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Nonself-Adjoint Degenerate Differential-Operator Equations of Higher Order

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This article deals with the Dirichlet problem for a degenerate nonself-adjoint differential-operator equation of higher order. We prove existence and uniqueness of the generalized solution as well as establish some analogue of the Keldysh theorem for the corresponding one-dimensional equation.

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1. Introduction

The main object of the present paper is the degenerate differential-operator equation

$$Lu \equiv (-1)^m \big(t^\alpha u^{(m)}\big)^{(m)} + A\big(t^{\alpha-1} u^{(m)}\big)^{(m-1)} + Pt^\beta u = f(t), \tag{1}$$

where $m \in \mathbb{N}$, t belongs to the finite interval $(0, b)$, $\alpha \geq 0, \alpha \neq 1, 3, \ldots, 2m-1$, $\beta \geq \alpha - 2m$, A and P are linear operators (in general unbounded) in the separable Hilbert space H, $f \in L_{2,-\beta}((0,b), H)$, i.e.,

$$\|f\|^2_{L_{2,-\beta}((0,b),H)} = \int_0^b t^{-\beta} \|f(t)\|^2_H \, dt < \infty.$$

We suppose that the operators A and P have common complete system of eigenfunctions $\{\varphi_k\}_{k=1}^\infty$, $A\varphi_k = a_k\varphi_k, P\varphi_k = p_k\varphi_k, k \in \mathbb{N}$, which form a Riesz basis in H, i.e., for any $x \in H$ there is a unique representation

$$x = \sum_{k=1}^\infty x_k \varphi_k$$

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and there are constants $c_1, c_2 > 0$ such that

$$c_1 \sum_{k=1}^{\infty} |x_k|^2 \leq ||x||^2 \leq c_2 \sum_{k=1}^{\infty} |x_k|^2.$$

If $m = 1$, the operator A is a multiplication operator, $Au = au, a \in \mathbb{R}$, $a \neq 0$ and $Pu = -u_{xx}$, $x \in (0, c)$ then we obtain the degenerate elliptic operator in the rectangle $(0, b) \times (0, c)$. The dependence of the character of the boundary conditions with respect to t for $t = 0$ on the sign of the number a was first observed by M.V. Keldish in [5] and next generalized by G. Jaiani in [4] (thus the statement of the boundary value problem depends on the "lower order" terms). The case $m = 1, \beta = 0, 0 \leq \alpha < 2$ was considered in [2], [6] (here $A = 0$) and the case $m = 2$, $\beta = 0$, $0 \leq \alpha \leq 4$ in [8]. In [9] the self-adjoint case of higher order degenerate differential-operator equations for arbitrary $\alpha \geq 0$, $\alpha \neq 1, 3, \ldots, 2m-1$ has been considered.

Our approach is based on the consideration of the one-dimensional equation (1), when the operators A and P are multiplication operators by numbers a and p respectively, $Au = au$, $Pu = pu$, $a, p \in \mathbb{C}$ (see [3]).

Observe that this method suggested by A.A. Dezin (see [3]) has been used for the degenerate self-adjoint operator equation on the infinite interval $(1, +\infty)$ in [12] and with arbitrary weight function on the finite interval in [11].

2. One-dimensional case

2.1. *Weighted Sobolev spaces* $\dot{W}_\alpha^m(0, b)$

Let $\dot{C^m}[0, b]$ denote the functions $u \in C^m[0, b]$, which satisfy the conditions

$$u^{(k)}(0) = u^{(k)}(b) = 0, k = 0, 1, \ldots, m-1. \tag{2}$$

Define $\dot{W}_\alpha^m(0, b)$ as the completion of $\dot{C^m}[0, b]$ in the norm

$$\|u\|^2_{\dot{W}_\alpha^m(0,b)} = \int_0^b t^\alpha |u^{(m)}(t)|^2 \, dt.$$

Denote the corresponding scalar product in $\dot{W}_\alpha^m(0, b)$ by $\{u, v\}_\alpha = (t^\alpha u^{(m)}, v^{(m)})$, where $(\cdot, \cdot)$ stands for the scalar product in $L_2(0.b)$.

Note that the functions $u \in \dot{W}_\alpha^m(0, b)$ for every $t_0 \in (\varepsilon, b]$, $\varepsilon > 0$ have the finite values $u^{(k)}(t_0), k = 0, 1, \ldots, m-1$ and $u^{(k)}(b) = 0, k = 0, 1, \ldots, m-1$ (see [1]).

For the proof of the following propositions we refer to [9] and [10].

Proposition 2.1: *For the functions* $u \in \dot{W}_\alpha^m(0, b), \alpha \neq 1, 3, \ldots, 2m-1$ *we have the following estimates*

$$|u^{(k)}(t)|^2 \leq C_1 t^{2m-2k-1-\alpha} \|u\|^2_{\dot{W}_\alpha^m(0,b)}, \; k = 0, 1, \ldots, m-1. \tag{3}$$

It follows from Proposition 2.1 that in the case $\alpha < 1$ (weak degeneracy) $u^{(j)}(0) = 0$ for all $j = 0, 1, \ldots, m-1$, while for $\alpha > 1$ (strong degeneracy) not all $u^{(j)}(0) = 0$.

More precisely, for $1 < \alpha < 2m-1$ the derivatives at zero $u^{(j)}(0) = 0$ only for $j = 0, 1, \ldots, s_\alpha$, where $s_\alpha = m - 1 - [\frac{\alpha+1}{2}]$ (here $[a]$ is the integral part of the a) and for $\alpha > 2m-1$ all $u^{(j)}(0)$, $j = 0, 1, \ldots, m-1$ in general may be infinite.
Denote $L_{2,\beta}(0,b) = \Big\{f, \int_0^b t^\beta |f(t)|^2\, dt < +\infty\Big\}$. Observe that for $\alpha \leq \beta$ we have $L_{2,\alpha}(0,b) \subset L_{2,\beta}(0,b)$.

Proposition 2.2: *For $\beta \geq \alpha - 2m$ we have a continuous embedding*

$$\dot{W}_\alpha^m(0,b) \subset L_{2,\beta}(0,b), \tag{4}$$

which is compact for $\beta > \alpha - 2m$.

Note that the embedding (4) in the case of $\beta = \alpha - 2m$ is not compact while for $\beta < \alpha - 2m$ it fails.
Denote $d(m,\alpha) = 4^{-m}(\alpha-1)^2(\alpha-3)^2 \cdots (\alpha-(2m-1))^2$. In Proposition 2.2 using Hardy inequality (see [7]) it was proved that

$$\int_0^b t^\alpha |u^{(m)}(t)|^2\, dt \geq d(m,\alpha) \int_0^b t^{\alpha-2m} |u(t)|^2\, dt. \tag{5}$$

Note that here $d(m,\alpha)$ is the exact number. Now it is easy to check that for $\beta \geq \alpha - 2m$

$$\|u\|^2_{\dot{W}_\alpha^m(0,b)} \geq b^{\alpha-2m-\beta} d(m,\alpha) \|u\|^2_{L_{2,\beta}(0,b)}. \tag{6}$$

2.2. *Nonself-adjoint degenerate equations*

In this subsection we consider one-dimensional version of equation (1)

$$Su \equiv (-1)^m \big(t^\alpha u^{(m)}\big)^{(m)} + a\big(t^{\alpha-1} u^{(m)}\big)^{(m-1)} + pt^\beta u = f(t), \tag{7}$$

where $\alpha \geq 0, \alpha \neq 1, 3, \ldots, 2m-1$, $\beta \geq \alpha - 2m$, $f \in L_{2,-\beta}(0,b)$, $a \neq 0$ and p are real constants.

Definition 2.3: A function $u \in \dot{W}_\alpha^m(0,b)$ is called a generalized solution of equation (7), if for arbitrary $v \in \dot{W}_\alpha^m(0,b)$ we have

$$\{u,v\}_\alpha + a(-1)^{m-1}\big(t^{\alpha-1}u^{(m)}, v^{(m-1)}\big) + p(t^\beta u, v) = (f,v). \tag{8}$$

Theorem 2.4: *Let the following condition be fulfilled*

$$\begin{aligned} a(\alpha-1)(-1)^m &> 0, \\ \gamma = b^{\alpha-2m-\beta}\big(d(m,\alpha) &+ \frac{a}{2}(\alpha-1)(-1)^m d(m-1,\alpha-2)\big) + p > 0. \end{aligned} \tag{9}$$

Then the generalized solution of equation (7) exists and is unique for every $f \in L_{2,-\beta}(0,b)$.

Proof: *Uniqueness.* To prove the uniqueness of the solution we set in equality (8) $f = 0$ and $v = u$. Let $\alpha > 1$ (in the case $\alpha < 1$ the proof is similar and we use

$\big(t^{\alpha-1}|u^{(m-1)}(t)|^2\big)\,|_{t=0}=0$, which follows from Proposition 2.1). Then integrating by parts we obtain

$$\big(t^{\alpha-1}u^{(m)},u^{(m-1)}\big)=-\frac{1}{2}\big(t^{\alpha-1}|u^{(m-1)}(t)|^2\big)|_{t=0}-\frac{\alpha-1}{2}\int_0^b t^{\alpha-2}|u^{(m-1)}(t)|^2\,dt.$$

It follows from the inequality (3) for $k=m-1$ that the value $\big(t^{\alpha-1}|u^{(m-1)}(t)|^2\big)\,|_{t=0}$ is finite. On the other hand, using inequality (5) we get

$$\int_0^b t^{\alpha-2}|u^{(m-1)}(t)|^2\,dt\geq d(m-1,\alpha-2)\int_0^b t^{\alpha-2m}|u(t)|^2\,dt.$$

Hence using inequality (6) we obtain

$$\begin{aligned}0=\{u,u\}_\alpha&+a(-1)^{m-1}\big(t^{\alpha-1}u^{(m)},u^{(m-1)}\big)+p(t^\beta u,u)\\&\geq\frac{a}{2}(-1)^m\big(t^{\alpha-1}|u^{(m-1)}(t)|^2\big)|_{t=0}+\gamma\int_0^b t^\beta|u(t)|^2\,dt.\end{aligned}$$

Now uniqueness of the generalized solution follows from condition (9).

Existence. To prove the existence of the generalized solution define a linear functional $l_f(v)=(f,v), v\in\dot{W}^m_\alpha(0,b)$. From the continuity of the embedding (4) it follows that

$$|l_f(v)|\leq\|f\|_{L_{2,-\beta}(0,b)}\|v\|_{L_{2,\beta}(0,b)}\leq c\|f\|_{L_{2,-\beta}(0,b)}\|v\|_{\dot{W}^m_\alpha(0,b)},$$

therefore the linear functional $l_f(v)$ is bounded on $\dot{W}^m_\alpha(0,b)$. Hence it can be represented in the form $l_f(v)=(f,v)=\{u^*,v\},u^*\in\dot{W}^m_\alpha(0,b)$ (this follows from the Riesz theorem on the representation of the linear continuous functional). The last two terms in the left hand-side of equality (8) also can be regarded as a continuous linear functional relative to u and represented in the form $\{u,Kv\}_\alpha, Kv\in\dot{W}^m_\alpha(0,b)$. In fact, using inequality (5) we may write

$$\begin{aligned}|a(-1)^{m-1}\big(t^{\alpha-1}u^{(m)},v^{(m-1)}\big)&+p(t^\beta u,v)|\\&\leq|a(t^{\frac{\alpha}{2}}u^{(m)},t^{\frac{\alpha}{2}-1}v^{(m-1)})|+|p(t^{\frac{\beta}{2}}u,t^{\frac{\beta}{2}}v)|\\&\leq c_1\|u\|_{\dot{W}^m_\alpha(0,b)}\left\{\int_0^b t^{\alpha-2}|v^{(m-1)}(t)|^2\,dt\right\}^{1/2}\\&+c_2\|u\|_{L_{2,\alpha-2m}(0,b)}\|v\|_{L_{2,\alpha-2m}(0,b)}\\&\leq\frac{2c_1}{|\alpha-1|}\|u\|_{\dot{W}^m_\alpha(0,b)}\|v\|_{\dot{W}^m_\alpha(0,b)}+c_3\|u\|_{\dot{W}^m_\alpha(0,b)}\|v\|_{\dot{W}^m_\alpha(0,b)}\\&=c\|u\|_{\dot{W}^m_\alpha(0,b)}\|v\|_{\dot{W}^m_\alpha(0,b)}.\end{aligned}$$

From equality (8) we deduce that for any $v\in\dot{W}^m_\alpha(0,b)$ we have

$$\{u,(I+K)v\}_\alpha=\{u^*,v\}_\alpha.\tag{10}$$

Observe that the image of the operator $I+K$ is dense in $\dot{W}^m_\alpha(0,b)$. Indeed, if we have some $u_0 \in \dot{W}^m_\alpha(0,b)$ such that

$$\{u_0,(I+K)v\}_\alpha = 0$$

for every $v \in \dot{W}^m_\alpha(0,b)$, we obtain $u_0 = 0$, since we have already proved uniqueness of the generalized solution for equation (7).
Assume that $0 < \sigma d(m,\alpha) b^{\alpha-2m-\beta} \le \gamma$. Then we can write

$$\begin{aligned}\{u,(I+K)u\}_\alpha &\ge \sigma\{u,u\}_\alpha + \big(b^{\alpha-2m-\beta}\big((1-\sigma)d(m,\alpha)\\ &\quad + \frac{a}{2}(\alpha-1)(-1)^m d(m-1,\alpha-2)\big)+p\big)\int_0^b t^\beta |u(t)|^2\,dt\\ &= \sigma\{u,u\}_\alpha + \big(\gamma - \sigma d(m,\alpha) b^{\alpha-2m-\beta}\big)\int_0^b t^\beta|u(t)|^2\,dt\\ &\ge \sigma\{u,u\}_\alpha.\end{aligned}$$

Finally we get

$$\{u,(I+K)u\}_\alpha \ge \sigma\{u,u\}_\alpha. \tag{11}$$

From (11) it follows that $(I+K)^{-1}$ is defined on $\dot{W}^m_\alpha(0,b)$ and is bounded. Consequently there exist operator $I+K^*$ and $(I+K^*)^{-1} = ((I+K)^{-1})^*$ (here K^* means the adjoint operator). Hence from (10) we obtain

$$u = (I+K^*)^{-1}u^*.$$

□

Define an operator $S : D(S) \subset \dot{W}^m_\alpha(0,b) \subset L_{2,\beta}(0,b) \to L_{2,-\beta}(0,b)$.

Definition 2.5: We say that $u \in \dot{W}^m_\alpha(0,b)$ belongs to $D(S)$ if there exists $f \in L_{2,-\beta}(0,b)$ such that equality (8) is fulfilled for every $v \in \dot{W}^m_\alpha(0,b)$. In this case we write $Su = f$.

The operator S acts from the space $L_{2,\beta}(0,b)$ to $L_{2,-\beta}(0,b)$. It is easy to check that $\mathbb{S} := t^{-\beta}S, D(\mathbb{S}) = D(S)$, $\mathbb{S} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ is an operator in the space $L_{2,\beta}(0,b)$, since if $f \in L_{2,-\beta}(0,b)$ then $f_1 := t^{-\beta}f \in L_{2,\beta}(0,b)$ and $\|f\|_{L_{2,-\beta}(0,b)} = \|f_1\|_{L_{2,\beta}(0,b)}$.

Proposition 2.6: *Under the assumptions of Theorem* 2.4 *the inverse operator* $\mathbb{S}^{-1} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ *is continuous for* $\beta \ge \alpha - 2m$ *and compact for* $\beta > \alpha - 2m$.

Proof: For the proof first observe that for $u \in D(\mathbb{S})$ we have

$$\|u\|_{L_{2,\beta}(0,b)} \le c\|f\|_{L_{2,-\beta}(0,b)} = c\|f_1\|_{L_{2,\beta}(0,b)}.$$

In fact, setting $v = u$ in equality (8), using inequalities (6), (11) and applying considerations of Theorem 2.4, we get

$$\begin{aligned}\sigma b^{\alpha-2m-\beta} d(m,\alpha)\|u\|^2_{L_{2,\beta}(0,b)} &\le \sigma d(m,\alpha)\|u\|^2_{\dot{W}^m_\alpha(0,b)}\\ &\le \{(I+K)u,u\}_\alpha = (f,u)\\ &\le \|f\|_{L_{2,-\beta}(0,b)}\|u\|_{L_{2,\beta}(0,b)}\\ &= \|f_1\|_{L_{2,\beta}(0,b)}\|u\|_{L_{2,\beta}(0,b)}.\end{aligned}$$

Thus we obtain

$$\|\mathbb{S}^{-1}f_1\|_{L_{2,\beta}(0,b)} \le c\|f_1\|_{L_{2,\beta}(0,b)}, \tag{12}$$

consequently the continuity of $\mathbb{S}^{-1}$ for $\beta \ge \alpha-2m$ is proved. To show the compactness of $\mathbb{S}^{-1}$ for $\beta < \alpha-2m$ it is enough to apply the compactness of the embedding (4) for $\beta < \alpha - 2m$. □

Let us consider the following equation

$$Tv \equiv (-1)^m\big(t^\alpha v^{(m)}\big)^{(m)} - a\big(t^{\alpha-1}v^{(m-1)}\big)^{(m)} + pt^\beta v = g(t), \tag{13}$$

where $\alpha \ge 0, \alpha \ne 1, 3, \dots, 2m-1$, $\beta \ge \alpha - 2m$, $g \in L_{2,-\beta}(0,b)$, $a \ne 0$ and p are real constants.

Definition 2.7: We say that $v \in L_{2,\beta}(0,b)$ is a generalized solution of equation (13), if for every $u \in D(S)$ the following equality holds

$$(Su, v) = (u, g). \tag{14}$$

Let $g_1 := t^{-\beta}g$. Definition 2.7 of the generalized solution as above defines an operator $\mathbb{T} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$, $\mathbb{T} := t^{-\beta}T$. Actually we have defined the operator $\mathbb{T}$ as the adjoint to $\mathbb{S}$ operator in $L_{2,\beta}(0,b)$, i.e.,

$$\mathbb{T} = \mathbb{S}^*.$$

Theorem 2.8: *Under the assumptions of Theorem* 2.4 *the generalized solution of equation* (13) *exists and is unique for every* $g \in L_{2,-\beta}(0,b)$. *Moreover, the inverse operator* $\mathbb{T}^{-1} : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ *is continuous for* $\beta \ge \alpha - 2m$ *and compact for* $\beta > \alpha - 2m$.

Proof: Solvability of the equation $\mathbb{S}u = f_1$ for any $f_1 \in L_{2,-\beta}(0,b)$ (see Theorem 2.4) implies uniqueness of the solution of equation (13), while existence of the bounded inverse operator $\mathbb{S}^{-1}$ (see Proposition 2.6) implies solvability of (13) for any $g \in L_{2,-\beta}(0,b)$ (see, for instance, [13]). Since we have $(\mathbb{S}^*)^{-1} = (\mathbb{S}^{-1})^*$, boundedness and compactness of the operator $\mathbb{S}^{-1}$ imply boundedness and compactness of the operator $\mathbb{T}^{-1}$ for $\beta \ge \alpha - 2m$ and $\beta > \alpha - 2m$ respectively (see Proposition 2.6). □

Remark 1: For $\alpha > 1$ and for every generalized solution v of equation (13) we have

$$\left(t^{\alpha-1}|u^{(m-1)}(t)|^2\right)|_{t=0} = 0. \tag{15}$$

In fact, replacing g by Tv in equality (14), integrating by parts the second term and using equality (8) we obtain (15). Note also that for equation (7) the left-hand side of (15) is only bounded. This is some analogue of the Keldysh theorem (see [5]).

Remark 2: Note another interesting phenomenon connected with degenerate equations, namely appearing continuous spectrum. Assume that in equation (7) $a = p = 0$ and $\beta = \alpha - 2m$. In [10] it was proved that the spectrum of the operator

$$Bu := (-1)^m t^{2m-\alpha}\big(t^\alpha u^{(m)}\big)^{(m)}, B : L_{2,\alpha-2m}(0,b) \to L_{2,\alpha-2m}(0,b)$$

is purely continuous and coincides with the ray $[d(m,\alpha), +\infty)$. Note also that the spectrum of the operator $Qu := (-1)^m t^{-\beta}(t^\alpha u^{(m)})^{(m)}, Q : L_{2,\beta}(0,b) \to L_{2,\beta}(0,b)$ for $\beta > \alpha - 2m$ is discrete.

3. Dirichlet problem for degenerate differential-operator equations

In this section we consider the operator equation

$$Lu \equiv (-1)^m\big(t^\alpha u^{(m)}\big)^{(m)} + A\big(t^{\alpha-1}u^{(m)}\big)^{(m-1)} + Pt^\beta u = f(t), \tag{16}$$

where $\alpha \geq 0, \alpha \neq 1, 3, \ldots, 2m-1$, $\beta \geq \alpha - 2m$, A and P are linear operators in the separable Hilbert space H, $f \in L_{2,-\beta}((0,b), H)$.
By assumption linear operators A and P have common complete system of eigenfunctions $\{\varphi_k\}_{k=1}^\infty$, $A\varphi_k = a_k\varphi_k, P\varphi_k = p_k\varphi_k, k \in \mathbb{N}$, which forms a Riesz basis in H, i.e., we can write

$$u(t) = \sum_{k=1}^\infty u_k(t)\varphi_k, \quad f(t) = \sum_{k=1}^\infty f_k(t)\varphi_k. \tag{17}$$

Hence operator equation (16) can be decomposed into an infinite chain of ordinary differential equations

$$L_k u_k \equiv (-1)^m\big(t^\alpha u_k^{(m)}\big)^{(m)} + a_k\big(t^{\alpha-1}u_k^{(m)}\big)^{(m-1)} + p_k t^\beta u_k = f_k(t), k \in \mathbb{N}. \tag{18}$$

It follows from the condition $f \in L_{2,-\beta}((0,b), H)$ that $f_k \in L_{2,-\beta}(0,b), k \in \mathbb{N}$. For one-dimensional equations (18) we can define the generalized solutions $u_k(t), k \in \mathbb{N}$ (see Section 2).

Definition 3.1: A function $u \in L_{2,\beta}((0,b), H)$ admitting representation

$$u(t) = \sum_{k=1}^\infty u_k(t)\varphi_k,$$

where $u_k(t), k \in \mathbb{N}$ are the generalized solutions of the one-dimensional equations (18) is called a generalized solution of the operator equation (16).

Actually we have defined the operator L as the closure of the differential operation $L(D)$ originally defined on all finite linear combinations of functions $u_k(t)\varphi_k, k \in \mathbb{N}$, where $u_k \in D(L_k)$.
The following result is a consequence of the general results of A.A. Dezin (see [3]).

Theorem 3.2: *The operator equation* (16) *is uniquely solvable for every* $f \in L_{2,-\beta}((0,b),H)$ *if and only if the equations* (18) *are uniquely solvable for every* $f_k \in L_{2,-\beta}(0,b), k \in \mathbb{N}$ *and uniformly with respect to* $k \in \mathbb{N}$

$$\|u_k\|_{L_{2,\beta}(0,b)} \leq c\|f_k\|_{L_{2,-\beta}(0,b)}. \tag{19}$$

Theorems 2.4 and 2.8 shows us that a sufficient condition for relations (19) are the conditions

$$\gamma_k = b^{\alpha-2m-\beta}\big(d(m,\alpha)+\frac{a_k}{2}(\alpha-1)(-1)^m d(m-1,\alpha-2)\big)+p_k > \varepsilon > 0, k \in \mathbb{N}. \tag{20}$$

Here we assume that $a_k \neq 0$, a_k and p_k are real for $k \in \mathbb{N}$. Thus we get the following result.

Theorem 3.3: *Let the condition* (20) *be fulfilled. Then operator equation* (16) *has a unique generalized solution for every* $f \in L_{2,-\beta}((0,b),H)$.

Proof: Since the system $\{\varphi_k\}_{k=1}^{\infty}$ forms a Riesz basis in H then according to (19) we can write

$$\begin{aligned}\|u\|^2_{L_{2,\beta}((0,b),H)} &= \int_0^b t^\beta \|u(t)\|^2_H \, dt \\ &\leq c_1 \int_0^b t^\beta \sum_{k=1}^{\infty} |u_k(t)|^2 \, dt \\ &\leq c_2 \sum_{k=1}^{\infty} \|f_k\|^2_{L_{2,-\beta}(0,b)} \\ &\leq C\|f\|_{L_{2,-\beta}((0,b),H)}.\end{aligned} \tag{21}$$

□

It follows from inequality (21) that the inverse operator $L^{-1} : L_{2,-\beta}((0,b),H) \to L_{2,\beta}((0,b),H)$ is bounded for $\beta \geq \alpha - 2m$. In contrast to the one-dimensional case (see Proposition 2.6 and Theorem 2.8) this operator for $\beta > \alpha - 2m$ will not be compact (it will be a compact operator only in case when the space H is finite-dimensional). The operator L acts from the space $L_{2,\beta}((0,b),H)$ to the space $L_{2,-\beta}((0,b),H)$. As in one-dimensional case define an operator acting in the same space, which is necessary to explore spectral properties of the operators. Set $f = t^\beta g$. Then $\|f\|_{L_{2,-\beta}((0,b),H)} = \|g\|_{L_{2,\beta}((0,b),H)}$. Hence the operator $\mathbb{L} = t^{-\beta}L$ is an operator in the space $L_{2,\beta}((0,b),H)$. As a consequence of Theorem 3.3 we can state that $0 \in \rho(\mathbb{L})$, where $\rho(\mathbb{L})$ is the resolvent set of the operator $\mathbb{L}$.

Remark 1: The simplest example of the operators described in Introduction consists of the operators on the n-dimensional cube $V = [0, 2\pi]^n$, generated by differential expressions of the form

$$L(-iD)u \equiv \sum_{|\alpha| \leq m} a_\alpha D^\alpha u$$

with constant coefficients. Here $\alpha \in \mathbb{Z}_+^n$ is a multi-index. This class of operators is at the same time quite a large class. Let $\mathcal{P}^\infty$ be the set of smooth functions that are periodic in each variable. Let $s \in \mathbb{Z}^n$. To every differential operation $L(-iD)$ we can associate a polynomial $A(s)$ with constant coefficients such that

$$A(-iD)e^{is\cdot x} = A(s)e^{is\cdot x}, \quad s \cdot x = s_1x_1 + s_2x_2 + \ldots + s_nx_n.$$

We define the corresponding operator $A : L_2(V) \to L_2(V)$ to be the closure in $L_2(V)$ of the differential operation $A(-iD)$ first defined on $\mathcal{P}^\infty$. Such operators are called Π-operators and have many interesting properties. The role of the functions $\{\varphi_k\}_{k=1}^\infty$ is played by the functions $e^{is\cdot x}, s \in \mathbb{Z}^n$. For details see the book of A.A. Dezin [3].

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