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Extension of the Dirac Factorization and Relevant Applications

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We will review the Dirac-like factorization method in connection with root operators. Square, cube and quartic root operators will be analyzed in some detail. Then, we will illustrate some applications of the method, specifically in connection with the square and cube root operators.

Key words: Dirac equation, Evolution equation, Root operator, Fractional calculus.

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1. Introduction

Many physical processes are well modelled by *evolution equations*, i.e. partial differential equations of the type [1]

$$\frac{\partial}{\partial t}\mathcal{V}(x,t) = \mathsf{K}\cdot\mathcal{V}(x,t). \tag{1}$$

It specifies the rate of change with t of the variable $\mathcal{V}$, which, designated to characterize the physical state of the dynamical system of concern, is regarded as a function of the independent "space" and "time" variables (x, t).

The heat equation (HE) [2], the time-dependent Schrödinger equation (SE) [3] and the paraxial wave equation (PWE) [4] are substantive examples of evolution equation.

Many physical processes are equally well modelled by equations, which cannot be traced back to the scheme of Eq. (1): equations, for instance, involving higher-order derivatives of $\mathcal{V}$ with respect to the "time" variable t, or equations containing fractional order derivatives with respect to the "space" variable x.

The relativistic heat equation (RHE) [5]

$$\left[\frac{\alpha}{C^2}\frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t}\right]u(\boldsymbol{r},t) = \alpha\nabla^2 u(\boldsymbol{r},t), \tag{2}$$

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α and C being the thermal diffusivity and the speed of heat, the Klein-Gordon equation (KGE) [6]

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \left(\frac{mc}{\hbar}\right)^2\right]\psi(\boldsymbol{r},t) = 0, \tag{3}$$

m and c denoting the mass of the particle and the speed of light, and the homogeneous scalar wave equation (WE) [7]

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\varphi(\boldsymbol{r},t) = 0, \tag{4}$$

are typical examples of equations containing the second-order derivative with respect to time.

On the other hand, the relativistic Schrödinger equation (RSE) [6]

$$i\hbar\frac{\partial\psi(\boldsymbol{r},t)}{\partial t} = \sqrt{m^2c^4 - \hbar^2c^2\nabla^2}\psi(\boldsymbol{r},t), \tag{5}$$

involves the square-root of an operator containing the laplacian ∇^2.

The analysis developed in [8–11] frames within the context of an investigation aimed at establishing if, at which extent and in which form some properties of the evolution equations, like the aforementioned HE, SE and PWE, could be recovered to equations, that are not of evolution type or demand to deal with fractional differential operators, examples of which are the RHE, KGE,WE and RSE, reported above.

In this connection, as shown in [8, 9], the Dirac-like factorization approach conveys a valuable method to tackle with both kinds of difficulties. Correspodingly, a variety of methods has been proposed in [10, 11] to deal with evolution-like equations ruled by square-root operators, which are addressed to as *relativistic evolution equations*.

Here, we will mainly be concerned with the Dirac factorization procedure. Accordingly, in Sect. 2, we will review the Dirac-like factorization method in connection with square-root operators. Extension of the procedure to higher-degree root operators will be approached in Sect. 3, where in fact cube and quartic root operators will be treated in some detail. Then, in Sect. 4 we will illustrate some applications of the method. The concluding notes of Sect. 5 will close the paper.

2. Dirac-like factorization to disentangle root operator functions: square root operators

Let us recall that the Dirac equation [6, 12]

$$\left[i\hbar\boldsymbol{\gamma}^j\frac{\partial}{\partial x^j} - mc\boldsymbol{I}_4\right]\boldsymbol{\psi} = 0, \quad j = 0,1,2,3, \tag{6}$$

where $\boldsymbol{\gamma}^j$, $j = 0,1,2,3$, are the (4×4) Dirac matrices, $(x^0,x^1,x^2,x^3) \equiv (ct,x,y,z)$ and $\boldsymbol{I}_4$ the 4×4 unit matrix, offers in a sense the "evolution-like" alternative to the KGE. As is well known, it was originally formulated by Dirac when seeking a relativistically covariant evolution equation for the state function of a quantum particle in the Schrödinger-like form,

$$i\hbar\frac{\partial\psi}{\partial t} = \widehat{H}\psi,$$

with the Hamiltonian $\widehat{H}$ being a linear Hermitian operator [12]. However, it can also be understood as following from a "factorization" of the KGE

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 + \left(\frac{mc}{\hbar}\right)^2\right]\boldsymbol{I}_4\boldsymbol{\psi} =$$
$$\left[i\hbar\gamma^j\frac{\partial}{\partial x^j} - mc\boldsymbol{I}_4\right]\left[-i\hbar\gamma^j\frac{\partial}{\partial x^j} - mc\boldsymbol{I}_4\right]\boldsymbol{\psi} = \boldsymbol{0}.$$

Then, one can eventually deal with an equation containing a first-order derivative with respect to the evolution variable although in a system of four coupled linear differential equations for the 4-component state vector $\boldsymbol{\psi}$.

We will show how to exploit such a factorization method to deal with root operators. Let us firstly exemplify the procedure in the case of a square-root operator.

2.1. *Square root function*

It is evident that the identity

$$A^2 + B^2 = (A+B)^2 = (A+B)(A+B),$$

can not hold if A and B are numbers (real or complex). In contrast, it can hold if A and B are anticommuting operators or matrices, for which

$$\left\{\widehat{A},\widehat{B}\right\} = \widehat{A}\widehat{B} + \widehat{B}\widehat{A} = 0.$$

Thus, one is led to write down the square root function $\sqrt{A^2+B^2}$ in the "disentangled" form

$$\sqrt{A^2+B^2} = A\alpha + B\beta \ , \tag{7}$$

with α and β being such that

$$\begin{aligned} \alpha^2 = \beta^2 = 1, \\ \alpha\beta + \beta\alpha = 0, \end{aligned} \tag{8}$$

in order to satisfy the desired equality (7).

We see that α and β cannot simply be numbers; indeed, as a direct consequence of (8), they must be traceless matrices with eigenvalues equal to ± 1, and hence of order $2n \times 2n$, $n = 1, 2, ..$, and determinant equal to $(-1)^n$.

Evidently, the scalar nature of the original function $\sqrt{A^2+B^2}$ is lost, since indeed in the stated identity (7) it is understood as a multiple of the $2n \times 2n$ unit matrix $\boldsymbol{I}_{2n}$. In fact, Eq. (7) conveys a matrix identity in a proper $2n$-dimensional vector space, whose meaning and ultimate dimensions (following those of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$) are dictated by the problem at hand. Remarkably, however, one gains a root-free matrix form expression, that could facilitate the solution of the problem although it must be reinterpreted in the light of the gained degree (or, degrees) of freedom (naturally conveyed, as seen, by the procedure). Furthermore, the method can open new perspectives within the theory of fractional calculus, suggesting alternative formulations to already established treatments and/or definitions.

In particular, when up-to-three addends are involved in the square root, the smallest admissible dimension $2n = 2$ is enough to ensure that the desired matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be realized. We may identify them with any two of the Pauli matrices

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{9}$$

so that we can eventually write

$$\sqrt{\widehat{A}^2 + \widehat{B}^2}\boldsymbol{I}_2 = \widehat{A}\boldsymbol{\sigma}_j + \widehat{B}\boldsymbol{\sigma}_k\,, \;\; j \neq k \quad j,k = 1,2,3 \tag{10}$$

or, more in general,

$$\sqrt{\widehat{A}^2 + \widehat{B}^2 + \widehat{C}^2}\boldsymbol{I}_2 = \widehat{A}\boldsymbol{\sigma}_j + \widehat{B}\boldsymbol{\sigma}_k + \widehat{C}\boldsymbol{\sigma}_l, \;\; j \neq k \neq l \quad j,k,l = 1,2,3 \tag{11}$$

The correspondence of each of the operators involved in the square root with a specific Pauli matrix is a mere matter of convenience, possibly suggested by the problem under investigation. Therefore, the resulting matrix expression of the original (scalar) operator function is not unique. We will illustrate this in Sect. 4.1.1.

3. Dirac-like factorization to disentangle root operator functions: estension of the procedure to higher-order root operator functions

It is quite natural to address the question whether it could be possible to extend the procedure to higher-order root operator functions, thus allowing one to write down

$$\sqrt[m]{\widehat{A}^m + \widehat{B}^m}\,\boldsymbol{I} = \widehat{A}\boldsymbol{\alpha} + \widehat{B}\boldsymbol{\beta}\,, \tag{12}$$

or, more in general, with m operators involved in,

$$\sqrt[m]{\widehat{A}_1^m + \widehat{A}_2^m + .. + \widehat{A}_m^m}\,\boldsymbol{I} = \widehat{A}_1\boldsymbol{\alpha}_1 + \widehat{A}_2\boldsymbol{\alpha}_2 + .. + \widehat{A}_m\boldsymbol{\alpha}_m\,, \tag{13}$$

where the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ could be identified by suitable conditions analogous to (8).

3.1. *Cube root function*

Thus, for instance, the factorization

$$(\widehat{A}^3 + \widehat{B}^3)\,\boldsymbol{I} = (\widehat{A}\boldsymbol{\alpha} + \widehat{B}\boldsymbol{\beta})^3,$$

allowing for the disentanglement of the cube root operator as

$$\sqrt[3]{\widehat{A}^3 + \widehat{B}^3}\,\boldsymbol{I} = \widehat{A}\boldsymbol{\alpha} + \widehat{B}\boldsymbol{\beta}\,, \tag{14}$$

is possible for commuting operators $\widehat{A}$, $\widehat{B}$, and matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ such as to satisfy the three-term relations

$$\begin{gathered} \boldsymbol{\alpha}^3 = \boldsymbol{\beta}^3 = \boldsymbol{I}, \\ \left\{\,\boldsymbol{\alpha}, \boldsymbol{\beta}^2\right\} + \{\boldsymbol{\beta}, \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}\} = \mathbf{0}, \quad \left\{\boldsymbol{\beta}, \boldsymbol{\alpha}^2\right\} + \{\boldsymbol{\alpha}, \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}\} = \mathbf{0}. \end{gathered} \tag{15}$$

We see that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are traceless matrices, with eigenvalues conveyed by the third roots of unity:

$$\mu^3 = 1,$$

and hence

$$\mu_0 = 1, \quad \mu_1 = -\frac{1}{2}(1 - i\sqrt{3}), \quad \mu_2 = -\frac{1}{2}(1 + i\sqrt{3}).$$

Therefore, they must be of the order $3n \times 3n$, $n = 1, 2, ..$, with determinant

$$\Delta = (\mu_0\,\mu_1\,\mu_2)^n = 1.$$

The matrices

$$\boldsymbol{\tau}_1 = \begin{pmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,0\,0 \end{pmatrix}, \quad \boldsymbol{\tau}_2 = \begin{pmatrix} 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \\ 1 & 0 & 0 \end{pmatrix}, \tag{16}$$

of smallest admissible dimension, provide a suitable pair of matrices satisfying the required conditions. They are seen to span, by repeated commutators, an 8-dimensional Lie algebra. The expressions of the other $\boldsymbol{\tau}$-matrices result to be

$$\begin{gathered} \tau_3 = \begin{pmatrix} 0 & 0 & 1 \\ \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \end{pmatrix} = \tau_2^\top, \qquad \tau_4 = \begin{pmatrix} \mu_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_1 \end{pmatrix}, \qquad \tau_5 = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix} = \tau_4^*, \\ \tau_6 = \begin{pmatrix} 0 & \mu_2 & 0 \\ 0 & 0 & \mu_1 \\ 1 & 0 & 0 \end{pmatrix} = \tau_2^* = \tau_7^\top, \ \tau_7 = \begin{pmatrix} 0 & 0 & 1 \\ \mu_2 & 0 & 0 \\ 0 & \mu_1 & 0 \end{pmatrix} = \tau_3^* = \tau_6^\top, \ \tau_8 = \begin{pmatrix} 0\,0\,1 \\ 1\,0\,0 \\ 0\,1\,0 \end{pmatrix}. \end{gathered} \tag{17}$$

Each of them is such that $\boldsymbol{\tau}_j^3 = \boldsymbol{I}_3$, and $\Delta_j = 1$, $j = 1, .., 8$. Also, their commutators are

$$[\boldsymbol{\tau}_j, \boldsymbol{\tau}_k] = -i\sqrt{3} f_{jkl} \boldsymbol{\tau}_l,$$

the relevant structure constants f_{jkl} being

$$f_{123} = f_{134} = f_{142} = f_{246} = f_{268} = f_{284} = f_{358} = f_{382} =$$
$$f_{478} = f_{483} = f_{562} = f_{674} = 1,$$

$$f_{156} = f_{146} = f_{175} = f_{235} = f_{251} = f_{346} = f_{371} = f_{461} =$$
$$f_{573} = f_{587} = f_{685} = f_{786} = -1.$$

Of course, $f_{jkl} = -f_{kjl}$.

Interestingly, only 4 commutators vanish, i.e.

$$[\boldsymbol{\tau}_1, \boldsymbol{\tau}_8] = [\boldsymbol{\tau}_2, \boldsymbol{\tau}_7] = [\boldsymbol{\tau}_3, \boldsymbol{\tau}_6] = [\boldsymbol{\tau}_4, \boldsymbol{\tau}_5] = 0,$$

and accordingly, the involved pairs of matrices are the only ones that do not allow for the desired factorization *à la* Dirac of the sum of third-power operators. Therefore, 24 possible pairs of matrices suitable for the disentanglement of the cube root are conveyed by the set of $\boldsymbol{\tau}$-matrices.

If a third term is added in the sum, amounting to the linearization issue

$$\sqrt[3]{\widehat{A}^3 + \widehat{B}^3 + \widehat{C}^3}\, \boldsymbol{I}_{3\times 3} = \widehat{A}\boldsymbol{\alpha} + \widehat{B}\boldsymbol{\beta} + \widehat{C}\boldsymbol{\gamma}, \tag{18}$$

a triplet of matrices is needed, such that each and each pair of them satisfy the relations (15), in addition to the further one

$$\sum_{p \in S_3} (\boldsymbol{\alpha}\, \boldsymbol{\beta}\, \boldsymbol{\gamma}) = 0, \tag{19}$$

the sum being over all the six possible products of the three matrices obtained from all their permutations p ($\in S_3$).

One can see that 24 suitable triplets of matrices can be extracted from the set of $\boldsymbol{\tau}$-matrices, the choice being a matter of convenience in conformity to the problem under analysis.

It is worth noting that the $\boldsymbol{\tau}$-matrices have been already deduced in [13] in connection with the analysis of the fractional Dirac equation. There, the triplets of matrices allowing for (18) are explicitly signalized.

3.2. *Quartic root function*

The disentanglement of the quartic root as

$$\sqrt[4]{\widehat{A}^4 + \widehat{B}^4}\, \boldsymbol{I} = \widehat{A}\boldsymbol{\alpha} + \widehat{B}\boldsymbol{\beta}\,, \tag{20}$$

demands for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ matrices such as to satisfy the four-term relations

$$\begin{gathered}\boldsymbol{\alpha}^4 = \boldsymbol{\beta}^4 = \boldsymbol{I}, \\ \{\boldsymbol{\alpha}^2, \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}\} = \mathbf{0}, \quad \{\boldsymbol{\beta}^2, \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}\} = \mathbf{0}, \\ \{\boldsymbol{\alpha}^2, \boldsymbol{\beta}^2\} + \{\boldsymbol{\alpha}, \boldsymbol{\beta}\}^2 = \mathbf{0}.\end{gathered} \tag{21}$$

Thus, the desired matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are traceless and with eigenvalues conveyed by the quartic roots of unity:

$$\mu^4 = 1,$$

yielding

$$\mu_0 = 1, \quad \mu_1 = i, \quad \mu_2 = -i, \quad \mu_3 = -1$$

As a consequence, they must be of the order $4n \times 4n$, $n = 1, 2, ..$, with determinant

$$\Delta = (\mu_0 \, \mu_1 \, \mu_2 \mu_3)^n = (-1)^n \, .$$

The anticommutation relations in the second row of (21) suggest a correspondence of the matrices $\boldsymbol{\alpha}^2$, $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ and $\boldsymbol{\beta}^2$ with $\boldsymbol{\sigma}$-composed matrices. Thus, working with the smallest admissible dimension, i.e. $4n = 4$, we start taking

$$\boldsymbol{\rho}_1 = \begin{pmatrix} \mathbf{0}_2 & -i\sqrt{\boldsymbol{\sigma}_3} \\ i\sqrt{\boldsymbol{\sigma}_3} & \mathbf{0}_2 \end{pmatrix}, \tag{22}$$

with[1)]

$$\sqrt{\boldsymbol{\sigma}_3} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Evidently, $\boldsymbol{\rho}_1^4 = \boldsymbol{I}_4$. Then, following the conditions (21), the suitable matrix

$$\boldsymbol{\rho}_2 = \begin{pmatrix} \boldsymbol{\sigma}_+ & \boldsymbol{\sigma}_- \\ \boldsymbol{\sigma}_- & \boldsymbol{\sigma}_+ \end{pmatrix}, \tag{23}$$

is obtained, where

$$\boldsymbol{\sigma}_+ = \frac{1}{2}(\boldsymbol{\sigma}_1 + i\boldsymbol{\sigma}_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_- = \frac{1}{2}(\boldsymbol{\sigma}_1 - i\boldsymbol{\sigma}_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Note that $\boldsymbol{\sigma}_+^2 = \boldsymbol{\sigma}_-^2 = \mathbf{0}$, and $[\boldsymbol{\sigma}_+, \boldsymbol{\sigma}_-] = \boldsymbol{\sigma}_3$ whereas $\{\boldsymbol{\sigma}_+, \boldsymbol{\sigma}_-\} = \boldsymbol{I}_2$.

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1)We recall that given a 2×2 matrix $\boldsymbol{M}$ with eigenvalues $\mu_1 \neq \mu_2$, the function $f(\boldsymbol{M})$ can be evaluated according to [14]

$$f(\boldsymbol{M}) = \frac{\mu_1 f(\mu_2) - \mu_2 f(\mu_1)}{\mu_1 - \mu_2} \boldsymbol{I} + \frac{f(\mu_1) - f(\mu_2)}{\mu_1 - \mu_2} \boldsymbol{M}.$$

Of course, $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ do not commute with each other (as it must be in order for they to allow for (20)); in fact,

$$[\boldsymbol{\rho}_1, \boldsymbol{\rho}_2] = \sqrt{2}\boldsymbol{\rho}_3, \quad \boldsymbol{\rho}_3 = \; e^{-i\frac{\pi}{4}} \begin{pmatrix} \boldsymbol{\sigma}_- & -i\boldsymbol{\sigma}_+ \\ i\boldsymbol{\sigma}_+ & -\boldsymbol{\sigma}_- \end{pmatrix},$$

whilst

$$\{\boldsymbol{\rho}_1, \boldsymbol{\rho}_2\} = i\sqrt{2}\boldsymbol{\rho}_3 = i\,[\boldsymbol{\rho}_1, \boldsymbol{\rho}_2]\,.$$

It can be verified that $\boldsymbol{\rho}_3$ can be a suitable matrix to realize (20) accompanied by either $\boldsymbol{\rho}_1$ or $\boldsymbol{\rho}_2$.

By repeated commutators, we span a 15-dimensional Lie algebra of matrices $\{\boldsymbol{\rho}_j\}_{j=1,..,15}$ such that $\boldsymbol{\rho}_j^4 = \boldsymbol{I}_4$, $\forall j$. Their explicit expressions have been reported in [11], where further details about their features are also given.

It results that 48 possible pairs of $\boldsymbol{\rho}$-matrices allow for the quartic-root decomposition (20).

3.3. *m-th root function*

Evidently, with increasing m, the decomposition, as addressed in (12) and (13), becomes even more complex. However, on the basis of the previous analysis, we can try to draw some basic issues, at least for the two-term case exemplified in (12).

The identity (12) yields $m+1$ relations involving terms of degree m in the $mn \times mn$ matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Firstly, the latter come to be the m-th roots of the unit matrix, being

$$\boldsymbol{\alpha}^m = \boldsymbol{\beta}^m = \boldsymbol{I}, \tag{24}$$

and hence their eigenvalues can be written as

$$\mu_j = e^{i\frac{2\pi}{m}j}, \;\; j = 0, 2, ...m-1.$$

It is easy to see that

$$\sum_{j=0}^{m-1} \mu_j = 0,$$

thus implying

$$Tr(\boldsymbol{\alpha}) = Tr(\boldsymbol{\beta}) = \mathbf{0}.$$

Furthermore, since

$$\prod_{j=0}^{m-1} \mu_j = (-1)^{m-1},$$

the determinant of the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ comes to be

$$\Delta = (-1)^{n(m-1)}.$$

As to the other conditions ensuring that (12) be satisfied, they can be synthesised in the form

$$\sum_{p,q} \boldsymbol{\alpha}^{p_1}\boldsymbol{\beta}^{q_1}\boldsymbol{\alpha}^{p_2}\boldsymbol{\beta}^{q_2}....\boldsymbol{\alpha}^{p_\alpha}\boldsymbol{\beta}^{q_\beta} = \mathbf{0} \tag{25}$$

where the sum is intended to comprise all the powers p_j and q_i such that $\sum_j p_j = l$ and $\sum_i q_i = k$, for any choice of integers (l, k) such that $l \neq 0$, $k \neq 0$ and $l+k = m$, thus yielding $\frac{m!}{l!k!}$ terms of degree m.

We see that the $m \times m$ matrices

$$\boldsymbol{\nu}_1 = \begin{pmatrix} 0 & \mu_0 & 0 & \cdots & 0 \\ 0 & 0 & \mu_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_0 \\ \mu_0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \boldsymbol{\nu}_2 = \delta \begin{pmatrix} 0 & \mu_1 & 0 & \cdots & 0 \\ 0 & 0 & \mu_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mu_{m-1} \\ \mu_0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

with $\delta = \begin{cases} 1 & m \text{ odd} \\ e^{i\frac{\pi}{m}} & m \text{ even} \end{cases}$, represent a suitable pair of matrices (of smallest allowable order) which satisfy (24) and (25). Starting from them, one can span an (m^2-1)-dimensional Lie algebra of matrices $\{\boldsymbol{\nu}_j\}_{j=1,..,m^2-1}$ satisfying (24).

4. Dirac-like factorization to disentangle root operator functions: possible applications

The Dirac-like factorization procedure can be applied to various (physical and/or mathematical) contexts, and also be variously finalized.

It can be effectively applied to deal with evolution equations ruled by fractional differential operators, like that entering the RSE. In fact, the factorization procedure allows for the "disentanglement" of root operators into the sum of operators, and hence, under appropriate conditions, one can overcome the problem of working with fractional differential operators [8, 11].

In addition, the factorization approach to root operators may open new perspectives within the theory of fractional calculus, suggesting, for instance, alternative formulations to already well-established definitions and/or treatments [9, 11].

We will illustrate both issues in connection with root functions of differential operators, in particular, square and cube roots respectively of second-order and third-order differential operators.

4.1. *Square root of differential operators*

4.1.1. Solving relativistic evolution equations

Let us consider evolution equations ruled by square roots of second-order differential operators. A typical example is offered by the equation

$$\begin{aligned}\partial_\zeta \psi(\xi,\zeta) &= -\sqrt{1-\partial_\xi^2}\psi(\xi,\zeta), \\ \psi(\xi,0) &= \psi_0(\xi).\end{aligned} \tag{26}$$

Setting $\zeta = i\frac{ct}{\lambda_c}$ and $\xi = \frac{x}{\lambda_c}$, $\lambda_c = \frac{\hbar}{mc}$ being the Compton wavelength of the particle, the above equation would yield the (1+1)D version of the RSE (5), and accordingly $\psi(\xi,\zeta)$ would represent the particle wave function. We may refer to (26) as *relativistic evolution equation.*

The Dirac-like "linearization" procedure turns the problem of the solution of (26) into that of the solution of the system of two coupled linear homogeneous partial differential equation of the first order

$$\begin{aligned}\partial_\zeta \boldsymbol{\psi}(\xi,\zeta) &= -(\boldsymbol{\sigma}_j - i\partial_\xi\boldsymbol{\sigma}_k)\,\boldsymbol{\psi}(\xi,\zeta), \;\; j\neq k, \\ \boldsymbol{\psi}(\xi,0) &= \boldsymbol{\psi}_0(\xi),\end{aligned} \tag{27}$$

for the two component vector $\boldsymbol{\psi}(\xi,0)$.

The solution to the above "evolution equation" is indeed immediately written in the form

$$\boldsymbol{\psi}(\xi,\zeta) = \boldsymbol{U}(\xi,\zeta)\boldsymbol{\psi}_0(\xi), \tag{28}$$

the evolution matrix being

$$\boldsymbol{U}(\xi,\zeta) = e^{-\zeta(\boldsymbol{\sigma}_j - i\partial_\xi\boldsymbol{\sigma}_k)}. \tag{29}$$

Since the exponent $\boldsymbol{K} = -\zeta(\boldsymbol{\sigma}_j - i\partial_\xi\boldsymbol{\sigma}_k)$ in (29) is a traceless matrix, whatever be the specific Pauli matrices chosen in the factorization, $\boldsymbol{U}(\xi,\zeta)$ can be given the explicit expression

$$\boldsymbol{U}(\xi,\zeta) = \cosh[\zeta(\sqrt{1-\partial_\xi^2})]\boldsymbol{I}_2 - \frac{\sinh[\zeta(\sqrt{1-\partial_\xi^2})]}{\sqrt{1-\partial_\xi^2}}(\boldsymbol{\sigma}_j - i\partial_\xi\boldsymbol{\sigma}_k). \tag{30}$$

Let us apply such a result to specific initial data. We may consider, for instance, the input vector

$$\boldsymbol{\psi}_0(\xi) = \begin{pmatrix} e^{-\frac{\xi^2}{4}} \\ 0 \end{pmatrix}. \tag{31}$$

Then, with the specific choice of the Pauli matrices yielding

$$\boldsymbol{K} = -(\boldsymbol{\sigma}_3 - i\partial_\xi \boldsymbol{\sigma}_2),$$

the evolution of the vector $\boldsymbol{\psi}$ occurs according to

$$\boldsymbol{\psi}(\xi,\zeta) = \begin{pmatrix} \widehat{C} - \widehat{S} & -\partial_\xi \widehat{S} \\ \partial_\xi \widehat{S} & \widehat{C} - \widehat{S} \end{pmatrix} \boldsymbol{\psi}_0(\xi). \tag{32}$$

Here, $\widehat{C}$ and $\widehat{S}$ are the cosh- and sinch-operator functions entering (30), i.e

$$\widehat{C} = \cosh[\zeta(\sqrt{1-\partial_\xi^2})], \qquad \widehat{S} = \frac{\sinh[\zeta(\sqrt{1-\partial_\xi^2})]}{\sqrt{1-\partial_\xi^2}}.$$

Therefore, with $\boldsymbol{\psi}_0(\xi)$ given by (31), we obtain

$$\boldsymbol{\psi}(\xi,\zeta) = \begin{pmatrix} \widehat{C} - \widehat{S} \\ \partial_\xi \widehat{S} \end{pmatrix} e^{-\frac{\xi^2}{4}}, \tag{33}$$

graphically displayed in Figure 1.

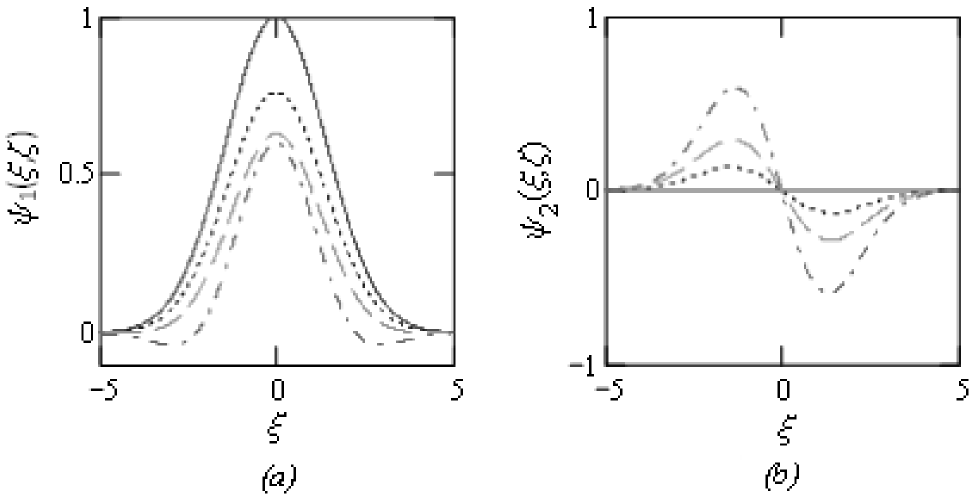

Figure 1. ζ-evolution of the two components *(a)* $\psi_1(\xi,\zeta)$ and *(b)* $\psi_2(\xi,\zeta)$ of the $\boldsymbol{\psi}$-vector for the input (31), shown at $\zeta = 0$ (solid line), $\zeta = 0.3$ (dotted line), $\zeta = 0.6$ (dashed line), and $\zeta = 1$ (dash-dotted line).

4.1.2. Suggesting alternative formulations in fractional calculus

As said, another possible context of application of the Dirac-like "linearization" procedure is that of the theory of the fractional calculus [13]. As an example, we consider the operator

$$\widehat{O} = \sqrt{a + \partial_x}, \tag{34}$$

a being an arbitrary constant. Resorting to the integral representation of the operator $\widehat{L}^{-\nu}$, $\Re(\nu) > 0$, as

$$\widehat{L}^{-\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty s^{\nu-1} e^{-s\widehat{L}} ds, \quad \Re(\nu) > 0 \tag{35}$$

which reproduces for operators the well-known Laplace-transform identity for c-numbers, the operator $\widehat{O}$ can be interpreted as

$$\widehat{O} f(x) = \frac{a+\partial_x}{\sqrt{a+\partial_x}} f(x) = (a+\partial_x) \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-1/2} e^{-as} e^{-s\partial_x} f(x) ds.$$

Alternatively, in the light of the analysis presented in Sect. 2.1, the scalar operator can be replaced by the operator matrix

$$\sqrt{a+\partial_x} \to \sqrt{a}\boldsymbol{\sigma}_j + \sqrt{\partial_x}\boldsymbol{\sigma}_k\,, \;\; j \neq k \quad j,k = 1,2,3, \tag{36}$$

for any specific choice of the inherent Pauli matrices, thus opening new perspectives within the theory of fractional calculus.

The operator nature of the l.h.s. would be conveyed by the matrix nature of r.h.s.; indeed, $\sqrt{\partial_x}$ may be seen as acting on 1, thus giving

$$\sqrt{\partial_x}\, 1 = \frac{1}{\sqrt{\pi x}},$$

according to the Euler definition of fractional derivative [13]

$$\partial_x^\nu x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\nu-\mu+1)} x^{\mu-\nu}.$$

Thus, the operator (34) can be regarded as acting in a 2D vector space through the matrix

$$\widehat{O} := \begin{pmatrix} 0 & \sqrt{a} - \frac{i}{\sqrt{\pi x}} \\ \sqrt{a} + \frac{i}{\sqrt{\pi x}} & 0 \end{pmatrix},$$

or also through

$$\widehat{O} := \begin{pmatrix} \frac{1}{\sqrt{\pi x}} & -i\sqrt{a} \\ i\sqrt{a} & -\frac{1}{\sqrt{\pi x}} \end{pmatrix}, \quad \widehat{O} := \begin{pmatrix} \frac{1}{\sqrt{\pi x}} & \sqrt{a} \\ \sqrt{a} & -\frac{1}{\sqrt{\pi x}} \end{pmatrix},$$

each matrix being obtained in correspondence with a specific choice of the Pauli matrices in (36). In our opinioin, this view deserves to be explored.

4.2. *Cube root of differential operators*

Paralleling the analysis developed in connection with Eq. (26), we may consider an evolution equation involving the cube root of the third-order differentiation operator ∂_ξ^3, namely

$$\begin{aligned}\partial_\zeta\psi(\xi,\zeta) &= \sqrt[3]{1+\partial_\xi^3}\psi(\xi,\zeta), \\ \psi(\xi,0) &= \psi_0(\xi).\end{aligned} \tag{37}$$

By the Dirac-like procedure, Eq. (37) is recast into the system of three coupled linear homogeneous partial differential equation of the first order for the three component vector $\boldsymbol{\psi}(\xi,0)$,

$$\begin{aligned}\partial_\zeta\boldsymbol{\psi}(\xi,\zeta) &= (\boldsymbol{\tau}_j + \partial_\xi\boldsymbol{\tau}_k)\,\boldsymbol{\psi}(\xi,\zeta), \;\; j \neq k, \\ \boldsymbol{\psi}(\xi,0) &= \boldsymbol{\psi}_0(\xi),\end{aligned} \tag{38}$$

for any choice of the suitable pairs of the $\boldsymbol{\tau}$-matrices.

The solution can formally be written as

$$\boldsymbol{\psi}(\xi,\zeta) = e^{\zeta(\boldsymbol{\tau}_j + \partial_\xi\boldsymbol{\tau}_k)}\,\boldsymbol{\psi}_0(\xi). \tag{39}$$

However, since the $\boldsymbol{\tau}$-matrices involved in the linearization of the original cube root operators should not commute with each other, in order to get an explicit expression for the evolution matrix

$$\boldsymbol{U}(\xi,\tau) = e^{\zeta(\boldsymbol{\tau}_j + \partial_\xi\boldsymbol{\tau}_k)} \tag{40}$$

entering (39), one needs to resort to appropriate ordering techniques.

Let us work, for instance, with the matrices (16). We apply the Zassenhaus formula [15, 16] giving the exponential of the sum of two operators as the in general infinite product of operators according to

$$e^{\widehat{X}+\widehat{Y}} = e^{\widehat{X}}e^{\widehat{Y}}\prod_{j=1}^{\infty} e^{\widehat{C}_j},$$

where the first terms in the product explicitly write as

$$\begin{aligned}\widehat{C}_1 &= -\frac{1}{2}\left[\widehat{X},\widehat{Y}\right], \\ \widehat{C}_2 &= \frac{1}{3}\left[\widehat{Y},\left[\widehat{X},\widehat{Y}\right]\right] + \frac{1}{6}\left[\widehat{X},\left[\widehat{X},\widehat{Y}\right]\right], \\ \widehat{C}_3 &= \frac{1}{8}\left\{\left[\widehat{Y},\left[\widehat{Y},\left[\widehat{X},\widehat{Y}\right]\right]\right] + \left[\widehat{Y},\left[\widehat{X},\left[\widehat{X},\widehat{Y}\right]\right]\right]\right\} - \frac{1}{24}\left[\widehat{X},\left[\widehat{X},\left[\widehat{X},\widehat{Y}\right]\right]\right].\end{aligned}$$

In the case of (40) we find that

$$\widehat{C}_1 = \frac{i\sqrt{3}}{2}\zeta^2\partial_\xi\boldsymbol{\tau}_3, \quad \widehat{C}_2 = \zeta^3\partial_\xi^2(\frac{1}{2}\boldsymbol{\tau}_4 - \boldsymbol{\tau}_5),$$

thus enabling us to write $\boldsymbol{U}(\xi, \tau)$ at the third order in the evolution parameter ζ as

$$\begin{aligned}\boldsymbol{U}(\xi,\zeta) &= e^{\zeta\boldsymbol{\tau}_1}e^{\zeta\partial_\xi\boldsymbol{\tau}_2}e^{\frac{i\sqrt{3}}{2}\zeta^2\partial_\xi\boldsymbol{\tau}_3}e^{\zeta^3\partial_\xi^2(\frac{1}{2}\boldsymbol{\tau}_4-\boldsymbol{\tau}_5)}+O(\zeta^4)\\ &= e^{\zeta\boldsymbol{\tau}_1}e^{\zeta\partial_\xi\boldsymbol{\tau}_2}e^{\frac{i\sqrt{3}}{2}\zeta^2\partial_\xi\boldsymbol{\tau}_3}e^{\frac{1}{2}\zeta^3\partial_\xi^2\boldsymbol{\tau}_4}e^{-\zeta^3\partial_\xi^2\boldsymbol{\tau}_5}+O(\zeta^4),\end{aligned}$$

the latter expression being allowed by the commutator relation $[\boldsymbol{\tau}_4, \boldsymbol{\tau}_5] = 0$.

Interestingly, each term in the above product of exponential matrices can be written in a form similar to (30). In fact, as a consequence of that $\boldsymbol{\tau}_j^3 = \boldsymbol{I}$, the exponential matrix $e^{a\boldsymbol{\tau}_j}$ turns out to be the sum of three terms; precisely,

$$e^{a\boldsymbol{\tau}_j} = \mathcal{A}_0(a)\boldsymbol{I}_{3\times 3} + \mathcal{A}_1(a)\boldsymbol{\tau}_j + \mathcal{A}_2(a)\boldsymbol{\tau}_j^2. \tag{41}$$

The coefficients $\mathcal{A}_j(a)$, $j = 0, 1, 2$ are given by

$$\begin{aligned}\mathcal{A}_0(a) &= {}_0F_2(-;\frac{1}{3},\frac{2}{3};\frac{a^3}{27}),\\ \mathcal{A}_1(a) &= a\,{}_0F_2(-;\frac{2}{3},\frac{4}{3};\frac{a^3}{27}),\\ \mathcal{A}_2(a) &= \frac{a^2}{2}\,{}_0F_2(-;\frac{4}{3},\frac{5}{3};\frac{a^3}{27}),\end{aligned} \tag{42}$$

where ${}_0F_2(\cdot)$ denotes the generalized hypergeometric function, formally represented by the series [17]

$${}_pF_q(a_1,..,a_p;b_1,..,b_q;z) = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_k}{\prod_{j=1}^{q}(b_j)_k}\frac{z^k}{k!}, \tag{43}$$

with $(a)_k \equiv \Gamma(a + k)/\Gamma(a)$ being the Pochhammer symbol. We recall that ${}_pF_q$ converges for all finite z if $p \leq q$. It is worth recalling that the relevant power series expressions of the $\mathcal{A}_j$s, conveyed by (43), have been investigated in [18] as pseudo-hyperbolic functions.

Accordingly, we can say that the exponentials $e^{a\boldsymbol{\tau}_j}$ belong to the algebra spanned (in general, over the complex field $\mathbb{C}$) by the $\{\boldsymbol{\tau}_j\}_{j=1,..,8}$ and the unit matrix.

5. Concluding notes

We have reviewed the square-root operator factorization method *à la* Dirac along with its extension to higher-degree root operators, as recently suggested in the literature [8, 9, 11]. Cube and quartic root operators have been investigated in some detail.

A variety of possible applications of the method have also been proposed and illustrated. In fact, solutions of equations involving root functions of differential operators, specifically, square and cube roots respectively of second-order and third-order differential operators, have been worked out. Also, it has been shown that the factorization approach to root operators may open new perspectives within the theory of fractional calculus.

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