;\; S_C, \tag{4.7}$$

$$\sum_{j=1}^{N} A_j\, \gamma_j^{(1)}(x) - \sum_{k=1}^{M} B_k\, \varphi_k^{(2)}(x) \approx f_1 \quad \text{on} \;\; S_T. \tag{4.8}$$

Rewrite these relations as follows

$$\sum_{j=1}^{N} A_j \begin{bmatrix} r_{_S}\partial_{n(x)}\gamma_j^{(1)}(x) \\ r_{_{S_C}}\partial_{n(x)}\gamma_j^{(1)}(x) \\ r_{_{S_T}}\gamma_j^{(1)}(x) \end{bmatrix} - \sum_{k=1}^{M} B_k \begin{bmatrix} r_{_S}\partial_{n(x)}\varphi_k^{(2)}(x) \\ -r_{_{S_C}}\partial_{n(x)}\varphi_k^{(2)}(x) \\ r_{_{S_T}}\varphi_k^{(2)}(x) \end{bmatrix} \approx \begin{bmatrix} F \\ F^{(+)} + F^{(-)} \\ f_1 \end{bmatrix}. \tag{4.9}$$

Note that

$$(F, F^{(+)} + F^{(-)}, f_1) \in B_p^{-\frac{1}{p}}(S) \times B_p^{-\frac{1}{p}}(S_C) \times B_p^{1-\frac{1}{p}}(S_T). \tag{4.10}$$

In what follows we show that approximation of type (4.9) is always possible in the space

$$\mathbb{H}_p := B_{p,p}^{-\frac{1}{p}}(S) \times B_{p,p}^{-\frac{1}{p}}(S_C) \times B_{p,p}^{1-\frac{1}{p}}(S_T). \tag{4.11}$$

To this end let us introduce the following set of vector functions

$$U_j^{(1)} = \begin{bmatrix} r_{_S}\partial_{n(x)}\gamma_j^{(1)}(x) \\ r_{_{S_C}}\partial_{n(x)}\gamma_j^{(1)}(x) \\ r_{_{S_T}}\gamma_j^{(1)}(x) \end{bmatrix}, \quad U_k^{(2)} = \begin{bmatrix} r_{_S}\partial_{n(x)}\varphi_k^{(2)}(x) \\ -r_{_{S_C}}\partial_{n(x)}\varphi_k^{(2)}(x) \\ r_{_{S_T}}\varphi_k^{(2)}(x) \end{bmatrix}, \quad j,k = 1,2,3,\cdots \tag{4.12}$$

and investigate its density properties.

Lemma 4.1: *The set of vector functions* $\{U_j^{(1)}, U_k^{(2)}\}_{j,k=1}^{\infty}$ *is linearly independent and dense in the space* $\mathbb{H}_p$ *for* $\frac{4}{3} < p < 4$.

Proof: First we prove the density property. To this end we have to show that if $h = (h_1, h_2, h_3)$ belongs to the adjoint space $(\mathbb{H}_p)^*$, i.e.

$$h = (h_1, h_2, h_3) \in (\mathbb{H}_p)^* := B^{\frac{1}{p}}_{p',p'}(S) \times \widetilde{B}^{\frac{1}{p}}_{p',p'}(S_C) \times \widetilde{B}^{-1+\frac{1}{p}}_{p',p'}(S_T), \ \frac{1}{p} + \frac{1}{p'} = 1, \quad (4.13)$$

and

$$\big\langle h \,,\, U_j^{(1)} \big\rangle = 0, \qquad \big\langle h \,,\, U_k^{(2)} \big\rangle = 0, \ \ j,k = 1,2,3,\cdots \quad (4.14)$$

then $h = 0$. Here $\langle \cdot \,,\, \cdot \rangle$ denotes the well defined duality relation between the mutually adjoint spaces $(\mathbb{H}_p)^*$ and $\mathbb{H}_p$ which extends the usual $\big(L_{p'} \,,\, L_p\big)$ duality.

Keeping in mind that $z^{(j)}$ and $y^{(k)}$ are dense subsets of S_e and S_i respectively, from (4.14) we deduce

$$\int\limits_S [\partial_{n(x)} \Gamma(x-y, \varkappa_1)]\, h_1(x)\, dS_x + \int\limits_{S_C} [\partial_{n(x)} \Gamma(x-y, \varkappa_1)]\, h_2(x)\, dS_x$$
$$+ \int\limits_{S_T} \Gamma(x-y, \varkappa_1)\, h_3(x)\, dS_x = 0, \qquad y \in S_e, \quad (4.15)$$

and

$$\big(\partial_{n(y)} + a\big)\Big[\int\limits_S [\partial_{n(x)} \Gamma(x-y, \varkappa_2)]\, h_1(x)\, dS_x - \int\limits_{S_C} [\partial_{n(x)} \Gamma(x-y, \varkappa_2)]\, h_2(x)\, dS_x$$
$$+ \int\limits_{S_T} \Gamma(x-y, \varkappa_2)\, h_3(x)\, dS_x \Big] = 0, \qquad y \in S_i, \quad (4.16)$$

Now we see that the function in the right hand side of (4.15) is a C^∞ regular radiating solution to the exterior Dirichlet problem in the domain Ω_e for the Helmholtz equation, while the function in the right hand side of (4.16) is a C^∞ regular solution to the interior Robin problem in the domain Ω_i for the Helmholtz equation.

Applying the corresponding uniqueness theorems and keeping in mind that these functions are analytic functions of real variable y in $\mathbb{R}^3 \setminus S$, we conclude that

$$w_1(y) := \int\limits_S [\partial_{n(x)} \Gamma(x-y, \varkappa_1)]\, h_1(x)\, dS_x + \int\limits_{S_C} [\partial_{n(x)} \Gamma(x-y, \varkappa_1)]\, h_2(x)\, dS_x$$
$$+ \int\limits_{S_T} \Gamma(x-y, \varkappa_1)\, h_3(x)\, dS_x = 0, \qquad y \in \Omega_1, \quad (4.17)$$

and

$$w_2(y) := \int\limits_S [\partial_{n(x)}\Gamma(x-y,\varkappa_2)]\, h_1(x)\, dS_x - \int\limits_{S_C} [\partial_{n(x)}\Gamma(x-y,\varkappa_2)]\, h_2(x)\, dS_x + \int\limits_{S_T} \Gamma(x-y,\varkappa_2)\, h_3(x)\, dS_x = 0, \qquad y \in \Omega_2. \quad (4.18)$$

Evidently both functions w_j, $j = 1, 2$, are solutions of the Helmholtz equation and due to the properties of the layer potentials we have:

$$w_1, w_2 \in H^1_{p'}(\Omega_1), \quad w_1, w_2 \in H^1_{p'}(\Omega_2) \cap \mathcal{S}(\Omega_2). \qquad (4.19)$$

Note that if $\frac{4}{3} < p < 4$, then p' satisfies the same inequality

$$\frac{4}{3} < p' < 4. \qquad (4.20)$$

From (4.17), (4.18), and (4.13) we find:

$$\left.\begin{array}{l} \{w_1\}^+ - \{w_1\}^- = h_1 \quad \text{on} \quad S_T \\ \{w_2\}^+ - \{w_2\}^- = h_1 \quad \text{on} \quad S_T \end{array}\right\} \Rightarrow \{w_1\}^+ + \{w_2\}^- = 0 \quad \text{on} \quad S_T, \qquad (4.21)$$

$$\left.\begin{array}{l} \{\partial_{n(y)}w_1\}^- - \{\partial_{n(y)}w_1\}^+ = h_3 \quad \text{on} \quad S_T \\ \{\partial_{n(y)}w_2\}^- - \{\partial_{n(y)}w_2\}^+ = h_3 \quad \text{on} \quad S_T \end{array}\right\} \Rightarrow \{\partial_{n(y)}w_1\}^+ + \{\partial_{n(y)}w_2\}^- = 0 \text{ on } S_T, \qquad (4.22)$$

$$\{\partial_{n(y)}w_1\}^+ - \{\partial_{n(y)}w_1\}^- = 0 \quad \text{on} \quad S_C \quad \Rightarrow \quad \{\partial_{n(y)}w_1\}^+ = 0 \quad \text{on} \quad S_C, \qquad (4.23)$$

$$\{\partial_{n(y)}w_2\}^+ - \{\partial_{n(y)}w_2\}^- = 0 \quad \text{on} \quad S_C \quad \Rightarrow \quad \{\partial_{n(y)}w_2\}^- = 0 \quad \text{on} \quad S_C, \qquad (4.24)$$

$$\left.\begin{array}{l} \{w_1\}^+ - \{w_1\}^- = h_1 + h_2 \quad \text{on} \quad S_C \\ \{w_2\}^+ - \{w_2\}^- = h_1 - h_2 \quad \text{on} \quad S_C \end{array}\right\} \Rightarrow \{w_1\}^+ + \{w_2\}^- = 2h_2 \quad \text{on} \quad S_C. \qquad (4.25)$$

From conditions (4.21), (4.22), (4.23), and (4.24) it follows that the pair

$$(w_1, -w_2) \in H^1_{p'}(\Omega_1) \times \big[H^1_{p'}(\Omega_2) \cap \mathcal{S}(\Omega_2)\big]$$

solves the homogeneous mixed transmission problem with p' satisfying the inequalities (4.20). Therefore due to Theorem 3.2 we have $w_1 = 0$ in Ω_1 and $w_2 = 0$ in Ω_2. In view of (4.21)-(4.25) we have $h = (h_1, h_2, h_3) = 0$. Thus the set of vector functions $\big\{U^{(1)}_j, U^{(2)}_k\big\}_{j,k=1}^{\infty}$ is dense in the space $\mathbb{H}_p$ for $\frac{4}{3} < p < 4$.

Now we show that any finite subsequence of the system $\big\{U^{(1)}_j, U^{(2)}_k\big\}_{j,k=1}^{\infty}$ is linearly independent. Let for some complex constants A_j and B_k the following linear combinations vanish

$$\sum_{j=1}^{N} A_j\, U_j^{(1)} - \sum_{k=1}^{N} B_k\, U_k^{(2)} = 0, \tag{4.26}$$

which means that the components of the vector (4.26) vanish on the corresponding surfaces

$$\sum_{j=1}^{N} A_j\, \big[U_j^{(1)}\big]_1 - \sum_{k=1}^{N} B_k\, \big[U_k^{(2)}\big]_1 = 0 \quad \text{on} \;\; S, \tag{4.27}$$

$$\sum_{j=1}^{N} A_j\, \big[U_j^{(1)}\big]_2 - \sum_{k=1}^{N} B_k\, \big[U_k^{(2)}\big]_2 = 0 \quad \text{on} \;\; S_C, \tag{4.28}$$

$$\sum_{j=1}^{N} A_j\, \big[U_j^{(1)}\big]_3 - \sum_{k=1}^{N} B_k\, \big[U_k^{(2)}\big]_3 = 0 \quad \text{on} \;\; S_T. \tag{4.29}$$

Define the functions

$$v_1^{(N)}(x) = \sum_{j=1}^{N} A_j\, \gamma_j^{(1)}(x), \quad x \in \mathbb{R}^3 \setminus \{z^{(1)}, \cdots, z^{(N)}\}, \tag{4.30}$$

$$v_2^{(N)}(x) = \sum_{k=1}^{N} B_k\, \varphi_k^{(2)}(x), \quad x \in \mathbb{R}^3 \setminus \{y^{(1)}, \cdots, y^{(N)}\}. \tag{4.31}$$

Evidently the functions v_1 and v_2 are analytic functions of the real variable x in their domains of definition, satisfy the Sommerfeld radiation conditions, and in accordance with the relations (4.27)-(4.29) satisfy the transmission conditions

$$\{\partial_n v_1\}^+ - \{\partial_n v_2\}^- = 0 \quad \text{on} \;\; S, \tag{4.32}$$

$$\{\partial_n v_1\}^+ + \{\partial_n v_2\}^- = 0 \quad \text{on} \;\; S_C, \tag{4.33}$$

$$\{v_1\}^+ + \{v_2\}^- = 0 \quad \text{on} \;\; S_T. \tag{4.34}$$

In view of the inclusions $v_1 \in C^\infty(\overline{\Omega}_1)$ and $v_1 \in C^\infty(\overline{\Omega}_2) \cap \mathcal{S}(\Omega_2)$ by Theorem 3.2 we conclude that $v_1(x) = 0$ for $x \in \Omega_1$ and $v_2(x) = 0$ for $x \in \Omega_2$, implying

$$v_1^{(N)}(x) = 0, \quad x \in \mathbb{R} \setminus \{z^{(1)}, \cdots, z^{(N)}\}, \tag{4.35}$$

$$v_2^{(N)}(x) = 0, \quad x \in \mathbb{R} \setminus \{y^{(1)}, \cdots, y^{(N)}\}, \tag{4.36}$$

due to the analyticity of the functions (4.30)-(4.31). Whence the equalities $A_j = 0$ and $B_k = 0$ follow immediately which completes the proof. □

From the above results it follows that approximation of the solution pair (u_1, u_2) to the mixed transmission problem is reduced to the approximation of the given functions $F, F^{(+)} + F^{(-)}$, and f_1 on the interface by the dense set of functions $\{U^{(j)}, U^{(k)}\}_{j,k=1}^{\infty}$ defined in (4.11).

5. Appendix A: Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with boundary

Here we recall some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary, in both Bessel potential and Besov spaces. These are the main tools for proving existence theorems for mixed boundary value, boundary–transmission and crack type problems using the potential method. They can be found, e.g., in [14], [16], [38].

Let $\overline{\mathcal{M}} \in C^\infty$ be a compact, n–dimensional, non-self-intersecting manifold with boundary $\partial\mathcal{M} \in C^\infty$, and let $\mathbf{A}$ be a strongly elliptic $N \times N$ matrix pseudodifferential operator of order $\nu \in \mathbb{R}$ on $\overline{\mathcal{M}}$. Denote by $\mathfrak{S}(\mathbf{A}; x, \xi)$ the principal homogeneous symbol matrix of the operator $\mathbf{A}$ in some local coordinate system $(x \in \overline{\mathcal{M}},\ \xi \in \mathbb{R}^n \setminus \{0\})$.

Let $\lambda_1(x), \cdots, \lambda_N(x)$ be the eigenvalues of the matrix

$$[\,\mathfrak{S}(\mathbf{A}; x, 0, \cdots, 0, +1)\,]^{-1}[\,\mathfrak{S}(\mathbf{A}; x, 0, \cdots, 0, -1)\,], \quad x \in \partial\mathcal{M}.$$

Introduce the notation

$$\delta_j(x) = \operatorname{Re}\big[\,(2\pi\, i)^{-1} \ln \lambda_j(x)\,\big],\ j = 1, \cdots, N,$$

where $\ln\zeta$ denotes the branch of the logarithm analytic in the complex plane cut along $(-\infty, 0]$. Due to the strong ellipticity of $\mathbf{A}$ we have the strict inequality

$$-1/2 < \delta_j(x) < 1/2 \quad \text{for} \quad x \in \overline{\mathcal{M}},\ \ j = 1, \cdots, N.$$

The numbers $\delta_j(x)$ do not depend on the choice of the local coordinate system. Note that in particular cases, when $\mathfrak{S}(\mathbf{A}; x, \xi)$ is an even matrix function in ξ or $\mathfrak{S}(\mathbf{A}; x, \xi)$ is a positive definite matrix for every $x \in \overline{\mathcal{M}}$ and $\xi \in \mathbb{R}^n \setminus \{0\}$, we have $\delta_j(x) = 0$ for $j = 1, \cdots, N$, since all the eigenvalues $\lambda_j(x)$ $(j = 1, \cdots, N)$ are positive numbers for any $x \in \overline{\mathcal{M}}$.

The Fredholm properties of strongly elliptic pseudodifferential operators on manifolds with boundary are characterized by the following theorem.

Theorem 5.1 : *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le t \le \infty$, and let $\mathbf{A}$ be a strongly elliptic pseudodifferential operator of order $\nu \in \mathbb{R}$, that is, there is a positive constant c_0 such that*

$$\operatorname{Re}\mathfrak{S}(\mathcal{A}; x, \xi)\,\eta \cdot \eta \ge c_0\,|\eta|^2$$

for $x \in \overline{\mathcal{M}}$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\eta \in \mathbb{C}^N$.

Then the operators

$$\mathbf{A} : \big[\widetilde{H}^s_p(\mathcal{M})\big]^N \to \big[H^{s-\nu}_p(\mathcal{M})\big]^N, \tag{5.1}$$

$$\mathbf{A} : \big[\widetilde{B}^s_{p,t}(\mathcal{M})\big]^N \to \big[B^{s-\nu}_{p,t}(\mathcal{M})\big]^N, \tag{5.2}$$

are Fredholm with zero index if

$$\frac{1}{p} - 1 + \sup_{x \in \partial\mathcal{M},\, 1\le j\le N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x \in \partial\mathcal{M},\, 1\le j\le N} \delta_j(x). \tag{5.3}$$

Moreover, the null–spaces and indices of the operators (5.1) *and* (5.2) *are the same* (*for all values of the parameter* $t \in [1, +\infty]$) *provided* p *and* s *satisfy the inequality* (5.3).

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