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Characterization of the Conjugate Functions by Moduli of Smoothness in the Space of Continuous Functions

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In the present paper we give our latest results, concerning to the estimates of the partial moduli of continuity of different orders of conjugate functions of many variables in some functional classes of the space of continuous functions.

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1. Some notations and definitions

Let $\mathbb{R}^n$ $(n = 1, 2, \ldots; \mathbb{R}^1 \equiv \mathbb{R})$ be the n-dimensional Euclidean space of points $\overline{x} = (x_1, \ldots, x_n)$ with real coordinates. Let B be an arbitrary non-empty subset of the set $M = \{1, \ldots, n\}$. Denote by $|B|$ the cardinality of B. Let x_B be such a point in $\mathbb{R}^n$ whose coordinates with indices in $M\backslash B$ are zero.

As usual $\mathbb{C}(T^n)$ $(\mathbb{C}(T^1) \equiv \mathbb{C}(T))$, where $T = [-\pi, \pi]$, denotes the space of all continuous functions $f : \mathbb{R}^n \to \mathbb{R}$ that are 2π-periodic in each variable, endowed with the norm

$$\|f\| = \max_{\overline{x}\in T^n} |f(\overline{x})|.$$

If $f \in L(T^n)$, then following Zhizhiashvili [1, p. 182], we call the expression

$$\widetilde{f}_B(\overline{x}) = \Big(-\frac{1}{2\pi}\Big)^{|B|} \int_{T^{|B|}} f(\overline{x} + s_B) \prod_{i\in B} \operatorname{ctg} \frac{s_i}{2}\, ds_B$$

the conjugate function of n variables with respect to those variables whose indices form the set B (with $\widetilde{f}_B \equiv \widetilde{f}$ for $n = 1$).

Suppose that $f \in \mathbb{C}(T^n)$, $1 \le i \le n$, and $h \in T$. Then for each $\overline{x} \in T^n$ let us consider the difference of k-th order

$$\Delta_i^k(h)\, f(\overline{x}) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x_1, \ldots, x_{i-1}, x_i + j\,h, x_{i+1}, \ldots, x_n)$$

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and define the partial modulus of continuity of k-th order of the function f with respect to the variable x_i by the equality

$$\omega_{k,i}\,(f;\delta) = \sup_{|h|\le\delta} \left\|\Delta_i^k(h)\,f\right\|.$$

($\omega_{k,i}\,(f;\delta) \equiv \omega_k\,(f;\delta)$ and $\omega_1\,(f;\delta) \equiv \omega\,(f;\delta)$ for $n = 1$).

Definition 1.1: A function $\omega_k : [0,\pi] \to \mathbb{R}$ $(k \ge 1, \omega_1 \equiv \omega)$ which satisfies the following four conditions:
1. $\omega_k(0) = 0$,
2. ω_k is nondecreasing,
3. ω_k is continuous,
4. $\frac{\omega_k(t)}{t^k}$ is almost decreasing in $[0,\pi]$,

we call the modulus of continuity of k-th order.

Definition 1.2: We say that the modulus of continuity of k-th order ω_k satisfies Zygmund's condition if

$$\int_0^\delta \frac{\omega_k(t)}{t}dt + \delta^k \int_\delta^\pi \frac{\omega_k(t)}{t^{k+1}}dt = O(\omega_k(\delta)),\ \delta \to 0+\,.$$

Let ω_k be a modulus of continuity of k-th order. Then we denote by $H_i(\omega_k;\mathbb{C}(T^n))$ $(i = 1,\ldots,n)$ the set of all functions $f \in \mathbb{C}(T^n)$ such that

$$\omega_{k,i}(f;\delta) = O\big(\omega_k(\delta)\big), \quad \delta \to 0+, \quad i = 1,\ldots,n.$$

We set

$$H\big(\omega_k;\mathbb{C}(T^n)\big) = \bigcap_{i=1}^{n} H_i\big(\omega_k;\mathbb{C}(T^n)\big).$$

If $\omega(\delta) = \delta^\alpha (0 < \delta < 1)$ then we denote the class $H\big(\omega;\mathbb{C}(T^n)\big)$ by $Lip(\alpha,\mathbb{C}(T^n))$ when $n > 1$ and by $Lip\alpha$ when $n = 1$.

2. Main results

Moduli of smoothness play a basic role in approximation theory, Fourier analysis and their applications. For a given function f, they essentially measure the structure or smoothness of the function via the k-th difference $\Delta_i^k(h)\,f(\overline{x})$. In fact, for the functions f belonging to the Lebesgue space $L^p(1 \le p < +\infty)$ or the space of continuous functions $\mathbb{C}$, the classical k-th modulus of continuity has turned out to be a rather good measure for determining the rate of convergence of best approximation. In this direction one could see books by V. K. Dzyadyk, I. A. Shevchuk [2] and by R. Trigub, E. Belinsky [3].

In the theory of functions of real variables there is a well-known theorem of Privalov [4] on the invariance of the functional class $Lip(\alpha, C(T))(0 < \alpha < 1)$ under the conjugate function $\widetilde{f}$. If $\alpha = 1$ the invariance of the functional class fails.

Later in 1924 Zygmund [5] obtained a stronger result:
If $f \in C(T)$ and

$$\int_0^{\pi} \frac{\omega(f,t)}{t} dt < \infty,$$

then $\widetilde{f} \in C(T)$and

$$\omega(\widetilde{f},\delta) \leq A[\int_0^{\delta} \frac{\omega(f,t)}{t} dt + \delta \int_{\delta}^{\pi} \frac{\omega(f,t)}{t^2} dt], \delta \in (0, \frac{\pi}{2}).$$

In 1945 Zygmund [6] established that the analog of the Privalov theorem is valid in the case $\alpha = 1$ for the modulus of continuity of the second order. Afterwards, in 1955 Bari and Stechkin [7] obtained resuls connected with behavior of the moduli of continuity of k-th order of the function f and its conjugate function.They obtained the necessary and sufficient condition on the modulus of continuity of k-th order ω_k(that ω_k satisfies Zygmund's condition) for the invariance of $H(\omega_k;\mathbb{C}(T))$ class under the conjugate function $\widetilde{f}$.

As to the functions of many variables, the first result in this direction belongs to Cesari [8]. He showed that the class $Lip(\alpha, C(T^2))(0 < \alpha < 1)$ is not invariant under the conjugate operators of two variables. Zhak [9] proved that Cesari results are exact. In [10] Zhak proved that Zygmund's well-known result [6] is not valid in two dimensional case. Later Lekishvili ([11],[12]) obtained the sharp estimates for partial moduli of continuity of conjugate functions in the spaces $Lip(\alpha, C(T^n))$ and $H(\omega, C(T^n))$ when the modulus of continuity ω satisfies Zygmund's condition.The case of moduli of continuity of second order was considered in [13]. In [14] Okulov considered the analogous problem for the class $H(\omega, C(T^n))$ in general case. In [15] and [16] we obtained the analogous results in the classes $H(\omega_k, C(T^n))$ $(k \geq 2)$ in general case. The following theorem is valid.

Theorem 2.1 :
a) *Let* $f \in H(\omega_k, \mathbb{C}(T^n))$ $(k \geq 2)$ *and for each* $B \subseteq M$

$$\int_{\left[0,\frac{\pi}{2}\right]^{|B|}} \min_{i\in B} \omega_k(s_i) \prod_{i\in B} \frac{ds_i}{s_i} < \infty.$$

Then

$$\omega_{k,j}(\widetilde{f}_B;\delta) = O(\int_{[0,\pi]^{|B|}} \min(\delta^k, s_j^k) s_j^{-k} \min_{i\in B} \omega_k(s_i) \prod_{i\in B} s_i^{-1} ds_i \ ), \ \ j \in B, \ \ \delta \to 0+, \tag{1}$$

$$\omega_{k,j}(\widetilde{f}_B;\delta) = O(\int_{[0,\pi]^{|B|}} \min\{ \min_{i\in B} \omega_k(s_i), \omega_k(\delta)\} \prod_{i\in B} s_i^{-1} ds_i \ ), \ \ j \in M \setminus B, \ \ \delta \to 0+. \tag{2}$$

b) *For each* $B \subseteq M$ *there exist functions* F *and* G *such that* $F, G \in H(\omega_k;\mathbb{C}(T^n))$

and

$$\omega_{k,j}(\widetilde{F}_B;\delta) \geq C \int_{[0,\pi]^{|B|}} \min(\delta^k, s_j^k) s_j^{-k} \min_{i\in B} \omega_k(s_i) \prod_{i\in B} s_i^{-1} ds_i \ ), \ j \in B, \ 0 \leq \delta \leq \delta_0, \tag{3}$$

$$\omega_{k,j}(\widetilde{G}_B;\delta) \geq C \int_{[0,\pi]^{|B|}} \min\{ \min_{i\in B} \omega_k(s_i), \omega_k(\delta)\} \prod_{i\in B} s_i^{-1} ds_i \ , \ \ j \in M\backslash B, \ 0 \leq \delta \leq \delta_0, \tag{4}$$

where C *and* δ_0 *are positive constants.*

Note that from this theorem follow all the above mentioned results an high dimensional case.

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