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The Unconditional Convergence of Fourier-Haar Series

Martin Grigoryan *

Faculty of Physics, Yerevan State University, Al.Manukyan 1,0025,Yerevan,Armenia;
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In this paper considered the question of the absolute and unconditional convergence of Fourier-Haar series.

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1. Introduction

Let $H = \{h_n(x)\}_{n=0}^{\infty}$, $x \in [0,1]$ denote the Haar system normalized in $L^2_{[0,1]}$ (see [1]). We recall that the Haar system is a basis in space $L^p_{[0,1]}$, p$\geq$ 1(see [2], [3]), i.e. each function $f(x) \in L^p_{[0,1]}$ can be represented by a unique series

$$\sum_{n=1}^{\infty} c_n(f)h_n(x) \ , \tag{1}$$

which converges to $f(x)$ in the $L^p_{[0,1]}-$ norm. Note that in (1)

$$c_n(f) = \int_0^1 f(x)h_n(x)dx \ , n \geq 1, \tag{2}$$

and the Fourier-Haar series (2) of each function $f(x) \in L^1_{[0,1]}$ converges to $f(x)$ almost everywhere on $[0, 1]$ (a.e.). It is known that the Haar system is not an unconditional basis in $L^1[0,1]$ (see[4]) i.e. there exists a function $f(x) \in L^1_{[0,1]}$, whose Fourier-Haar series $\sum_{k=1}^{\infty} c_k(f)h_k(x)$ can be so rearranged as to become divergent in $L^1[0,1]$.

A.M. Olevskii [5] has constructed a function $f(x) \in L^{\infty}_{[0,1]}$, whose Fourier-Haar series $\sum_{k=1}^{\infty} c_k(f)h_k(x)$ can be so rearranged as to become divergent almost everywhere on $[0, 1]$.

Note that P.L.Ul'yanov and E.M.Nikishin in [6] proved: if Haar series unconditionally is convergent almost everywhere on [0,1] then it absolutely convergent almost everywhere.

*Email: gmarting@ysu.am

The spectrum of $f(x)$ (denoted by $\Lambda(f) = spec(f)$) is the support of the sequence of Fourier coefficients $\{c_k(f)\}$ of the function $f(x)$ in the Haar's system, i.e. the set of integers where $c_k(f)$ is non-zero.

In this paper we prove the following results, communicated at the International Conference on Fourier Analysis and Approximation Theory dedicated to the 80th birthday of Academician Levan Zhizhiashvili (see [20]):

Theorem 1.1: *For every $\epsilon > 0$, there exists a measurable set $E \subset [0,1]$ with $|E| > 1-\varepsilon$, such that for every function $f(x) \in L_{[0,1]}$ one can find a function $\tilde{f}(x) \in L_{[0,1]}$, $\tilde{f}(x)=f(x)$, $x \in E$, whose Fourier-Haar series is unconditionally convergent almost everywhere on $[0,1]$, and the sequence $\{c_k(\tilde{f})\ ,\ k \in spec(\tilde{f}(x))\} \searrow 0$(.i.e. the nonzero terms of the sequence of Fourier coefficients $\{c_k(\tilde{f})\}$ of the function $\tilde{f}(x)$ in the Haar system is monotonically decreasing and converges to zero.)*

Note that P.L.Ul'yanov in [7] constructed a function $f_0(x) \in L^1_{[0,1]}$, whose Fourier-Haar coefficients diverge unboundedly .

Theorem 1 is equivalent to the following:

Theorem 1.2: *For every $\epsilon > 0$, there exists a measurable set $E \subset [0,1]$ with $|E| > 1-\varepsilon$, such that for every function $f(x) \in L_{[0,1]}$ one can find a function $g(x) \in L_{[0,1]}$, $g(x) = f(x)$, $x \in E$, whose Fourier-Haar series is absolutely convergent almost everywhere on $[0, 1]$, and the sequence $\{c_k(g)\ ,\ k \in spec(g)\} \searrow 0$.*

Note that Theorems 1 and 2 are not true for the trigonometric system.

For the trigonometric and Walsh systems. interesting results in this direction were obtained by many mathematicians (see for example [8]-[19]).

The following questions remain open.

Question 1. Is it possible to take the modified function $g(x)$ in theorem 2 such that its Fourier-Haar series absolutely converges in the $L^1[0,1]$ norm?

Question 2. Is it possible to take the modified function $\tilde{f}(x)$ such that its Fourier series in the trigonometric system unconditionally converges in the $L^1[0,1]$ norm ?

2. Basic lemmas

At first we recall the definition of the Haar system(see [1]). It is a system of functions $H = \{h_n(x)\}_{n=0}^{\infty}$, $x \in [0,1]$, in which $h_1(x) \equiv 1$, $x \in [0,1]$ and for $n = 2^k+m$; $k = 0,1,...$; $m = 1,2,...,2^k$

$$h_n(x) = h_k^{(m)}(x) = h_{2^k+m}(x) = \begin{cases} 2^{k/2} & if \ \frac{m-1}{2^k} < x < \frac{2m-1}{2^{k+1}}, \\ -2^{k/2} & if \ \frac{2m-1}{2^{k+1}} < x < \frac{m}{2^k}, \\ 0 & for \ x \notin [\frac{m-1}{2^k}, \frac{m}{2^k}]. \end{cases} \tag{3}$$

The values taken by these functions in the discontinuity points are not essential in the present work, hence we do not give them.

By $\Delta_n = \Delta_k^{(i)}$, $n = 2^k+i$ $(n \geq 2)$, we denote the support of the function $h_n(x) = h_k^{(i)}(x)$. An interval $\Delta_n = \Delta_k^{(i)} = (\frac{i-1}{2^k}, \frac{i}{2^k})$, $n = 2^k+i$; $k = 0,1,...$; $i = 1,2,...,2^k$, is termed a dyadic interval.

For a set E we denote its characteristic function by $\chi_{E(x)}$.

Lemma 2.1: *For any given numbers* $\gamma \neq 0$, $N_0 > 1$, $q, q_0, (q > q_0 > 2)$, $\delta \in (0,1)$ *and interval* $\Delta \subset [0,1]$ *of the form* $\Delta = \ \Delta_k^{(s)} = (\frac{i-1}{2^\nu}, \frac{i}{2^\nu})$, $\ i \in [1; 2^\nu]$ *there exists a measurable set* $G \subset E \subset \Delta$ *and a polynomial* $Q(x)$ *by* H *of the form*

$$Q(x) = \sum_{k=N_0}^{N} a_k h_k(x)$$

which satisfy the conditions:

$$|E| = (1 - 2^{-q})|\Delta|,$$

$$Q(x) = \begin{cases} \gamma, & x \in E; \\ 0, & x \notin \Delta. \end{cases}$$

$$\int_0^1 |Q(x)|\, dx < 2|\gamma||\Delta| \cdot$$

$$\sum_{k=N_0}^{N} a_k |h_k(x)| < 2^{q_0}|\gamma|, x \in G$$

$$|G| = (1 - 2^{-q_0})|\Delta|,$$

$$0 \le a_k < \delta,$$

and nonzero coefficients in $\{a_k\}_{k=N_0}^{N}$ *are arranged in the decreasing order.*

Proof: Chosen a subsequence $\{l_i\}$ so that

$$l_{i+1} - l_i \ge 2 \ \ \forall \ \ i \in N, \tag{4}$$

and a natural j so large that

$$l_j \ge 2 \ \log_2 \frac{|\ \gamma\ |}{\delta} + log_2 N_0 + \nu, \tag{5}$$

We define a polynomial $Q_1(x)$ in the following way

$$Q_1(x) = 2^{-\frac{l_j}{2}} \mid \gamma \mid \sum_{s\ (\Delta_{l_j}^{(s)} \subset \Delta)} h_{l_j}^{(s)}(x).$$

The polynomial$Q_1(x)$ on Δ takes values γ and $-\gamma$. We denote by E_1 a set, on which $P_1(x)$ is equal to $-\gamma$.

By induction we define polynomials $Q_2(x), Q_3(x), ..., Q_q(x)$ and the sets $E_2, E_3, ..., E_q$ in the following way

$$Q_{i+1}(x) = 2^{i-\frac{l_{j+i}}{2}} \mid \gamma \mid \sum_{s\ (\Delta^{(s)}_{l_{j+i}} \subset E_i)} h^{(s)}_{l_{j+i}}(x), \tag{6}$$

$$E_{i+1} = \{t \in E_i \quad : \quad Q_{i+1}(t) \neq 2^i \gamma\}, \tag{7}$$

It is clear that

$$\mid Q_{i+1}(x) \mid = 2^{i-\frac{l_{j+i}}{2}} \mid \gamma \mid \sum_{s\ (\Delta^{(s)}_{l_{j+i}} \subset E_i)} \mid h^{(s)}_{l_{j+i}}(x), \mid = \begin{cases} 2^i \mid \gamma \mid \forall x \in E_i. \\ 0, \quad x \notin E_i. \end{cases} \tag{8}$$

$$\mid E_1 \mid = \frac{\mid \Delta \mid}{2} \quad and \quad \mid E_{i+1} \mid = \frac{\mid E_i \mid}{2} \quad for \ \ all \quad i = 1, 2, \ldots, q-1, \tag{9}$$

and

$$E_0 = \Delta \supset E_1 \supset E_2 \ldots \supset E_{q_0} \supset .. \supset E_q, \tag{10}$$

Define a polynomial $Q(x)$ and a sets E and G as follows

$$Q(x) = \sum_{i=1}^{q} Q_i(x) \tag{11}$$

$$E = \Delta \setminus E_q, \qquad G = \Delta \setminus E_{q_0}, \tag{12}$$

From (8)-(12) we have

$$\mid G \mid = \mid \Delta \mid - \mid E_{q_0} \mid = (1 - 2^{-q_0}) \mid \Delta \mid,$$

$$\mid E \mid = \mid \Delta \mid - \mid E_q \mid = (1 - 2^{-q}) \mid \Delta \mid,$$

$$Q(x) = \begin{cases} \gamma, \forall x \in E. \\ -(2^q - 1)\gamma, \ \forall \ x \in E_q. \\ 0, \forall \ x \notin \Delta \end{cases}$$

From this we get

$$\int_0^1 |Q(x)| \, dx < 2|\gamma||\Delta|$$

That is, the statements 1)-3) and 5) of Lemma 2.1 are satisfied.Now we will check the fulfillment of statement (4) of Lemma 2.1.

Further, by (5),(6) and (11) the polynomial Q(x) is of the form

$$Q(x)=\sum_{k=N_0}^{N}a_kh_k(x),a_k=\int_0^1 Q(x)h_k(x)dx, \tag{13}$$

All coefficients in decomposition of polynomials $Q_i(x)$ are nonnegative; consequently coefficients a_k will be also nonnegative. All nonzero coefficients of the polynomial $Q_i(x)$ are equal

$$2^{i-1-\frac{l_{j+i-1}}{2}}\mid\gamma\mid,$$

and from (4) we have

$$2^{i-1-\frac{l_{j+i-1}}{2}}\mid\gamma\mid\geq 2^{i-\frac{l_{j+i}}{2}}\mid\gamma\mid,$$

hence nonzero numbers in $\{a_k\}_{k=N_0}^{N}$ are arranged in the decreasing order. For the proof termination it is necessary to notice that (see (5)

$$2^{i-\frac{l_{j+i}}{2}}\mid\gamma\mid\leq 2^{-\frac{l_j}{2}}\mid\gamma\mid<\delta$$

Taking relations (8),(10) and (12) for all $x\in G$ and each $i>q_0$ we obtain $Q_i(x)=0$.

Therefore, by (8)-(11) and (13) for all $x\in G$ we have

$$\sum_{k=N_0}^{N}a_k|h_k(x)|=\sum_{i=1}^{q}|Q_i(x)|=\sum_{i=1}^{q_0}|Q_i(x)|<2^{q_0}|\gamma|,x\in G$$

□

Lemma 2.1 is proved.

Lemma 2.2: *Let numbers $k_0\geq 1,\epsilon\in(0,1)$ and a Haar polynomial $f(x)$ with $\int_0^1|f(x)|\,dx<1$ be given.Then one can find a measurable set $G\subset E\subset\Delta$ and a polynomial $P(x)$ in the Haar system H of the form*

$$Q(x)=\sum_{k=k_0+1}^{\bar{k}}a_kh_{s_k}(x)\ ,\ s_k\nearrow\ ,$$

that satisfy the following conditions:

1) $\mid E\mid>1-\epsilon$;

2) $\mid G\mid>1-\sqrt{\int_0^1|f(x)|\,dx}$;

3) $Q(x) = f(x) \quad E;$

4) $\epsilon > a_k \geq a_{k+1} > 0, k \in [k_0; \bar{k});$

5) $\int_0^1 |Q(x)|\, dx \leq 2 \int_0^1 |f(x)|\, dx;$

6) $\sum_{k=k_0+1}^{\bar{k}} a_k |h_{s_k}\ (x)| < \frac{4|f(x)|}{\sqrt{\int_0^1 |f(x)| dx;}} \qquad if \qquad x \in G,$

Proof: Let

$$f(x) = \sum_{j=0}^{j_0} b_j h_j(x) = \sum_{\nu=1}^{\mu_0} \gamma_\nu \cdot \chi_{\Delta_\nu(x)} \tag{14}$$

where $\Delta_\nu$are dyadic intervals of the form$\Delta_k^{(s)} = (\frac{i-1}{2^{\nu_0}}, \frac{i}{2^{\nu_0}}), \quad i \in [1; 2^{\nu_0}]$
Let:

$$q_0 = 2 - \left[\log_2 \sqrt{\int_0^1 |f(x)|\, dx;} \right], q = q_0 + \left[\log_2 \frac{1}{\epsilon} \right] \tag{15}$$

Repeated application of Lemma 1 yields a sequence of measurable sets $\{E_\nu\}_{\nu=1}^{\mu_0}$, $\{G_\nu\}_{\nu=1}^{\mu_0}$and a sequence of polynomials $\{Q_\nu(x)\}_{\nu=1}^{\mu_0}$ in the Haar system of the form

$$Q_\nu = \sum_{k=m_{\nu-1}}^{m_\nu - 1} a_k^{(\nu)} h_{s_k}(x),\ \nu = 1, 2, \ldots, \mu_0, m_0 = k_0 + 1, \tag{16}$$

such that

$$Q_\nu(x) = \begin{cases} \gamma_\nu, & x \in E_\nu; \\ 0, & x \notin \Delta_\nu. \end{cases} \tag{17}$$

$$\epsilon > a_{m_{\nu-2}}^{(\nu-1)} \geq ... \geq a_k^{(\nu-1)} \geq a_{k+1}^{(\nu-1)} \geq a_{m_{\nu-1}-1}^{(\nu-1)}$$

$$> a_{m_{\nu-1}}^{(\nu)} \geq ... \geq a_k^{(\nu)} \geq a_{k+1}^{(\nu)} \geq ... \geq a_{m_\nu - 1}^{(\nu)} \quad > 0, 1 \leq \nu \leq \mu_0, \tag{18}$$

$$G_\nu \subset E_\nu \subset \Delta_\nu|, 1 \leq \nu \leq \mu_0, \tag{19}$$

$$|E_\nu| = (1 - 2^{-q})|\Delta_\nu, \tag{20}$$

$$|G_\nu| = (1 - 2^{-q_0})|\Delta_\nu|, \tag{21}$$

$$\int_0^1 \mid Q_\nu(x) \mid dx < 2 \mid \gamma_\nu \mid\mid \Delta_\nu \mid, \tag{22}$$

$$\sum_{k=m_{\nu-1}}^{m_\nu-1} a_k^{(\nu)} |h_{s_k}(x)| < \begin{cases} 2^{q_0+1}|\gamma_\nu|, & x \in G_\nu, \\ 0, & x \notin \Delta_\nu. \end{cases} \tag{23}$$

We put

$$Q(x) = \sum_{\nu=1}^{\mu_0} Q_\nu(x) = \sum_{k=k_0+1}^{\bar{k}} a_k h_{n_k}, \tag{24}$$

where

$$a_k = a_k^{(\nu)}, k \in [m_{\nu-1}, m_\nu), 1 \le \nu \le \mu_0(m_{\mu_0} - 1), \tag{25}$$

$$E = \bigcup_{\nu=1}^{\mu_0} E_\nu, \text{ and} G = \bigcup_{\nu=1}^{\mu_0} G_\nu. \tag{26}$$

From this and (24) we obtain

$$Q(x) = f(x) \ x \in E,$$

$$\epsilon > a_k \ge a_{k+1} > 0, \quad k \in (k_0, \bar{k}),$$

$$\int_0^1 |Q(x)|\, dx \le 2 \sum_{\nu=1}^{\nu_0} |\gamma_\nu||\Delta_\nu| = 2 \int_0^1 |f(x)|\, dx$$

$$|E| > 1 - \epsilon, |\ G \ | > 1 - \sqrt{\int_0^1 |f(x)|\, dx};$$

Taking relations (15),(23)-(25) for all $x \in G$ we have

$$\sum_{k=k_0+1}^{\bar{k}} a_k |h_{s_k}(x)| = \sum_{\nu=1}^{\mu_0} \sum_{k=m_{\nu-1}}^{m_\nu-1} a_k |h_{s_k}(x)| \le \frac{2\ |f(x)|}{\sqrt{\int_0^1 |f(x)|\, dx}}$$

□

Lemma 2.2 is proved.

Proof (of Theorem 1.1): Let $\epsilon \in (0,1)$and let

$$\{, f_n(x)\}_{n=1}^{\infty}, \quad x \in [0,1) \tag{27}$$

be a sequence of Haar polynomials with rational coefficients.

Applying Lemma 2.2 consecutively, we can find a sequences $\{G_n\}$, $\{E_n\}$ of sets and a sequence of polynomials in the Haar system of the form

$$Q_n(x) = \sum_{s_k \in [m_{n-1}, m_n)} a_{s_k} h_{s_k}(x),\ n \geq 1,\ m_n \nearrow,\ (\ a_{s_k} > 0,\ s_k \nearrow), \tag{28}$$

which satisfy the conditions:

$$Q_n(x) = f_n(x), \quad x \in E_n, n \geq 1 \tag{29}$$

$$|E_n| > 1 - \epsilon \cdot 4^{-8(n+2)}, \tag{30}$$

$$\int_0^1 |Q_n(x)|\, dx < 2 \int_0^1 |f_n(x)|\, dx, \tag{31}$$

$$\frac{1}{n} > a_{s_k} \geq a_{s_{k+1}} > a_{s_{m_n}} > 0, \quad \forall n \geq 1, \quad \forall s_k, s_{k+1} \in [m_{n-1}, m_n - 1). \tag{32}$$

$$\sum_{s_k \in [m_{n-1}, m_n)} a_{s_k} |h_{s_k}(x)| \leq \frac{4\ |f_n(x)|}{\sqrt{\int_0^1 |f_n(x)|\, dx}}, \forall x \in G_n, n \geq 1 \tag{33}$$

$$|G_n| > 1 - \sqrt{\int_0^1 |f_n(x)|\, dx}, n \geq 1. \tag{34}$$

We put

$$a_i = a_{s_k}, \forall i \in [s_k, s_{k+1}), \forall k \geq 1. \tag{35}$$

and

$$E = \bigcap_{n=1}^{\infty} E_n. \tag{36}$$

It is clear (see (30),(34))

$$|E| > 1 - \epsilon, a_i \searrow 0 (a_i > 0)$$

Let $f(x) \in L^1(0,1)$. It is not hard to see that one can find a subsequence $\{f_{n_k}(x)\}_{n=1}^{\infty}$ from sequence (27) such that

$$\lim_{N\to\infty}\int_0^1\left|\sum_{k=1}^{N} f_{n_k}(x)-f(x)\right|dx=0. \tag{37}$$

$$\lim_{N\to\infty}\sum_{k=1}^{N} f_{n_k}(x)=f(x).a.e.on[0,1]. \tag{38}$$

and

$$\overline{\epsilon}\cdot 4^{-4(k+3)}\leq\int_0^1|f_{n_k}(x)|\,dx\leq\overline{\epsilon}\cdot 4^{-4(k+2)},\quad k\geq 2\ |. \tag{39}$$

$$\overline{\epsilon}=\min\left\{\frac{\epsilon}{2},\int_E|f(x)|dx\right\}$$

Let

$$B_k=\left\{x\subset[0,1];|f_{n_k}(x)|\,dx\leq 4^{-3(k+2)}\right\}.k\geq 2. \tag{40}$$

From this and (39) we have

$$|[0,1)\setminus B_k|\cdot 4^{-3(k+2)}\leq\int_{[0,1)\setminus B_k}|f_{n_k}(x)|\,dx\leq\overline{\epsilon}\cdot 4^{-4(k+2)},\quad k\geq 2\ ,\quad .$$

Then

$$|B_k|>1-\epsilon\cdot 4^{-(k+2)}. \tag{41}$$

We put

$$B=\bigcup_{\nu=1}^{\infty}\bigcap_{k=\nu}^{\infty}(B_k\cap G_{n_k}). \tag{42}$$

From (31), (34), (39), (41) and (42) we obtain

$$\int_0^1\left|\sum_{k=1}^{\infty}Q_{n_k}(x)\right|dx\leq 2\sum_{k=1}^{\infty}\int_0^1|f_{n_k}(x)|\,dx<\infty \tag{43}$$

$$|B|=1.$$

Let the function $\tilde{f}(x)$ and the series $\sum_{i=1}^{\infty} \delta_i a_i \varphi_i(x)$ be defined as follows:

$$\tilde{f}(x) = \sum_{k=1}^{\infty} Q_{n_k}(x) = \sum_{k=1}^{\infty} \sum_{s_j \in [m_{n_k-1}, m_{n_k})} a_{s_j} h_{s_j}(x). \tag{44}$$

$$\sum_{i=1}^{\infty} \delta_i a_i h_i(x) = \sum_{k=1}^{\infty} \sum_{s_j \in [m_{n_k-1}, m_{n_k})} a_{s_j} h_{s_j}(x). \tag{45}$$

where

$$\delta_i = \begin{cases} 1, & \text{for } \ i = s_j \ , \quad \text{where } \ s_j \in \cup_{k=1}^{\infty} [m_{n_k-1}, m_{n_k}) \ . \\ 0, & \text{otherwise} \ . \end{cases}$$

From this and (29), (31), (36), (39), (43)-(45) we have

$$\tilde{f}(x) \in L^1(0,1); \quad \tilde{f}(x) = f(x) \ , \ x \in E,$$

$$\lim_{k \to \infty} \int_0^1 \left| \sum_{i=1}^{m_{n_k}-1} \delta_i a_i h_i(x) - \tilde{f}(x) \right| dx = 0.$$

and therefore

$$\delta_i a_i = \int_0^1 \tilde{f}(x) h_i(x) dx, i \geq 1$$

Let $x \in B$. Then for some k_0 (see (42)) we have $x \subset B_k \cap G_{n_k} \forall k \geq k_0$.
From (39), (40) we obtain

$$\sum_{s_j \in [m_{n_k-1}, m_{n_k})} a_{s_j} |h_{s_j}(x)| \leq \frac{4 \ |f_{n_k}(x)|}{\sqrt{\int_0^1 |f_{n_k}(x)| \, dx}}$$

$$\leq \frac{4.2^{-3(k+2)}}{2^{-2(k+2)}} \to 0, \qquad k \to \infty.$$

Further, from (44), (45), and (43) it follows that the series (45) *absolutely* (unconditionally) converges *almost everywhere on* $[0, 1]$ to $\tilde{f}(x)$.

i.e.

$$\sum_{i=1}^{\infty} \delta_i a_i |h_i(x)| = \sum_{k=1}^{\infty} \sum_{s_j \in [m_{n_k-1}, m_{n_k})} a_{s_j} |h_{s_j}(x)| < \infty, x \in B.$$

$$\lim_{N\to\infty}\sum_{i=1}^{N}\delta_i a_i h_i(x)=\tilde{f}(x)$$

□

Theorem 1.1 is proved.

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