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Even Order Singular Elliptic Equations

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Contents

Abstract. The present Lecture Notes is devoted to singular partial differential equations, i.e., to partial differential equations with the order degeneracy.
It is foreseen as a Lecture Course for the Advanced Courses of TICMI and the Elective Course within the framework of master programs in Mathematics and in Applied Mathematics. The results stated in the course are applied in investigations of cusped prismatic shells and bars and of motion of fluids in angular ducts.

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Singular Elliptic Equations; Second Order Elliptic Equations with Order Degenerations; Weighted Boundary Value Problems; Explicit Solutions; BVPs with discontinuous Data, Singular Generalized Analytic Functions; Weighted Boundary Value Problems for Higher Order PDFs; Higher (Even) Order Degenerate Elliptic Equations; Generalized Poisson Integral; Generalization of Schwartz Formula

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Chapter 1

Introduction

Historically first works concerning singular partial differential equations [PDE (in other words PDEs with unbounded, i.e., singular, coefficients)] were devoted to particular cases of elliptic Euler-Poisson-Darboux (EPD) equation

$$E^{(a,b)}u := y(u_{xx} + u_{yy}) + au_x + bu_y = 0, \tag{1.1}$$

where a and b are constans unless otherwise stated.

Equation (1.1) is elliptic in the half-plane

$$\mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 : y > 0\}$$

and has the order degeneration on x-axis which we denote by R^1 (the same symbol R^1 we use for the one-dimensional Euclidean space of real numbers; R^2 denotes at the same time a plane and two-dimensional Euclidean space of pairs of real numbers; analogously is defined $\mathbb{R}^p, \quad p \geq 3$).

By means of the operators

$$\partial_z := \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y),$$

where

$$z = x + iy \quad \text{and} \quad \overline{z} = x - iy$$

are complex numbers and their conjugates, respectively, equation (1.1) can be rewritten as

$$(z - \overline{z})\partial^2_{z\overline{z}}u - \frac{b - ia}{2}\partial_z u + \frac{b + ia}{2}\partial_{\bar{z}}u = 0 \tag{1.2}$$

in the complex form. In the scientific literature equation (1.2) (see G. Darboux [32]) is called either Euler-Poisson-Darboux or Euler-Poisson, or Euler equation.

The particular case $a = 0$, i.e., equation

$$y(u_{xx} + u_{yy}) + bu_y = 0 \tag{1.3}$$

is called either Euler-Poisson-Darboux equation or equation of the generalized theory of axialsymmetric potentials, or Weinstein equation.

(1.3) we may also rewrite as

$$\frac{\partial}{\partial x}\Big(y^b\frac{\partial u}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(y^b\frac{\partial u}{\partial y}\Big)=0$$

which indicates the existence of a stream-function v defined by the generized Stokes-Beltrami equations (see A. Weinstein [187])

$$y^b u_x = v_y, \quad y^b u_y = -v_x, \quad b \geq 0.$$

It is easily seen, that axialsymmetric with respect to x_1-axis solutions of the Laplace equation

$$u_{x_1x_1}+\cdots+u_{x_px_p}=0, \;\; p\geq 3,$$

satisfy equation (1.3), where

$$x := x_1, \;\; y := \sqrt{x_2^2+\cdots+x_p^2}, \;\; b := p-2. \tag{1.4}$$

The case $b = 1$ (i.e., $p = 3$) was the object of investigation already in the Laplace time. Here essential steps was made by Stokes and Beltrami ([3]). W. Arndt ([2]) pointed out the importance of investigating the case $b = 3$ ($p = 5$) for consideration of the torsion problem for Shafts of revolution. For arbitrary b the study was started by A. Weinstein ([183]-[187]).

Qualitative and structural properties of solutions of (1.3), their relation to degenerate first order elliptic systems, and boundary value problems (BVP) are considered by L. Bers, A. Gelbart ([6]), M.P. Brousse ([16], [17]) A. Huber ([68]-[70]]), A. Erdélyi ([41]), P. Henrici ([65],[66]), M.A. Hyman ([71]), R.P. Gilbert ([60]-[62]), A. Vasharin, P.I. Lizorkin ([172]), P.R. Garabedian ([56]), I.P. Krivenkov ([124]-[128]), K.B. Ranger ([158]), S.V. Parter ([155]), B. Brelot-Collin, M. Brelot ([14],[15]) L.E. Vostrova ([180]), M.V. Korshavina ([122]), A.R. Khvoles ([119], [120]), L.G. Mikhailov, N. Rajabov ([141]), A. Sattarov ([162]), V.I. Evsin ([42], [43]), A.J. Fryant ([55]), T.V. Chekmariov ([24]]), M.G. Muskhelishvili ([146], [147]), G.P. Kapoor, A. Nautiyal ([115]), G. Jaiani [99]), J. Šimkovič [163], N. Chinchaladze, A. Sakevarashvili [31], N. Chinchaladze [28], [29], G. Jaiani [108], [107], [106], [109], and others.

After transformation of variables

$$x=\xi, \quad y=\frac{\eta^{\frac{m+2}{2}}}{\frac{m+2}{2}}, \quad \eta>0, \tag{1.5}$$

where

$$m=\frac{2b}{1-b}\;\;\Big(b=\frac{m}{m+2}\Big), \;\; 0<b<1,$$

from (1.1) we obtain

$$\eta^m u_{\xi\xi}+u_{\eta\eta}+\frac{m+2}{2}a\eta^{\frac{m}{2}-1}u_\eta=0. \tag{1.6}$$

Such [equation (1.6) when $a = 0$ is an equation with noncharacteristic degeneracy, sometimes it is also called the equation with weak degeneracy] and more general second order equations with variable nonsingular coefficients by

$$u_\xi, u_\eta, u,$$

and noncharacteristic degeneration is considered by F. Tricomi ([169], [170]), E. Holmgren ([67]), I. Vekua ([173]), A. Bitsadze ([8], [9], [10], [11]), S. Gellerstedt ([57]-[59]]), F. Frankl ([51], [52]), S.G.Mikhlin ([142]) (s. also [10]], [164], [165], and references therein), and others.

After transformation of variables

$$x = \xi, \quad y = 2\sqrt{\eta}, \quad \eta > 0, \tag{1.7}$$

from (1.1) we have

$$u_{\xi\xi} + \eta u_{\eta\eta} + \frac{a}{2\sqrt{\eta}} u_\xi + \frac{b+1}{2} u_\eta = 0. \tag{1.8}$$

Such [equation (1.8) when $a = 0$ is an equation with characteristic degeneracy, sometimes it is also called the equation with strong degeneracy] and more general second order equations with variable nonsingular coefficients by

$$u_\xi, u_\eta, u,$$

and characteristic degeneration is considered by M.V. Keldysh ([117]), O.A. Oleinik ([153]), N.D. Vvedenskaya ([182]), M.I. Vishik ([176], [177]), S.G. Mikhlin ([143], [144]), I.L. Karol ([116]), S.A. Tersenov ([167], [168]), Khe Kan Cher ([118]), and others.

All the above-mentioned equations belong to the class of PDEs with nonnegative characteristic form. The unified theory of which belongs to G.Fichera ([44]-[46]). Different problems for this class of PDEs are investigated by E. Magenes ([135]), O.A. Oleinik, E.V. Radkevich ([154]), H. Yamada ([188]), J.J. Kohn, L. Nierenberg ([121]), V.P. Glushko ([63]) (see also a survey V.P. Glushko, I.B. Savchenko [64]), M.I. Freidlin ([53]), L.I. Kaminin ([113]), V.A. Malovichko ([136], [137]) V.F. Moss ([145]), H. Okumura ([152]),L.I. Kaminin, B.N. Khimchenko ([114]), G. Jaiani [96], [100], [102], O.I. Marichev [138], [139], A.C. Cavalheiro [23] and others.

Divergent form second order degenerate PDEs are considered by M. Franciosi ([49], [50]), B. Franchi, E. Lanconelli ([48]), L.D. Kudrjavtsev ([129]), G. Porru ([156]]), I.V. Rybalov ([161]), and others.

K.O. Friedrichs ([54]) has investigated BVPs for symmetric operators independent of their type.

Analytical theory of elliptic equations with order degeneration is given in a monograph of A.I. Yanushauskas ([189]).

For equation (1.1) I. Vekua (see [174], pp.27,28 and [175]) has constructed the complex Riemann function (see [174], pp.53-54 and also [123], pp.36-45) and by its mean obtained a representation of all the regular (i.e., of C^2 class) solutions of equation (1.1) in any domain lying inside the upper half-plane $\mathbb{R}^2_+$.

V.I. Evsin ([42]) constructed a fundamental solution of equation (1.1) and solved the Holmgren problem, when

$$a \in \mathbb{R}^1, b \in]0, 1[.$$

Fuchs' method developed for ordinary differential equations (ODE) is applied by N. Rajabov, K. Boltaev ([157]) in order to investigate more general than equation (1.1) (s. also K. Boltaev, N. Rajabov ([13]). Equation (1.1) is investigated in works of G. Jaiani ([72], [74]-[76], [77]-[80], [81], [82], [83], [84], [85]-[94], [95], [97]) (s. also Chap. 3 of the present work).

For a fourth order degenerate PDE in some cases A.Narchaev ([148]) proved a uniqueness theorem for the Dirichlet BVP, when on the boundary only unknown function (in contrast to non-degenerate fourth order PDE, when also its derivative should be given) is prescribed. In certain function classes S.M.Nikol'skii and P.I.Lizorkin ([150]) proved the existence and uniqueness theorems for the Dirichlet BVP for a degenerate on the whole boundary $2m$ order PDE of the divergent form, when on the boundary $k < m$ conditions are given. P.Bolley, J.Camus ([12]) studied the Dirichlet and Neumann problems for a strongly degenerate higher order PDE. M.Troisi ([171]) investigated a general BVP for an (in general) non-divergent form higher order PDE with order degeneration on a part of the boundary. General BVPs for for a strongly degenerate higher order PDE were studied by J.A. Roitberg, Z.G. Sheftel ([159], [160]). In a paper of M.I. Vishik and V.V. Grushin ([178]) a survey of some other investigations devoted to higher order degenerate on the boundary PDEs is given. In investigations of well-posedness [in the sense that which part of the boundary for which order derivatives of unknown functions included itself should be freed from boundary conditions (BC)] of BVPs for degenerate higher order PDEs the crucial part play theory of weighted spaces. To embedding theorems for weighted spaces and in some cases to their applications to BVPs are devoted works of V.K. Zakharov ([190]), R.D. Meyer ([140]), O.V. Besov, V.P. Il'in, L.D. Kudrjavtsev, P.I. Lizorkin, S.M. Nikol'skii ([7]), P.I. Lizorkin, S.M. Nikol'skii ([133]), L.D. Kudrjavtsev ([130]), S.M. Nikol'skii ([149]), S.M. Nikol'skii, P.I. Lizorkin ([150]), A. Kufner ([131]), A. Kufner, B. Opic ([132]), G.Jaiani [101] and others (s. also a survey paper of S.M. Nikol'skii, P.I. Lizorkin, N.V. Miroshin ([151]). Higher order elliptic-parabolic equations are studied by A. Canfora ([18]-[22]), M.L. Benevento, T. Bruno, L.Castelano ([5]), M.L. Benevento ([4]), V.P. Glushko ([63]), A.S. Fokht ([47]), M. Franciosi ([49]), M.A. Malovichko ([137]). M.M. Smirnov ([166]) studied a model fourth order mixed type equation. To iterated EPD equation are devoted works of G. Jaiani [72], [73], [74], [77]-[79], [83], [84], [98], [103]. More general, than the iterated EPD equation, higher order PDEs with order degeneration are investigated by G. Jaiani [98], [105] and N. Chinchaladze [25]-[27].

A system of second order mixed type equations was investigated by V.P. Didenko ([35],[36],[38]), V.N. Vragov [181], and others. A system of PDEs with order degeneration was investigated by V.P. Didenko ([37]). J.A. Roitberg, Z.G. Sheftel ([160]) considered general BVPs (in general) with singular coefficients by less order derivatives and the right hand side, when elliptic system in the sense of Douglis-Nierenberg together with BCs satisfy the Lopatinski condition [s. e.g.,

S. Agmon, A. Douglis, L. Nierenberg ([1]), L.P. Volevich ([179]), and also I.B. Lopatinski ([134]). The system of second order degenerate equations, in particular, with order degeneration considered by J. Dufner ([39], [40]). G. Jaiani ([96], [104]) proved existence of weak and uniqueness of classical solutions of BVPs posed in manner of G. Fichera, studying systems of second order PDEs only with the order degeneration, under less restrictions than in the above-mentioned works of J. Dufner ([39], [40]). To some systems of second order PDEs with order degeneration are devoted works of G. Devdariani, G. Jaiani, S. Kharibegashvili, D. Natroshvili ([34]), G. Devdariani ([33]), G. Jaiani, B.-W. Schulze ([111], [112]), G. Jaiani, S. Kharibegashvili, D. Natroshvili, W.L. Wendland ([110]), N. Chinchaladze, R.P. Gilbert, G. Jaiani, S. Kharibegashvili, D. Natroshvili ([30]).

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Chapter 2

Auxiliary Statements

This chapter contains generalizations of the classical L'hopital rules and Cauchy integral of principal value in the forms systematically used in Chapters 3. The properties of a class of special functions in integral form and their relation to the so-called classical Euler Gamma and Beta functions are studied. Throughout the work, especially in Part II, such special functions play an essential part in representing solutions of weighted BVPs as explicit integral expressions.

2.1 Some auxiliary results

Theorem 2.1.1 *Let defined in the domain* $|x-x_0| < \overset{*}{x}$, $0 < y < \overset{*}{y}$, $x_0, \overset{*}{x}, \overset{*}{y} = \text{const}$, *functions* $f(x,y)$ *and* $g(x,y)$ *satisfy the following conditions:*

1^0. $\lim\limits_{y\to 0+} g(x,y) = +\infty$ *uniformly when* $|x-x_0| < \overset{*}{x}$;

2^0. *for any fixed* $\eta > 0$ *functions* $f(x,\eta)$ *and* $g(x,\eta)$ *be bounded when* $|x-x_0| < \overset{*}{x}$;

3^0. *there exist finite derivatives* $f'_y(x,y)$ *and* $g'_y(x,y) \neq 0$ *when* $|x-x_0| < \overset{*}{x}$, $0 < y < \overset{*}{y}$;

4^0. *there exists a finite limit*

$$\lim_{(x,y)\to(x_0,0+)} \frac{f'_y(x,y)}{g'_y(x,y)} = c.$$

Then

$$\lim_{(x,y)\to(x_0,0+)} \frac{f(x,y)}{g(x,y)} = c.$$

Proof. Since $g'_y(x,y) \neq 0$, according to Darboux theorem (see, e.g., [1]) for a fixed x it conserves the sign in the interval $]0, \overset{*}{y}[$ and the function $g(x,y)$ is monotonic one. From 1^0 it is clear that for a fixed x the derivative $g'_y(x,y) < 0$ and, hence, by decreasing y the function $g(x,y)$, monotonically increasing, tends to $+\infty$. Therefore, we can assume that $g(x,y) > 0$.

$\forall\varepsilon > 0$, by virtue of 4^0, we can find such $\eta(x_0) > 0$ that for $|x - x_0| < \eta(x_0)$ (we can always suppose that $\eta(x_0) < \overset{*}{x}$) and $0 < y < \eta(x_0)$ we have

$$\left|\frac{f'_y(x,y)}{g'_y(x,y)} - c\right| < \frac{\varepsilon}{2}$$

Applying to the segment $[y, \eta]$ the Cauchy formula

$$\frac{f(x,y) - f(x,\eta)}{g(x,y) - g(x,\eta)} = \frac{f'_y\,(x, \xi(x,y,\eta))}{g'_y\,(x, \xi(x,y,\eta))},$$

where $y < \xi(x,y,\eta) < \eta(x_0)$, we obtain

$$\left|\frac{f(x,y) - f(x,\eta)}{g(x,y) - g(x,\eta)} - c\right| = \left|\frac{f'_y\,(x, \xi(x,y,\eta))}{g'_y\,(x, \xi(x,y,\eta))} - c\right| < \frac{\varepsilon}{2} \tag{2.1}$$

for $|x - x_0| < \eta(x_0)$, $0 < y < \eta(x_0)$.

Let us consider the identity

$$\frac{f(x,y)}{g(x,y)} - c = \frac{f(x,\eta) - cg(x,\eta)}{g(x,y)} + \left[1 - \frac{g(x,\eta)}{g(x,y)}\right]\left[\frac{f(x,y) - f(x,\eta)}{g(x,y) - g(x,\eta)} - c\right].$$

Hence,

$$\left|\frac{f(x,y)}{g(x,y)} - c\right| \le \left|\frac{f(x,\eta) - cg(x,\eta)}{g(x,y)}\right| + \left|\frac{f(x,y) - f(x,\eta)}{g(x,y) - g(x,\eta)} - c\right|.$$

According to 1^0, 2^0 the fist summand tends to zero and there exists such $\delta(x_0)$ [without loss of generality we can assume that $\delta(x_0) < \eta(x_0)$] that for $|x - x_0| < \delta(x_0)$ and $0 < y < \delta(x_0)$ the first summand will be less than $\frac{\varepsilon}{2}$. By virtue of (2.1), the second summand will be less than $\frac{\varepsilon}{2}$ for $|x - x_0| < \delta(x_0)$ $0 < y < \delta(x_0)$ as well.

Thus,

$$\left|\frac{f(x,y)}{g(x,y)} - c\right| \quad \text{for} \quad |x - x_0| < \delta(x_0) \quad \text{and} \quad 0 < y < \delta(x_0).$$

So, the theorem is proved. □

Corollary 2.1.2 *Theorem 2.1.1 will be valid for $c = +\infty$, provided*

$$\lim_{y\to 0+} f(x,y) = +\infty \quad \text{uniformly for} \quad |x - x_0| < \overset{*}{x}.$$

Proof. In this case, obviously, $f'_y(x,y) \neq 0$ at least in some neighborhood of the point $(x_0, 0)$. Changing places of f and g, we have

$$\lim_{(x,y)\to(x_0,0+)} \frac{g'_y(x,y)}{f'_y(x,y)} = 0.$$

Therefore,
$$\lim_{(x,y)\to(x_0,0+)} \frac{g(x,y)}{f(x,y)} = 0,$$
and, finally,
$$\lim_{(x,y)\to(x_0,0+)} \frac{f(x,y)}{g(x,y)} = +\infty,$$
since at least in some neighborhood of the point $(x_0, 0)$ we have $f(x,y) > 0$ and $f(x,y) > 0$. □

Theorem 2.1.3 *Let the functions $f(x,y)$ and $g(x,y)$ are defined in the domain stated in Theorem 2.1.1 and satisfy the following conditions:*

1^0. $\lim\limits_{y\to 0+} g(x,y) = \lim\limits_{y\to 0+} g(x,y) = 0$ *for* $|x - x_0| < \overset{*}{x}$*;*

2^0. *there exist finite derivatives* $f'_y(x,y)$, $g'_y(x,y)$ *and* $g(x,y), g'_y(x,y) \neq 0$ *for* $|x - x_0| < \overset{*}{x}$, $0 < y < \overset{*}{y}$*;*

3^0 *there exists a finite or infinite limit*
$$\lim_{(x,y)\to(x_0,0+)} \frac{f'_y(x,y)}{g'_y(x,y)} = c.$$
Then
$$\lim_{(x,y)\to(x_0,0+)} \frac{f(x,y)}{g(x,y)} = c.$$

Proof. Assuming
$$f(x,0+) := \lim_{y\to 0+} f(x,y) = 0, \quad |x - x_0| < \overset{*}{x};$$
$$g(x,0+) := \lim_{y\to 0+} g(x,y) = 0, \quad |x - x_0| < \overset{*}{x},$$
we get that the functions $f(x,y)$ and $g(x,y)$ are continuous from the right at point $y = 0$ with respect to y for fixed x when $|x - x_0| < \overset{*}{x}$. This property together with the properties indicated in Theorem 2.1.3 allows us to apply to the functions $f(x,y)$ and $g(x,y)$ the Cauchy formula
$$\frac{f(x,y)}{g(x,y)} = \frac{f(x,y) - f(x,0+)}{g(x,y) - g(x,0+)} = \frac{f'_y\,(x,\xi(x,y))}{g'_y\,(x,\xi(x,y))}, \quad |x - x_0| < \overset{*}{x},$$
where $0 < \xi(x,y) < y < \overset{*}{y}$.

According to 3^0, there exists a finite or infinite limit
$$\lim_{(x,y)\to(x_0,0+)} \frac{f'_y\,(x,\xi(x,y))}{g'_y\,(x,\xi(x,y))} = c$$
and, therefore, a limit
$$\lim_{(x,y)\to(x_0,0+)} \frac{f(x,y)}{g(x,y)} = c.$$
□

Remark 2.1.4 *Theorems 2.1.1 and 2.1.3 are valid for a complex-valued function $f(x,y)$ as well.*

Remark 2.1.5 *Theorems 2.1.1 and 2.1.3 are also valid in the case when limits are considered along a fixed ways.*

Definition 2.1.6 *Let $f(\xi)$ be integrable on any finite interval of $\mathbb{R}^1$ and for a given in a neighborhood of $+\infty$ function $\psi(R)$ the limit*

$$\lim_{R\to+\infty}\psi(R)=+\infty.$$

If the limit

$$\lim_{R\to+\infty}\int\limits_{-R}^{\psi(R)} f(\xi)d\xi$$

exists it will be called the generalized principal value of the (in general, divergent) integral

$$\int\limits_{-\infty}^{+\infty} f(\xi)d\xi$$

and it will be denoted by

$$\text{p.v.}\psi.\int\limits_{-\infty}^{+\infty} f(\xi)d\xi.$$

If the integral converges in a usual sense, then its value coincides with the generalized principal value for any ψ.

If $\psi(R)\equiv R$, then the principal value and the generalized principal value coincide.

Theorem 2.1.7 *If $x\in\left[\overset{0}{x},\overset{*}{x}\right]$, $t\in]-\infty,+\infty[$, and a function $f(x,t)$ is continuous with its derivative f'_y on the strip $\left[\overset{0}{x},\overset{*}{x}\right]\times]-\infty,+\infty[$; moreover, the integral*

$$\chi(x):=\text{p.v.}\psi.\int\limits_{-\infty}^{+\infty} f(x,t)dt$$

exists for a certain $x\in\left[\overset{0}{x},\overset{*}{x}\right]$, while the integral

$$\chi_1(x):=\int\limits_{-\infty}^{+\infty} f'_x(x,t)dt$$

is uniformly convergent with respect to $x\in\left[\overset{0}{x},\overset{*}{x}\right]$, then the function χ is defined for all $x\in\left[\overset{0}{x},\overset{*}{x}\right]$ and

$$\frac{d}{dx}\left[\text{p.v.}\psi.\int\limits_{-\infty}^{+\infty} f(x,t)dt\right]=\int\limits_{-\infty}^{+\infty} f'_x(x,t)dt.$$

Proof. For $R > 0$ let

$$\chi^R(x) := \int\limits_{-R}^{\psi(R)} f(x,t)dt,$$

then

$$\frac{d\chi^R(x)}{dx} = \int\limits_{-R}^{\psi(R)} f'_x(x,t)dt =: \chi_1^R(x), \tag{2.2}$$

since f and f'_x are continuous on the rectangle $\left[\overset{0}{x}, \overset{*}{x}\right] \times [-R, \psi(R)]$. According to the conditions of the theorem for a certain $x \in \left[\overset{0}{x}, \overset{*}{x}\right]$ we have

$$\chi^R(x) \to \chi(x), \quad R \to +\infty. \tag{2.3}$$

Besides, in view of uniform convergence of the integral $\chi_1(x)$ and equality (2.2), we get

$$\lim_{R\to+\infty} \frac{d\chi^R(x)}{dx} = \lim_{R\to+\infty} \chi_1^R(x) = \int\limits_{-\infty}^{+\infty} f'_x(x,t)dt = \chi_1(x) \tag{2.4}$$

uniformly on $\left[\overset{0}{x}, \overset{*}{x}\right]$.

But if the assertions (2.3), (2.4) are valid, then as it is well known (see e.g., [5], p.125) the function $\chi^R(x)$ tends uniformly on $\left[\overset{0}{x}, \overset{*}{x}\right]$ to the differentiable function $\chi(x)$ as $R \to +\infty$, and

$$\frac{d\chi(x)}{dx} = \chi_1(x)$$

on the segment $\left[\overset{0}{x}, \overset{*}{x}\right]$. $\square$

2.2 Properties of a special function $M_k(a,b,j,m)$

Let us consider (see G. Jaiani [4])

$$M_k(a,b,j,m) := y^{b+m-k-1} \int\limits_{-\infty}^{+\infty} (\xi - x)^k \frac{\partial^m \theta^j e^{a\theta} \rho^{-b}}{\partial y^m} d\xi \tag{2.5}$$

$$= y^{b+m} \int\limits_{-\infty}^{+\infty} t^k \frac{\partial^m \theta^j e^{a\theta} \rho^{-b}}{\partial y^m}\Big|_{\xi=x+yt} dt = y^{b+m} \int\limits_{0}^{\pi} (-\cot\theta)^k \frac{\partial^m \theta^j e^{a\theta} \rho^{-b}}{\partial y^m}\Big|_{t=-\cot\theta} sin^{-2}\theta d\theta,$$

where

$$\theta = \arg(z-\xi),\ \rho = |z-\xi|^{\frac{1}{2}},\ j,k,m \in \mathbb{N}^0,\ z \in \mathbb{R}^2_+,\ \xi \in \mathbb{R}^1,$$

are polar coordinates with the pole at point $(x, y) = (\xi, 0)$,
a and b are complex constants, $\mathbb{R}^2_+$ ia the upper half-plane of the complex plane of the variable $z = x + iy$, $\mathbb{R}^1$ is the axis of the real numbers,

$$\theta \in [0, \pi],\ \mathbb{N}^0 := \mathbb{N} \cup \{0\},$$

$\mathbb{N}$ is the set of natural numbers. $\mathbb{N}_1$ and $\mathbb{N}_2$ denote the sets of the odd and even natural numbers, respectively, $\mathbb{N}^0_2 := \mathbb{N} \cup \{0\}$.

Theorem 2.2.1 *The function $M_k(a, b, j, m)$ is defined [i.e.,the integral (2.5) is convergent], and is independent of x, y:*
when

$$\mathrm{Re} b + m - k - 1 > 0 \tag{2.6}$$

and either $a \neq 0,\ m \in \mathbb{N}^0$, or $a = 0,\ j \neq 0,\ m \in \mathbb{N}^0$, or $a = j = m = 0$, or $a = j = 0,\ b \neq 0, -2, \ldots, -2\left(m - \left[\frac{m}{2}\right] - 1\right),\ m \in \mathbb{N}_2$;
or when

$$\mathrm{Re} b + m - k > 0 \tag{2.7}$$

and $a = j = 0,\ b \neq 0, -2, \ldots, -2\left(m - \left[\frac{m}{2}\right] - 1\right),\ m \in \mathbb{N}_1$.

If $a = j = 0$ and either $b \in \left\{0, -2, \ldots, -2\left(m - \left[\frac{m}{2}\right] - 1\right)\right\},\ m \in \mathbb{N}$, or (2.7) is fulfilled for $m, k \in \mathbb{N}_1$, or (2.6) is fulfilled for $k \in \mathbb{N}_1,\ m \in \mathbb{N}^0_2$, then

$$M_k(0, b, 0, m) = 0 \tag{2.8}$$

Proof. Using the method of the mathematical induction, we prove that

$$\frac{\partial^m e^{a\theta}\rho^{-b}}{\partial y^m} = \sum_{\kappa=1}^{\left[\frac{m}{2}\right]+1} B_\kappa\left(b, m; a(x-\xi), y\right) e^{a\theta}\rho^{-b-2(m-\kappa+1)}, \tag{2.9}$$

where

$$B_1\left(b, m; a(x-\xi), y\right) = \prod_{l=1}^{m} \left\{(x-\xi) - [b + 2(l-1)]\, y\right\} \tag{2.10}$$

$$\begin{aligned} &B_\kappa\left(b, m; a(x-\xi), y\right) \\ &= \sum_{\alpha_{\kappa-1}=2\kappa-3}^{m-1} \left(\prod_{j=1}^{\kappa-2} \sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right) \left\{\prod_{k=1}^{\kappa-1} \left[b + 2\left(\alpha_k - k\right)\right]\left(\alpha_k - m\right)\right. \\ &\left.\times \prod_{\substack{l=1 \\ l\neq\alpha_i-i+1 \\ i=1,2,\ldots,\kappa-1}}^{m-\kappa+1} \left\{a(x-\xi) - [b + 2(l-1)]y\right\}\right\}, \end{aligned} \tag{2.11}$$

$$\kappa = 2, \ldots, \left[\frac{m}{2}\right] + 1,$$

$$\prod_{j=1}^{\kappa-2}\sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2} := \sum_{\alpha_{\kappa-2}=2\kappa-5}^{\alpha_{\kappa-1}-2}\ \sum_{\alpha_{\kappa-3}=2\kappa-7}^{\alpha_{\kappa-2}-2}\cdots\sum_{\alpha_2=3}^{\alpha_3-2}\sum_{\alpha_1=1}^{\alpha_2-2},\quad \prod_{j=1}^{l-1}(\cdot)\equiv 1. \tag{2.12}$$

The last product in (2.11) we take equal to 1 if none of l are admissible.

Indeed, it is easily seen that (2.9) is true for $m = 1, \dots, 6$. Now assuming that it takes place for $m = n-1$ and $m = n$, we prove its validity for $m = n+1$. Evidently,

$$\begin{aligned}
&\frac{\partial^{n+1} e^{a\theta}\rho^{-b}}{\partial y^{n+1}} = \frac{\partial^n \left[a(x-\xi)-by\right] e^{a\theta}\rho^{-b-2}}{\partial y^n}\\
&\left[a(x-\xi)-by\right]\frac{\partial^n e^{a\theta}\rho^{-b-2}}{\partial y^n} - nb\frac{\partial^{n+1} e^{a\theta}\rho^{-b-2}}{\partial y^{n+1}}\\
&=\sum_{\kappa=1}^{\left[\frac{n}{2}\right]+1}\left[a(x-\xi)-by\right]B_\kappa\left(b+2,n;a(x-\xi),y\right)e^{a\theta}\rho^{-b-2-2(n-\kappa+1)}\\
&-\sum_{\kappa=1}^{\left[\frac{n-1}{2}\right]+1} nbB_\kappa\left(b+2,n-1;a(x-\xi),y\right)e^{a\theta}\rho^{-b-2-2(n-\kappa)}\\
&=\sum_{\kappa=1}^{\left[\frac{n}{2}\right]+1}\left[a(x-\xi)-by\right]B_\kappa\left(b+2,n;a(x-\xi),y\right)e^{a\theta}\rho^{-b-2-2(n-\kappa+1)}\\
&-\sum_{\kappa=2}^{\left[\frac{n+1}{2}\right]+1} nbB_{\kappa-1}\left(b+2,n-1;a(x-\xi),y\right)e^{a\theta}\rho^{-b-2-2(n-\kappa+1)}\\
&=\sum_{\kappa=2}^{\left[\frac{n}{2}\right]+1}\left\{\left[a(x-\xi)-by\right]B_\kappa\left(b+2,n;a(x-\xi),y\right)\right.\\
&-nbB_{\kappa-1}\left(b+2,n-1;a(x-\xi),y\right)e^{a\theta}\rho^{-b-2-2(n-\kappa+2)}\\
&+\left[a(x-\xi)-by\right]B_1\left(b+2,n;a(x-\xi),y\right)e^{a\theta}\rho^{-b-2(n+1)}
\end{aligned}$$

$$+\begin{cases} 0 & \text{for}\quad n\in\mathbb{N}_2\\ -nbB_{\frac{n+1}{2}}\left(b+2,n-1;a(x-\xi),y\right)e^{a\theta}\rho^{-b-n-1} & \text{for}\quad n\in\mathbb{N}_1.\end{cases} \tag{2.13}$$

By virtue of (2.10),

$$\begin{aligned}
&\left[a(x-\xi)-by\right]B_1\left(b+2,n;a(x-\xi),y\right)\\
&=\left[a(x-\xi)-by\right]\prod_{l=0}^{n}\left\{a(x-\xi)-\left[b+2+2(l-1)\right]y\right\}\\
&=\prod_{l=0}^{n}\left[a(x-\xi)-(b+2l)y\right]=\prod_{l=0}^{n+1}\left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}\\
&\qquad\qquad = B_1\left(b,n+1;a(x-\xi),y\right).
\end{aligned} \tag{2.14}$$

In view of (2.11), (2.12) for $n \in \mathbb{N}_1$ we have

$$
\begin{aligned}
&-nbB_{\frac{n+1}{2}}\left(b+2, n-1; a(x-\xi), y\right) \\
&= -nb \sum_{\alpha_{\frac{n+1}{2}-1}=n-2}^{n-2} \; \sum_{\alpha_{\frac{n+1}{2}-2}=n-4}^{n-4} \cdots \sum_{\alpha_2=3}^{3} \; \sum_{\alpha_2=1}^{1} \left\{ \prod_{k=1}^{\frac{n+1}{2}-1} [b+2 \right. \\
&\left. +2\,(\alpha_k-k)](\alpha_k-n+1) \prod_{\substack{l=1 \\ l\neq \alpha_i-i+1 \\ i=1,2,\dots,\frac{n+1}{2}-1}}^{\frac{n-1}{2}} \{a(x-\xi)-[b+2(l-1)]y\} \right\} \\
&= (-b)n(-b-2)(n-2)(-b-4)(n-4)\cdots(-b-n+1)
\end{aligned}
$$

$$
= (-1)^{\frac{n+1}{2}} n!! \prod_{k=1}^{\frac{n+1}{2}} [b+2(k-1)]\,. \tag{2.15}
$$

On the other hand, because of (2.11) for $m = n+1 \in \mathbb{N}_2$ we have

$$
\begin{aligned}
&B_{\frac{n+1}{2}+1}\left(b, n+1; a(x-\xi), y\right) \\
&= \sum_{\alpha_{\frac{n+1}{2}}=n}^{n} \left(\prod_{j=1}^{\frac{n+1}{2}-1} \; \sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2} \right) \left\{ \prod_{k=1}^{\frac{n+1}{2}} [b+2\,(\alpha_k-k)]\,(\alpha_k-n-1) \right. \\
&\left. \times \prod_{\substack{l=1 \\ l\neq \alpha_i-i+1 \\ i=1,2,\dots,\frac{n+1}{2}}}^{\frac{n+1}{2}} \{a(x-\xi)-[b+2(l-1)]\,y\} \right\}
\end{aligned}
$$

$$
= (-1)^{\frac{n+1}{2}} n!! \prod_{k=1}^{\frac{n+1}{2}} [b+2(k-1)]\,, \tag{2.16}
$$

since

$$
\alpha_{\frac{n+1}{2}-1} = n-2, \dots,\ \alpha_2 = 3,\ \alpha_1 = 1.
$$

From the equality of the right hand sides of (2.15) and (2.16) there follows the equality of the left hand sides

$$
-nbB_{\frac{n+1}{2}}\left(b+2, n-1; a(x-\xi), y\right) = B_{\frac{n+1}{2}+1}\left(b, n+1; a(x-\xi), y\right). \tag{2.17}
$$

According to (2.11) we have

$$\begin{aligned}
&[a(x-\xi)-by]\, B_\kappa\,(b+2,n;a(x-\xi),y) - nbB_{\kappa-1}\,(b+2,n-1;a(x-\xi),y)\\
&=[a(x-\xi)-by] \sum_{\alpha_{\kappa-1}=2\kappa-3}^{n-1} \left(\prod_{j=1}^{\kappa-2} \sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right) \left\{ \prod_{k=1}^{\kappa-1} [b+2+2\,(\alpha_k-k)]\,(a_k-n)\right.\\
&\left.\times \prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\dots,\kappa-1}}^{n-\kappa-1} \{a(x-\xi)-[b+2+2(l-1)]y\}\right\}\\
&-nb \sum_{\alpha_{k-2}=2\kappa-5}^{n-2} \left(\prod_{j=1}^{\kappa-3} \sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right) \left\{ \prod_{k=1}^{\kappa-2} [b+2+2\,(\alpha_k-k)]\,(a_k-n+1)\right.\\
&\qquad\left.\times \prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\dots,\kappa-2}}^{n-\kappa+1} \{a(x-\xi)-[b+2+2(l-1)]y\}\right\}, \qquad (2.18)
\end{aligned}$$

$$\kappa = 2,3,\dots,\left[\frac{n}{2}\right]+1$$

(we assume that $\prod\limits_{\substack{l=p\\ l\neq\alpha_i-i+1\\ i=k,k+1,\dots,m}}^{q} (\cdot) \equiv \prod\limits_{l=p}^{q}(\cdot)$ for $m<k$ and $\sum \left(\prod\limits_{j=l}^{l-2}(\cdot)\right) \{\cdot\} \equiv \{\cdot\}$).

It is easily seen that

$$\prod_{k=1}^{\kappa-1} [b+2+2\,(\alpha_k-k)]\,(a_k-n) = \prod_{k=1}^{\kappa-1} [b+2\,(\alpha_k'-k)]\,(\alpha_k'-n-1)\,, \qquad (2.19)$$
$$\alpha_k' = \alpha_k+1;$$

$$\begin{aligned}
&[a(x-\xi)-by] \prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\dots,\kappa-1}}^{n-\kappa+1} \{a(x-\xi)-[b+2+2(l-1)]\,y\}\\
&=[a(x-\xi)-by] \prod_{\substack{l=2\\ l\neq\alpha_i-i+2\\ i=1,2,\dots,\kappa-1}}^{n+1-\kappa+1} \{a(x-\xi)-[b+2(l-1)]\,y\}\\
&=\prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\dots,\kappa-1}}^{n+1-\kappa+1} \{a(x-\xi)-[b+2(l-1)]\,y\}\,; \qquad (2.20)
\end{aligned}$$

$$
\begin{aligned}
&-nb\prod_{k=1}^{\kappa-2}\left[b+2+2\left(\alpha_k-k\right)\right]\left(\alpha_k-n+1\right)\\
&=-nb\prod_{k=2}^{\kappa-1}\left[b+2+2\left(\alpha_{k-1}-k+1\right)\right]\left(\alpha_{k-1}-n+1\right)\\
&=-nb\prod_{k=2}^{\kappa-1}\left[b+2\left(\alpha^{"}_k-k\right)\right]\left(\alpha^{"}_k-n-1\right), \quad \alpha^{"}_{k+1}=\alpha_k+2; \qquad (2.21)
\end{aligned}
$$

$$
\begin{aligned}
&\prod_{\substack{l=1\\ l\neq\alpha_i-i+1\\ i=1,2,\dots,\kappa-2}}^{n-\kappa+1}\left\{a(x-\xi)-\left[b+2+2(l-1)\right]y\right\}\\
&=\prod_{\substack{l=2\\ l\neq\alpha_i-i+2\\ i=1,2,\dots,\kappa-2}}^{n+1-\kappa+1}\left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}\\
&=\prod_{\substack{l=2\\ l\neq\alpha^{"}_{i+1}-1\\ i=1,2,\dots,\kappa-2}}^{n+1-\kappa+1}\left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}\\
&=\prod_{\substack{l=2\\ l\neq\alpha^{"}_{i+1}-1\\ i=2,\dots,\kappa-1}}^{n+1-k+1}\left\{a(x-\xi)-\left[b+2(l-1)\right]y\right\}. \qquad (2.22)
\end{aligned}
$$

Substituting (2.19)-(2.22) into (2.18), we get

$$
\begin{aligned}
&\left[a(x-\xi)-by\right]B_\kappa\left(b+2,n;a(x-\xi),y\right)-nbB_{\kappa-1}\left(b+2,n-1;a(x-\xi),y\right)\\
&=\sum_{\alpha'_{\kappa-1}=2\kappa-2}^{n-1}\left(\prod_{j=1}^{\kappa-2}\sum_{\alpha'_j=2j}^{\alpha'_{j+1}-2}\right)\left\{\prod_{k=1}^{\kappa-1}\left[b+2\left(\alpha'_k-k\right)\right]\left(a'_k-n-1\right)\right.\\
&\qquad\left.\times\prod_{\substack{l=1\\ l\neq\alpha'_i-i+1\\ i=1,2,\dots,\kappa-1}}^{n+1-\kappa+1}\left\{a(x-\xi)-[b+2(l-1)]y\right\}\right\}\\
&+\sum_{\alpha^{"}_{\kappa-2}=2\kappa-3}^{n}\left(\prod_{j=1}^{\kappa-3}\sum_{\alpha^{"}_{j+1}=2j+1}^{\alpha^{"}_{\kappa+2}-2}\right)\left\{-nb\prod_{k=2}^{\kappa-1}\left[b+2\left(\alpha^{"}_k-k\right)\right]\left(a^{"}_k-n-1\right)\right.
\end{aligned}
$$

$$\left.\times \prod_{\substack{l=2 \\ l\neq\alpha_i^{"}-i+1 \\ i=2,3,\dots,\kappa-1}}^{n+1-\kappa+1} \{a(x-\xi)-[b+2(l-1)]y\} \right\}, \tag{2.23}$$

$$\kappa = 2,3,\dots,\left[\frac{n}{2}\right]+1.$$

On the other hand, by virtue of (2.11), if we separate the sum corresponding to $\alpha_1 = 1$, then for $m = n+1$ we have

$$\begin{aligned}
&B_\kappa\left(b, n+1; a(x-\xi), y\right) \\
&= \sum_{\alpha_{\kappa-1}=2\kappa-2}^{n} \left(\prod_{j=1}^{\kappa-2}\sum_{\alpha_j=2j}^{\alpha_{j+1}-2}\right)\left\{\prod_{k=1}^{\kappa-1}\left[b+2\left(\alpha_k-k\right)\right]\left(\alpha_k-n-1\right)\right. \\
&\left.\times \prod_{\substack{l=1 \\ l\neq\alpha_i'-i+1 \\ i=1,2,\dots,\kappa-1}}^{n+1-\kappa+1} \{a(x-\xi)-[b+2(l-1)]\,y\}\right\} \\
&+ \sum_{\alpha_{\kappa-1}=2\kappa-3}^{n} \left(\prod_{j=2}^{\kappa-2}\sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right)\left\{-nb\prod_{k=2}^{\kappa-1}\left[b+2\left(\alpha_k-k\right)\right]\left(\alpha_k-n-1\right)\right.
\end{aligned}$$

$$\left.\times \prod_{\substack{l=2 \\ l\neq\alpha_i-i+1 \\ i=2,3,\dots,\kappa-1}}^{n+1-\kappa+1} \{a(x-\xi)-[b+2(l-1)]\,y\}\right\}, \tag{2.24}$$

$$\kappa = 2,3,\dots,\left[\frac{n+1}{2}\right]+1,$$

since in the first group of sums none of equalities

$$\alpha_j = 2j-1,\ \ j=2,3,\dots,\kappa-1,$$

are possible, otherwise we would obtain that $\alpha_1 = 1$ but such terms we have separated in the second group. Let us note that the last products in (2.24) begins from $l = 2$ because of $l \neq \alpha_1 - 1 + 1 = \alpha_1 = 1$.

If we compare (2.23) (where α_j' and $\alpha_j^{"}$ we can denote by α_j) and (2.24) and take into account that

$$\prod_{j=2}^{\kappa-2}\sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2} \equiv \prod_{j=1}^{\kappa-3}\sum_{\alpha_{j+1}=2j+1}^{\alpha_{j+1}-2},\ \ \kappa\geq 4,$$

from the equality of the right-hand sides there follows the equality of the left-hand sides

$$\begin{aligned}
&[a(x-\xi)-by]\, B_\kappa\left(b+2,n;a(x-\xi),y\right)\\
&-nbB_{\kappa-1}\left(b+2,n-1;a(x-\xi),y\right)\\
&=B_\kappa\left(b,n+1;a(x-\xi),y\right), \quad \kappa=2,3,\dots,\left[\frac{n}{2}\right]+1.
\end{aligned}\tag{2.25}$$

Substituting (2.14), (2.17), and (2.25) into (2.13), we get

$$\frac{\partial^{n+1} e^{a\theta}\rho^{-b}}{\partial y^{n+1}}=\sum_{\kappa=1}^{\left[\frac{n+1}{2}\right]+1} B_\kappa\left(b,n+1;a(x-\xi),y\right) e^{a\theta}\rho^{-b-2(n-\kappa+2)}.$$

So, equality (2.9) is proved.

It is well-known (s. [1], pp. 235-236), that

$$\frac{d^m \arctan\ \tau}{d\tau^m}=(-1)^{m-1}(m-1)!\left(1-\tau^2\right)^{-\frac{m}{2}}\sin\left(m\ \arctan\frac{1}{\tau}\right),\quad \tau\neq 0.\tag{2.26}$$

But since,

$$\operatorname{arccot}\frac{1}{\tau}=\begin{cases}\arctan\ \tau & \text{for}\quad \tau>0;\\ \arctan\ \tau+\pi & \text{for}\quad \tau<0,\end{cases}$$

we have

$$\frac{d^m \operatorname{arccot}\ \frac{1}{\tau}}{d\tau^m}=\frac{d^m \arctan\ \tau}{d\tau^m},\quad \tau\neq 0.$$

Introducing y by the relation

$$\tau=\frac{y}{x-\xi},\quad y>0,\quad x\neq\xi,$$

where $x,\xi\in\mathbb{R}^1$ are parameters, in view of (2.26), we get

$$\begin{aligned}
\frac{\partial^m\theta}{\partial y^m}&=(x-\xi)^{-m}\left.\frac{\partial^m\theta}{\partial\tau^m}\right|_{\tau=\frac{y}{x-\xi}}=(x-\xi)^{-m}\left.\frac{\partial^m \arctan\ \tau}{\partial\tau^m}\right|_{\tau=\frac{y}{x-\xi}}\\
&=(-1)^{m-1}(m-1)!(x-\xi)^{-m}\left[1+\frac{y^2}{(x-\xi)^2}\right]^{-\frac{m}{2}}\sin\left(m\ \arctan\frac{x-\xi}{y}\right)\\
&=(-1)^{m-1}(m-1)!\rho^{-m}\left[\operatorname{sign}(x-\xi)\right]^{-m}\sin\left(m\ \arctan\frac{x-\xi}{y}\right).
\end{aligned}\tag{2.27}$$

Using (2.27), by means of the mathematical induction with respect to j we can prove that

$$
\frac{\partial^m \theta^j}{\partial y^m} = (-1)^{m-j} \left[\operatorname{sign}(x-\xi)\right]^{-m} \rho^{-m} \sum_{\kappa_j=0}^{m} \left(\prod_{k=1}^{j-2} \sum_{\kappa_{k+1}=0}^{\kappa_{k+2}} \right) \left\{ \left(\begin{array}{c} m \\ \kappa_j \end{array} \right) \right.
$$

$$
\times \prod_{k=1}^{j-2} \left(\begin{array}{c} \kappa_{k+2} \\ \kappa_{k+1} \end{array} \right) (\kappa_2 - 1)!\,(m - \kappa_j - 1)!
$$

$$
\times \prod_{k=1}^{j-2} (\kappa_{k+2} - \kappa_{k+1} - 1)! \sin\left(\kappa_2 \arctan \frac{x-\xi}{y}\right) \sin\left[(m-\kappa_j) \arctan \frac{x-\xi}{y}\right]
$$

$$
\left. \times \prod_{k=1}^{j-2} \sin\left[(\kappa_{k+2} - \kappa_{k+1}) \arctan \frac{x-\xi}{y}\right] \right\}, \quad j \geq 2. \tag{2.28}
$$

According to the Leibnitz formula,

$$
\frac{\partial^m \theta^j e^{a\theta} \rho^{-b}}{\partial y^m} = \sum_{\kappa=0}^{m} \left(\begin{array}{c} m \\ \kappa \end{array} \right) \frac{\partial^\kappa \theta^j}{\partial y^\kappa} \frac{\partial^{m-\kappa} e^{a\theta} \rho^{-b}}{\partial y^{m-\kappa}}. \tag{2.29}
$$

By virtue of (2.9)-(2.11), and (2.28) it is easy to check that for a fixed z belonging to the closure of an arbitrary bounded domain lying inside $\mathbb{R}^2_+$, we have

$$
\frac{\partial^m e^{a\theta} \rho^{-b}}{\partial y^m} = O\left(|x-\xi|^{-\operatorname{Re} b - m}\right), \quad |\xi| \to +\infty, \quad a \neq 0, \tag{2.30}
$$

$$
\frac{\partial^m \rho^{-b}}{\partial y^m} = \begin{cases} O\left(|x-\xi|^{-\operatorname{Re} b - 2\left(m - \left[\frac{m}{2}\right]\right)}\right), & |\xi| \to +\infty, \\ 0, b \in \left\{0, -2, \ldots, -2\left(m - \left[\frac{m}{2}\right] - 1\right)\right\}, & m \in \mathbb{N}, \end{cases} \tag{2.31}
$$

$$
\frac{\partial^m \theta^j}{\partial y^m} = O\left(|x-\xi|^{-m}\right), \quad |\xi| \to +\infty, \quad j \in \mathbb{N}. \tag{2.32}
$$

After substitution $\xi = x + yt$ formulas (2.30)-(2.32) we may rewrite in the following forms

$$
\left. \frac{\partial^m e^{a\theta} \rho^{-b}}{\partial y^m} \right|_{\xi = x + yt} = O\left(t|^{-\operatorname{Re} b - m}\right), \quad |t| \to +\infty, \quad a \neq 0; \tag{2.33}
$$

$$
\left. \frac{\partial^m \rho^{-b}}{\partial y^m} \right|_{\xi = x + yt} = O\left(|t|^{-\operatorname{Re} b - 2\left(m - \left[\frac{m}{2}\right]\right)}\right), \quad |t| \to +\infty; \tag{2.34}
$$

$$
\left. \frac{\partial^m \theta^j}{\partial y^m} \right|_{\xi = x + yt} = O\left(|t|^{-m}\right), \quad |t| \to +\infty, \quad j \in \mathbb{N}. \tag{2.35}
$$

In view of (2.30)-(2.35), in the above mentioned domain from (2.29) we get

$$
\frac{\partial^m \theta^j e^{a\theta}\rho^{-b}}{\partial y^m}
= \begin{cases}
O\left(|x-\xi|^{-\mathrm{Re}b-m}\right) = O\left(|t|^{-\mathrm{Re}b-m}\right), \ |\xi||t| \to +\infty, \\
when \ \ either \ \ a \neq 0, \ m \in \mathbb{N}^0, \ \ or \ \ a=0, \ j\neq 0, \ m \in \mathbb{N}^0, \\
or \ \ a=j=m=0, \\
or \ \ a=j=0, \ b \neq 0,-2,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right), \ m \in \mathbb{N}_2; \\
O\left(|x-\xi|^{-\mathrm{Re}b-m-1}\right) = O\left(|t|^{-\mathrm{Re}b-m-1}\right), \ |\xi|,|t| \to +\infty, \\
when \ \ a=j=0, \ b\neq 0,-2,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right), \ m \in \mathbb{N}_1; \\
0, \ \ when \ \ a=j=0, \ b \in \left\{0,-2,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right)\right\}, \ m \in \mathbb{N}.
\end{cases} \tag{2.36}
$$

Indeed, the cases $a \neq 0$ and $a=j=m=0$ are obvious. In the cases $a=j=0$ $m \in \mathbb{N}$ and $a=0$, $j\neq 0$, $m \in \mathbb{N}_0$ we have to take into account

$$
2\left(m-\left[\frac{m}{2}\right]\right) = \begin{cases}
2\left(m-\dfrac{m}{2}\right) = m, \ \ m \in \mathbb{N}_2; \\
2\left(m-\dfrac{m-1}{2}\right) = m+1, \ \ m \in \mathbb{N}_1,
\end{cases}
$$

and

$$
\left|\frac{\partial^m \theta^j \rho^{-b}}{\partial y^m}\right| = \left|\sum_{\kappa=0}^{m}\binom{m}{\kappa}\frac{\partial^\kappa \theta^j}{\partial y^\kappa}\cdot\frac{\partial^{m-\kappa}\rho^{-b}}{\partial y^{m-\kappa}}\right| \le \sum_{\kappa=0}^{m} C_\kappa |x-\xi|^{-\mathrm{Re}b-m-\left(m-\kappa-2\left[\frac{m-\kappa}{2}\right]\right)}
$$

$$
= \sum_{\kappa=0}^{m} C_\kappa \begin{cases} |x-\xi|^{-\mathrm{Re}b-m} \ \ for \ \ m-\kappa \in \mathbb{N}_2; \\ |x-\xi|^{-\mathrm{Re}b-m-1} \ \ for \ \ m-\kappa \in \mathbb{N}_1, \end{cases}
$$

$$
\le C|x-\xi|^{-\mathrm{Re}b-m}, \ \ |\xi| \to +\infty, \ \ C, C_\kappa = \mathrm{const}\,,
$$

respectively.

If

$$
a=j=0, \ \ b \in \left\{0,-2,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right)\right\} \ \ m \in \mathbb{N},
$$

then

$$
\frac{\partial^m \theta^j e^{a\theta}\rho^{-b}}{\partial y^m} = \frac{\partial^m \rho^{-b}}{\partial y^m} = 0, \tag{2.37}
$$

because of

$$
m > 0, 2, \dots, 2\left(m-\left[\frac{m}{2}\right]-1\right),
$$

since

$$
2\left(m-\left[\frac{m}{2}\right]-1\right) = \begin{cases} m-2, \ \ \ m \in \mathbb{N}_2; \\ m-1, \ \ \ m \in \mathbb{N}_1. \end{cases} \tag{2.38}
$$

From (2.38) it is easily seen, that

$$b \in \left\{0, -2, \ldots, -2\left(m - \left[\frac{m}{2}\right] - 1\right)\right\},\ m \in \mathbb{N}.$$

may be rewritten as

$$b = -2n, \quad n = \begin{cases} 0, 1, \ldots, \dfrac{m-2}{2}, & m \in \mathbb{N}_2, \\ 0, 1, \ldots, \dfrac{m-1}{2}, & m \in \mathbb{N}_1. \end{cases}$$

From (2.31) it follows (2.8).

According to (2.9)-(2.11) and (2.27), (2.28), we have

$$\left.\frac{\partial^m e^{a\theta}\rho^{-b}}{\partial y^m}\right|_{\xi=x+yt}$$

$$= y^{-b-m} \sum_{\kappa=1}^{\left[\frac{m}{2}\right]+1} \tilde{B}_\kappa(b, m; at) e^{a\cdot\operatorname{arccot}(-t)} \left(1+t^2\right)^{-\frac{b}{2}-m+\kappa-1}, \tag{2.39}$$

where

$$\tilde{B}_1(b, m; at) = (-1)^m \prod_{l=1}^{m} \left[at + b + 2(l-1)\right],$$

$$\tilde{B}_\kappa(b, m; at) = (-1)^{m-2\kappa+2}$$

$$\times \sum_{\alpha_{\kappa-1}=2\kappa-3}^{m-1} \left(\prod_{j=1}^{\kappa-2} \sum_{\alpha_j=2j-1}^{\alpha_{j+1}-2}\right) \left\{ \prod_{k=1}^{\kappa-1} \left[b + 2\left(\alpha_k - k\right)\right]\right.$$

$$\left.\times \left(m - \alpha_k\right) \prod_{\substack{l=1 \\ l \neq \alpha_i - i + 1 \\ i = 1, 2, \ldots, \kappa-1}}^{m-\kappa+1} \left[at + b + 2(l-1)\right]\right\}, \quad \kappa = 2, 3, \ldots, \left[\frac{m}{2}\right] + 1;$$

and

$$\left.\frac{\partial^m \theta^j}{\partial y^m}\right|_{\xi=x+yt}$$

$$= \begin{cases} (-1)^{m-j}\left[\operatorname{sign}(-t)\right]^{-m} y^{-m}\left(1+t^2\right)^{-\frac{m}{2}} \displaystyle\sum_{\kappa_j=0}^{m} \left(\prod_{k=1}^{j-2} \sum_{\kappa_{k+1}=0}^{\kappa_k+2}\right) \\ \times \left\{ \begin{pmatrix} m \\ \kappa_j \end{pmatrix} \displaystyle\prod_{k=1}^{j-2} \left(\ \kappa_{k+2}\kappa_{k+1}\ \right) (\kappa_2 - 1)!\,(m - \kappa_j - 1)! \right. \\ \times \displaystyle\prod_{k=1}^{j-2} (\kappa_{k+2} - \kappa_{k+1} - 1)! \sin\left[\kappa_2 \arctan(-t)\right] \sin\left[(m - \kappa_j)\right. \\ \left. \times \arctan(-t)\right] \displaystyle\prod_{k=1}^{j-2} \sin\left[(\kappa_{k+2} - \kappa_{k+1}) \arctan(-t)\right] \Bigg\}, \quad j \geq 2; \\ (-1)^{m-1}(m-1)!\,y^{-m}\left(1+t^2\right)^{-\frac{m}{2}} \\ \times \left[\operatorname{sign}(-t)\right]^{-m} \sin\left[m \cdot \arctan(-t)\right], \quad j = 1, \end{cases} \tag{2.40}$$

respectively.

If (2.7) is fulfilled and $m, k \in \mathbb{N}_1$, then

$$M_k(0,b,0,m) = y^{b+m-k-1} \int\limits_{-\infty}^{+\infty} (x-\xi)^k \frac{\partial^m \rho^{-b}}{\partial y^m} d\xi$$

$$= y^{b+m} \int\limits_{-\infty}^{+\infty} t^k \left. \frac{\partial^m \rho^{-b}}{\partial y^m} \right|_{\xi=x+yt} dt = 0,$$

since the integrand, in view of (2.39), is an odd function with respect to t and the integral is convergent because of (2.7). So, (2.8) is proved.

After substitution $\xi = x + yt$ the expression (2.5) we may rewrite as

$$M_k(a,b,j,m) = y^{b+m} \int\limits_{-\infty}^{+\infty} t^k \left. \frac{\partial^m \theta^j e^{a\theta} \rho^{-b}}{\partial y^m} \right|_{\xi=x+yt} dt.$$

Hence, by virtue of (2.29), (2.39), (2.40) it is evident that the right-hand side of the last equality and therefore, the function $M_k(a,b,j,m)$ is independent of x, y.

Thus, Theorem 2.2.1 is proved. $\square$

Remark 2.2.2 *In view of (2.36), if condition (2.6) is fulfilled, the function $M_k(a,b,j,m)$ is defined. When*

$$a = j = 0, \; b \neq 0, -2, \dots, -2\left(m - \left[\frac{m}{2}\right] - 1\right), \; m \in \mathbb{N}_1 \tag{2.41}$$

it is defined under the weaker restriction (2.7).

Let

$$M_k(a,b,m) := M_k(a,b,0,m), \;\; M(a,b,m) := M_0(a,b,m), \tag{2.42}$$

$$\Lambda_k(a,b) := M_k(a,2-b,0), \;\; \Lambda(a,b) := \Lambda_0(a,b) = \int\limits_0^{\pi} e^{a\theta} \sin^{-b}\theta d\theta, \;\; b<1; \tag{2.43}$$

$$\overset{*}{\Lambda}(a,b) := M_0(a,2-b,1,0) = \int\limits_0^{\pi} \theta e^{a\theta} \sin^{-b}\theta d\theta, \;\; b<1, \tag{2.44}$$

$$(\alpha, m) := \alpha(\alpha+1)\cdots(\alpha+m-1), \;\; m>1; \;\; (\alpha,0) \equiv 1.$$

Theorem 2.2.3 *Under restrictions of Theorem 2.2.1*

$$M_k(a,b,j,m+1) = (k-b-m+1) M_k(a,b,j,m). \tag{2.45}$$

The following equalities are valid:

$$M_k(a,b,j,m) = (-1)^m (b-k-1,m) M_k(a,b,j,0) \tag{2.46}$$

and

$$M_k(a,b,m) = (-1)^m (b-k-1,m)\Lambda_k(a,2-b) \tag{2.47}$$

for $\mathrm{Re}b > 1+k,\ k,m \in \mathbb{N}^0$*;*

$$\Lambda(a,2-b-2m) = \frac{a^2+(b+2m)^2}{(b+2m)(b+2m+1)}\Lambda(a,-b-2m), \tag{2.48}$$

for $\mathrm{Re}b > 1-2m$*;*

$$M(a,b,m) = \frac{(-1)^m \prod\limits_{\kappa=1}^{m}\left\{a^2+[b+2(\kappa-1)]^2\right\}}{(b+m-1,m)}\Lambda(a,2-b-2m), \tag{2.49}$$

when either $\mathrm{Re}b > 1-m,\ m \in \mathbb{N}$ *or if* $a=0$*, when* $\mathrm{Re}b > m,\ m \in \mathbb{N}_1$ *(in the last case in (2.49)* $b = 1-m$ *is allowed if the right-hand side we consider as a corresponding limit which will be equal to zero);*

$$M(a,b,m) = (-1)^{m-1}(b,m-1)M(a,b,1) \ \text{ for } \ \mathrm{Re}b > 0, \ \ m \in \mathbb{N}. \tag{2.50}$$

When $a = 0,\ m \in \mathbb{N}_1$ *(2.50) is valid also for* $\mathrm{Re}b > -1$*.*

Proof. Equality (2.45) we may obtain as follows

$$\begin{aligned}
M_k(a,b,j,m+1) &= y^{b+m-k}\int\limits_{-\infty}^{+\infty}(\xi-x)^k\frac{\partial^{m+1}\theta^j e^{a\theta}\rho^{-b}}{\partial y^m+1}d\xi \\
&= y^{b+m-k}\frac{\partial}{\partial y}\int\limits_{-\infty}^{+\infty}(\xi-x)^k\frac{\partial^{m}\theta^j e^{a\theta}\rho^{-b}}{\partial y^m}d\xi \\
&= \frac{\partial}{\partial y}\left[y^{b+m-k}\int\limits_{-\infty}^{+\infty}(\xi-x)^k\frac{\partial^{m}\theta^j e^{a\theta}\rho^{-b}}{\partial y^m}d\xi\right] \\
&\quad -(b+m-k)y^{b+m-k-1}\int\limits_{-\infty}^{+\infty}(\xi-x)^k\frac{\partial^{m}\theta^j e^{a\theta}\rho^{-b}}{\partial y^m}d\xi \\
&= \frac{\partial}{\partial y}\left[yM_k(a,b,j,m)\right]-(b+m-k)M_k(a,b,j,m) \\
&= (1+k-b-m)M_k(a,b,j,m).
\end{aligned}$$

Using l-times formula (2.45), when $\mathrm{Re}b+m-l-k-1>0$ (when $a=0$, $j=0$, $m-l\in\mathbb{N}_1$, we may take $\mathrm{Re}b+m-l-k>0$), we get

$$M_k(a,b,j,m) = (2+k-b-m,l)M_k(a,b,j,m-l).$$

Therefore, in particular, for $j=k=0$, $l=m-1$, we obtain (2.50), because of

$$(2-b-m,m-1) = (-1)^{m-1}(b,m-1),$$

while for $l = m$, we have

$$M_k(a,b,j,m) = (2+k-b-m,m)M_k(a,b,j,0) = (-1)^m(b-k-1,m)M_k(a,b,j,0),$$

i.e., (2.46). Hence, we get (2.47) since

$$M_k(a,b,0,0) = M_k(a,b,0) = \Lambda_k(a,2-b).$$

For $\mathrm{Re}b > 1$, $a \neq 0$, bearing in mind (2.43), (2.42), (2.5) and using twice integration by parts, we get

$$\Lambda(a,-b) = \int\limits_0^\pi e^{a\theta}\sin^b\theta d\theta = \frac{b(b-1)}{a^2}\Lambda(a,2-b) - \frac{b^2}{a^2}\Lambda(a,2-b). \tag{2.51}$$

Thus,

$$\Lambda(a,2-b) = \frac{a^2+b^2}{b(b-1)}\Lambda(a-b). \tag{2.52}$$

The last remains valid also in the case $a = 0$, what immediately follows from the following equalities

$$\Lambda(0,2-b) = -\int\limits_0^\pi \sin^b\theta d\cot\theta = b\int\limits_0^\pi \sin^{b-2}\theta\cos^2\theta d\theta = b\int\limits_0^\pi \sin^{b-2}\theta d\theta - b\int\limits_0^\pi \sin^b\theta d\theta.$$

If we replace b by $b+2m$ we get (2.48).

When $m = 1$, because of

$$\Lambda_1(a,b) = -\frac{a}{b}\Lambda(a,b) \;\text{ for }\; \mathrm{Re}b < 0, \tag{2.53}$$

evidently,

$$\begin{aligned} M(a,b,1) &= y^b \int\limits_{-\infty}^{+\infty} \frac{\partial e^{a\theta}\rho^{-b}}{\partial y} d\xi = -\int\limits_{-\infty}^{+\infty} (at+b)e^{a\cdot\mathrm{arccot}(-t)}\left(1+t^2\right)^{-\frac{b}{2}-1} dt \\ &= -a\Lambda_1(a,-b) - b\Lambda(a,-b) = -\frac{a^2+b^2}{b}\Lambda(a,-b), \quad \mathrm{Re}b > 0. \end{aligned}$$

In the case $a = 0$ we may assume $\mathrm{Re}b > -1$. So, formula (2.49) is true for $m = 1$. Now, we assume its validity for $m = n$ and consider $M(a,b,n+1)$. By virtue of (2.45), when either $j = k = 0$ and $\mathrm{Re} > 1-n$, $n \in \mathbb{N}$ or $a = 0$ and $\mathrm{Re}b > -n$, $n \in \mathbb{N}_1$, we have

$$\begin{aligned} &M(a,b,n+1) = (1-b-n)M(a,b,n) \\ &= (1-b-n)\frac{(-1)^n \prod\limits_{\kappa=1}^n \left\{a^2 + [b+2(\kappa-1)]^2\right\}}{(b+n-1,n)}\Lambda(a,2-b-2n) \\ &= \frac{(-1)^{n+1} \prod\limits_{\kappa=1}^n \left\{a^2 + [b+2(\kappa-1)]^2\right\}}{(b+n,n-1)}\Lambda(a,2-b-2n). \end{aligned}$$

Whence, taking into account (2.48) for $m = n$, we get

$$M(a,b,n+1) = (-1)^{n+1}\frac{\prod\limits_{\kappa=1}^{n}\left\{a^2+[b+2(\kappa-1)]^2\right\}}{(b+n,n+1)}\Lambda(a,-b-2n).$$

But, both the sides of this equality are analytic functions with respect to b when either $\mathrm{Re}b > -n$ or $a = 0$, $\mathrm{Re}b > -n-1$, $n+1 \in \mathbb{N}_1$ (in the last case points $b = -n$, $n \in \mathbb{N}_2$ are removable points of singularity for the right-hand side), which coincide either for $\mathrm{Re}b > 1-n$, $n \in \mathbb{N}$ or in case $a = 0$ for $\mathrm{Re}b > -n$, $n \in \mathbb{N}_1$. Then, according to the uniqueness theorem of analytic function both the sides coincide in the whole domain of their analyticity. Thus, Theorem 2.2.3 is proved. □

Now, we give some useful formulas.

It is well known that (see e.g., [2], pp. 491 and 386):

$$\Lambda_k(a,b) = \int\limits_{-\infty}^{+\infty} t^k e^{a\cdot\mathrm{arccot}(-t)}(1+t^2)^{\frac{b}{2}-1}dt$$
$$= (-1)^k\frac{2^b\pi e^{\pi\left(\frac{a}{2}-i\frac{k}{2}\right)}\sum\limits_{n=0}^{k}\frac{(-1)^n(-k,n)\left(\frac{a}{2i}+\frac{b}{2},n\right)}{\left(1+\frac{a}{2i}-\frac{b}{2}-k,n\right)(1,n)}}{(1-b-k)\tilde{B}\left(1-\frac{a}{2i}-\frac{b}{2},1+\frac{a}{2i}-\frac{b}{2}-k\right)},\quad \mathrm{Re}b < 1-k, \tag{2.54}$$

whence,

$$\Lambda_k(a,b) = e^{a\pi}\Lambda_k(-a,b);$$

$$A(c,b) := \int\limits_0^{\pi}\cos(c\theta)\sin^{-b}\theta d\theta = \frac{2^b\pi\cos\frac{c\pi}{2}}{(1-b)\tilde{B}\left(\frac{2+c-b}{2},\frac{2-c-b}{2}\right)},\quad \mathrm{Re}b<1,\quad c\in\mathbb{R}^1; \tag{2.55}$$

$$B(c,b) := \int\limits_0^{\pi}\sin(c\theta)\sin^{-b}\theta d\theta = \frac{2^b\pi\sin\frac{c\pi}{2}}{(1-b)\tilde{B}\left(\frac{2+c-b}{2},\frac{2-c-b}{2}\right)},\quad \mathrm{Re}b<1,\quad c\in\mathbb{R}^1; \tag{2.56}$$

where

$$\tilde{B}(x,y) := \int\limits_0^1 t^{x-1}(1-t)^{y-1}dt,\quad \mathrm{Re}\,x>0,\quad \mathrm{Re}\,y>0,$$

is the Euler Beta function (see [6], pp. 962-964). Evidently, when b and c are real numbers

$$\tilde{B}\left(\frac{2+c-b}{2},\frac{2-c-b}{2}\right) > 0 \ \text{ for } \ b < 2\pm c.$$

It is easily seen that

$$A(c,2-b) = \frac{(b+1)(b+2)}{(b+2)^2-a^2}A(c,-b),\quad b>-1,$$

$$B(c,2-b)=\frac{(b+1)(b+2)}{(b+2)^2-a^2}B(c,-b),\quad b>-1,$$

$$A(c,b)=A(c,b)\cos(c\pi)+B(c,b)\sin(c\pi),\quad A(1,b)=0,\quad b<1,$$

$$B(c,b)=A(c,b)\sin(c\pi)-B(c,b)\cos(c\pi),\quad b<1.$$

Taking into account the last one, from (2.55) and (2.56) we conclude that

$$A(2k+1,b)=0,\quad A(c,b)\neq 0,\quad c\neq 2k+1,\quad k=0,\pm1,\pm2,\dots;$$

$$B(2k,b)=0,\quad B(c,b)\neq 0,\quad c\neq 2k,\quad k=0,\pm1,\pm2,\dots;$$

$$A^2(c,b)+B^2(c,b)=2^{2b}(1-b)^{-2}\pi^2\tilde{B}^{-2}\left(\frac{2+c-b}{2},\frac{2-c-b}{2}\right)>0.$$

Besides, from (2.55) and (2.56) it immediately follows

$$A(c,b)=\cot\left(\frac{c\pi}{2}\right)B(c,b).$$

Theorem 2.2.4 *For complex numbers* a, b, *and* $k\in\mathbb{N}^0$, $\operatorname{Re} b<1-k$, *the inequality*

$$\Lambda_k(a,b)\neq 0$$

is valid if and only if when

$$b-ia,\quad b+ia+2k\bar{\in}\mathbb{N}_2, \tag{2.57}$$

$$\sum_{n=0}^{k}\frac{(-1)^n(-k,n)\left(\frac{a}{2i}+\frac{b}{2},n\right)}{(1,n)\left(1+\frac{a}{2i}-\frac{b}{2}-k,n\right)}\neq 0,+\infty. \tag{2.58}$$

For $a,b\in\mathbb{R}^1$, $b<1-k$ *and* $k\in\mathbb{N}^0$ *we have*

$$\Lambda_k(a,b)\begin{cases}>0, & \text{when either } k\in\mathbb{N}_2^0,\ \text{ or } a>0,\ k\in\mathbb{N}_1;\\ <0, & \text{when } a<0,\ k\in\mathbb{N}_1;\\ =0, & \text{when } a=0,\ k\in\mathbb{N}_1,\end{cases} \tag{2.59}$$

while, for $k\in\mathbb{N}$ *we have*

$$(k+1)\Lambda_k(a,-k-1)+a\Lambda_{k+1}(a,-k-1)$$

$$\begin{cases}>0, & \text{when either } k\in\mathbb{N}_2,\ \text{ or } a>0,\ k\in\mathbb{N}_1;\\ <0, & \text{when } a<0,\ k\in\mathbb{N}_1;\\ =0, & \text{when } a=0,\ k\in\mathbb{N}_1,\end{cases} \tag{2.60}$$

Proof. Substituting

$$\tilde{B}\left(1-\frac{a}{2i}-\frac{b}{2},1+\frac{a}{2i}-\frac{b}{2}-k\right)=\frac{\Gamma\left(1-\frac{a}{2i}-\frac{b}{2}\right)\Gamma\left(1+\frac{a}{2i}-\frac{b}{2}-k\right)}{\Gamma(2-b-k)}$$

into (2.54) and taking into account that the Euler function

$$\Gamma(z)=\frac{1}{z}\prod_{n=1}^{\infty}\left\{\left(1+\frac{1}{n}\right)^z\left(1+\frac{z}{n}\right)^{-1}\right\}$$

[s. below 2.66)] is not equal to zero in the complex plane but the points $z = 0, -1, -2, \dots,$ where it has poles (see, e.g., [6], p. 16), the first part of the theorem becomes clear.

From

$$\begin{aligned}\Lambda_k(a,b) &= \int\limits_{-\infty}^{+\infty} t^k e^{a\cdot\operatorname{arccot}(-t)}\left(1+t^2\right)^{\frac{b}{2}-1} dt \\ &= \int\limits_{0}^{+\infty} \left[e^{a\cdot\operatorname{arccot}(-t)} + (-1)^k e^{a\cdot\operatorname{arccot} t}\right] t^k \left(1+t^2\right)^{\frac{b}{2}-1} dt\end{aligned} \tag{2.61}$$

it is obvious [see (2.43), (2.5)] that if $k \in \mathbb{N}_2^0$, then $\Lambda_k(a,b) > 0$; if $a = 0$, $k \in \mathbb{N}_1$, then $\Lambda_k(0,b) = 0$; if $a \neq 0$ then (2.59) is valid because of inequalities

$$\operatorname{arccot}(-t) > \operatorname{arccot} t \ \text{ for } \ t \in]0,+\infty[; \tag{2.62}$$

$$e^{a\cdot\operatorname{arccot}(-t)} - e^{a\cdot\operatorname{arccot} t} \begin{cases} >0, & a>0; \\ <0, & a<0, \end{cases} \quad t \in]0,+\infty[. \tag{2.63}$$

In view of (2.61), we have

$$\begin{aligned}&(k+1)\Lambda_k(a,-k-1) + a\Lambda_{k+1}(a,-k-1) \\ &= (k+1)\int\limits_0^{+\infty} \left[e^{a\cdot\operatorname{arccot}(-t)} + (-1)^k e^{a\cdot\operatorname{arccot} t}\right] t^k \left(1+t^2\right)^{\frac{k+1}{2}-1} dt \\ &+a\int\limits_0^{+\infty} \left[e^{a\cdot\operatorname{arccot}(-t)} + (-1)^{k+1} e^{a\cdot\operatorname{arccot} t}\right] t^k \left(1+t^2\right)^{\frac{k+1}{2}-1} dt.\end{aligned}$$

Whence, using inequalities (2.62) and (2.63) and separately considering the cases: a is arbitrary, $k \in \mathbb{N}_2$; $a > 0$, $k \in \mathbb{N}_1$; $a < 0$, $k \in \mathbb{N}_1$; $a = 0$, $k \in \mathbb{N}_1$ it is easily seen that (2.60) is valid. $\square$

Corollary 2.2.5 *For complex numbers a and b the inequality*

$$M(a,b,m) \neq 0$$

is valid if and only if when $2 - b - 2m \pm ia \bar{\in} N_2$ and either $\operatorname{Re} b > 1 - m$, $m \in \mathbb{N}^0$, $a^2 + [b + 2(\kappa - 1)]^2 \neq 0$, $\kappa = 1, ..., m$; or if $a = 0$, when $\operatorname{Re} b > -m$, $m \in \mathbb{N}_1$, $b \neq 0, -2, ..., -2\Big(m - \left[\frac{m}{2}\right] - 1\Big)$.

If $a, b \in \mathbb{R}^1$ then

$$M(a,b,m) \neq 0$$

when either $a \neq 0$, $b > 1 - m$, $m \in \mathbb{N}^0$; or $a = 0$, $b \neq 0, -2, ..., -2\Big(m - \left[\frac{m}{2}\right] - 1\Big)$ and either $b > 1 - m$, $m \in \mathbb{N}_2$ or $b > -m$, $m \in \mathbb{N}_1$.

Proof. According to formula (2.49) and Theorem 2.2.4 it is not difficult to prove the corollary 2.9. It should be only mentioned that for $a, b \in \mathbb{R}^1$ the numbers $2-b-2m \pm ia$ can not be even positive ones, since when $a \neq 0$ it is pure complex number, while when $a = 0$ we have $1 - m \leq 0$ and either $1 - b - m < 0$, or $1 - b - m < 1$, i.e, in both the cases $2 - b - 2m < 1$. □

Theorem 2.2.6 *For* $\mathrm{Re}b < 1 - k$ *and* $k \geq 2$ *we have*

$$\Lambda_k(a,b) = -\frac{\Lambda_0(a,b)}{(b,k)} \begin{cases} \sum\limits_{j=0}^{\frac{k-2}{2}} \overset{(k)}{c}_{2j} a^{2j} - a^k, & k \in \mathbb{N}_2; \\ \sum\limits_{j=0}^{\frac{k-3}{2}} \overset{(k)}{c}_{2j+1} a^{2j+1} + a^k, & k \in \mathbb{N}_1 \backslash \{1\}, \end{cases} \tag{2.64}$$

where $\overset{(k)}{c}_j$ *are independent of* a.

Proof. Let us prove in advance that

$$\Lambda_k(a,b) = \frac{-1}{b+k-1}\left[(k-1)\Lambda_{k-2}(a,b) + a\Lambda_{k-1}(a,b)\right], \tag{2.65}$$
$$\mathrm{Re}b < 1-k,\ k \in \mathbb{N}_1 \backslash \{1\}.$$

Indeed,

$$\Lambda_k(a,b) = \int\limits_{-\infty}^{+\infty} t^{k-1} e^{a\cdot\mathrm{arccot}(-t)} d\frac{(1+t^2)^{\frac{b}{2}}}{b}$$
$$= -\frac{1}{b}\int\limits_{-\infty}^{+\infty} (k-1)t^{k-2} e^{a\cdot\mathrm{arccot}(-t)} \left(1+t^2\right)^{\frac{b}{2}} dt$$
$$-\frac{a}{b}\int\limits_{-\infty}^{+\infty} t^{k-1} e^{a\cdot\mathrm{arccot}(-t)} \left(1+t^2\right)^{\frac{b}{2}-1} dt$$
$$= -\frac{k-1}{b}\Lambda_{k-2}(a,b+2) - \frac{a}{b}\Lambda_{k-1}(a,b)$$
$$= -\frac{k-1}{b}\Lambda_{k-2}(a,b) - \frac{k-1}{b}\Lambda_k(a,b) - \frac{a}{b}\Lambda_{k-1}(a,b), \tag{2.66}$$

since, it is easily seen, that

$$\Lambda_{k-2}(a,b+2) = \int\limits_{-\infty}^{+\infty} t^{k-2} e^{a\cdot\mathrm{arccot}(-t)} \left(1+t^2\right)^{\frac{b}{2}} dt$$
$$= \int\limits_{-\infty}^{+\infty} t^{k-2} e^{a\cdot\mathrm{arccot}(-t)} \left(1+t^2\right)^{\frac{b}{2}-1} dt$$
$$+ \int\limits_{-\infty}^{+\infty} t^{k} e^{a\cdot\mathrm{arccot}(-t)} \left(1+t^2\right)^{\frac{b}{2}-1} dt = \Lambda_{k-2}(a,b) + \Lambda_k(a,b). \tag{2.67}$$

(2.65) immediately follows from (2.66).

Because of (2.65) we have

$$\begin{aligned}\Lambda_2(a,b) &= -\frac{1}{b+1}\left[\Lambda_0(a,b)+a\Lambda_1(a,b)\right]\\ &= -\frac{1}{b+1}\left[\Lambda_0(a,b)-\frac{a^2}{b}\Lambda_0(a,b)\right] = -\frac{\Lambda_0(a,b)}{b(b+1)}\left(b-a^2\right); \quad (2.68)\\ \Lambda_3(a,b) &= -\frac{1}{b+1}\left[2\Lambda_1(a,b)+a\Lambda_2(a,b)\right]\\ &= -\frac{\Lambda_0(a,b)}{b(b+1)(b+2)}\left[-a(3b+2)+a^3\right], \quad (2.69)\end{aligned}$$

i.e., formula (2.64) is true for $k=2,3$. Assuming that it is valid for $2,3,...,k$ let us prove its validity for $k+1$.

By virtue of (2.64) and (2.65), for $\operatorname{Re} b<-k$ we have

$$\begin{aligned}\Lambda_{k+1}(a,b) &= -\frac{1}{b+1}\left[k\Lambda_{k-1}(a,b)+a\Lambda_k(a,b)\right]\\ &= -\frac{\Lambda_0(a,b)}{b+k}\left[-\frac{k}{(b,k-1)}\begin{cases}\sum\limits_{j=0}^{\frac{k-3}{2}} \overset{(k-1)}{c_{2j}}\, a^{2j}-a^{k-1}, & k-1\in\mathbb{N}_2;\\ \sum\limits_{j=0}^{\frac{k-4}{2}} \overset{(k-1)}{c_{2j+1}}\, a^{2j+1}+a^{k-1}, & k-1\in\mathbb{N}_1,\end{cases}\right.\\ &\left.-\frac{a}{(b,k)}\begin{cases}\sum\limits_{j=0}^{\frac{k-3}{2}} \overset{(k)}{c_{2j+1}}\, a^{2j+1}+a^{k}, & k\in\mathbb{N}_1;\\ \sum\limits_{j=0}^{\frac{k-2}{2}} \overset{(k)}{c_{2j}}\, a^{2j}-a^{k}, & k\in\mathbb{N}_2,\end{cases}\right]\end{aligned}$$

$$= -\frac{\Lambda_0(a,b)}{(b,k+1)}\begin{cases}\sum\limits_{j=0}^{\frac{k-1}{2}} \overset{(k+1)}{c_{2j}}\, a^{2j}-a^{k+1}, & k\in\mathbb{N}_1, \text{ i.e., } k+1\in\mathbb{N}_2;\\ \sum\limits_{j=0}^{\frac{k-2}{2}} \overset{(k-1)}{c_{2j+1}}\, a^{2j+1}+a^{k+1}, & k\in\mathbb{N}_2, \text{ i.e., } k+1\in\mathbb{N}_1,\end{cases}$$

i.e., formula (2.65) is valid for $k+1$. □

Remark 2.2.7 *It is well known that (see (2.44) and [2], p. 460),*

$$\overset{*}{\Lambda}(0,b)=\int\limits_0^{\pi}\theta\sin^{-b}\theta d\theta=2^{b-1}\pi^2\Gamma(1-b)\Gamma^{-2}\left(1-\frac{b}{2}\right), \quad (2.70)$$

where

$$\Gamma(z) := \int\limits_0^\infty e^{-t}t^{z-1}dt, \quad \mathrm{Re} z > 0, \tag{2.71}$$

is the Euler Gamma function (see (2.43) and [2], pp. 947-951). From (2.54) we get

$$\begin{aligned}
\Lambda(0,b) &= \int\limits_0^\pi \sin^{-b}\theta d\theta \\
&= \frac{2^b\pi}{1-b}\tilde{B}^{-1}\left(1-\frac{b}{2}, 1-\frac{b}{2}\right) = \frac{2^b\pi\Gamma(2-b)}{(1-b)\Gamma^2\left(1-\frac{b}{2}\right)} \\
&= 2^b\pi\Gamma(1-b)\Gamma^{-2}\left(1-\frac{b}{2}\right),
\end{aligned} \tag{2.72}$$

since (see [2], pp. 951 and 964)

$$\Gamma(2-b) = (1-b)\Gamma(1-b) \text{ and } \tilde{B}(a,b) = \frac{\Gamma(a)\Gamma(B)}{\Gamma(a+b)} = \tilde{B}(b,a).$$

From (2.70) and (2.72) it is easy to conclude that

$$\overset{*}{\Lambda}(0,b) = \frac{\pi}{2}\Lambda(0,b), \quad b < 1. \tag{2.73}$$

Because of [see (2.43) and [2], p. 490 and also (2.54) for $k = 0$]

$$\begin{aligned}
\Lambda(a,b) &= \int\limits_0^\pi e^{a\theta}\sin^{-b}\theta d\theta \\
&= \frac{2^b\pi e^{a\frac{\pi}{2}}}{(1-b)B\left(\frac{2-ia-b}{2}, \frac{2+ia-b}{2}\right)}, \quad Re\, b < 1,
\end{aligned} \tag{2.74}$$

whence,

$$\Lambda(-a,b) = e^{-a\pi}\Lambda(a,b), \quad Re\, b < 1. \tag{2.75}$$

From (2.55), (2.56) we conclude

$$A(c,b) = \cot\frac{c\pi}{2}B(c,b), \quad B(c,b) = \tan\frac{c\pi}{2}A(c,b), \quad Re\, b < 1. \tag{2.76}$$

Further, after some transformations we derive

$$\begin{aligned}
&A(c,b) = A(c,b)\cos(c\pi) + B(c,b)\sin(c\pi), \quad A(1,b) = 0, \quad Re\, b < 1, \\
&B(c,b) = A(c,b)\sin(c\pi) - B(c,b)\cos(c\pi), \quad Re\, b < 1.
\end{aligned} \tag{2.77}$$

From (2.76) it follows that for $c \neq \pm 1, \pm 3, \cdots$

$$
\begin{aligned}
&A^2(c,b)+B^2(c,b)=A^2(c,b)+\tan^2\frac{c\pi}{2}A^2(c,b)\\
&=A^2(c,b)(1+\tan^2\frac{c\pi}{2})=A^2(c,b)\cos^{-2}\frac{c\pi}{2},\quad Re\, b<1.
\end{aligned}
\tag{2.78}
$$

Similarly, for $c\neq 0,\pm 2,\pm 4,\cdots$

$$
A^2(c,b)+B^2(c,b)=B^2(c,b)\sin^{-2}\frac{c\pi}{2},\quad Re\, b<1. \tag{2.79}
$$

2.3 Mathematical moments

Let $f(x_1,x_2,x_3)$ be a given function in $\overline{\Omega}\subset\mathbb{R}^3$ having integrable partial derivatives, let ω be the projection on $x_3=0$ of Ω bounded by the surfaces $x_3=\overset{(+)}{h}(x_1,x_2)$, $x_3=\overset{(-)}{h}(x_1,x_2)$, $(x_1,x_2)\in\omega$, and the cylindrical surface paralel to x_3-axis, let f_r be its r-th order moment defined as follows

$$
f_r(x_1,x_2):=\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} f(x_1,x_2,x_3)P_r(ax_3-b)dx_3,\quad (x_1,x_2)\in\omega,
$$

where

$$
a(x_1,x_2):=\frac{1}{h(x_1,x_2)},\quad b(x_1,x_2):=\frac{\widetilde{h}(x_1,x_2)}{h(x_1,x_2)},
$$

$$
2h(x_1,x_2)=\overset{(+)}{h}(x_1,\,x_2)-\overset{(-)}{h}(x_1,\,x_2)>0,
$$

$$
2\widetilde{h}(x_1,x_2)=\overset{(+)}{h}(x_1,\,x_2)+\overset{(-)}{h}(x_1,\,x_2)>0,
$$

and

$$
P_r(\tau)=\frac{1}{2^r r!}\frac{d^r(\tau^2-1)^r}{d\tau^r},\quad r=0,1,\cdots,
$$

are the r-th order Legendre polynomials with the orhogonality property

$$
\int\limits_{-1}^{+1}P_m(\tau)P_n(\tau)d\tau=\frac{2}{2m+1}\delta_{mn}.
$$

From here, substituting

$$
\tau=ax_3-b=\frac{2}{\overset{(+)}{h}(x_1,\,x_2)-\overset{(-)}{h}(x_1,\,x_2)}x_3-\frac{\overset{(+)}{h}(x_1,\,x_2)+\overset{(-)}{h}(x_1,\,x_2)}{\overset{(+)}{h}(x_1,\,x_2)-\overset{(-)}{h}(x_1,\,x_2)},
$$

we have

$$
\Big(m+\frac{1}{2}\Big)a\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)}P_m(ax_3-b)P_n(ax_3-b)dx_3=\delta_{mn}.
$$

Using the well-known formulas of integration by parts (with respect to x_3) and differentiation with respect to a parameter of integrals depending on parameters (x_α), taking into account $P_r(1) = 1$, $P_r(-1) = (-1)^r$, we deduce

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b) f_{,3}\, dx_3 = -a \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P'_r(ax_3 - b) f dx_3 + \overset{(+)}{f} - (-1)^r \overset{(-)}{f}, \quad (2.80)$$

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3 - b) f_{,\alpha}\, dx_3 = f_{r,\alpha} - \overset{(+)}{f}\overset{(+)}{h}_{,\alpha} + (-1)^r \overset{(-)}{f}\overset{(-)}{h}_{,\alpha}$$

$$- \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P'_r(ax_3 - b)(a_{,\alpha}\, x_3 - b_{,\alpha}) f dx_3, \quad \alpha = 1, 2, \qquad (2.81)$$

where superscript prime means differentiation with respect to the argument $ax_3 - b$, subscripts preceded by a comma mean partial derivatives with respect to the corresponding variables, $\overset{(\pm)}{f} := f[x_1, x_2, \overset{(\pm)}{h}(x_1, x_2)]$. Applying the following relations from the theory of the Legendre polynomials (see e.g. [3], p. 299 or p. 338-339 of the second edition)

$$P'_r(\tau) = \sum_{s=0}^{r} (2s+1) \frac{1 - (-1)^{r+s}}{2} P_s(\tau)\ {}^{1},$$

$$\tau P'_r(\tau) = r P_r(\tau) + P'_{r-1}(\tau) = r P_r(\tau) + \sum_{s=0}^{r-1} (2s+1) \frac{1 + (-1)^{r+s}}{2} P_s(\tau)\ {}^{2} \qquad (2.82)$$

and, in view of $\dfrac{a_{,\alpha}}{a} = (\ln a)' = -\dfrac{h_{,\alpha}}{h}$, $\quad \dfrac{a_{,\alpha}}{a} b = \widetilde{h} a_{,\alpha}$, $\quad b_{,\alpha} = (\widetilde{h} a)_{,\alpha}$, it is easily seen that

$$P'_r(ax_3 - b)(a_{,\alpha}\, x_3 - b_{,\alpha}) = \frac{a_{,\alpha}}{a}(ax_3 - b) P'_r(ax_3 - b) + (\frac{a_{,\alpha}}{a} b - b_{,\alpha}) P'_r(ax_3 - b)$$

$$= -h_{,\alpha}\, h^{-1}(ax_3 - b) P'_r(ax_3 - b) - \widetilde{h}_{,\alpha}\, h^{-1} P'_r(ax_3 - b)$$

$$= -\overset{r}{a}_{\alpha \underline{r}} P_r(ax_3 - b) - \sum_{s=0}^{r-1} \overset{r}{a}_{\alpha s} P_s(ax_3 - b)\ {}^{3}, \qquad (2.83)$$

---

[1]on the top of the symbol $\sum$ both $r - 1$ and r are true since the last term equals zero.
[2]on the top of the symbol $\sum$ both $r - 2$ and $r - 1$ are true since the last term equals zero.
[3]since

$$\sum_{s=0}^{r-1} (2s+1) \left[\frac{h_{,\alpha} + (-1)^{r+s} h_{,\alpha}}{2h} + \frac{\tilde{h}_{,\alpha} - (-1)^{r+s} \tilde{h}_{,\alpha}}{2h} \right] P_s(ax_3 - b)$$

where

$$\overset{r}{a}_{\alpha\underline{r}} := r\frac{h_{,\alpha}}{h}, \quad \overset{r}{a}_{\alpha s} := (2s+1)\frac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}, \quad s \neq r.$$

Now, bearing in mind (2.83) and (2.82), from (2.80) and (2.81) we have

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,\alpha}\,dx_3$$

$$= f_{r,\alpha} + \sum_{s=0}^{r}\overset{r}{a}_{\alpha s}f_s - \overset{(+)}{f}\overset{(+)}{h}_{,\alpha} + (-1)^r\overset{(-)}{f}\overset{(-)}{h}_{,\alpha}, \quad \alpha = 1,2, \tag{2.84}$$

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b)f_{,3}\,dx_3 = \sum_{s=0}^{r}\overset{r}{a}_{3s}f_s + \overset{(+)}{f} - (-1)^r\overset{(-)}{f}, \tag{2.85}$$

respectively. Here

$$\overset{r}{a}_{3s} := -(2s+1)\frac{1-(-1)^{s+r}}{2h}.$$

Let

$$f(x_1,x_2,x_3) = \sum_{r=0}^{\infty} a\Big(r+\frac{1}{2}\Big)f_r(x_1,x_2)P_r(ax_3-b), \tag{2.86}$$

then

$$\overset{(\pm)}{f} := f(x_1,x_2,\overset{(\pm)}{h}(x_1,x_2)) = \sum_{s=0}^{\infty} a\Big(s+\frac{1}{2}\Big)f_s(\pm 1)^s$$

$$= \sum_{s=0}^{\infty}\frac{(\pm 1)^s(2s+1)}{2h}f_s, \quad i = \overline{1,3}, \tag{2.87}$$

whence

$$\overset{(+)}{f} - (-1)^r\overset{(-)}{f} = -\sum_{s=0}^{\infty}\overset{r}{a}_{3s}f_s, \quad i = \overline{1,3}, \tag{2.88}$$

$$\overset{(+)}{f}\overset{(+)}{h}_{,\alpha} - (-1)^r\overset{(-)}{f}\overset{(-)}{h}_{,\alpha} = \sum_{s=0}^{\infty}\overset{r}{a}{}^{*}_{\alpha s}f_s, \quad i = \overline{1,3}, \quad \alpha = 1,2, \tag{2.89}$$

$$= \sum_{s=0}^{r-1}\frac{(2s+1)}{2h}\Bigg(\frac{\overset{(+)}{h}_{,\alpha} - \overset{(-)}{h}_{,\alpha} + \overset{(+)}{h}_{,\alpha}(-1)^{r+s} - \overset{(-)}{h}_{,\alpha}(-1)^{r+s}}{2}$$

$$+\frac{\overset{(+)}{h}_{,\alpha} + \overset{(-)}{h}_{,\alpha} - \overset{(+)}{h}_{,\alpha}(-1)^{r+s} - \overset{(-)}{h}_{,\alpha}(-1)^{r+s}}{2}\Bigg)P_s(ax_3-b)$$

$$= \sum_{s=0}^{r-1}(2s+1)\frac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}P_s(ax_3-b)$$

where
$$\overset{r}{a}{}^*_{\alpha s} = \overset{r}{a}_{\alpha s}, \quad s \neq r, \quad \overset{r}{a}{}^*_{\alpha \underline{r}} = (2r+1)\frac{h_{,\alpha}}{h}.$$

Substituting (2.89) and (2.88) into (2.84) and (2.85), respectively, we get

$$\begin{aligned}\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b) f_{,\alpha}\, dx_3 &= f_{r,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s} f_s - \sum_{s=0}^{\infty} \overset{r}{a}{}^*_{\alpha s} f_s \\ &= f_{r,\alpha} + \sum_{s=r}^{\infty} \overset{r}{b}_{\alpha s} f_s, \end{aligned} \tag{2.90}$$

where
$$\overset{r}{b}_{js} := -\overset{r}{a}_{js}, \ s > r; \ \ \overset{r}{b}_{js} = 0, \ s < r;$$
$$\overset{r}{b}_{\alpha \underline{r}} := \overset{r}{a}_{\alpha \underline{r}} - \overset{r}{a}{}^*_{\alpha \underline{r}} = -(r+1)\frac{\overset{(+)}{h}_{,\alpha} - \overset{(-)}{h}_{,\alpha}}{2h}, \quad \overset{r}{b}_{3\underline{r}} = 0,$$

and

$$\begin{aligned}\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b) f_{,3}\, dx_3 &= \sum_{s=0}^{r} \overset{r}{a}_{3s} f_s - \sum_{s=0}^{\infty} \overset{r}{a}_{3s} f_s \\ &= -\sum_{s=r+1}^{\infty} \overset{r}{a}_{3s} f_s, \end{aligned} \tag{2.91}$$

respectively.

If $\overset{(+)}{f}$ and $\overset{(-)}{f}$ are known (prescribed), then from (2.84) and (2.85), correspondingly, we obtain

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b) f_{,\alpha}\, dx_3 = f_{r,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{\alpha s} f_s$$
$$+ \overset{(+)}{f}\overset{(+)}{n}_\alpha \sqrt{1+(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2} + (-1)^r \overset{(-)}{f}\overset{(-)}{n}_\alpha \sqrt{1+(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2} \tag{2.92}$$

and

$$\int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} P_r(ax_3-b) f_{,3}\, dx_3 = \sum_{s=0}^{r} \overset{r}{a}_{3s} f_s$$
$$+ \overset{(+)}{f}\overset{(+)}{n}_3 \sqrt{1+(\overset{(+)}{h}_{,1})^2+(\overset{(+)}{h}_{,2})^2} + (-1)^r \overset{(-)}{f}\overset{(-)}{n}_3 \sqrt{1+(\overset{(-)}{h}_{,1})^2+(\overset{(-)}{h}_{,2})^2}, \tag{2.93}$$

because of

$$\overset{(\pm)}{n}_{\alpha} = \frac{\mp \overset{(\pm)}{h}_{,\alpha}}{\sqrt{1 + (\overset{(\pm)}{h}_{,1})^2 + (\overset{(\pm)}{h}_{,2})^2}}, \quad \overset{(\pm)}{n}_{3} = \frac{\pm 1}{\sqrt{1 + (\overset{(\pm)}{h}_{,1})^2 + (\overset{(\pm)}{h}_{,2})^2}}.$$

References to Chapter 2

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Chapter 3

Weighted Boundary Value Problems for Second Order Degenerate Partial Differential Equations

This chapter is, mainly, devoted to the elliptic Euler-Poisson-Darboux (EPD) equation. This equation has an order degeneration on the straight line $y = 0$. We thoroughly study weighted, in general, BVPs for the half-plane and in a finite domain, containing an interval of line $y = 0$ as a part of the boundary of the domain, when on the line of degeneration the m-th order derivative of the regular solution of EPD equation with the corresponding weight is prescribed. These weights are dependent on m and on both the constant coefficients a and b of EPD equation. The explicit integral representation of solutions of the Dirichlet ($m = 0, -\infty < b < +\infty$) and Neumann ($m = 1, 0 < b < +\infty$) problems were obtained by G. Jaiani [9], [15]. Later, actually, the same representations were obtained by O. Marichev [27], where G. Jaiani [9] is cited (see also O.I. Marichev, A. A. Kilbas, O. A. Repin [28]]), using the another method. He additionally investigated the Neumann problem for $-\infty < b \leq 0$. The general case ($m \geq 0, -\infty < b < +\infty$) was studied by G. Jaiani [12], [13], [15], [16]. EPD equation can be considered as a model equation for PDEs of general form with order degeneration from the point of view of well-posedness of weighted BVPs when the fixed order of the derivative of the solution assigned on the boundary with the corresponding weight is larger, in general, than the order of the degenerate equation under consideration. Moreover, we investigate the behavior of solutions of BVPs at the points of discontinuity of the boundary data by approaching them along different ways. It turns out that the behavior depends on the angle between tangents to the above ways at these points and the x-axis. in the case of the half-plane we express solutions of the above-mentioned weighted BVPs in explicit integral forms. In the particular case $a = 0, b = 0$ these formulas contain the well-known Poisson integral, representing the regular solution of the Dirichlet problem of the L'aplace equation for the half-plane. We apply the above-mentioned integral representation of solutions of BVPs for the half-plane to approximate solving corresponding BVPs in a finite domain when on the curvilinear part of the boundary of the domain the the homogeneous Dirichlet data are prescribed. Moreover, we state some results, following immediately from the previous results for a degenerate equation that is more general than the EPD equation in two-dimensional domain and for a degenerate equation in a p-dimensional domain for $p > 2$. In the final part of this chapter we study the canonocal form of degenerate equations of second order in two independent variables. Those contain , in particular, elliptic equations which may have non-characteristic, characteristic (Keldysh equation), order and/or type degeneration. We prove a theorem for the above-mentioned equation, which gives us criteria when the Dirichlet and when the Keldysh problems are well-posed in the classical sense. From this theorem we obtain Keldysh' theorem in the case of the Keldysh equation.

3.1 Green's formula. The correspondence, maximum, and weighted Zaremba-Giraud principles. Fundamental solutions depending only on the polar angle

From Green's general formula if $u, \nu \in C^2(G) \bigcap C^1(G \bigcup \partial G)$[1] for an arbitrary bounded domain $G \subset \mathbb{R}^2$ with a piecewise smooth boundary $\Gamma := \partial G$, entirely contained in $\mathbb{R}^2_+$, in the case of the operator

$$L^{(a,b)}u := y^{b-1}E^{(a,b)}u = (y^b u_x)_x + (y^b u_y)_y + ay^{b-1}u_x,$$

we obtain

$$\iint\limits_G (vL^{(a,b)}u - uL^{(-a,b)}v)dG = -\int\limits_{\partial G} \left[y^b \left(\nu \frac{\partial u}{\partial v} - u\frac{\partial v}{\partial v} \right) + ay^{b-1}uv\cos(\nu, x) \right] d\Gamma, \tag{3.1}$$

where ν is the inward normal to ∂G, the operator $L^{(-a,b)}$ is conjugate to the operator $L^{(a,b)}$ and let double integral be convergent, in general, as an improper one.

Let

$$K_\delta := \{(x, y) \in K_R : y \geq \delta\},$$

and

$$K_R := \{(x, y) \in \mathbb{R}^2_+ : x^2 + y^2 < R^2\}.$$

For arbitrarily small $\delta > 0$ the (3.1) is valid for K_δ. In particular, for $v \equiv 1$ and $L^{(a,b)}u = 0$ (we denote a solution of the last equation by $u^{(a,b)}(x, y)$) if we substitute into (3.1) u^2 instead of u we get

$$\iint\limits_{K_\delta} y^b(u_x^2 + u_y^2)dK_\delta = -\int\limits_{\partial K_\delta} y^b u \frac{\partial u}{\partial \nu} d\partial K_\delta + \frac{a}{2} \int\limits_{C_R \cap \{y \geq \delta\}} y^{b-1}u^2 dy, \tag{3.2}$$

[1] C^2 and C^1 are the classes of twice and once continuously differentiable functions in G and $G \bigcup \partial G$, respectively.

where

$$C_R := \{(x, y) \in \mathbb{R}^2_+ : x^2 + y^2 = R^2\}.$$

From the identity

$$y^{1-b} E^{(a,2-b)}(y^{b-1}u) \equiv E^{(a,b)}u \tag{3.3}$$

we have the following correspondence principle

$$u^{(a,b)} = y^{1-b} u^{(a,2-b)}. \tag{3.4}$$

According to correspondence principle (3.4) each solution $u^{(a,b)}$ generates a solution $u^{(a,2-b)}$ and vice-versa. This principle was proved by A. Weinstein [37] for the case when $a = 0$. Where the following second principle is proved as well.

$$u_y^{(0,b)} = y u^{(0,2+b)}, \tag{3.5}$$

is proved as well. (3.5) immediately follows from the identity

$$\frac{\partial}{\partial y} \frac{E^{(0,b)}u}{y} = E^{(0,2+b)} \left(\frac{1}{y} \frac{\partial u}{\partial y} \right), \quad y \neq 0. \tag{3.6}$$

According to the principle (3.5) each solution $u^{(0,b)}$ generates a solution $u^{(0,2+b)}$ and vice-versa. The validity of the correspondence principles

$$u^{(b)} = y^{1-b} u^{(2-b)} \tag{3.7}$$

and

$$u_y^{(b)} = y u^{(2+b)} \tag{3.8}$$

was shown by Weinstein [38] also for solutions $u^{(b)}$ of the equation

$$L_b u := u_{yy} + \frac{b}{y} u_y + X(u) = 0,$$

where $X(u)$ is an arbitrary linear operator independent of the variable y (clearly, in particular, $X(u)$ maybe $u_{xx} + a u_x$ while the last equation is not covered by equation (1.1)). It is easily seen that from (3.7) it follows (3.4) but from (3.8) it does not follow (3.5) for $a \neq 0$. Evidently,

$$X(y^{b-1}u) = y^{b-1} X(u) \quad \text{and} \quad X\left(\frac{1}{y} \frac{\partial u}{\partial y} \right) = \frac{1}{y} \frac{\partial}{\partial y} X(u).$$

Hence,

$$y^{1-b} X(y^{b-1}u) = X(u) \quad \text{and} \quad \frac{\partial}{\partial y} X(u) = y X\left(\frac{1}{y} \frac{\partial u}{\partial y} \right).$$

Now summing, the last identities with the identities (3.3) and (3.6), respectively, we obtain two new identities:

$$y^{1-b} \left(E^{(a,2-b)} + X \right) (y^{b-1}u) = \left(E^{(a,b)} + X \right)(u)$$

and

$$\frac{\partial}{\partial y}\left(\frac{1}{y}E^{(0,b)}+X\right)(u)=\left(E^{(0,2+b)}+yX\right)\left(\frac{1}{y}\frac{\partial u}{\partial y}\right).$$

which prove that the correspondence principles (3.4) and (3.5) remain true for solutions of equations

$$\left(E^{(a,b)}+X\right)u=0$$

and

$$\left(E^{(0,2+b)}+yX\right)u=0,$$

respectively.

In the polar coordinate system with the pole at the point $(x,y)=(\xi,0)$

$$\rho\equiv\rho_\xi:=\sqrt{(x-\xi)^2+y^2},\ \ \theta:=arg\,z=arc\,ctg\frac{x-\xi}{y}$$

$$(z:=x+iy\equiv(x,y))\in\mathbb{R}^2_+,\text{a fixed }\xi\in\mathbb{R}^1)$$

EPD equation (1.1) has the form

$$\rho^2\frac{\partial^2 w}{\partial\rho^2}+\rho\frac{\partial w}{\partial\rho}+\frac{\partial^2 w}{\partial\theta^2}+a\rho\operatorname{ctg}\theta\frac{\partial w}{\partial\rho}-a\frac{\partial w}{\partial\theta}+b\rho\frac{\partial w}{\partial\rho}+b\operatorname{ctg}\theta+\frac{\partial w}{\partial\theta}=0.$$

From here it is clear that solutions depending only on θ satisfy the equation

$$\frac{\partial^2 w(\theta)}{\partial\theta^2}+(b\operatorname{ctg}\theta-a)\frac{dw(\theta)}{d\theta}=0,$$

i.e.

$$\frac{\partial}{\partial\theta}\ln\frac{dw(\theta)}{d\theta}\equiv a-b\text{ctg}\theta,\ \ \ln\frac{dw(\theta)}{d\theta}=\int\limits_{\theta_0}^{\theta}(a-b\text{ctg}\theta)d\theta+\ln C_3,$$

$$\frac{dw(\theta)}{d\theta}=\exp\Big[\int\limits_{\theta_0}^{\theta}(a-b\text{ctg}\theta)d\theta+\ln C_3\Big]=C_3\exp\Big[a(\theta-\theta_0)-b\ln\sin\theta\Big]$$

$$=C_1e^{a\theta}\sin^{-b}\theta,\quad C_1:=C_3e^{-a\theta_0},$$

and have the form

$$C_1\Omega(\theta;\theta_0,a,b)+C_2,\ \ C_1,C_2=\text{const}\,, \tag{3.9}$$

where $\theta_0\in]0,\pi[$ and

$$\Omega(\theta,\theta_0,a,b):=\int\limits_{\theta_0}^{\theta}e^{a\tau}\sin^{-b}\tau d\tau.$$

We can immediately verify that the expression

$$\frac{d}{d\xi}\Omega(\theta;\theta_0,a,b) = \frac{e^{a\theta}\sin^{1-b}\theta}{\rho} = y^{1-b}e^{a\theta}\rho^{b-2} \quad (3.10)$$

[2]

is a solution of equation (1.1) in $\mathbb{R}^2_+$ for a fixed $\xi \in \mathbb{R}^1$. If we replace in (3.10) "b" by "$2-b$" then, clearly, we get solution $u^{(a,2-b)} = y^{b-1}e^{a\theta}\rho^{-b}$. Therefore, the first correspondence principle (3.4) gives another solution of equation (1.1)

$$e^{a\theta}\rho^{-b}. \quad (3.11)$$

If $b \in]-\infty, 1[$, in the particular case

$$C_1 = 1,\ C_2 = 0,\ \theta_0 = 0$$

(3.9) has the form

$$\Omega(x-\xi, y) := \Omega(\theta; 0, a, b) = \int\limits_0^\theta e^{a\tau}\sin^{-b}\tau d\tau. \quad (3.12)$$

Whence,

$$\Omega(\theta;0,a,b)|_{y=0} = \begin{cases} 0,\ \text{when } x > \xi, \\ \Lambda(a,b) := \int\limits_0^\pi e^{a\tau}\sin^{-b}\tau d\tau,\ \text{when } x < \xi, \end{cases}$$

since

$$\lim_{y\to 0+}\theta = \lim_{y\to 0+}\operatorname{arc\,ctg}\frac{x-\xi}{y} = \begin{cases} 0,\ \text{when } x > \xi, \\ \pi,\ \text{when } x < \xi. \end{cases}$$

It is easily seen that

$$\Omega(\theta;\theta_0,a,b) = O(1), \quad \theta \to 0+, \pi-, \quad (3.13)$$

when

$$\theta_0 \in [0,\pi], \quad z \in \mathbb{R}^2_+,\ \xi \in \mathbb{R}^1,\ b \in]-\infty, 1[;$$

$$\Omega(\theta;\theta_0,a,b) = \begin{cases} O(\ln\theta),\ \theta \to 0+, \\ O(\ln(\pi-\theta)),\ \theta \to \pi-, \\ O(\ln|\xi|),\ |\xi| \to +\infty, \\ O(\ln|t|),\ |t| \to +\infty, \end{cases} \quad (3.14)$$

---

[2]

$$\sin\alpha \equiv \sin\theta = \frac{y}{\rho}$$

$$\frac{\partial\theta}{\partial\xi} = \frac{y}{\rho^2}$$

$$\frac{\partial\theta}{\partial y} = \frac{x-\xi}{\rho^2}$$

when

$$\theta_0 \in]0,\pi[, \quad z \in \mathbb{R}^2_+,\ \xi \in \mathbb{R}^1,\ b = 1;$$

$$\Omega(\theta;\theta_0,a,b) = \begin{cases} O(\theta^{1-b}),\ \theta \to 0+, \\ O((\pi-\theta)^{1-b}),\ \theta \to \pi-, \\ O(|\xi|^{b-1}),\ |\xi| \to +\infty, \\ O(|t^{b-1}|),\ |t| \to +\infty, \end{cases} \tag{3.15}$$

when

$$\theta_0 \in]0,\pi[, \quad z \in \mathbb{R}^2_+,\ \xi \in \mathbb{R}^1,\ b \in]1,+\infty[.$$

Indeed, if $b \in]-\infty, 1[$, then, keeping in mind (2.43), (2.42), (2.5),

$$\Omega(\theta_0; 0, a, b) \le \Lambda(a,b) = \text{const}\,;$$

if $b = 1$, then

$$\lim_{\theta\to 0+} \frac{\Omega(\theta;\theta_0,a,1)}{\ln\theta} = \lim_{\theta\to 0+} \frac{e^{a\theta}\sin^{-1}\theta}{\theta^{-1}} = 1,$$

$$\lim_{\theta\to\pi-} \frac{\Omega(\theta;\theta_0,a,1)}{-\ln(\pi-\theta)} = \lim_{\theta\to\pi-} \frac{e^{a\theta}\sin^{-1}\theta}{(\pi-\theta)^{-1}} = \lim_{\theta\to\pi-} \frac{e^{a\theta}\sin^{-1}(\pi-\theta)}{(\pi-\theta)^{-1}} = e^{a\pi};$$

if $b \in]1,+\infty[$, then

$$\lim_{\theta\to 0+} \frac{\Omega(\theta;\theta_0,a,b)}{\dfrac{\theta^{1-b}}{1-b}} = \lim_{\theta\to 0+} \frac{e^{a\theta}\sin^{-b}\theta}{\theta^{-b}} = 1,$$

$$\lim_{\theta\to\pi-} \frac{\Omega(\theta;\theta_0,a,b)}{\dfrac{(\pi-\theta)^{1-b}}{b-1}} = \lim_{\theta\to\pi-} \frac{e^{a\theta}\sin^{-b}\theta}{(\pi-\theta)^{-b}} = \lim_{\theta\to\pi-} \frac{e^{a\theta}\sin^{-b}(\pi-\theta)}{(\pi-\theta)^{-b}} = e^{a\pi}.$$

Further, taking into account

$$\theta|_{\xi=x+yt} = \operatorname{arc\,ctg}\ \frac{x-\xi}{y}\bigg|_{\xi=x+yt} = \operatorname{arc\,ctg}\,(-t), \quad z \in \mathbb{R}^2_+, \quad \xi, \quad t \in \mathbb{R}^1, \tag{3.16}$$

$$\lim_{\substack{\xi\to-\infty \\ \theta\to-\infty}} \theta = 0, \quad \lim_{\substack{\xi\to+\infty \\ \theta\to+\infty}} (\pi-\theta) = 0, \quad z \in \mathbb{R}^2_+, \tag{3.17}$$

we have

$$\lim_{\xi\to-\infty} \frac{\theta}{-y\xi^{-1}} = \lim_{\xi\to+\infty} \frac{\dfrac{y}{(x-\xi)^2+y^2}}{y\xi^{-2}} = \lim_{\xi\to+\infty} \frac{1}{\left(\frac{x}{\xi}-a\right)^2 + \left(\frac{y}{\xi}\right)^2} = 1,$$

$$\lim_{t\to-\infty}\frac{\theta}{-t^{-1}}=\lim_{t\to-\infty}\frac{\dfrac{1}{1+t^2}}{t^{-2}}=1,$$

$$\lim_{\xi\to+\infty}\frac{\pi-\theta}{y\xi^{-1}}=\lim_{\xi\to+\infty}\frac{\dfrac{-y}{(x-\xi)^2+y^2}}{-y\xi^{-2}}=1,$$

$$\lim_{\xi\to+\infty}\frac{\pi-\theta}{t^{-1}}=\lim_{\xi\to+\infty}\frac{\dfrac{-1}{1+t^2}}{-t^{-2}}=1,$$

using these equalities, we obtain

$$\lim_{\xi\to-\infty}\frac{\ln\theta}{-\ln(-\xi)}=\lim_{\xi\to-\infty}\frac{\dfrac{1}{\theta}\cdot\dfrac{y}{(x-\xi)^2+y^2}}{-\xi^{-1}}=\lim_{\xi\to-\infty}\frac{\dfrac{-1}{y\xi^{-1}}\cdot\dfrac{y}{(x-\xi)^2+y^2}}{-\xi^{-1}}=1,$$

$$\lim_{t\to-\infty}\frac{\ln\theta}{-\ln(-t)}=\lim_{t\to-\infty}\frac{\dfrac{1}{\theta}\cdot\dfrac{1}{1+t^2}}{-t^{-1}}=1,$$

$$\lim_{\xi\to+\infty}\frac{\ln(\pi-\theta)}{-\ln\xi}=\lim_{\xi\to+\infty}\frac{\dfrac{1}{(\pi-\theta)}\cdot\dfrac{-y}{(x-\xi)^2+y^2}}{-\xi^{-1}}=1,$$

$$\lim_{t\to+\infty}\frac{\ln(\pi-\theta)}{-\ln t}=\lim_{t\to+\infty}\frac{\dfrac{1}{(\pi-\theta)}\cdot\dfrac{-1}{(1+t^2)}}{-t^{-1}}=1.$$

From the above assertions there follow (3.14),(3.15).

Let

$$u(z):=u(x,y).$$

Maximum Principle 3.1.1 *If* $u\in C^2(\mathbb{R}^2_+)\bigcap C(\mathbb{R}^2_+\bigcup\mathbb{R}^1)$ *is a solution of* (1.1) *and*

$$u=\begin{cases} O(1),\ r\to+\infty,\ \text{when or } a\in\mathbb{R}^1,\ b\in]-\infty,0[,\ \text{or } a=0,b=0;\\ o(1),\ r\to+\infty,\ \text{when or } a\in\mathbb{R}^1,\ b\in]0,1[,\ \text{or } a\neq 0,b=0,\end{cases}$$

then

$$\sup_{z\in\mathbb{R}^2_+\cup\mathbb{R}^1}|u(z)|=\sup_{x\in\mathbb{R}^1}|u(x,0)|. \tag{3.18}$$

Proof. Let $b\in[0,1[$ and $\varepsilon>0$ is less than the value of $|u|$ at a certain point $(x_0,0)$. Let us choose so large R that the half-circle $\bar K_R$ contain the above point and $|u(z)|<\varepsilon$ in $\bar{\mathbb{R}}^2_+\backslash K_R$. Then

$$\max_{\mathbb{R}^2_+\cup\mathbb{R}^1}|u(z)|=\max_{\bar K_R}|u(z)|=\max_{\partial K_R}|u(z)|=\max_{[-R,+R]}|u(x,0)|=\max_{\mathbb{R}^1}|u(x,0)|. \tag{3.19}$$

Let now, $b \in]-\infty, 0[$,

$$M := \sup_{\mathbb{R}^1} u(x,0).$$

Let us fix $\epsilon > 0$ and consider

$$U(z) = M + \varepsilon e^{a\overset{0}{\theta}} r^{-b}, \quad \overset{0}{\theta} := \operatorname{arc\,ctg} \frac{x}{y}.$$

By virtue of (3.11), $U(z)$ satisfies equation (1.1) in $\mathbb{R}^2_+$. Evidently,

$$\lim_{r\to+\infty} U(z) = +\infty.$$

Let us choose R so large that on the boundary of the half-circle K_R the difference

$$U(z) - u(z)$$

be nonnegative. We can it always achieve on the segment $[-R, R]$ since

$$U(x,0) = M + \varepsilon e^{a\overset{*}{\theta}} |x|^{-b} \geq M \geq u(x,0), \quad \overset{*}{\theta} = \begin{cases} \pi, & x < 0, \\ 0, & x > 0, \end{cases}$$

while on the half-circle C_R, in view of boundedness of u, by means of appropriate choice of R we will have

$$U(z) \geq u(z).$$

Therefore, according to the strong extremum principle for the elliptic equations (see e.g. [7], p.74) we conclude that at any point of K_R the function $U(z) - u(z)$ is nonnegative. Hence, since by fixed z and $\varepsilon \to 0$ function $U(z) \to M$, at any point of the half-circle K_R we have

$$u(z) \leq M.$$

By virtue of the strong extremum principle, nonconstant function $u(z)$ at points of domain K_R cannot take its maximal value M. Therefore, we have strong inequality

$$u(z) < M \quad \text{in } K_R. \tag{3.20}$$

Since for any point $z \in \mathbb{R}^2_+$ we can choose a half-circle K_R containing z, the equality (3.20) is valid in $\mathbb{R}^2_+$.

If

$$m := \inf_{\mathbb{R}^1} u(x,0),$$

then

$$-m = \sup_{\mathbb{R}^1} [-u(x,0)]$$

and, according to the above proved, we get

$$-u(z) < -m \text{ when } z \in \mathbb{R}^2_+,$$

i.e.,

$$m < u(z) \text{ when } z \in \mathbb{R}^2_+. \tag{3.21}$$

From (3.20), (3.21) we conclude

$$|u(z)| < \max\{|m|, |M|\}, \quad \text{when} \quad z \in \mathbb{R}^2_+.$$

But the last relation and relation (3.18) are equivalent. □

The case $a = b = 0$ is classical one (see, e.g., [24], p.83).

Let S be simply connected domain with the boundary $\partial S = \varsigma \bigcup I$, where ς is the Jordan open arc lying in $\mathbb{R}^2_+$ with the ends $\varsigma_1 := (\xi_1, 0)$, $\varsigma_n := (\xi_n, 0)$ and $\bar{I} := [\xi_1, \xi_n]$ is the segment of the axis $\mathbb{R}^1$.

Generalized Weighted Zaremba-Giraud Principle 3.1.2 *Let function* $u \in C^2(S) \bigcap C(\bar{S})$ *satisfy the inequality*

$$E^{(a,b)} u \geq 0 (\leq 0), \quad b \in]0, 1[,$$

and attains maximal positive (minimal negative) value at an inner point $x_0 \in I$, *i.e.,*

$$u_{|\bar{\varsigma}} < (>)\, u(x_0, 0).$$

Then

$$\lim_{x \to x_0} (-1)^{j-1} y^{b+j-1} \frac{\partial^j u(z)}{\partial y^j} < 0\ (> 0), \quad z \in \mathbb{R}^2_+, \quad j = 1, ..., l, \quad l \in N,$$

provided,

$$y^{b+l-1} \frac{\partial^l u}{\partial y^l} \in C(SUI).$$

Proof. Let first, $l = 1$. Without loss of generality we assume that

$$u(x_0, 0) = 1$$

is a maximal value. Evidently,

$$\lim_{z \to x_0} y^b \frac{\partial u}{\partial y} > 0. \tag{3.22}$$

is excluded. Let

$$\lim_{z \to x_0} y^b \frac{\partial u}{\partial y} = 0. \tag{3.23}$$

By change of variables

$$x = \xi, \quad y = \frac{\eta^{\frac{m+2}{2}}}{\frac{m+2}{2}}, , \eta > 0, \ m = \frac{2b}{1-b}, \ \left(b = \frac{m}{m+2}\right)\ 0 < b < 1\ (m > 0),$$

the operator $E^{(a,b)}$ takes the form

$$E^{(a,b)} u = \frac{2}{m+2} \eta^{1-m/2}\, \widetilde{E}\, \widetilde{u},$$

where

$$\widetilde{E}\,\widetilde{u} := \eta^m\,\widetilde{u}_{\,\xi\xi} + \widetilde{u}_{\,\eta\eta} + \frac{m+2}{2}a\eta^{\frac{m}{2}-1}\,\widetilde{u}_{\,\xi}, \ \ \widetilde{u}(\xi\eta) := u\left(\xi, \frac{\eta^{\frac{m+2}{2}}}{\frac{m+2}{2}}\right). \tag{3.24}$$

□

According to the conditions of Principle 3.1.2 and (3.18), we obtain

$$\widetilde{E}\,\widetilde{u} \geq 0, \quad \widetilde{u}(x_0,0) = 1, \quad \lim_{(\xi,\eta)\to(x_0,0+)} \frac{\partial\,\widetilde{u}}{\partial\eta} = 0. \tag{3.25}$$

Let us consider the function (compare with [32], p.34)

$$\nu(\xi,\eta) = \frac{\varepsilon\,\widetilde{u}(\xi,\eta)}{e^{\widetilde{A}} - \varepsilon e^{\eta}},$$

where $\widetilde{A}$ is the maximum of ordinates of points of the image $\widetilde{\zeta}$ of the arc ζ. Since the point $(x_0,0)$ does not coincide with the ends of the arc $\widetilde{\zeta}$, by our assumption,

$$\max_{\widetilde{\zeta}}\widetilde{u} \leq 1-\varepsilon, \quad 0<\varepsilon = \text{const } < 1.$$

Therefore,

$$\begin{aligned} \nu|_{\widetilde{\zeta}} &\leq \frac{\varepsilon(1-\varepsilon)}{e^{\widetilde{A}} - \varepsilon e^{\widetilde{A}}} = \frac{\varepsilon}{e^{\widetilde{A}}} < \frac{\varepsilon}{e^{\widetilde{A}}-\varepsilon}, \\ \nu|_{I\setminus\{x_0\}} &= \frac{\varepsilon\,\widetilde{u}(x,0)}{e^{\widetilde{A}}-\varepsilon} \leq \frac{\varepsilon}{e^{\widetilde{A}}-\varepsilon}, \ \ \nu(x_0,0) = \frac{\varepsilon}{e^{\widetilde{A}}-\varepsilon}. \end{aligned} \tag{3.26}$$

By virtue of the first relation of (3.25), taking into account (3.24),

$$\eta^m\nu_{\xi\xi} + \nu_{\eta\eta} + \frac{m+2}{2}a\eta^{\frac{m}{2}-1}\nu_{\xi} - \frac{2\varepsilon e^{\eta}}{e^{\widetilde{A}}-\varepsilon e^{\eta}}\nu_{\eta} - \frac{\varepsilon e^{\eta}}{e^{\widetilde{A}}-\varepsilon e^{\eta}} \geq 0.$$

whence, bearing in mind the weak maximum principle for elliptic equations (see [7], p.75) ν attains a positive maximum on the boundary ∂S, i.e. by (3.26), at point $(x_0,0)$.

On the other hand, in view of (3.25),

$$\lim_{(\xi,\eta)\to(x_0,0+)}\frac{\partial\nu}{\partial\eta} = \lim_{(\xi,\eta)\to(x_0,0+)}\left[\frac{\varepsilon}{e^{\widetilde{A}}-\varepsilon e^{\eta}}\frac{\partial\,\widetilde{u}}{\partial\eta} + \frac{\varepsilon^2 e^{\eta}}{\left(e^{\widetilde{A}}-\varepsilon e^{\eta}\right)^2}\widetilde{u}\right] = \frac{\varepsilon^2}{\left(e^{\widetilde{A}}-\varepsilon\right)^2} > 0.$$

But it contradicts with the fact that ν attains its positive maximum at point $(x_0,0)$. Thus, along with (3.22) also (3.23) is excluded and remains only

$$\lim_{z\to x_0} y^b\frac{\partial u}{\partial y} < 0.^3$$

[3] by deriving this inequality $b=0$ is also admissible.

The case of a negative minimum can be reduced to the previous case considering the function $-u(x,y)$. So, the principle is proved in the case $l=1$.

Using Theorem 2.1.1, the validity of the principle for the arbitrary l follows from the following equalities

$$\lim_{z\to x_0} y^b \frac{\partial u}{\partial y} = \frac{-1}{b} \lim_{z\to x_0} \frac{\frac{\partial^2 u}{\partial y^2}}{y^{-b-1}} = \frac{(-1)^{j-1}}{(b,j-1)} \lim_{z\to x_0} y^{b+j-1} \frac{\partial^j u}{\partial y^j}$$
$$= \frac{(-1)^{l-1}}{(b,l-1)} \lim_{z\to x_0} y^{b+l-1} \frac{\partial^l u}{\partial y^l},$$

since the last limit exists according to the corresponding condition of the principle and $(b,j-1) := b(b+1)\cdots(b+j-2) > 0$ for $j=2,...,l$.

Maximum Principle 3.1.3 *Let* $b \in]0,1[$*. If* $u \in C^2(S) \bigcap C\left(S \cup \partial S \setminus \overset{*}{I}\right)$ ($\overset{*}{I} \subset \bar{I}$, $\overset{*}{I}$ *is a finite set of first kind discontinuity points* ξ_j *of function* $u(\zeta)$, $\zeta \in \partial S$), *then*

$$\sup_{z\in S\cup\partial S\setminus \overset{*}{I}} |u(z)| = \sup_{\zeta\in \partial S\setminus \overset{*}{I}} |u(\zeta)|. \tag{3.27}$$

This equality is also valid for $S \equiv \mathbb{R}^2_+$ *under additional condition*

$$\lim_{|z|\to\infty} u(x) = 0, \quad z \in \mathbb{R}^2_+. \tag{3.28}$$

Proof. Let

$$M := \sup_{\zeta\in \partial S\setminus \overset{*}{I}} u(\zeta).$$

Fix arbitrary positive ε and consider the function

$$U(z) = M + \varepsilon \sum_{j=1}^{n} e^{a\cdot \operatorname{arctg}\frac{x-\xi_j}{y}} \rho_j^{-b},$$

where

$$\xi_j \in \overset{*}{I}, \quad \rho_j^2 := |z-\xi_j|^2, \quad j=\overline{1,n}.$$

$U(z)$, by virtue of (3.11), satisfies equation (1.1), is greater than M and continuous in $S\cup\partial S\setminus\overset{*}{I}$; moreover, when z approaches ξ_j, $U(z)$ tends to $+\infty$. Circumscribe at all the points ξ_j semi-circles in $\mathbb{R}^2$ with sufficiently small radius δ. Denote by $\tilde{S}_\delta$ a domain which we obtain as an intersection of all the corresponding semi-discs with the domain S. The difference $U(z)-u(z)$ is nonnegative on the common boundary of S and $\tilde{S}_\delta$. It is also nonnegative on the above semi-circles with the radius δ for sufficiently small δ[4], since $u(z)$ is bounded and values of $U(z)$ are unboundedly increasing on semi-circles by $\delta \to 0$. Therefore, bearing in mind the strong maximum principle (see, Maximum Principle 3.1.1), we conclude that

[4]If we had for points ξ_j difference δ_j then we took $= \delta := \min_j\{\delta_j\}$ the last minimum exists since we have a finite number of points ξ_j.

at any point of $\widetilde{S}_\delta$ and, hence, of S (since any point of S belongs to $\widetilde{S}_\delta$ for sufficiently small δ) the function $U(z) - u(z)$ is nonnegative. But for a fixed z and $\varepsilon \to 0$, $U(z) \to M$. Then

$$u(z) \le M$$

and according to the weak maximum principle,

$$u(z) < M.$$

If

$$m := \inf_{\zeta \in \partial S \setminus \overset{*}{I}} u(\zeta),$$

then

$$-m := \sup_{\zeta \in \partial S \setminus \overset{*}{I}} [-u(\zeta)]$$

and accordng to the above-proved,

$$-u(z) < -m \quad \text{when} \quad z \in S.$$

Hence,

$$u(z) > m \quad \text{when} \quad z \in S.$$

Thus,

$$m < u(z) < M \quad \text{when} \quad z \in S.$$

Finally,

$$|u(z)| < \max\{|m|, |M|\}, \quad \text{when} \quad z \in S,$$

which is equivalent to (3.27). So, we have proved Maximum Principle 3.1.3 for the finite domain S.

Let us now consider the case of the half-plane $\mathbb{R}^2_+$. For arbitrary

$$\varepsilon \in \left] 0, \sup_{\zeta \in \mathbb{R}^1 \setminus \overset{*}{I}} |u(\zeta)| \right[,$$

in view of (3.28), we can find such a large R, that all

$$\xi_j \in \bar{K}_R,$$

where

$$K_R := \{(x, y) : \mathbb{R}^2_+ : x^2 + y^2 < \mathbb{R}^2\}$$

and

$$|u(z)| < \varepsilon \quad \text{when} \quad z \in \bar{\mathbb{R}}^2_+ \setminus K_R. \tag{3.29}$$

On the other hand, using Maximum Principle 3.1.3 for the finite domain, we have

$$\sup_{z \in \bar{K}_R \setminus \overset{*}{I}} |u(z)| = \sup_{\zeta \in \partial K_R \setminus \overset{*}{I}} |u(\zeta)|. \tag{3.30}$$

From (3.29) and (3.30) there follows

$$\sup_{z\in\mathbb{R}^2_+\cup\mathbb{R}^1\setminus\overset{*}{I}}|u(z)|=\sup_{z\in\bar{K}_R\setminus\overset{*}{I}}|u(z)|=\sup_{\zeta\in\partial K_R\setminus\overset{*}{I}}|u(\zeta)|=\sup_{\zeta\in\mathbb{R}^1\setminus\overset{*}{I}}|u(\zeta)|.$$

□

Maximum Principle 3.1.4 *Let $a=b=0$. If bounded $u\in C^2(S)\bigcap C\left(S\cup\partial S\setminus\overset{*}{I}\right)$ is a solution of equation* (1.1) (*which in this case becomes Laplace equation for* $y\neq 0$), *then the relation* (3.27) *holds. The case $S\equiv\mathbb{R}^2_+$ is admissible as well.*

Proof. For a finite S the proof of this principle there follows from the extremum principle proved in M.A. Lavrentyev and B.V. Shabat [26] (see p.211, Theorem 5):

If in S bounded harmonic function $u(z)$ on the boundary $\partial(S)$ takes the piece-wise continues values $u(\zeta)$ with a finite number of the first kind discontinuity points ζ_k, $k=\overline{1,n}$, then inside S the values of $u(z)$ are confined between minimal and maximal values of $u(\zeta)$ (one-sided values of $u(\zeta)$ at points of discontinuity ζ_k, $k=\overline{1,n}$, are not considered).

In the case of $\mathbb{R}^2_+$ it can be proved by means of the function

$$\nu=M+\varepsilon\ln\left[x^2+(y+1)^2\right]^{\frac{1}{2}},$$

using the method developed by proof of Maximum Pinciple 3.1.3. □

3.2 The weighted Dirichlet problem in the half-plane

Let in this section a and b be complex numbers. By $y\to 0+$ as functions of y solutions of equation (1.1), which have not isolated singularities on the straight line $y=0$, principally behave as solutions of the ordinary differential equation (ODE)

$$yu''(y)+bu'(y)=0 \tag{3.31}$$

It is easily seen that general solution of equation (3.31) has the form

$$u(y)=\begin{cases} C_1y^{1-b}+C_2, & \text{when}\quad b\neq 1;\\ C_1\ln y+C_2, & \text{when}\quad b=1,\end{cases}$$

where C_1 and C_2 arbitrary complex constans. Therefore, the solutions for $y\to 0+$ behave as follows:

(i) for $\mathrm{Re}b\in]-\infty,1[$ all the solutions are modulo bounded;

(ii) for $b=1$ among them there exist unbounded solutions, while all the solutions multiplied by $(\ln y)^{-1}$ are bounded;

(iii) for $\mathrm{Re}b\in]1,+\infty[$ among them there exist unbounded solutions, while all the solutions multiplied by y^{b-1} are bounded.

Bearing in mind the above assertions, set the following weighted Dirichlet problem:

Problem 3.2.1. *In $\mathbb{R}^2_+$ find $u \in C^2(\mathbb{R}^2_+)$, satisfying equation (1.1) and one from the following boundary conditions:*

$$\lim_{z\to x_0} u(z) = f(x_0) \quad \text{when} \quad \mathrm{Re}b \in]-\infty, 1[; \tag{3.32}$$

$$\lim_{z\to x_0} (-\ln y)^{-1} u(z) = -f(x_0) \quad \text{when} \quad b = 1, \tag{3.33}$$

$$\lim_{z\to x_0} y^{b-1} u(z) = f(x_0) \quad \text{when} \quad \mathrm{Re}b \in]1, +\infty[, \tag{3.34}$$

where $z \in \mathbb{R}^2_+$, $x_0 \in \mathbb{R}^1 \setminus \overset{*}{I}$, f is piecewise continuous, bounded, in general complex-valued function defined on $\mathbb{R}^1$, $\overset{*}{I}$ is a set of points of discontinuity of function f.

Remark 3.2.2 *Evidently, the corresponding to Problem 3.2.1 BVPs for ODE (3.31) are explicitly solvable except for* $\mathrm{Re}b = 1$, $\mathrm{Im}b \neq 0$. *Also in the last case the general solution*

$$\begin{aligned} C_1 y^{-i\mathrm{Im}b} + C_2 &= C_1 (e^{\ln y})^{-i\mathrm{Im}b} + C_2 = C_1 e^{-i\mathrm{Im}b \ln y} + C_2 \\ &= C_1[\cos(\mathrm{Im b \ln y}) - i \sin(\mathrm{Im b \ln y})] + C_2 \end{aligned}$$

is bounded, while its limit does not exist by $y \to 0+$, since it is oscillating solution. Therefore, it does not exist a solution taking prescribed value for $y = 0$. More precisely, if we take $C_1 = 0$, we arrive at the trivial, i.e., insignificant solution of BVP, when the prescribed constant value at $y \to 0+$ itself will be solution.

Theorem 3.2.3 *A solution of Problem 3.2.1 has the form*

$$\frac{y^{1-b}}{\Lambda(a,b)} \int_{-\infty}^{\infty} f(\xi) e^{a\theta} \rho^{b-2} d\xi, \quad \mathrm{Re}b \in]-\infty, 1[, \quad \Lambda(a,b) \neq 0 \quad [\text{see}(2.59)]; \tag{3.35}$$

$$\frac{1}{1+e^{\pi}} \int_{-\infty}^{\infty} f(\xi) e^{a\theta} \rho^{-1} d\xi, \quad b = 1; \tag{3.36}$$

$$\frac{1}{\Lambda(a, 2-b)} \int_{-\infty}^{\infty} f(\xi) e^{a\theta} \rho^{-b} d\xi, \quad \mathrm{Re}b \in]1, \infty[, \quad \Lambda(a, 2-b) \neq 0 \quad [\text{see}(2.59)], \tag{3.37}$$

where for $b = 1$ the function f should satisfy a condition ensuring convergence of the integral (3.36), e.g., the condition

$$f(\xi) = O(|\xi|^{-\varepsilon}), \quad |\xi| \to \infty, \quad \varepsilon = \text{const} > 0.$$

Proof. Function [see (3.10)]

$$y^{1-b}e^{a\theta}\rho^{b-2} \tag{3.38}$$

is a solution of equation (1.1) for complex constants a and b too.

Let us consider the integral (3.35). Since Re$b < 1$, it can be shown that the integral (3.35), its derivatives of any order with respect to x and y are absolutely and uniformly convergent in any bounded closed domain lying in $\mathbb{R}^2_+$.

Hence, the integral (3.35) represents a solution of equation (1.1) since the integrand coincides with the solution (3.38) multiplied by $\dfrac{f(\xi)}{\Lambda(a,b)}$. After substitution [see (3.16)] $\xi = x + yt$ from (3.35) we obtain

$$u(x,y) = \frac{1}{\Lambda(a,b)} \int\limits_{-\infty}^{+\infty} f(x+yt)e^{a\cdot\operatorname{arc\,ctg}(-t)}(1+t^2)^{\frac{b}{2}-1}dt.$$

Whence,

$$|u(x,y)| \leq \frac{M}{|\Lambda(a,b)|} \int\limits_{-\infty}^{+\infty} e^{a\cdot\operatorname{arc\,ctg}(-t)}|(1+t^2)^{\frac{b}{2}-1}|dt = \frac{M\Lambda(\operatorname{Re}a, \operatorname{Re}b)}{|\Lambda(a,b)|},$$

where

$$M := \sup_{\xi\in\mathbb{R}^1} |f(\xi)|. \tag{3.39}$$

Evidently,

$$u(x,y) - f(x_0) = \frac{1}{\Lambda(a,b)} \int\limits_{-\infty}^{+\infty} |f(x+yt) - f(x_0)|e^{a\cdot\operatorname{arc\,ctg}(-t)}(1+t^2)^{\frac{b}{2}-1}dt, \tag{3.40}$$

here x_0 is a point of continuity of the function f.

Because of (3.39), we have

$$|f(x+yt) - f(x_0)| < 2M \ \text{ when } \ z \in \mathbb{R}^+_2, \ \ x_0, t \in \mathbb{R}^1.$$

Assume $\varepsilon > 0$ arbitrary small. Then, by virtue of uniform convergence of the integral (3.40), with respect to $z \in \mathbb{R}^2_+$, $x_0 \in \mathbb{R}^1$, there exists sufficiently large $R(\varepsilon) > 1$ such that

$$\begin{aligned} &\frac{2M}{|\Lambda(a,b)|} \int\limits_{-\infty}^{-R} e^{\operatorname{Re}a\cdot\operatorname{arc\,ctg}(-t)}(1+t^2)^{\frac{\operatorname{Re}b}{2}-1}dt < \frac{\varepsilon}{3}, \\ &\frac{2M}{|\Lambda(a,b)|} \int\limits_{R}^{+\infty} e^{\operatorname{Re}a\cdot\operatorname{arc\,ctg}(-t)}(1+t^2)^{\frac{\operatorname{Re}b}{2}-1}dt < \frac{\varepsilon}{3}. \end{aligned} \tag{3.41}$$

Since the function f is continuous at the point x_0, we can find such a $\delta_1(\varepsilon, x_0)$ that

$$|f(x+yt)-f(x_0)| < \frac{\varepsilon}{3}\,\frac{|\Lambda(a,b)|}{|\Lambda(\mathrm{Re}a,\mathrm{Re}b)|}, \;\; when \;\; |x+yt-x_0| < \delta_1.$$

Now if $|t| < R,\; |x-x_0| < \delta$ and $0 < y < \delta$, where

$$\delta = \frac{\delta_1}{2R} < \frac{\delta_1}{2},$$

we have

$$|x+yt-x_0| \le |x-x_0| + y|t| < \delta + \delta R < \frac{\delta_1}{2} + \frac{\delta_1}{2R}R = \delta_1,$$

i.e.,

$$|f(x+yt)-f(x_0)| < \frac{\varepsilon}{3}\,\frac{|\Lambda(a,b)|}{|\Lambda(\mathrm{Re}\,a,\mathrm{Re}\,b)|} \;\; \text{when} \;\; |t| < R,\; |x-x_0| < \delta,\;\; 0 < y < \delta. \tag{3.42}$$

If we present the integral (3.35) as a sum of three integrals with integration limits $-\infty, -R$; $\;-R, +R$; $\;+R, +\infty$, according to the inequalities (3.41) and (3.42), we get

$$|u(x,y)-f(x_0)| < \frac{2\varepsilon}{3} + \frac{\varepsilon}{3\Lambda(\mathrm{Re}\,a,\mathrm{Re}\,b)}\int\limits_{-\infty}^{+\infty} e^{\mathrm{Re}\,a\cdot\mathrm{arc\,ctg}\,(-t)}(1+t^2)^{\frac{\mathrm{Re}\,b}{2}-1}dt = \varepsilon, \tag{3.43}$$

when $|x-x_0| < \delta(\varepsilon, x_0),\;\; 0 < y < \delta(\varepsilon, x_0)$, i.e., the expression (3.35) satisfies BC (3.32). □

Remark 3.2.4 *If function f is uniformly continuous on $\mathbb{R}^1$, then inequality* (3.43) *will be fulfilled uniformly. Therefore, by $z \to x_0$ the integral* (3.35) *tends to $f(x_0)$ uniformly.*

Remark 3.2.5 *For* $a = 0,\; b = \frac{m}{m+2},\; m = \mathrm{const} > 0$ *from* (3.35) *we obtain the well-known result of I. Vekua [34] for the Gellerstadt equation, i.e. for homogeneous equation with the operator (3.24) with $a = 0$ on the left-hand side.*

Remark 3.2.6 *If f is a piecewise constant function with complex values $c_1, c_2, ..., c_n$ correspondingly on intervals* $(-\infty, x_1), (x_1, x_2), ..., (x_n, +\infty)$, $\;(x_i < x_{i+1}, i = 1, ..., n-1)$, *then in view of* (3.10), (3.12), *from* (3.35) *we obtain (compare with I. Vekua [34])*

$$u(x,y) = \frac{1}{\Lambda(a,b)}\left\{c_1\int\limits_{-\infty}^{x_1}\frac{d}{d\xi}\Omega(x-\xi,y)d\xi + c_2\int\limits_{x_1}^{x_2}\frac{d}{d\xi}\Omega(x-\xi,y)d\xi\right.$$

$$\left. +... + c_{n+1}\int\limits_{x_n}^{+\infty}\frac{d}{d\xi}\Omega(x-\xi,y)d\xi\right\} = \left\{\frac{1}{\Lambda(a,b)}\{c_1\Omega(x-x_1,y)\right.$$

$$\left. +c_{n+1}[\Lambda(a,b)-\Omega(x-x_n,y)] + \sum_{k=2}^{n} c_k[\Omega(x-x_k,y)-\Omega(x-x_{k-1},y)]\right\}.$$

Here we have taken into account (3.12) and (3.17), i.e.,

$$\lim_{\xi\to-\infty}\Omega(x-\xi,y)=\int\limits_0^0 e^{a\tau}\sin^{-b}\tau d\tau=0,$$

and

$$\lim_{\xi\to+\infty}\Omega(x-\xi,y)=\int\limits_0^\pi e^{a\tau}\sin^{-b}\tau d\tau=\Lambda(a,b).$$

Let us now consider the integral (3.36). Evidently, it satisfies equation (1.1) and, bearing in mind Remark 2.1.4, BC (3.33). Indeed,

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{1}{1+e^{a\pi}}\int\limits_{-\infty}^{+\infty} f(\xi)e^{a\theta}\rho^{-1}d\xi=\frac{-y}{1+e^{a\pi}}\lim_{z\to x_0}\int\limits_{-\infty}^{+\infty} f(\xi)[a(x-\xi)-y]e^{a\theta}\rho^{-3}d\xi$$

$$=\frac{1}{1+e^{a\pi}}\lim_{z\to x_0}\int\limits_{-\infty}^{+\infty} f(x+yt)(at+1)e^{a\cdot\operatorname{arc\,ctg}(-t)}(1+t^2)^{-\frac{3}{2}}dt=f(x_0),$$

since according to (2.54) and (2.53),

$$\int\limits_{-\infty}^{+\infty} e^{a\cdot\operatorname{arc\,ctg}(-t)}(at+1)(1+t^2)^{-\frac{3}{2}}dt=a\Lambda_1(a,-1)+\Lambda(a,-1)$$

$$=(a^2+1)\Lambda(a,-1)=(a^2+1)\int\limits_0^\pi e^{a\theta}\sin\theta d\theta=1+e^{a\pi},$$

because of

$$\int\limits_0^\pi e^{a\theta}\sin\theta d\theta=\frac{e^{a\pi}+1}{a^2+1},$$

which is easily seen[5]

Finally, let $\operatorname{Re} b>1$. Introduce a new unknown function

$$U(z)=y^{b-1}u(z). \tag{3.44}$$

[5]Indeed,

$$I=\int\limits_0^\pi e^{a\theta}\sin\theta=\frac{1}{a}\int\limits_0^\pi \sin\theta de^{a\theta}=\sin\theta e^{\pi\theta}\Big|_0^\pi-\frac{1}{a}\int\limits_0^\pi e^{a\theta}\cos\theta d\theta$$

$$=-\frac{1}{a^2}\int\limits_0^\pi\cos\theta de^{a\theta}=-\frac{1}{a^2}\Big[\cos\theta e^{a\theta}\Big|_0^\pi+\int\limits_0^\pi e^{a\theta}\sin\theta d\theta\Big]$$

$$=-\frac{1}{a^2}\Big[(-e^{a\pi}-1)+I\Big]=\frac{e^{a\pi}+1}{a^2}-\frac{1}{a^2}I.$$

By virtue of the correspondence principle (3.4) and (3.39), we see that the function $U(z)$ satisfies equation

$$E^{(a,b^*)} = 0, \quad b^* = \mathrm{Re}(2-b) < 1, \tag{3.45}$$

and BC

$$\lim_{z \to x_0} U(z) = f(x_0), \quad z \in \mathbb{R}^2_+, \quad x_0 \in \mathbb{R}^1 \setminus \overset{*}{I}.$$

After representation of a solution of the last BVP by means of the formula (3.35), where "b" should be replaced by "$2-b$" and returning via equality (3.44) to $u(z)$ we get (3.37).

Remark 3.2.7 *Let* $a, b \in \mathbb{R}^1$. *Then*

$$\Lambda(a,b) \neq 0 \ (b < 1), \ \Lambda(a, 2-b) \neq 0 \ (b > 1)$$

If $a, b \bar{\in} \mathbb{R}^1$, i.e., they are complex numbers, the expressions (3.35) and (3.37) will have sense if and only if $b \pm ia \bar{\in} \mathbb{N}_2$ (see Theorem 2.2.4). If $a \in \mathbb{R}^1$, then $1 + e^{a\pi} \neq 0$ and the expression (3.36) has a sense. If $a \bar{\in} \mathbb{R}^1$, then the expression (3.36) will have sense if and only if

$$a \neq (2k+1)i,$$

when k is an arbitrary integer.

3.3 The weighted Neumann type problem in the half-plane

The limit as $y \to 0+$ of the m-th order derivative of the general solution of equation (3.31)

$$\frac{d^m u(y)}{dy^m} = \left\{ \begin{array}{l} (-1)^m C_1(b-1,m) y^{1-b-m} \ \text{ for } \ b \neq 1, 0, -1, , 2-m, \ \text{ when } m \geq 1; \\ (-1)^{m-1} C_1(b-1,m) y^{-m} \ \text{ for } \ b = 1, \ \text{ when } m \geq 1; \\ 0 \ \text{ for } \ b = 0, -1, , 2-m, \ \text{ when } m \geq 2, \end{array} \right. \tag{3.46}$$

does not give so complete information about the behaviour by $y \to 0+$ of the m-th order derivative

$$\frac{\partial^m u(x,y)}{\partial y^m}$$

of the solution of equation (1.1) as we it had in the case of the behaviour by $y \to 0+$ of the solution $u(x,y)$ of equation (1.1).

As it will be shown below the behaviour of $\frac{\partial^m u(x,y)}{\partial y^m}$ as $y \to 0+$ depends not only on values of b but also on the coefficient a and oddness and evenness of

m. Therefore, in this sense (1.1) becomes a model equation for PDEs with order degeneration. Nevertheless, some preliminary conclusions we are able to make according to (3.46), all the m-th order derivatives for $y \to 0+$ behave as follows:

(i) if $b \in]-\infty, 1-m[$, all become zero (under assumption of boundedness of solutions at infinity all the derivatives are identically zero for $0 \leq y < +\infty$);

(ii) if $b = 1-m$, all are constants (under assumption of boundedness of solutions at infinity all the derivatives are identically zero for $0 \leq y < +\infty$);

(iii) if $b \in]1-m, +\infty[$, $b \neq 0, -1, ..., 2-m$, all are unbounded (under assumption of boundedness of solutions at infinity all the derivatives are identically zero for $0 \leq y < +\infty$), provided $b \in]-\infty, 1[$.

On the other hand all the m-th order derivatives multiplied by y^{b+m-1} remain bounded, while for $b = 0, -1, ..., 2-m$, $m \geq 2$ they are identically zero. Thus, if $b \in]1-m, +\infty[\cap]1, +\infty[\equiv]1, +\infty[$ only one thing is clear that as the weight of the m-th order derivative of the solution of equation (1.1) will serve y^{m+b-1}. when $y \to 0+$.

In this section, by means of solutions of the form (3.10) and (3.11) we construct solutions of equation (1.1) *in* $\mathbb{R}_2^+$, when on the boundary m-th order derivative of the solution with the corresponding weight is prescribed. More general problems for non-degenerate PDEs of the canonical form are studied by I. Vekua (see [35], p. 113 and pp. 138-149, and also analogues problems in works of I. Vekua [33] and N. Muskhelishvili [31], p. 260).

Let us consider the n-th order antiderivative of the function $f(\xi)$

$$f^{(-m)}(\xi) := \int\limits_{\xi_0}^{\xi} \frac{(\xi-\tau)^{m-1}}{(m-1)!} f(\tau) d\tau + \sum_{k=0}^{m-1} c_k \xi^k, \quad \xi_0, \xi \in \mathbb{R}^1,$$

where

$$c_k = \text{const}, \quad m \in \mathbb{N}^0, \quad f^{(0)}(\xi) := f(\xi),$$

f is an integrable on $]\xi_0, \xi[$ function. We denote by $\underset{*}{f}^{(-m)}$ bounded functions among functions $f^{(-m)}$ and by $\underset{0}{f}^{(-m)}$ vanishing as $|\xi| \to +\infty$ functions.

Let us introduce the following function classes:

$C^m(G)$ is a set of functions with continuous partial derivatives of order $\leq m$ in $G \subseteq \mathbb{R}^p$;

$\underset{*}{C}^m(G) \subset C^m(G)$ is a subset of bounded functions with bounded partial derivatives;

$C^0(G) := C(G)$ is a set of continuous in G functions;

$\underset{*}{C}^{-m}$, $m \in \mathbb{N}$, is a set of continuous and bounded functions in $\mathbb{R}^1$ with $\underset{*}{f}^{(-j)}$, $j = 1, \dots, m$;

$\underset{0}{C}^{-m} \subset \underset{*}{C}^{-m}$, $m \in \mathbb{N}$, is a set of functions f with $\underset{0}{f}^{(-m)}$;

$\underset{*}{C} \equiv \underset{*}{C}^0 \supset \underset{0}{C}^0$ is a set of continuous and bounded functions.

$T^m\big(\gamma_m(y), G, \overset{*}{I}\big)$, $G \subseteq \mathbb{R}^2_+$, $m \in \mathbb{N}^0$, is a class of functions

$$u \in C^2(G) \backslash C(\bar{G} \backslash \overset{*}{I} \cap \mathbb{R}^2_+),$$

satisfying equation (1.1) in G and the following condition: bounded in neighborhoods of points of a set $\overset{*}{I} \subset \partial G \backslash \mathbb{R}^1$ of finite number of isolated points function

$$\gamma_m(y) \frac{\partial^m u}{\partial y^m} \in C((\bar{G} \backslash \overset{*}{I}) \cap \mathbb{R}^2_\varepsilon)$$

where

$$\mathbb{R}^2_\varepsilon := \{(x, y) : \ x \in \mathbb{R}^1, \ 0 \le y \le \varepsilon = \text{const } < 1\},$$

$$\gamma_m(y) := \begin{cases} 1 \text{ when } (a, b) \in i_{1,m} \cup i_{6,m}, \\ y^{-1} \ \text{ when } \ (a, b) \in i_{5,m} \\ (y \ln \frac{1}{y})^{(-1)} \ \text{ when } \ (a, b) \in i_{4,m} \\ (\ln \frac{1}{y})^{(-1)} \ \text{ when } \ (a, b) \in i_{3,m}, \\ y^{b+m-1} \ \text{ when } \ (a, b) \in i_{2,m}; \end{cases}$$

$$i_{1,m} := \left\{(a, b) : b \in] - \infty, 1 - m[\wedge a \ne 0, \ m \in \mathbb{N}^0 \vee a = 0, \ m \in \mathbb{N}^0_2\right\};$$

$$i_{2,m} := \{(a, b) : b \in] - m, 1 - m[, a = 0, \ m \in \mathbb{N}_1 \vee$$
$$b \in]1 - m, +\infty[, \ m \in \mathbb{N}^0 \wedge a \ne 0 \vee a = 0, \ b \ne -2n > 1 - m, \ n \in \mathbb{N}^0$$
$$(i.e., \ b \ne 0, -2, ..., -2(m - [\frac{m}{2}] - 1) \ \ for \ \ m \in \mathbb{N} \backslash \{1\})\};$$
$$i_{3,m} = \left\{(a, b) : b = 1 - m \ \wedge \ a \ne 0, \ m \in \mathbb{N}^0 \ \vee \ a = 0, \ m \in \mathbb{N}^0_2\right\};$$
$$i_{4,m} := \{(a, b) : a = 0, \ \ b = -m, \ \ m \in \mathbb{N}_1\};$$
$$i_{5,m} := \{(a, b) : a = 0, \ \ b \in] - \infty, -m[, \ \ m \in \mathbb{N}_1\};$$
$$i_{6,m} := \{(a, b) : a = 0, \ b = -2n \ge 1 - m, \ n \in \mathbb{N}^0, \ m \in \mathbb{N}$$
$$(i.e., \ b = 0, -2, ..., -2(m - [\frac{m}{2}] - 1) \ \ for \ \ m \in \mathbb{N} \backslash \{1\})\};$$

$\mathbb{N} := \{1, 2, ...\}, \ \mathbb{N}^0 := \mathbb{N} \cup \{0\}, \ \mathbb{N}_1 := \{1, 3, ...\}, \ \mathbb{N}_2 := \{2, 4, ...\}, \ \mathbb{N}^0_2 := \mathbb{N}_2 \cup \{0\},$

$\wedge$ is conjunction, $\vee$ is disjunction. $\overset{*}{I}$ may be an empty set as well.

We introduce the following classes of functions as well:

$$T^m\left(\gamma_m(y), G\right) := T^m\left(\gamma_m(y), G, \varnothing\right);$$

$$T^m\left(\gamma_m(y)\right) := T^m\left(\gamma_m(y), \mathbb{R}^2_+\right);$$

$T^m_n\left(\gamma_m(y)\right) \subset T^m(\gamma_m(y))$, $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, be a class of functions $u(x, y)$ with properties $u \in C\left(\mathbb{R}^2_+ \bigcup \mathbb{R}^1\right)$ and $u(x, 0) \in \underset{*}{C^n}(\mathbb{R}^1)$;

$\underset{0}{T^m_n}\left(\gamma_m(y)\right) \subset T^m_n(\gamma_m(y))$, $n \in \mathbb{N}$, be a class of functions $u(x, y)$ with

$$\lim_{|x|\to+\infty} u(x,0)=0;$$

$$\underset{0}{T_0^m}\left(\gamma_m(y)\right)\subset T_0^m\left(\gamma_m(y)\right)\subset T^m\left(\gamma_m(y)\right).$$

Let

$$f(\xi)\in\begin{cases}\underset{*}{C}^{-m} \ \text{when}\ \ (a,b)\in i_{1,m}\vee(i_{3,m},\ m>0);\\[2ex] \underset{*}{C}^{-m-1}\ \text{when}\ \ (a,b)\in i_{4,m}\vee i_{5,m};\\[2ex] \underset{*}{C}^{-1}\ \text{when}\ \ (a,b)\in i_{3,0};\\[2ex] \underset{0}{C}^{0}\ \text{and}\ f(\xi)=O(|\xi|^{-\alpha}),\ \ |\xi|\to+\infty,\ \ \alpha>1-b,\\[2ex] \qquad \text{for}\ \ b\in]-\infty,1[\ \ \text{when}\ \ (a,b)\in i_{2,m};\\[2ex] \underset{*}{C}^{-j},\ 0\le j\le m,\ \ \text{and}\ \ \underset{0}{f}^{k-m}(\xi)=O(|\xi|^{-\alpha_k}),\\[2ex] \qquad |\xi|\to+\infty,\ \ \text{for}\ \ \alpha_k>k,\ \ 0\le k\le m\ \ \text{when}\ \ (a,b)\in i_{6,m}.\end{cases}$$

Let

$$\alpha_m(a,b):=\begin{cases}0\ \ \text{when}\ \ (a,b)\in i_{1,m}\vee i_{3,m},\\ 1\ \ \text{when}\ \ (a,b)\in i_{4,m}\vee i_{5,m},\\ -m\ \ \text{when}\ \ (a,b)\in i_{2,m}\vee i_{6,m}.\end{cases}$$

Main Proposition. The BVP for elliptic EPD equation (1.1) with the boundary condition (BC)

$$\lim_{y\to0+}\gamma_m(y)\frac{\partial^m u}{\partial y^m}=f(x)$$

is always solvable in $T^m_{m+\alpha_m}(\gamma_m(y))$. If $(a,b)\in i_{1,0}\vee i_{3,0}\vee i_{2,0}\vee i_{6,m}$, it is uniquely solvable and if $(a,b)\in i_{1,m}\vee i_{3,m}\vee i_{4,m}\vee i_{5,m},\ \ m>0$, it is solvable up to an additive constant under some restrictions at infinity [e.g., boundedness if $(a,b)\in i_{5,m}\vee i_{4,m}\vee i_{1,m},\ \ b<0$, and $u=O(y^{1-b}),\ \ x^2+y^2\to+\infty$, if $(a,b)\in i_{2,m},\ \ b>2$]. In $\underset{0}{T^m_{m+\alpha_m}}(\gamma_m(y))$ the solution is unique. The solutions have been constructed in the explicit form (see G. Jaiani [14], [15], [16], [20] and below).

In order to prove the main proposition we consider particular BVPs composing the general BVP formulated in Main Proposition.

Problem 3.3.1. *Let $(a,b)\in i_{1,m}$. Find $u\in T^m_m(1)$ satisfying BC*

$$\lim_{z\to x_0}\frac{\partial^m u(z)}{\partial y^m}=f\left(x_0\right),\ z\in\mathbb{R}^2_+,\quad x_0\in\mathbb{R}^1, \tag{3.47}$$

where $f \in \underset{*}{C}^{(-m)}$ *and the condition at infinity*

$$u = \begin{cases} O(1), r \to +\infty, & \text{when either } \ a \in R^1, \ b \in]-\infty, 0[\ , \ \text{or } a = 0, \ b = 0; \\ o(1), r \to +\infty, & \text{when either } \ a \in R^1, \ b \in]0, 1[\ , \ \text{or } a \neq 0, \ b = 0. \end{cases} \tag{3.48}$$

Problem 3.3.2. *Let* $a = 0, \ b \in]-\infty, -m[, \ m \in \mathbb{N}_1$. *Find* $u \in T^m_{m+1}\left(y^{-1}\right)$ *satisfying BC*

$$\lim_{z \to x_0} y^{-1} \frac{\partial^m u(z)}{\partial y^m} = f\left(x_0\right), \ z \in \mathbb{R}^2_+, \ \left(x_0\right) \in \mathbb{R}^1, \tag{3.49}$$

where $f \in \underset{*}{C}^{-m-1}$ *and the condition at infinity.*

$$u = O(1), \ r \to +\infty \tag{3.50}$$

Problem 3.3.3. *Let* $a = 0, \ b = -m, \ m \in \mathbb{N}_1$. *Find* $u \in T^m_{m+1}\left(\left(y\ln\frac{1}{y}\right)^{-1}\right)$ *satisfying BC*

$$\lim_{z \to x_0} \left(y\ln\frac{1}{y}\right)^{-1} \frac{\partial^m u(z)}{\partial y^m} = -f\left(x_0\right), \ z \in \mathbb{R}^2_+, \ \left(x_0\right) \in \mathbb{R}^1, \tag{3.51}$$

where $f \in \underset{*}{C}^{-m-1}$ *and the condition at infinity (3.50).*

Problem 3.3.4. *Let* $(a, b) \in i_{2,m}$. *Find* $u \in T^m_m\left(y^{b+m-1}\right)$ *satisfying BC*

$$\lim_{z \to x_0} y^{b+m-1} \frac{\partial^m u(z)}{\partial y^m} = f\left(x_0\right), \ z \in \mathbb{R}^2_+, \ x_0 \in \mathbb{R}^1, \tag{3.52}$$

where $f \in \underset{*}{C}^0$ *[when* $b \in]-\infty, 1[$, *we assume*

$$f(\xi) = O(|\xi|^{-\alpha}), \ \ |\xi| \to +\infty, \ \ \alpha > 1 - b,$$

or when $b \in]0, 1[$, *we assume* $f \in \underset{*}{C}^{-1}$*] and the corresponding condition from the following ones:*

$$u = O\left(y^{1-b}\right), \ r \to +\infty, \tag{3.53}$$

when either $a \in \mathbb{R}^1, \ \ b \in]2, +\infty[$, *or* $a = 0, \ \ b = 2$*;*

$$u = o\left(y^{1-b}\right), \ r \to +\infty, \tag{3.54}$$

when either $a \in \mathbb{R}^1, \ \ b \in]1, 2[$, *or* $a \neq 0, \ \ b = 2$*;*

$$u = O\left(r^{-1}\right), \ u_x, u_y = O\left(r^{-2}\right), \ r \to +\infty, \tag{3.55}$$

and

$$\lim_{y\to 0+}\int\limits_{-\infty}^{+\infty} u\cdot u_y dx=0$$

if

$$\lim_{y\to 0+}(\ln y)^{-1}u=0,\ x\in\mathbb{R}^1, \tag{3.56}$$

when $a\in\mathbb{R}^1,\ \ b=1$*;*

$$u=O\left(\mathbb{R}^2_+ U\mathbb{R}^1\right),\ u=o(1),\ r\to+\infty, \tag{3.57}$$

when $a\in\mathbb{R}^1,\ \ b\in]0,1[$*;*

$$u=\overset{(1)}{I}+O\left(r^{-1}\right),\ u_x=\overset{(1)}{I}_x+O\left(r^{-2}\right),\ u_y=\overset{(1)}{I}_y+O\left(r^{-2}\right),\ r\to+\infty, \tag{3.58}$$

where

$$\overset{(1)}{I}:=M_0^{-1}(a,b,0,m)\int\limits_{-\infty}^{+\infty} f(\xi)e^{a\theta}\rho^{-b}d\xi,$$

and

$$\lim_{y\to 0+}y^b\int\limits_{-\infty}^{+\infty} u\cdot u_y dx=0$$

if

$$\lim_{y\to 0+}y^{b+m-1}\frac{\partial^m u}{\partial y^m}=0,\ x\in\mathbb{R}^1, \tag{3.59}$$

when $a\in\mathbb{R}^1,\ b\in]-\infty,0]$.

Problem 3.3.5. *Let* $(a,b)\in i_{3,m}$*. Find* $u\in T^m_m\in\left(\left(\ln\frac{1}{y}\right)^{-1}\right)$ *satisfying BC*

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{\partial^m u(z)}{\partial y^m}=-f\left(x_0\right),\ z\in\mathbb{R}^2_+,\ (x_0)\in\mathbb{R}^1, \tag{3.60}$$

where $f\in \underset{*}{C}^{-m}$ *for* $m>0$*, while for* $m=0$ *either* $f\in \underset{*}{C}^{-1}$ *or* f *is a continuous function,*

$$f(\xi)=O\left(|\xi|^{-\alpha}\right),\ |\xi|\to+\infty,\ \alpha>0, \tag{3.61}$$

and the conditions (3.48) and (3.55), (3.56) correspondingly for $m>0$ *and* $m=0$.

Problem 3.3.6. *Let* $a=0,\ b=0,\ m\in\ \mathbb{N}$*. Find* $u\in\ T^m(1)$ *with the bounded m-th order derivative with respect to y, satisfying BC (3.47), where*

$$f(\xi)=O\left(|\xi|^{-\alpha}\right),\ |\xi|\to+\infty,\ \alpha>m,$$

is a continuous function.

Problem 3.3.7. *Let* $a=0,\ b=-2n\geq 1-m,$ *where* $n\in\mathbb{N}^0,\ m\in\mathbb{N}\backslash\{1,2\}$ (*i.e.* $b\in]1-m,+\infty[\cap\left\{-2,-4,\dots,-2\left(m-\left[\frac{m}{2}\right]-1\right)\right\},\ m\in\mathbb{N}\backslash\{1,2,3\}$) *or* $a=0,\ b=1-m,\ m\in\mathbb{N}_1\backslash 1.$ *Find* $u\in T^m(1)$ *satisfying BC (3.47), where*

$$f\in\underset{0}{C}^{-j},\ j=\overline{0,m};\ \underset{0}{f}^{(k-m)}(\xi)=O(|\xi|^{-\alpha_k}),\ \xi\to+\infty,\ \alpha_k>k,\ k=\overline{0,m}. \quad (3.62)$$

Theorem 3.3.8 *The solutions of Problems 3.3.1-3.3.6 have, correspondingly, the following forms*

$$u_1(x,y)=\Lambda_m^{-1}(a,b)y^{1-b}\int\limits_{-\infty}^{+\infty}\underset{*}{f}^{(-m)}(\xi)e^{a\theta}\rho^{b-2}d\xi,\ \Lambda_m(a,b)\neq 0 \text{ [see (2.59)]}; \quad (3.63)$$

$$u_2(x,y)=\Lambda_{m+1}^{-1}(0,b)y^{1-b}\int\limits_{-\infty}^{+\infty}\underset{*}{f}^{(-m-1)}(\xi)\rho^{b-2}d\xi,\ \Lambda_{m+1}(0,b)\neq 0 \text{ [see (2.59)]}; \quad (3.64)$$

$$u_3(x,y)=(m+2)^{-1}\Lambda_{m+1}^{-1}(0,-m-2)y^{m+1}\int\limits_{-\infty}^{+\infty}\underset{*}{f}^{(-m-1)}(\xi)\rho^{-m-2}d\xi, \quad (3.65)$$

$$\Lambda_{m+1}(0,-m-2)\neq 0 \ \text{ [see (2.59)]};$$

$$u_4(x,y)=M^{-1}(a,b,m)\int\limits_{-\infty}^{+\infty}f(\xi)e^{a\theta}\rho^{-b}d\xi, \quad (3.66)$$

$M(a,b,m)\neq 0$ [see the second part of Corollary 2.2.5];

$$u_5(x,y)=d_m^{-1}(a)y^m\int\limits_{-\infty}^{+\infty}\underset{*}{f}^{(-m)}(\xi)e^{a\theta}\rho^{-m-1}d\xi,\ \ d_m\neq 0 \ \text{ [see (2.60)]}, \quad (3.67)$$

where

$$d_m(a)=\begin{cases}(m+1)\Lambda_m(a,-m-1)+a\Lambda_{m+1}(a,-m-1),\ m>0,\\ 1+e^{a\pi},\ m=0,\end{cases}$$

$d_m(a)\neq 0$ [see (2.60)];

$$u_6(x,y)=\frac{1}{\pi(m-1)!}\int\limits_{-\infty}^{+\infty}f(\xi)\left\{\ln\rho\sum_{l=0}^{\left[\frac{m+1}{2}\right]-1}\begin{pmatrix} m-1\\ 2l+\dfrac{1+(-1)^m}{2}\end{pmatrix}\right.$$

$$\times(-1)^{\left[\frac{m+1}{2}\right]-l-1}y^{2l+\frac{1+(-1)^m}{2}}(x-\xi)^{2\left(\left[\frac{m}{2}\right]-l-\frac{1+(-1)^m}{2}\right)}$$

$$+\mathrm{arctg}\frac{y}{x-\xi}\sum_{l=0}^{\left[\frac{m}{2}\right]-1}\left(\begin{array}{c} m-1 \\ 2l+\dfrac{1+(-1)^m}{2}\end{array}\right)$$

$$\times(-1)^{\left[\frac{m}{2}\right]-l+1}y^{2l+\frac{1-(-1)^m}{2}}(x-\xi)^{2\left(\left[\frac{m}{2}\right]-l\right)-1}\Big\}\,d\xi+\sum_{l=0}^{m-1}Q_l(x)y^l, \qquad (3.68)$$

where

$$\sum_{l=0}^{-1}(\cdots):=0;$$

$$Q_{m-1}(x)=\overset{m-1}{C_1}x+\overset{m-1}{C_2},$$

$$Q_{m-2}(x)=\overset{m-2}{C_1}x+\overset{m-2}{C_2},$$

$$Q_l(x)=-(l+1)(l+2)\int\limits_{x_0}^{x}(x-t)Q_{l+2}(t)dt+\overset{l}{C_1}+\overset{l}{C_2},\ l=0,\ldots,m-3,$$

$$\overset{l}{C_\alpha}=\mathrm{const}\,,\ \alpha=1,2,\ l=0,\ldots,m-1; \qquad (3.69)$$

$$u_7(x,y)=\frac{\sum\limits_{k=0}^{n}\overset{n}{a}_k\Gamma(k+1)(-y)^k}{\pi\prod\limits_{j=1}^{n}[m-(2j-1)]}\int\limits_{-\infty}^{+\infty}\overset{0}{f}{}^{(-m)}(\xi)\left\{\begin{array}{c}(-1)^{\frac{m}{2}}\mathrm{Re}\\(-1)^{\frac{m+1}{2}}\mathrm{Im}\end{array}\right\} \qquad (3.70)$$
$$\times\left[y+i(\xi-x)\right]^{k+1}\rho^{-2(k+1)}d\xi,$$

here Re and Im correspond to $m\in N_2$ and $m\in N_1$ correspondingly,

$$\overset{n}{a}_k=\overset{n-1}{a}_{k-1}-(2n-1-k)\overset{n-1}{a}_k,$$

$$k=\overline{0,n};\ \overset{l}{a}_{-1}=0,\ \overset{l}{a}_j=0\ (l<j),\ \overset{j}{a}_j=1,\ l=\overline{0,n-1},\ j=\overline{0,n}.$$

Solutions of the problems 3.3.1, 3.3.5 for $m=0$, and Problem 2.3.4 are unique.

Solutions of the problems 3.3.1, 3.3.5 for $m>0$, and 3.3.2, 3.3.3 are determined up to an additive constant.

A solution of Problem 3.3.6 is determined up to the additive

$$\sum_{l=0}^{m-1}Q_1(x)y^l,$$

which contains $2m$ arbitrary constants

$$\overset{l}{C_\alpha},\quad l=\overline{0,m-1},\quad \alpha=1,2.$$

If $f \in \underset{0}{C}^{-m}$ in the cases of the problems 3.3.1, 3.3.5 for $m > 0$ and if $f \in \underset{0}{C}^{-m-1}$ in the cases of the problems 3.3.2, 3.3.3, then solutions of the above-mentioned problems are unique in the classes

$$\underset{0}{T}{}^m_m(1), \quad \underset{0}{T}{}^m_m\left(\left(\ln\frac{1}{y}\right)^{-1}\right), \quad \underset{0}{T}{}^m_{m+1}(1)\left(y^{-1}\right), \quad \underset{0}{T}{}^m_{m+1}(1)\left(\left(y\ln\frac{1}{y}\right)^{-1}\right),$$

respectively. In this cases in the expressions (3.63)-(3.65) and (3.67) the stars should be replaced by the zeroes.

Bounded and vanishing at infinity solutions of Problem 3.3.6 do not exist, in general. In the cases of their existence bounded solutions are determined up to an additive constant, while vanishing at infinity solutions are determined uniquely.

The solution of Problem 3.3.7 is unique under the condition

$$u = O(r^{-1}), \;\; u_x, u_y = O(r^{-2}), \quad r \to +\infty,$$

and

$$\lim_{y\to 0+} y^b \int\limits_{-\infty}^{+\infty} u_y dx = 0$$

if

$$\lim_{y\to 0+} \frac{\partial^m u}{\partial y^m} = 0, \quad x \in \mathbb{R}^1.$$

It is easily seen by means of Green's formula.

Remark 3.3.9 *The cases $m = 0$, $a, b \in \mathbb{R}^1$, and $m = 1$, $a \in \mathbb{R}^1$, $b \in]0, +\infty[$ are contained in the works of G. Jaiani [9], [10], [11]. Later the cases $m = 0, 1$ were considered by O. Marichev [27], [28] in an another way, where the question of uniqueness of solutions are not considered for all the values of coefficients. The case of arbitrary $m \in \mathbb{N}^0$ was studied by G. Jaiani [14], [15], [16].*

Remark 3.3.10 *If f is a bounded piece-wise continuous function, omitting the stars in formulas (3.63)-(3.65), (3.67) and in the cases of the problems 3.3.3 and 3.3.5 assuming*

$$f^{(-m-1)}(\xi) = O\left(|\xi|^{m-\alpha+1}\right), \; |\xi| \to +\infty, \; \alpha > 0, \; f^{(-1)} \in \underset{*}{C}^0$$

and

$$f^{(-m)}(\xi) = O\left(|\xi|^{m-\alpha}\right), \; |\xi| \to +\infty, \; \alpha > 0, \; f^{(-1)} \in \underset{*}{C}^0,$$

respectively, then the expressions (3.63)-(3.67), (3.70) satisfy equation (1.1) in $\mathbb{R}^+_2$ and, correspondingly, BCs (3.47), (3.49), (3.51), (3.52), (3.60), (3.47), (3.47) at points of continuity of f even if a and b are complex numbers and $Re\, b$ meet conditions demanded from the real constant "b" (except of the problems 3.3.3, 3.3.5 where the constant b we always assume real). Of course we exclude those complex values of a and b when the denominators of (3.63), (3.64), (3.66), (3.67) become zero (see Section 2.2, the first parts of Theorem 2.2.4 and Corollary 2.2.5).

Remark 3.3.11 *Because of (3.3), (3.4) we may reduce the following problem 3.3.12 to the problems 2.3.1-2.3.6.*

Problem 3.3.12. *Find $u(x,y) \in C^2(\mathbb{R}^2_+)$ satisfying equation (1.1) in $\mathbb{R}^2_+$ and one of the following BCs:*

$$\lim_{y\to x_0} \frac{\partial^m y^{b-1}u}{\partial y^m} = f\left(x_0\right) \quad if \quad (a, 2-b) \in i_{1,m};$$

$$\lim_{y\to x_0} y^{-1}\frac{\partial^m y^{b-1}u}{\partial y^m} = f\left(x_0\right) \quad if \quad a=0,\ b \in]2+m,+\infty[,\ m \in \mathbb{N}_1;$$

$$\lim_{y\to x_0} \left(y\ln\frac{1}{y}\right)^{-1} \frac{\partial^m y^{m+1}u}{\partial y^m} = -f\left(x_0\right) \quad if \quad a=0,\ b=2+m,\ m \in \mathbb{N}_1;$$

$$\lim_{y\to x_0} y^{1+m-b}\frac{\partial^m y^{b-1}u}{\partial y^m} = f\left(x_0\right) \quad if \quad (a, 2-b) \in i_{2,m};$$

$$\lim_{y\to x_0} \left(\ln\frac{1}{y}\right)^{-1} \frac{\partial^m y^{m}u}{\partial y^m} = -f\left(x_0\right) \quad if \quad (a, 2-b) \in i_{3,m};$$

$$\lim_{y\to x_0} \frac{\partial^m y^{-1}u}{\partial y^m} = f\left(x_0\right) \quad if \quad a=0,\ b=2+m,\ m \in \mathbb{N},$$

where f is a piecewise continuous function bounded on $\mathbb{R}^1$, $x_0 \in \mathbb{R}^1$.

Additional conditions on f along with the conditions for uniqueness of solutions may be easily reformulated.

According to the correspondence principle (3.3), (3.4), from the solutions (3.63)-(3.67) of the problems 3.3.1-3.3.7, respectively, we immediatly get representation of the solution of Problem 3.3.12 under the corresponding BC stated in Problem 3.3.12. To this end in the expressions (3.63)-(3.67) "b" should be replaced by "$2-b$" and the expressions obtained should be multiplied by y^{1-b}.

Let us modify the problems 3.3.3 and 3.3.5 (see G. Jaiani [15], pp. 48-53):

Problem 3.3.3* Let $a=0$, $b=-m$, $m \in \mathbb{N}_1$. Find

$$u \in T^m\left(\left(y\ln\frac{1}{y}\right)^{-1}\right)$$

satisfying BC (3.51), where f is a continuous function

$$f(\xi) = O\left(|\xi|^{-\alpha}\right),\ |\xi| \to +\infty,\ \alpha > m+1, \tag{3.71}$$

and

$$u = \overset{(2)}{I} + O\left(r^{-1}\right),\ u_x = \overset{(2)}{I}_x + O\left(r^{-2}\right),\ u_y = \overset{(2)}{I}_y + O\left(r^{-2}\right),\ r \to +\infty,$$

where

$$\overset{(2)}{I} := M^{-1}(0,-m,m+2)\int\limits_{-\infty}^{+\infty} f(\xi)\rho^m d\xi,$$

and

$$\lim_{y\to 0+} y^{-m}\int\limits_{-\infty}^{+\infty} u\cdot u_y dx = 0 \quad if \quad \lim_{y\to 0+}(y\ln\frac{1}{y})^{-1}\frac{\partial^m u}{\partial y^m} = 0,\ x \in \mathbb{R}^1.$$

Problem 3.3.5* Let $(a,b)\in i_{3,m}$. Find

$$u\in T^m\left(\left(\ln\frac{1}{y}\right)^{-1}\right)$$

satisfying BC (3.60), where f is a continuous function,

$$f(\xi)=O\left(|\xi|^{-\alpha}\right),\ |\xi|\to+\infty,\ \alpha>m, \tag{3.72}$$

and

$$u=\overset{(3)}{I}+O\left(r^{-1}\right),\ u_x=\overset{(3)}{I}_x+O\left(r^{-2}\right),\ u_y=\overset{(3)}{I}_y+O\left(r^{-2}\right),\ r\to+\infty,$$

here

$$\overset{(3)}{I}:=M^{-1}(a,1-m,1+m)\int\limits_{-\infty}^{+\infty}f(\xi)e^{a\theta}\rho^{m-1}d\xi,$$

and

$$\lim_{y\to0+}y^{1-m}\int\limits_{-\infty}^{+\infty}u\cdot u_y dx=0,\quad if\quad \lim_{y\to0+}(\ln y)^{-1}\frac{\partial^m u}{\partial y^m}=0,\ x\in\mathbb{R}^1.$$

Theorem 3.3.13 *Unique solutions of the problems 3.3.3* and 3.3.5* have the forms*

$$\overset{*}{u}_3=M^{-1}(0,-m,m+2)\int\limits_{-\infty}^{+\infty}f(\xi)\rho^m d\xi, \tag{3.73}$$

$$M(0,-m,m+2)\neq0,\ \text{[see second part of Corollary 2.2.5]}$$

and

$$\overset{*}{u}_5=M^{-1}(a,1-m,1+m)\int\limits_{-\infty}^{+\infty}f(\xi)e^{a\theta}\rho^{m-1}d\xi, \tag{3.74}$$

$$M(a,1-m,1+m)\neq0,\ \text{[see second part of Corollary 2.2.5]},$$

respectively.

Remark 3.3.14 *Pairs of solutions (3.65) and (3.73), (3.67) and (3.74) of equation (1.1) satisfy the same BCs (3.51) and (3.60). But if $m>0$, at infinity and by $y\to0+$ they (but not their m-th order derivatives with respect to y) behave differently. The analogues remark should also be made with respect to solutions (3.68) and (3.69) by $a=0,\ \ b=0$.*

Remark 3.3.15 *If in the solution (3.73) (or (3.74)) the function f is piece-wise smooth, then (3.73) ((3.74)) satisfies equation (1.1) in $\mathbb{R}^2_+$ and BC (3.51) ((3.60)) at points of continuity of f even if a is a complex number.*

Remark 3.3.16 *The solutions (3.66), (3.68), (3.73), (3.74) are unbounded, in general, at infinity. E.g., if $f(\xi) \geq 0$ is a finite function and $b < 0$, then because of (3.176) (see below),*

$$u_4 = M^{-1}(a,b,m) \int\limits_{-\gamma}^{+\gamma} f(\xi) e^{a\theta} \rho^{-b} d\xi$$
$$\geq r^{-b} M^{-1}(a,b,m)(1-\overset{*}{\varepsilon})^{-\frac{b}{2}} \int\limits_{-\gamma}^{+\gamma} f(\xi) e^{a\theta} d\xi \underset{r\to+\infty}{\to} +\infty.$$

If $m = 1$, then (3.68) is bounded, provided

$$\int\limits_{-\infty}^{+\infty} f(\xi) d\xi = 0.$$

Remark 3.3.17 *Denote by $\overset{(k)}{u}$ a solution of Problem 3.3.4, when the k-th order derivative is prescribed in BC.*

$$\overset{(k)}{u}(x,y) = (-1)^j (b+k-1, j) \overset{(k+j)}{u}(x,y), \quad k, j \in \mathbb{N}^0,$$

if $Re b > 1-k$, since

$$\lim_{z\to x_0} y^{b+k-1} \frac{\partial^k u}{\partial y^k} = \frac{(-1)^j}{(b+k-1,j)} \lim_{z\to x_0} y^{b+k+j-1} \frac{\partial^{k+j} u}{\partial y^{k+j}}, \ z \in \mathbb{R}^2_+, \ x_0 \in \mathbb{R}^1.$$

Let $b = 1$ and $\overset{*}{u}$ be a solution in the case of BC (3.33), then

$$\frac{(-1)^m}{(m-2)!} \overset{*}{u}(x,y) = \overset{m}{u}(x,y), \ m \in \mathbb{N},$$

since

$$\lim_{z\to x_0} \left(\frac{1}{y}\right)^{-1} u = \frac{(-1)^m}{(m-2)!} \lim_{z\to x_0} y^m \frac{\partial^m u}{\partial y^m}, \ z \in \mathbb{R}^2_+, \ x_0 \in \mathbb{R}^1.$$

Solution of Problem 3.3.1 Let u be a solution of Problem 3.3.1, then since $b < 1-m \leq 1$ and $u \in C(\mathbb{R}^2_+ \cup \mathbb{R}^1)$, u will also be a solution of Problem 3.2.1 which takes values $u(x,0)$ at the boundary. According to Theorem 3.2.3 it can be represented as

$$u(x,y) = \frac{y^{1-b}}{\Lambda(a,b)} \int\limits_{-\infty}^{+\infty} u(\xi,0) e^{a\theta} \rho^{b-2} d\xi \tag{3.75}$$

Because of $u(x,0) \in \underset{*}{C^m}(\mathbb{R}^1)$ after substitution $\xi = x + yt$ in (3.75) we may differentiate the obtained integral m-times with respect to y under the integral sign and

$$\lim_{z\to x_0} \frac{\partial^m u(x,y)}{\partial y^m} = \frac{\Lambda_m(a,b)}{\Lambda(a,b)} \frac{\partial^m u(x_0,0)}{\partial x^m}, \ x_0 \in \mathbb{R}^1.$$

Whence, by virtue of (3.47), we have

$$\frac{\Lambda_m(a,b)}{\Lambda(a,b)}\frac{\partial^m u(x_0,0)}{\partial x^m} = f\left(x_0\right), \ x_0 \in \mathbb{R}^1,$$

i.e.,

$$\frac{\partial^m u(x,0)}{\partial x^m} = \frac{\Lambda(a,b)}{\Lambda_m(a,b)} f(x), \ x \in \mathbb{R}^1.$$

Finally,

$$u(x,0) = \frac{\Lambda(a,b)}{\Lambda_m(a,b)} \underset{*}{f}^{(-m)}(x).$$

Substituting the last into (3.75) we arrive at (3.63).

Now, it is directly easily seen that (3.63) belongs to the class $T_m^m(1)$ and satisfies BC (3.47).

Solution of Problem 3.3.2 According to the conditions of Problem 3.3.2,

$$b < -m = 1 - (m+1).$$

Under this condition Problem 3.3.1 is solvable for $m+1$ instead of m, i.e., there exists $u \in T_{m+1}^{m+1}(1)$ satisfying BC

$$\lim_{z\to x_0} \frac{\partial^{m+1} u}{\partial y^{m+1}} = f\left(x_0\right), \ z \in \mathbb{R}^2_+, \ x_0 \in \mathbb{R}^1.$$

Therefore,

$$\lim_{z\to x_0} y^{-1}\frac{\partial^m u}{\partial y^m} = \lim_{z\to x_0} \frac{\partial^{m+1} u}{\partial y^{m+1}}, \ z \in \mathbb{R}^2_+, \ x_0 \in \mathbb{R}^1, \tag{3.76}$$

provided

$$\lim_{z\to x_0} \frac{\partial^m u}{\partial y^m} = 0, \ z \in \mathbb{R}^2_+, \ x_0 \in \mathbb{R}^1, \tag{3.77}$$

But u as the solution of Problem 3.3.1 admits the representation (3.64) which it follows from (3.63), replacing there m by $m+1$. Because of oddness of m (3.64) meets the condition (3.77). Indeed[6],

$$\lim_{z\to x_0} \frac{\partial^m u_2}{\partial y^m} = \lim_{z\to x_0} \Lambda^{-1}_{m+1}(0,b) \int\limits_{-\infty}^{+\infty} \underset{*}{f}^{(-1)}(x+yt) t^m (1+t^2)^{\frac{b}{2}-1} dt$$

---

[6]Since, after substituting $\xi = x + yt$,

$$\lim_{z\to x_0} \frac{\partial^m}{\partial y^m}\left[\Lambda^{-1}_{m+1}(0,b) y^{1-b} \int\limits_{-\infty}^{+\infty} \underset{*}{f}^{(-m-1)}(\xi)\rho^{b-2} d\xi\right]$$

$$= \lim_{z\to x_0} \frac{\partial^m}{\partial y^m} \Lambda^{-1}_{m+1}(0,b) \int\limits_{-\infty}^{+\infty} \underset{*}{f}^{(-m-1)}(x+yt)(1+t^2)^{\frac{b}{2}-1} dt$$

$$= \Lambda_{m+1}^{-1}(0,b)\underset{*}{f}^{(-1)}(x)\int\limits_{-\infty}^{+\infty} t^m(1+t^2)^{\frac{b}{2}-1}dt = 0.$$

So, by virtue of (3.76), $u_2 \in T_{m+1}^m(1)$ satisfies BC (3.49).
Solution of Problem 3.3.4 Let first $b > 1$ and u be a solution of Problem 3.3.4. Then, in view of Theorem 2.1.1, taking into account (3.52), we have

$$\lim_{z\to x_0} y^{b-1}u = \lim_{z\to x_0} \frac{(-1)^m}{(b-1,m)} y^{b+m-1}\frac{\partial^m u}{\partial y^m} = \frac{(-1)^m}{(b-1,m)} f\left(x_0\right). \tag{3.78}$$

Hence, u as a solution of Problem 3.2.1 in the case of BC (3.34), where $f(x_0)$ should be replaced by $\frac{(-1)^m}{(b-1,m)} f\left(x_0\right)$, which by virtue of (2.47) for $k=0$ and (3.37), where $f(\xi)$ should be replaced by $\frac{(-1)^m}{(b-1,m)} f\left(\xi\right)$, admits the representation

$$u = \frac{(-1)^m}{(b-1,m)\Lambda(a,2-b)}\int\limits_{-\infty}^{+\infty} f(\xi)e^{a\theta}\rho^{-b}d\xi = M^{-1}(a,b,m)\int\limits_{-\infty}^{+\infty} f(\xi)e^{a\theta}\rho^{-b}d\xi,$$

i.e., the representation (3.66). Now, if we take into account the integral representation of $M(a,b,m)$ and (2.36), (2.39), we directly verify, that under the conditions of Problem 3.3.4 with respect to $f,\ a$, and b, (3.66) belongs to $T^m(y^{m+b-1})$ and satisfies BC (3.52).
Solution of Problem 3.3.5* Since according to the conditions of Problem 3.3.5* $b = 1-m$, evidently,

$$b > -m = 1-(m+1)$$

and Problem 3.3.4, where $b = 1-m$ is solvable for $m+1$, instead of m i.e., exists $u \in T^{m+1}(y)$ such that

$$\lim_{z\to x_0} y\frac{\partial^{m+1}u}{\partial y^{m+1}} = f\left(x_0\right),\ z\in\mathbb{R}_+^2,\ x_0\in\mathbb{R}^1.$$

By virtue of (3.66), where "m" and "b" should be replaced by "$m+1$" and "$1-m$", respectively, it as a solution of Problem 3.3.4, allows the representation

$$u(x,y) = M^{-1}(a,1-m,1+m)\int\limits_{-\infty}^{+\infty} f(\xi)e^{a\theta}\rho^{-b}d\xi,$$

where f should fulfill the condition (3.72).

On the other hand

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{\partial^m u}{\partial y^m} = \lim_{z\to x_0}\Big[-\ln y\Big]^{-1}\frac{\partial^m u}{\partial y^m} = -y\lim_{z\to x_0}\frac{\partial^{m+1}u}{\partial y^{m+1}} = -f\left(x_0\right).$$

Thus, (3.74) belongs to $T^m((ln\frac{1}{y})^{-1})$ and meets BC (3.60).
Solution of Problem 3.3.5 Let u be a solution of Problem 3.3.5. Then, since $b = 1-m < 1$ when $m\in\mathbb{N}$ and $u\in C(\mathbb{R}_+^2\cup\mathbb{R}^1)$, the function u will also be a

solution of Problem 3.2.1 with the boundary values $u(x,0)$. Therefore, according to Theorem 3.2.3, it admits the representation (3.35), where $b=1-m$. Since $u(x,0)\in \underset{*}{C}^m(\mathbb{R}^1)$ (i.e.,

$$\frac{\partial^m u(x,0)}{\partial x^m}\in \underset{*}{C}^{-1}$$

and the integral

$$\int\limits_{-\infty}^{+\infty} \left.\frac{\partial^m u(\xi,0)}{\partial \xi^m}\right|_{\xi=x+yt} t^m e^{a\cdot \operatorname{arctg}(-t)}\left(1+t^2\right)^{-\frac{m+1}{2}}dt \tag{3.79}$$

is uniformly convergent in $\overline{G}\subset\mathbb{R}^2_+$), substituting $\xi=x+yt$ in (3.75) with $b=1-m$, differentiating m-times with respect to y and making stated below transformations, bearing in mind (2.54) we obtain

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{\partial^m u}{\partial y^m}$$

$$=\Lambda^{-1}(a,1-m)\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\int\limits_{-\infty}^{+\infty}\left.\frac{\partial^m u(\xi,0)}{\partial\xi^m}\right|_{\xi=x+yt} t^m e^{a\cdot\operatorname{arctg}(-t)}\left(1+t^2\right)^{-\frac{m+1}{2}}dt$$

$$=\Lambda^{-1}(a,1-m)\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\int\limits_{-\infty}^{+\infty}\frac{\partial^m u(\xi,0)}{\partial\xi^m}(\xi-x)^m e^{a\theta}\rho^{-m-1}d\xi$$

$$=\Lambda^{-1}(a,1-m)\lim_{z\to x_0}(-y)\int\limits_{-\infty}^{+\infty}\frac{\partial^m u(\xi,0)}{\partial\xi^m}(\xi-x)^m\left[a(\xi-x)-(m+1)y\right]e^{a\theta}\rho^{-m-3}d\xi$$

$$=-\Lambda^{-1}(a,1-m)\lim_{z\to x_0}\int\limits_{-\infty}^{+\infty}\left.\frac{\partial^m u(\xi,0)}{\partial\xi^m}\right|_{\xi=x+yt}(m+1+at)t^m e^{a\cdot\operatorname{arctg}(-t)}\left(1+t^2\right)^{-\frac{m+3}{2}}dt$$

$$=\frac{(m+1)\Lambda_m(a,-m-1)+a\Lambda_{m+1}(a,-m-1)}{\Lambda(a,1-m)}\frac{\partial^m u\left(x_0,0\right)}{\partial x^m}=-f\left(x_0\right),$$

since, as we assumed u is the solution of the problem under consideration. Whence,

$$\frac{\partial^m u\left(x_0,0\right)}{\partial x^m}=\frac{\Lambda(a,1-m)}{(m+1)\Lambda_m(a,-m-1)+a\Lambda_{m+1}(a,-m-1)}f\left(x_0\right),$$

i.e.,

$$u(x,0)=\frac{\Lambda(a,1-m)}{(m+1)\Lambda_m(a,-m-1)+a\Lambda_{m+1}(a,-m-1)}\underset{*}{f}^{(-m)}(x).$$

Substituting the last expression into (3.75), we get (3.67) for $m>0$. Now, it is easily seen that (3.67) for $m>0$ belongs to $T^m_m\left((\ln\frac{1}{y})^{-1}\right)$ and fulfills BC (3.60).

The case $m=0$ we have considered in Section 3.2.

Solution of Problem 3.3.3* Since according to the conditions of Problem 3.3.3* $b=-m$, evidently, $b=-m=1-(m+1)$ and Problem 3.3.5* is solvable when the order of the derivative in BC (3.60) is $m+1$. Hence, exists $u\in T^{m+1}_{m+1}\left(\left(y\ln\frac{1}{y}\right)^{-1}\right)$ which fulfills BC

$$\lim_{z\to x_0}\big(y\ln\frac{1}{y}\big)^{-1}\frac{\partial^{m+1}u}{\partial y^{m+1}}=f(x_0),\quad z\in\mathbb{R}^2_+,\quad x_0\in\mathbb{R}^1.$$

By virtue of (3.74), it as solution of Problem 3.3.5* admits the representation

$$\overset{*}{u}_3(x,y)=M^{-1}(0,-m,m+2)\int\limits_{-\infty}^{+\infty}f(\xi)\rho^m d\xi,$$

where

$$f(\xi)=O(|\xi|^{-\alpha}),\quad |\xi|\to+\infty,\quad \alpha>m+1.$$

But on the other hand[7]

$$\lim_{z\to x_0}\big(y\ln\frac{1}{y}\big)^{-1}\frac{\partial^m u}{\partial y^m}=\lim_{z\to x_0}\big(\ln\frac{1}{y}\big)^{-1}\frac{\partial^{m+1}u}{\partial y^{m+1}},$$

provided

$$\lim_{z\to x_0}\frac{\partial^m u}{\partial y^m}=0,\quad z\in\mathbb{R}^2_+,\quad x_0\in\mathbb{R}^1.$$

The constructed solution, in view of (2.39), satisfies the last condition. Indeed,

$$\begin{aligned}
&\lim_{z\to x_0}\frac{\partial^m\overset{*}{u}_3}{\partial y^m}\\
&=M^{-1}(0,-m,m+2)\lim_{z\to x_0}y\sum_{k=1}^{\frac{m+1}{2}}\tilde{B}_k(-m,m;0)\int\limits_{-\infty}^{+\infty}f(x+yt)(1+t^2)^{-\frac{m}{2}+k-1}dt\\
&=-M^{-1}(0,-m,m+2)\tilde{B}_{\frac{m+1}{2}}(-m,m;0)\lim_{z\to x_0}y\int\limits_{-\infty}^{+\infty}f(x+yt)(1+t^2)^{-\frac{1}{2}}dt\\
&=M^{-1}(0,-m,m+2)\tilde{B}_{\frac{m+1}{2}}(-m,m;0)\lim_{z\to x_0}y\int\limits_{-\infty}^{+\infty}f(\xi)\rho^{-1}d\xi\\
&=M^{-1}(0,-m,m+2)\tilde{B}_{\frac{m+1}{2}}(-m,m;0)\lim_{z\to x_0}\frac{-y\int\limits_{-\infty}^{+\infty}f(\xi)\rho^{-3}d\xi}{-y^{-2}}\\
&=M^{-1}(0,-m,m+2)\tilde{B}_{\frac{m+1}{2}}(-m,m;0)\lim_{z\to x_0}y\int\limits_{-\infty}^{+\infty}f(x+yt)(1+t^2)^{-\frac{3}{2}}dt=0,
\end{aligned}$$

since

$$\left|y\int\limits_{-\infty}^{+\infty}f(x+yt)(1+t^2)^{-\frac{m}{2}+k-1}dt\right|\le y\max_{\xi\in\mathbb{R}^1}|f(\xi)|\int\limits_{-\infty}^{+\infty}(1+t^2)^{-\frac{m}{2}+k-1}dt\underset{y\to0+}{\longrightarrow}0,$$

[7] $\big(y\ln\frac{1}{y}\big)'=-\ln y-1\sim-\ln y$ as $y\to0+$.

when $1 < k < \frac{m+1}{2}$, and analogously for $k = 1$

$$\left| y \int\limits_{-\infty}^{+\infty} f(x+yt)(1+t^2)^{-\frac{m}{2}}\,dt \right| \le y \max_{\xi\in\mathbb{R}^1} |f(\xi)| \int\limits_{-\infty}^{+\infty} (1+t^2)^{-\frac{m}{2}}\,dt \underset{y\to 0+}{\longrightarrow} 0.$$

Thus, $\overset{*}{u}_3 \in T_m^m\Big(\big(y\ln\frac{1}{y}\big)^{-1}\Big)$ and satisfies BC (3.51). Note that [8]

$$\frac{m+1}{2} = \left[\frac{m}{2}\right] + 1 \ \text{ for } \ m \in \mathbb{N}_1.$$

Solution of Problem 3.3.3 Let u be a solution of Problem 3.3.3. Since $b = -m < 1$ and $u \in C(\mathbb{R}^2_+ \bigcup \mathbb{R}^1)$, u will also be a solution of Problem 2.2.1 which takes values $u(x,0)$ on the boundary $y = 0$. Therefore, according to Theorem 3.2.3 and formula (3.35), where $b = -m$, it admits the representation

$$u(x,y) = \frac{y^{1+m}}{\Lambda(0,-m)} \int\limits_{-\infty}^{+\infty} u(\xi,0)\rho^{-m-2}d\xi. \tag{3.80}$$

Because of $u(x,0) \in \underset{*}{C}^{m+1}(\mathbb{R}^1)$, taking into account the assertions (3.79) for $m+1$, we have

$$\begin{aligned}
&\lim_{z\to x_0} \big(y\ln\frac{1}{y}\big)^{-1}\frac{\partial^m u}{\partial y^m} = -\Lambda^{-1}(0,m)\lim_{z\to x_0}\frac{\int\limits_{-\infty}^{\infty}\frac{\partial^{m+1}u(\xi,0)}{\partial\xi^{m+1}}|_{\xi=x+yt}t^{m+1}(1+t^2)^{-\frac{m+2}{2}}dt}{\ln y + 1}\\
&= -\frac{m+2}{\Lambda(0,m)}\lim_{z\to x_0} y^2\int\limits_{-\infty}^{\infty}\frac{\partial^{m+1}u(\xi,0)}{\partial\xi^{m+1}}(\xi-x)^{m+1}\rho^{-m-4}d\xi\\
&= -(m+2)\frac{\Lambda_{m+1}(0,-m-2)}{\Lambda(0,-m)}\frac{\partial^{m+1}u(x_0,0)}{\partial x^{m+1}} = -f(x_0),
\end{aligned} \tag{3.81}$$

since we assumed that u is a solution of the problem under consideration. By calculations (3.81) we took into account

$$\lim_{z\to x_0}\frac{\partial^m a}{\partial y^m} = \lim_{z\to x_0}\int\limits_{-\infty}^{+\infty}\frac{\partial^m u(\xi,0)}{\partial\xi^m}|_{\xi=x+yt}t^m(1+t^2)^{-\frac{m+2}{2}}dt$$

$$= \frac{\partial^m u(x_0,0)}{\partial x^m}\int\limits_{-\infty}^{+\infty} t^m(1+t^2)^{-\frac{m+2}{2}}dt = 0,$$

since

$$\int\limits_{-\infty}^{+\infty} t^m(1+t^2)^{-\frac{m+2}{2}}dt = 0$$

[8]Indeed, $m = 2j+1$ for $j \in \mathbb{N}$, then $\frac{m+1}{2} = \frac{2j+2}{2} = j+1$; $\left[\frac{m}{2}\right]+1 = \left[\frac{2j+1}{2}\right]+1 = \left[j+\frac{1}{2}\right]+1 = j+1$.

because of oddness of m. Thus,

$$\frac{\partial^{m+1}u(x,0)}{\partial x^{m+1}} = \frac{\Lambda(0,-m)}{(m+2)\Lambda_{m+1}(0,-m-2)}f(x),$$

i.e.,

$$u(x,0) = \frac{\Lambda(0,-m)}{(m+2)\Lambda_{m+1}(0,-m-2)}\underset{*}{f}^{-m-1}(x).$$

Substituting the last expression into (3.80) we get (3.65). Now, it is easily seen that (3.65) belongs to $T^m_{m+1}\Big(\big(y\ln\frac{1}{y}\big)^{-1}\Big)$ and satisfies BC (3.51).

Solution of Problem 3.3.6 Let

$$v := \frac{\partial^m u}{\partial y^m}.$$

then

$$\Delta v = 0, \quad z \in \mathbb{R}^2_+,$$

and

$$v(x,0) = f(x), \quad x \in \mathbb{R}^1.$$

According to (3.35) for $a = b = 0$ we obtain

$$v(x,y) = \frac{y}{\pi}\int\limits_{-\infty}^{+\infty} f(\xi)\rho^{-2}d\xi, \quad z \in \mathbb{R}^2_+,$$

i.e.,

$$\frac{\partial^m u(x,y)}{\partial y^m} = \frac{y}{\pi}\int\limits_{-\infty}^{+\infty} f(\xi)\frac{d\xi}{(x-\xi)^2+y^2}.$$

Whence,

$$u = \frac{1}{\pi}\int\limits_{-\infty}^{+\infty} f(\xi)\Big[\int\limits_{y_0}^{y}\frac{(y-\tau)^{m-1}}{(m-1)!}\frac{\tau d\tau}{(x-\xi)^2+\tau^2}\Big]d\xi + \sum_{k=0}^{m-1}\tilde{Q}_k(x)y^k. \tag{3.82}$$

Further, using the formulas, which we prove by mathematical induction

$$\int\frac{\tau^{2k+1}d\tau}{a^2+\tau^2} = \sum_{j=1}^{k}(-1)^{k+j}\frac{a^{2(k-j)}\tau^{2j}}{2j} + (-1)^k\frac{a^{2k}}{2}\ln(a^2+\tau^2),^{9}\ k \in \mathbb{N}^0,\ \sum_{j=1}^{0}(\cdots) \equiv 0,$$

[9]In particular cases, for $k = 0,1,2,3$ this formula is correct (see Dwight [4] Formulas 121.1, 123.1, 125.1, 127.1). Assuming correctness for k, we prove correctness for $k+1$, indeed,

$$\int\frac{\tau^{2k+3}d\tau}{a^2+\tau^2} = \int\frac{\tau^{2k+1}(\tau^2+a^2)d\tau}{a^2+\tau^2} - \int\frac{\tau^{2k+1}a^2d\tau}{a^2+\tau^2}$$

$$= \frac{\tau^{2k+2}}{2k+2} - a^2\Big[\sum_{j=1}^{k}(-1)^{k+j}\frac{a^{2(k-j)}\tau^{2j}}{2j} + (-1)^k\frac{a^{2k}}{2}\ln(a^2+\tau^2)\Big]$$

$$= \sum_{i=1}^{k+1}(-1)^{k+1+j}\frac{a^{2(k+1-j)}\tau^{2j}}{2j} + (-1)^{k+1}\frac{a^{2(k+1)}}{2}\ln(a^2+\tau^2).$$

$$\int \frac{\tau^{2k} d\tau}{a^2+\tau^2} = \sum_{j=1}^{k} (-1)^{k+j+1} \frac{a^{2(k-j-1)}\tau^{2j+1}}{2j+1} + (-1)^{k+1} a^{2k-2}\tau$$
$$+(-1)^k a^{2k-1} \arctan \frac{\tau}{a},^{10} \;\; k \in \mathbb{N},$$

and choosing the functions $\tilde{Q}_k(x)$ in such a way that (3.82) be a harmonic function in $\mathbb{R}^2_+$, after some transformations and simplifications we get (3.68).

Solution of Problem 3.3.7 First of all let us note that the representation (3.70) is true for

$$a = 0,\; b = -2n \geq 1 - m, n \in \mathbb{N}^0,\; m \in \mathbb{N}\backslash\{1\}.$$

From (3.62) for $k = m$ it follows that

$$\underset{0}{f}^{(0)}(\xi) \equiv f(\xi) = O\left(|\xi|^{-\alpha_m}\right),\; |\xi| \to +\infty,\; \alpha_m > m \geq 1,$$

i.e., function $f(\xi)$ is absolutely integrable on $\mathbb{R}^1$. Assume that for all $y > 0$ functions

$$\frac{\partial^j u(x,y)}{\partial y^j},\; j = \overline{0,m},$$

are absolutely integrable with respect to x on the interval $]-\infty,+\infty[$, uniformly with respect to y, moreover,

$$u, u_x \to 0 \;\; as \;\; |x| \to +\infty,$$

and $u \in C^2\left(R^2_+\right)$; in order to apply the Fourier transformation after multiplying by $\dfrac{e^{ixt}}{\sqrt{2\pi}}$ and then integrating with respect to x, from

$$y\left(u_{xx}+u_{yy}\right) - 2nu_y = 0 \tag{3.83}$$

and

$$\left.\frac{\partial^m u(x,y)}{\partial y^m}\right|_{y=0} = f(x), \tag{3.84}$$

we get

$$y\frac{\partial^2 U(t,y)}{\partial y^2} - 2n\frac{\partial U(t,y)}{\partial y} - t^2 y U(t,y) = 0 \tag{3.85}$$

[10] In particular cases, for $k = 1,2,3,4$ this formula is correct (see Dwaight [4] Formulas 122.1, 124.1, 126.1, 128.1). Assuming correctness for k, we prove correctness for $k+1$, indeed,

$$\int \frac{\tau^{2k+2}d\tau}{a^2+\tau^2} = \int \frac{\tau^{2k}(\tau^2+a^2)d\tau}{a^2+\tau^2} - \int \frac{\tau^{2k}a^2 d\tau}{a^2+\tau^2}$$

$$= \frac{\tau^{2k+1}}{2k+1} - a^2\Big[\sum_{j=1}^{k-1}(-1)^{k+1+j}\frac{a^{2(k-1-j)}\tau^{2j+1}}{2j+1} + (-1)^{k+1}a^{2k-2}\tau + (-1)^k a^{2k-1} arctg\frac{x}{a}\Big]$$

$$= \sum_{j=1}^{k}(-1)^{k+2+j}\frac{a^{2(k-j)}\tau^{2j+1}}{2j+1} + (-1)^{k+2}a^{2k}\tau + (-1)^{k+1}a^{2k+1}\arctan\frac{\tau}{a},\;\; \sum_{1}^{0}(\cdots) \equiv 0.$$

and

$$\left.\frac{\partial^m U(t,y)}{\partial y^m}\right|_{y=0} = F(x), \tag{3.86}$$

respectively, where

$$U(t,y) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} u(x,y)e^{ixt}dx, \tag{3.87}$$

$$F(t) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} f(x)e^{ixt}dx. \tag{3.88}$$

It is well-known that the solution of (3.85) has the form

$$U(t,y) = y^{2n+1}\left(y^{-1}\frac{\partial}{\partial y}\right)^n \left(\frac{V(t,y)}{y}\right), \tag{3.89}$$

where $V(t,y)$ is a solution of

$$\frac{\partial^2 V(t,y)}{\partial y^2} - t^2 V(t,y) = 0.$$

The general solution of the last equation looks like

$$V(t,y) = \begin{cases} C_1(t)e^{y|t|} + C_2(t)e^{-y|t|}, \ t \neq 0; \\ C_1 y + C_2, \ t = 0. \end{cases} \tag{3.90}$$

Let

$$C_1(t) \equiv 0, \ C_1 = 0, \ C_2 = C_2(0). \tag{3.91}$$

It is easily seen that

$$\left(y^{-1}\frac{\partial}{\partial y}\right)^n \left(\frac{e^{-y|t|}}{y}\right) = y^{-(2n+1)}e^{-y|t|}\sum_{k=0}^{n} \overset{n}{a}_k \left(-y|t|\right)^k. \tag{3.92}$$

Taking into account (3.90)-(3.92), from (3.89) we get

$$U(t,y) = C_2(t)e^{-y|t|}\sum_{k=0}^{n} \overset{n}{a}_k \left(-|t|\right)^k y^k \tag{3.93}$$

which is a solution of (3.85). Now, we choose $C_2(t)$ in order to satisfy BC (3.86) as follows:

$$\lim_{y\to 0+}\frac{\partial^m U(t,y)}{\partial y^m}$$
$$= C_2(t)\sum_{k=0}^{n} \overset{n}{a}_k \left(-|t|\right)^k \sum_{l-0}^{k} \left(\begin{array}{c} m \\ l \end{array}\right)(-1)^l(-k,l)\left(-|t|\right)^{m-l} \lim_{y\to 0+} e^{-y|t|}y^{k-l}$$
$$= C_2(t)\sum_{k=0}^{n} \overset{n}{a}_k \left(\begin{array}{c} m \\ k \end{array}\right) k! \left(-|t|\right)^m = F(t).$$

Therefore, if $t \neq 0$, then

$$C_2(t) = \frac{F(t)\,(-|t|)^{-m}}{\prod\limits_{j=1}^{n}[m-(2j-1)]}, \tag{3.94}$$

since easily can be verified, that

$$\sum_{k=0}^{n} \overset{n}{a}_k \begin{pmatrix} m \\ k \end{pmatrix} k! = \sum_{k=0}^{n} \overset{n}{a}_k \frac{m!}{(m-k)!} = \prod_{j=1}^{n}[m-(2j-1)] \neq 0.$$

From the condition (3.62) it follows that

$$\int\limits_{+\infty}^{-\infty} x^k f(x)dx = 0,\ k = \overline{0, m-1}.$$

Hence, after differentiation of (3.88) k-times, $k = \overline{0, m-1}$, with respect to t under integral sign and substituting $t = 0$, we obtain

$$F^{(k)}(0) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} x^k f(x)dx = 0,\ k = \overline{0, m-1}.$$

Now, assuming $x^m f(x)$ absolutely integrable on $\mathbb{R}^1$, we arrive at

$$C_2(0) = \frac{F^{(m)}(0)}{(-1)^m m! \prod\limits_{j=1}^{n}[m-(2j-1)]}, \tag{3.95}$$

since after applying the L'hopital rool m-times, we have

$$\lim_{t\to 0} \frac{F(t)}{(-|t|)^m} = \frac{F^{(m)}(0)}{(-1)^m m!}.$$

Substituting (3.94) (where for $t = 0$ as the value of $C_2(t)$ we take the limit of (3.93) as $t \to 0$, i.e., (3.95)) into (3.93), we get

$$U(t, y) = \frac{F(t)\,(-|t|)^{-m}}{\prod\limits_{j=1}^{n}[m-(2j-1)]} e^{-y|t|} \sum_{k=0}^{n} \overset{n}{a}_k\,(-y|t|)^k.$$

Whence, assuming that $u(x, y)$ and $f(x)$ meet the Dirichlet conditions with respect to x, by virtue of (3.87) and (3.88), we obtain

$$\begin{aligned} u(x,y) & \\ &= \frac{1}{2\pi \prod\limits_{j=1}^{n}[m-(2j-1)]} \int\limits_{-\infty}^{+\infty} \left[\int\limits_{-\infty}^{+\infty} f(\xi) e^{i\xi t} d\xi \right] (-|t|)^m\, e^{-y|t|} \\ &\times \sum_{k=0}^{n} \overset{n}{a}_k\,(-y|t|)^k\, e^{-ixt} dt. \end{aligned} \tag{3.96}$$

Since, because of (3.62),

$$\int_{-\infty}^{+\infty} f(\xi)e^{i\xi t}d\xi = (-it)^m \int_{-\infty}^{+\infty} \overset{-m}{\underset{0}{f}}(\xi)e^{i\xi t}d\xi,$$

(3.96) we can rewrite in the form (3.70).

Now, we can directly verify that (3.70) satisfies equation (3.83) and BC (3.47), provided (3.62) is fulfilled.

Since for $y>0$ we can differentiate (3.70) under integral sign with respect to x and y as much as desired, it is easily seen that (3.70) satisfies (3.83). To this and we need to use the following equalities

$$2(n-k)\overset{n}{a}_k + (k+1)(2n-k)\overset{n}{a}_{k+1} = 0,\ n\in\mathbb{N},\ k=\overline{0,n-1},$$

which are easy to prove.

In order to prove that (3.70) satisfies equation (3.83) we need to make substitution $\xi = x+yt$ in the integral, then for $y>0$ differentiate with respect to y under integral sign which is allowed, by virtue of (3.62). Then come back to the variable ξ and by the calculation of the limit use m-times the L'hopital rule. Finally, we get $f(x)$ as the limit. In this connection we apply the following equalities:

$$\begin{aligned}
&\sum_{k=0}^{n} \overset{n}{a}_k(-1)^k\Gamma(k+1)\sum_{\delta=0}^{k+1}\begin{pmatrix} k+1\\ \delta\end{pmatrix}\left\{\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\right\} i^{k-\delta+1}\\
&\times \sum_{l=0}^{\min\{m,k+\delta\}}\begin{pmatrix} m\\ l\end{pmatrix}(-1)^l(-k-\delta,l)\\
&\times \sum_{\kappa=1}^{\left[\frac{m-l}{2}\right]+1} B_\kappa\left(2(k+1),m-l,0,1\right)\Lambda_{m+k-\delta+1}\left(0,-2(k+m-l-\kappa+1)\right)\\
&=\left\{\begin{matrix}(-1)^{\frac{m}{2}}\\ (-1)^{\frac{m+1}{2}}\end{matrix}\right\} m!\pi\prod_{j=1}^{n}\left[m-(2j-1)\right],\ m\geq 2n-1,\ n\in\mathbb{N}^0,\ m\in\left\{\begin{matrix}\mathbb{N}_2^0\\ \mathbb{N}_1\end{matrix}\right\},
\end{aligned}$$

$$\begin{aligned}
&\lim_{y\to 0+}\int_{-\infty}^{+\infty} f(\xi)(\xi-x)^m\sum_{k=0}^{n}\overset{n}{a}_k(-1)^k\Gamma(k+1)\\
&\times\sum_{\delta=0}^{k+1}\left\{\begin{matrix}\mathrm{Re}\\ \mathrm{Im}\end{matrix}\right\} i^{k-\delta+1}(\xi-x)^{k-\delta+1}\sum_{l=0}^{\min\{\gamma,k+\delta\}}\begin{pmatrix}\gamma\\ l\end{pmatrix}(-1)^l(-k-\delta,l)y^{k+\delta-l}\\
&\times\sum_{k=1}^{\left[\frac{\gamma-l}{2}\right]+1} B_\kappa\left(2(k+1),\gamma-l,0,y\right)\rho^{-2(k+\gamma-l-\kappa+2)}d\xi = 0,\ m\in\left\{\begin{matrix}\mathbb{N}_2^0\\ \mathbb{N}_1\end{matrix}\right\}.
\end{aligned}$$

The constructed solution is unique under the conditions

$$u = O\left(r^{-1}\right),\ u_x,u_y = O\left(r^{-2}\right), r\to+\infty,$$

and

$$\lim_{y\to 0+} y^b\int_{-\infty}^{+\infty} uu_y dx = 0,$$

provided

$$\lim_{y\to 0+}\frac{\partial^m u}{\partial y^m}=0,\ x\in\mathbb{R}^1.$$

It can be proved by applying Green's formula.

In order to finish the proof of the Theorems 3.3.8 and 3.3.13 let us investigate the question of uniqueness of solutions.

According to Maximum Principle 3.1.1, Problem 3.3.1 is uniquely solvable in the case $m=0$, provided the conditions (3.48) are fulfilled. Therefore, as it follows from the method of construction of the solution of Problem 3.3.1 in the case $m\in\mathbb{N}$, under the conditions (3.48) it is uniquely determined by means of $\underset{*}{f}^{(-m)}$. But $\underset{*}{f}^{(-m)}$ itself is determined up to an additive constant, i.e., u_1 is determined up to an additive constant

$$\frac{y^{1-b}}{\Lambda_m(a,b)}\int\limits_{-\infty}^{+\infty}Ce^{a\theta}\rho^{b-2}d\xi=\frac{\Lambda(a,b)}{\Lambda_m(a,b)}C=\text{const}\,.$$

If we are looking for solutions in $\underset{0}{T}{}_m^m$ (1), then it is uniquely determined by $f_0^{(-m)}$. But $f_0^{(-m)}$ itself is uniquely determined. Whence, the constructed solution is unique.

In the same manner we study the question of uniqueness for Problem 3.3.5, with $m\in\mathbb{N}$, Problem 3.3.2, and Problem 3.3.3.

In the case of Problem 3.3.5 by $m=0$, we use the formula (3.2) by $b=1$ for the difference of two possible solutions. Taking into account (3.55) and tending R to $+\infty$, we obtain

$$\iint\limits_{y>\delta}y(u_x^2+u_y^2)dxdy=-\delta\int\limits_{-\infty}^{+\infty}uu_ydx, \tag{3.97}$$

$$\left|yu\frac{\partial u}{\partial v}d\varsigma\right|\Big|_{r=R}\le CRR^{-1}R^{-2}Rd\overset{0}{\theta}=CR^{-1}d\overset{0}{\theta},$$

$$\left|u^2\right|\Big|_{r=R}\le CR^{-2}Rd\overset{0}{\theta}=CR^{-1}d\overset{0}{\theta},$$

where $\overset{0}{\theta}\in[0,\pi]$, $C=\text{const}\,$. Now, tending δ to 0, by virtue of (3.56) and (3.97), we get

$$\iint\limits_{\mathbb{R}^2_+}y(u_x^2+u_y^2)dxdy=0. \tag{3.98}$$

Hence $u=\text{const}\ =0$, since u is vanishing at infinity. In order to show uniquely solvability of Problem 3.3.4 we consider separately the cases $b\in]1,+\infty[$, $b=1$, $b\in]0,1[$, $b\in]-\infty,0]$.

In the first case ($b>1$) if $m=0$, according to the correspondence principle (3.7), the uniqueness conditions (3.53) and (3.54) follow from the uniqueness conditions of Problem 3.3.1 by $m=0$. If $m \in N$, then since, by virtue of (3.78), each solution of Problem 3.3.4, when $m \in N$, at the same time is a solution of Problem 3.3.1, by $m=0$, it is unique under conditions (3.53) and (3.54) as well.

If $b=1$, in view of

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1} u = -\lim_{z\to x_0}(-1)^{m-1}(1,m-1)^{-1}y^m\frac{\partial^m u}{\partial y^m}, \quad z\in\mathbb{R}^2_+, \quad x_0\in\mathbb{R}^1,$$

a solution of problem under consideration will be a solution of Problem 3.3.5 by $m=0$. Therefore, it will be unique under conditions (3.55), (3.56).

Let $b \in]0,1[$. For the difference v of two possible solutions, vanishing at infinity, on a semicircle C_R of a sufficiently big radius $R(\varepsilon)$ we have

$$|v(x,y)| < \varepsilon, \tag{3.99}$$

where $\varepsilon > 0$ is small as much as desired.

According to Generalized Weighted Zaremba-Giraud Principle, since

$$\lim_{z\to x_0} y^{b+m-1}\frac{\partial^m v}{\partial y^m} = 0, \quad z\in\mathbb{R}^2_+, \quad x_0 \in]-R,+R[,$$

the function v cannot attain maximal positive and minimal negative values on the interval $]-R,+R[\in\mathbb{R}^1$. On the other hand, according to the strong extremum principle, the function v cannot attain extremal values in a half-disk K_R. So, (3.99) holds on $\bar{K}_R$. But v is independent of ε, hence, $v(x,y)\equiv 0$ when $z\in\mathbb{R}^2_+$.

Let $b \in]\infty, 0[$. Like of the case of Problem 3.3.5, when $m=0$, since for the difference v of two possible we have the following estimates (see (3.58), (3.59))

$$\left|y^b v\frac{\partial v}{\partial v}d\varsigma\right| \le C_1\delta^b R^{-1}R^{-2}Rd\overset{0}{\theta} < C_2^\delta R^{-2}d\overset{0}{\theta},$$

$$\left|y^{b-1}v^2 d\varsigma\right| \le C_1\delta^{b-1}R^{-2}Rd\overset{0}{\theta} < C_2^\delta R^{-1}d\overset{0}{\theta}, \quad C_1, C_2^\delta = \text{const}\,,$$

first we receive (3.97) and then (3.98), where y and δ should be to the power b. Therefore, we conclude $v\equiv 0$ when $z\in R^2_+$.

The solution of Problem 3.3.6 under assumption of boundedness of $\frac{\partial^m u}{\partial y^m}$ is determined up to the additive

$$\sum_{k=0}^{m-1} Q_k(x)y_k \tag{3.100}$$

with (3.69) containing $2m$ arbitrary constants. Considering the difference of two possible solutions it will have the form (3.100) satisfying the conditions

$$Q_k(x)\equiv 0, \quad k=\overline{1,m-1},$$

and $Q_0(x)$ will be bounded function. But it is possible only if

$$\overset{l}{C}_\alpha = 0, \quad \alpha=1,2, \quad l=\overline{1,m-1}, \quad \overset{0}{C}_1 = 0.$$

Hence, it remains only an arbitrary constant $\overset{0}{C}_2$ and the solution will be determined up to this additive constant.

3.4 Behavior of the solution (and of its derivatives) of the general boundary value problem on the boundary

The behavior of the solutions and their derivatives of the arbitrary order of the problems 3.3.1-3.3.6 is characterized by the following limits (see [20]):

$$\lim_{z\to x_0}\frac{\partial^p u_1}{\partial y^j\partial x^{p-j}}=\frac{\Lambda_j(a,b)}{\Lambda_m(a,b)}\underset{*}{f}^{(p-m)}(x_0),\quad f\in \underset{*}{C}^{p-m},\quad z\in\mathbb{R}^2_+,\quad x_0\in\mathbb{R}^1,\tag{3.101}$$

when $(a,b)\in i_{1,m}$, $b\in]-\infty,1-j[$, $j,p\in\mathbb{N}^0$. Moreover, if $a=0$ and $j\in\mathbb{N}_1$, then

$$\lim_{z\to x_0} y^{-1}\frac{\partial^p u_1}{\partial y^j\partial x^{p-j}}=\frac{\Lambda_{j+1}(0,b)}{\Lambda_m(0,b)}\underset{*}{f}^{(p-m+1)}(x_0),\quad f\in \underset{*}{C}^{p-m+1},\tag{3.102}$$

when $b<-j$, and

$$\lim_{z\to x_0}\left(y\ln\frac{1}{y}\right)^{-1}\frac{\partial^p u_1}{\partial y^j\partial x^{p-j}}=\frac{(j+2)\Lambda_{j+1}(0,-2-j)}{\Lambda_m(0,-j)}\underset{*}{f}^{(p-m+1)}(x_0),\quad f\in \underset{*}{C}^{p-m+1},\tag{3.103}$$

when $b=-j$.

$$\lim_{z\to x_0} y^j\frac{\partial^p u_1}{\partial y^j\partial x^{p-j}}=\begin{cases}0 \ \text{ if } \ j>0,\\ \frac{\Lambda(a,b)}{\Lambda_m(a,b)}\underset{*}{f}^{(p-m)}(x_0) \ \text{ if } \ j=0,\end{cases}\tag{3.104}$$

when $(a,b)\in i_{1,m}$, $b\in[1-j,+\infty]$, $j,p\in\mathbb{N}^0$, $f\in\underset{*}{C}^{p-m}$.

$$\lim_{z\to x_0}\frac{\partial^p u_2}{\partial y^j\partial x^{p-j}}=\frac{\Lambda_j(0,b)}{\Lambda_{m+1}(0,b)}\underset{*}{f}^{(p-m-1)}(x_0),\quad f\in \underset{*}{C}^{p-m-1},\tag{3.105}$$

when $b\in]-\infty,-m[\cap]-\infty,1-j[$. $j,p\in\mathbb{N}^0$, $a=0$, $m\in\mathbb{N}_1$,$j\in\mathbb{N}_1$. Moreover, if $j\in\mathbb{N}_1$, then

$$\lim_{z\to x_0} y^{-1}\frac{\partial^p u_2}{\partial y^j\partial x^{p-j}}=\frac{\Lambda_{j+1}(0,b)}{\Lambda_{m+1}(0,b)}\underset{*}{f}^{(p-m)}(x_0),\quad f\in \underset{*}{C}^{p-m},\tag{3.106}$$

when $b<-j$,

$$\lim_{z\to x_0}\left(y\ln\frac{1}{y}\right)^{-1}\frac{\partial^p u_2}{\partial y^j\partial x^{p-j}}=\frac{(j+2)\Lambda_{j+1}(0,-2-j)}{\Lambda_{m+1}(0,-j)}\underset{*}{f}^{(p-m)}(x_0),\quad f\in \underset{*}{C}^{p-m},\tag{3.107}$$

when $b=-j$.

$$\lim_{z\to x_0} y^j\frac{\partial^p u_2}{\partial y^j\partial x^{p-j}}=\begin{cases}0 \ \text{ if } \ j>0,\\ \frac{\Lambda(0,b)}{\Lambda_{m+1}(0,b)}\underset{*}{f}^{(p-m-1)}(x_0) \ \text{ if } \ j=0,\end{cases}\tag{3.108}$$

when $b\in[1-j,-m]$, $j,p\in\mathbb{N}^0$, $a=0$, $m\in\mathbb{N}_1$, $f\in\underset{*}{C}^{p-m-1}$.

$$\lim_{z\to x_0} \frac{\partial^p u_3}{\partial y^j \partial x^{p-j}} = \frac{\Lambda_j(0,-m)}{\Lambda_{m+1}(0,-2-m)} \underset{*}{f}^{(p-m-1)}(x_0), \quad f \in \underset{*}{C}^{p-m-1}, \tag{3.109}$$

when $j < 1+m$, $j,p \in \mathbb{N}^0$, $a=0$, $b=-m$, $m \in \mathbb{N}_1$. Moreover, if $j \in \mathbb{N}_1$, then

$$\lim_{z\to x_0} y^{-1} \frac{\partial^p u_3}{\partial y^j \partial x^{p-j}} = \frac{\Lambda_{j+1}(0,-m)}{(m+2)\Lambda_{m+1}(0,-2-m)} \underset{*}{f}^{(p-m)}(x_0), \quad f \in \underset{*}{C}^{p-m}, \tag{3.110}$$

when $j < m$, and

$$\lim_{z\to x_0} \left(y \ln \frac{1}{y}\right)^{-1} \frac{\partial^p u_3}{\partial y^j \partial x^{p-j}} = -\underset{*}{f}^{(p-j)}(x_0), \quad f \in \underset{*}{C}^{p-j}, \tag{3.111}$$

when $j = m$.

$$\lim_{z\to x_0} y^j \frac{\partial^p u_3}{\partial y^j \partial x^{p-j}} = 0, \quad f \in \underset{*}{C}^{p-m-1}, \tag{3.112}$$

when $y \geq 1+m$, $j,p \in \mathbb{N}\backslash\{1\}$, $a=0$, $b=-m$, $m \in \mathbb{N}_1$.

$$\lim_{z\to x_0} y^{b+j-1} \frac{\partial^p u_4}{\partial y^j \partial x^{p-j}} = \frac{M(a,b,j)}{M(a,b,m)} \underset{*}{f}^{(p-j)}(x_0), \quad f \in \underset{*}{C}^{p-j}, \tag{3.113}$$

when $(a,b) \in i_{2,m} \cap i_{2,j}$, $p \in \mathbb{N}^0$.

$$\lim_{z\to x_0} \frac{\partial^p u_5}{\partial y^j \partial x^{p-j}} = d_m^{-1}(a)\Lambda_j(a,1-m) \underset{*}{f}^{(p-m)}(x_0), \quad f \in \underset{*}{C}^{p-m}, \tag{3.114}$$

when $(a,b) \in i_{3,m}$, $m > 0$, $j < m$, $p \in \mathbb{N}^0$. Moreover, if $a = 0$ and $j \in \mathbb{N}_1$, then

$$\lim_{z\to x_0} y^{-1} \frac{\partial^p u_5}{\partial y^j \partial x^{p-j}} = d_m^{-1}(0)\Lambda_j(0,1-m) \underset{*}{f}^{(p-m-1)}(x_0), \quad f \in \underset{*}{C}^{p-m+1}, \tag{3.115}$$

when $j < m-1$, and

$$\begin{aligned} &\lim_{z\to x_0} \left(y \ln \frac{1}{y}\right)^{-1} \frac{\partial^p u_5}{\partial y^j \partial x^{p-j}} \\ &= -(m+1)d_m^{-1}(0)\Lambda_m(0,-1-m) \underset{*}{f}^{(p-m+1)}(x_0), \quad f \in \underset{*}{C}^{p-m+1}, \end{aligned} \tag{3.116}$$

when $j = m-1$.

$$\lim_{z\to x_0} y^j \frac{\partial^p u_5}{\partial y^j \partial x^{p-j}} = \begin{cases} 0 \ \text{ if } \ m > 0, \ \ j \geq m, \\ \frac{M(a,b,j)}{1+e^{a\pi}} \underset{*}{f}^{(p-j)}(x_0) \ \text{ if } \ m = 0 \ j > 0, \end{cases} \tag{3.117}$$

when $(a,b) \in i_{3,m}$, $p \in \mathbb{N}$, $f \in \underset{*}{C}^{p-j}$

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{\partial^p u_5}{\partial x^p}=-\underset{*}{f}^{(p)}(x_0),\quad f\in \underset{*}{C}^p, \tag{3.118}$$

when $m=0$ (in this case $u_5\equiv\overset{*}{u}_5$), $p\in\mathbb{N}^0$.

$$\lim_{z\to x_0} y^{j-m}\frac{\partial^p u_6}{\partial y^j\partial x^{p-j}}=0,\quad f\in \underset{*}{C}^{p-j}, \tag{3.119}$$

when $j,p\in\mathbb{N}\backslash\{1\}$, $m\in\mathbb{N}$, $j>m$.

$$\lim_{z\to x_0}\frac{\partial^p u_6}{\partial y^m\partial x^{p-m}}=\underset{*}{f}^{(p-m)}(x_0),\quad f\in \underset{*}{C}^{p-m}, \tag{3.120}$$

when $m,p\in\mathbb{N}$.

$$\lim_{z\to x_0} y^{j-m-1}\frac{\partial^p\overset{*}{u}_3}{\partial y^j\partial x^{p-j}}=-\frac{M(0,-m,j)}{M(0,-m,2+m)}\underset{*}{f}^{(p-j)}(x_0),\quad f\in \underset{*}{C}^{p-j}, \tag{3.121}$$

when $j,\ p\in\mathbb{N}\backslash\{1\}$, $m\in\mathbb{N}_1$, $2j-2\left[\frac{j}{2}\right]>1+m$.

$$\lim_{z\to x_0} y^{j-m}\frac{\partial^p\overset{*}{u}_5}{\partial y^j\partial x^{p-j}}=-\frac{M(a,1-m,j)}{M(0,1-m,1+m)}\underset{*}{f}^{(p-j)}(x_0),\quad f\in \underset{*}{C}^{p-j}, \tag{3.122}$$

when $j,p\in\mathbb{N}$ and either $a\neq 0$, $m\in\mathbb{N}^0$, $j>m$ or $a=0$, $m\in\mathbb{N}_2^0$, $2j-2\left[\frac{j}{2}\right]>m$.

The equalities (3.101) - (3.103), (3.105) - (3.107), (3.109) - (3.111), (3.113) - (3.116), (3.118), (3.121), (3.122) we prove by means of the technique used in the sections 3.2 and 3.3. The proof of the equalities (3.104), (3.108), (3.112), (3.117) is somewhat different. E.g., we prove (3.104) when $j>0$:

$$\begin{aligned}&\lim_{z\to x_0} y^j\frac{\partial^p u_1}{\partial y^j\partial x^{p-j}}\\&=\Lambda_m^{-1}(a,b)\lim_{z\to x_0}\int_{-\infty}^{+\infty}\underset{*}{f}^{(p-j-m)}(x+yt)y^{j+1}\frac{\partial^j y^{1-b}e^{a\theta}\rho^{b-2}}{\partial y^j}\Big|_{\xi=x+yt}dt\\&=\Lambda_m^{-1}(a,b)\underset{*}{f}^{(p-j-m)}(x_0)y^j\int_{-\infty}^{+\infty}\frac{\partial^j y^{1-b}e^{a\theta}\rho^{b-2}}{\partial y^j}d\xi=0,\end{aligned}$$

since

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{\partial^{p+q}}{\partial x^p\partial y^q}\left[y^{1-b-k}(\xi-x)^k e^{a\theta}\rho^{b-2}\right]d\xi\\&=\frac{\partial^{p+q}}{\partial x^p\partial y^q}\left[y^{1-b-k}\int_{-\infty}^{+\infty}(\xi-x)^k e^{a\theta}\rho^{b-2}d\xi\right]=\frac{\partial^{p+q}\Lambda_k(a,b)}{\partial x^p\partial y^q}=0,\end{aligned} \tag{3.123}$$

by $z \in \mathbb{R}^2_+$, Re$b < 1-k$, $p, q, k \in \mathbb{N}^0$, $p^2+q^2 \neq 0$.

Let us prove (3.119) and (3.120). (3.68) we can rewrite in the form (3.82), where $\tilde{Q}_k(x)$ will be certain functions, taking into account the conditions (3.69). After differentiation of (3.82) m-times with respect to y we get

$$\lim_{z\to x_0} y^{j-m}\frac{\partial^p u_6}{\partial y^j \partial x^{p-j}} = \lim_{z\to x_0} y^{j-m}\frac{\partial^{p-m}\partial^m u_6}{\partial y^{j-m}\partial x^{p-j}\partial y^m}$$

$$= \lim_{z\to x_0} y^{j-m}\frac{\partial^{p-j}}{\partial x^{p-j}}\frac{\partial^{j-m}}{\partial y^{j-m}}\frac{y}{\pi}\int\limits_{-\infty}^{+\infty} f(\xi)\rho^{-2}d\xi$$

$$= \lim_{z\to x_0} y^{j-m}\pi^{-1}\int\limits_{-\infty}^{+\infty} \underset{*}{f}^{(p-j)}(\xi)\frac{\partial^{j-m}y\rho^{-2}}{\partial y^{j-m}}d\xi$$

$$= \underset{*}{f}^{(p-j)}(x_0)\frac{y^{j-m}}{\pi}\int\limits_{-\infty}^{+\infty}\frac{\partial^{j-m}y\rho^{-2}}{\partial y^{j-m}}d\xi$$

$$= \begin{cases} 0, & \text{if} \quad j>m, \ \text{because of (3.123)} \\ & \quad\ \ \text{for } a=0,\ b=0,\ p=0,\ q=j-m,\ k=0, \\ \underset{*}{f}^{(p-m)}(x_0), & \text{if} \quad j=m, \end{cases}$$

Thus, we have proved (3.119) and (3.120).

3.5 Boundary value problems in the finite domain

Let S be simply connected domain with the boundary $\partial S = \varsigma \cup \bar{I}$ consisting of the open smooth arcς lying in $\mathbb{R}^2_+$ with the ends $\zeta_1 = (\xi_1, 0)$, $\zeta_n = (\xi_n, 0)$ and the segment $\bar{I}$ ($I :=]\xi_1, \xi_n[$). Throughout the section $\zeta := (\xi, \eta) \equiv \xi + i\eta \in \partial S$, $z := (x, y) \equiv x + iy \in S$. A denotes the maximal ordinate of $\zeta \in \varsigma$.

Let f be a continuous function on ∂S. We consider the following BVPs (see [16], pp.25-30 and [22]).

Problem 3.5.1. *Let $b \in]-\infty, 1[$. Find the function $u \in T^0(1, S)$ satisfying BC*

$$u(\zeta) = f(\zeta), \ \ \zeta \in \partial S.$$

Problem 3.5.2. *Let $b = 1$. The arc ς is orthogonally rest on the x-axis with its small linear ends. The function $f(\xi) := f(\xi, 0)$ satisfies the Hölder condition at the left ξ_1 and the right ξ_n ends of the segment $\bar{I}$ and finite limits*

$$\lim_{\eta\to 0_+}\Big(\ln\frac{Ae}{\eta}\Big)f(\zeta), \ \frac{Ae}{\eta} \neq 1, \ \zeta = \xi + i\eta \in \varsigma \tag{3.124}$$

exist along ς. Find the function $u \in T^0\big((\ln\frac{Ae}{y})^{-1}, S\big)$ satisfying BC

$$\lim_{z\to\zeta}\Big(\ln\frac{Ae}{y}\Big)^{-1}u(z) = f(\zeta), \ \frac{Ae}{y} \neq 1, \ z \in S, \ \ \zeta \in \partial S. \tag{3.125}$$

Problem 3.5.3. *Let $b \in]1, +\infty[$. Find the function $u \in T^0(y^{b-1}, S)$ satisfying BC*

$$\lim_{z\to\zeta} y^{b-1}u(z) = f(\zeta), \ \ z \in S, \ \ \zeta \in \partial S.$$

Problem 3.5.4. *Let $b = 1$ and the arc ς meet the hypotheses of Problem 3.5.2. Find the function $u \in T^0\Big(\Big(\big(\ln \frac{Ae}{y}\big)^{-1}\Big), S\Big)$ satisfying BCs*

$$u(\zeta) = f(\zeta), \ \ \zeta \in \varsigma,$$

$$\lim_{z\to\xi} \Big(\ln \frac{Ae}{y}\Big)^{-1} u(z) = \varphi(\xi), \ \ z \in S, \ \ (\xi, 0) \in \bar{I},$$

where the function f is continuous on $\bar{\varsigma}$; the function φ is continuous on $\bar{I}$, vanishes at the ends ξ_1, ξ_n of the segment $\bar{I}$ and meets the Hölder condition at these points.

Problem 3.5.5. *Let $b \in]0, 1[$. The arc ς has the continuous curvature and meets the hypotheses of Problem 3.5.2,*

$$\frac{|a|}{2\pi} \cdot \frac{\Gamma^2\big(\frac{1}{2}\big)}{\Gamma(b)} \left\{ \frac{\pi\Gamma(b)}{\Gamma^2\big(b+\frac{1}{2}\big)} - (1-b)\sum_{k=1}^{+\infty} \frac{\Gamma(b+2k+1)\Gamma^2\big(k+\frac{1}{2}\big)}{\Gamma(2k+1)\Gamma^2\big(\frac{b+1}{2}+k\big)} \right\} < 1.$$

Find the function $u \in T^0(1, S) \cap T^m(y^{b+m-1}, S)$ satisfying BCs

$$u(\zeta) = f(\zeta), \ \ \zeta \in \bar{\varsigma}, \tag{3.126}$$

$$\lim_{z\to\xi} y^{b+m-1} \frac{\partial^m u(z)}{\partial y^m} = \varphi(\xi), \ \ z \in S, \ \ (\xi, 0) \in I \ \ m \in \mathbb{N}, \tag{3.127}$$

where the function f is continuous on $\bar{\varsigma}$, while the function φ is continuous on I, moreover φ may have at the ends of the segment $\bar{I}$ singularities of the order less than $1 - b$.

Theorem 3.5.6 *The problems 3.5.1-3.5.5 are uniquely solvable.*

Proof. In the case of Problem 3.5.1 Theorem 3.5.6 follows from the theorem proved for the more general equation in G. Jaiani [23] (see also Section 3.9).

According to the correspondence principle (3.4), we reduce Problem 3.5.3 to Problem 3.5.1.

In the cases of the problems 3.5.2 and 3.5.4 we prove Theorem 3.5.6 in much the same way. More precisely, these two BVPs, actually, coincide. Therefore, we restrict ourselves to examination of Problem 3.5.2.

By virtue of (3.124) and (3.125), we extend continuously the function f on the entire x-axis, assuming f equal to zero outside the segment $\bar{I}$.

In view of Theorem 3.3.8 (see the formula (3.67) for $m = 0$, i.e., (3.36)), the function

$$\tilde{w}(z) = (1 + e^{a\pi})^{-1} \int_{\xi_1}^{\xi_n} f(\xi) e^{a\theta} \rho^{-1} d\xi$$

is a solution of Problem 3.3.5 for $m = 0$.

Consider the following difference

$$\tilde{v}(z) = u(z) - \tilde{w}(z). \tag{3.128}$$

Let $u(z)$ be a solution of Problem 3.5.2, then $\tilde{v}(z)$ will be a solution of (1.1) satisfying BCs

$$\tilde{v}(\zeta) = \ln \frac{Ae}{\eta} f(\zeta) - \tilde{w}(\zeta) =: \tilde{\psi}(\zeta), \quad \zeta \in \varsigma, \tag{3.129}$$

$$\lim_{z \to \xi} \left(\ln \frac{Ae}{y} \right)^{-1} \tilde{v}(z) = 0, \ \ (\xi, 0) \in \bar{I}. \tag{3.130}$$

Now, we prove existence of finite limits along ς

$$\lim_{\zeta \to \xi_i} \tilde{w}(\zeta), \ \ \zeta \in \varsigma, \ \ i = 1, n.$$

Let for clearness $\zeta \to \xi_1$. Then, because of orthogonality of the linear ends of ς to the x-axis,

$$\lim_{\zeta \to (\xi_1, 0)} \tilde{w}(\zeta) = \lim_{\substack{\eta \to 0+ \\ \xi = \xi_1}} \tilde{w}(\xi_1, \eta) = (1 + e^{a\pi})^{-1} \lim_{\substack{\eta \to 0+ \\ \xi = \xi_1}} \left[\int\limits_{\xi_1}^{\tilde{\xi}} f(\tau) e^{a \cdot \arg(\zeta - \tau)} |\zeta - \tau|^{-1} d\tau \right.$$

$$\left. + (1 + e^{a\pi})^{-1} e^{a\pi} \int\limits_{\tilde{\xi}}^{\xi_n} f(\tau) |\xi_1 - \tau|^{-1} d\tau \right], \ \ \zeta \in \varsigma, \ \ \xi_1 < \tilde{\xi} < \xi_n.$$

In the last expression the first integral is uniformly convergent with respect to $\eta \in [0, \delta]$ if δ is as much small that the point ζ belongs to the linear ends of ς orthogonal to x-axis (i.e. $\xi = \xi_1$) and $\tilde{\xi}$ belongs to such a neighborhood of the point ξ_1, where $f(\xi)$ meets the Hölder condition. Indeed, in this case, because of $f(\xi_1) = 0$ [see (3.130)], the integrand

$$\begin{aligned} &\left| f(\tau) e^{a \arg(\zeta - \tau)} [(\xi_1 - \tau)^2 + \eta^2]^{-1/2} \right| \le e^{a\pi} |f(\tau)| \, |\xi_1 - \tau|^{-1} \\ &= e^{a\pi} |f(\tau) - f(\xi_1)| \, |\xi_1 - \tau|^{-1} \le C_2 |\tau - \xi_1|^{\mu - 1}, \ \ \tau \in]\xi, \tilde{\xi}[, \ \ \eta \in [0, \delta], \end{aligned}$$

where $C_2 = \text{const}$ and $\mu \in]0, 1]$ is the Hölder exponent. Therefore, exists integrable majorant. Thus,

$$\lim_{\zeta \to \zeta_1} \tilde{w}(\zeta) = (1 + e^{a\pi})^{-1} e^{a\pi} \int\limits_{\xi_1}^{\tilde{\xi}} f(\tau) |\xi_1 - \tau|^{-1} d\tau < +\infty, \ \ \zeta \in \varsigma.$$

Similarly, we prove

$$\lim_{\zeta \to \zeta_n} \tilde{w}(\zeta) = (1 + e^{a\pi})^{-1} e^{a\pi} \int\limits_{\tilde{\xi}}^{\xi_n} f(\tau) |\xi_n - \tau|^{-1} d\tau < +\infty, \ \ \zeta \in \varsigma.$$

If we assume these limit values as the values of the function $\tilde{w}$ at points ξ_1 and ξ_n, and take into account (3.124), then defined by the equality (3.128) function $\tilde{\psi}$ we can consider as the continuous on $\bar{\varsigma}$ function. Consequently, we can extend $\tilde{\psi}$ continuously on $\bar{S}$. After that, using the Wiener method (see, e.g., [1], p. 189), we construct the bounded solution of $\tilde{v}$ satisfying equation (1.1) and BC (3.127). Because of boundedness of $\tilde{v}$ BC (3.129) will be fulfilled as well. Now, evidently, from the equality (3.128) we find the solution u of Problem 3.5.2. So, we have proved the existence of the solution of Problem 3.5.2.

Let u be the difference of two possible solutions of Problem 3.5.2, then the function

$$\tilde{u} = \left(\ln\frac{Ae}{y}\right)^{-1} u \tag{3.131}$$

will be the solution of the equation

$$y\Delta\tilde{u} + a\tilde{u}_x + \left(1 - \frac{2}{\ln\frac{Ae}{y}}\right)\tilde{u}_y = 0 \tag{3.132}$$

satisfying BC

$$\tilde{u}(\zeta) = 0, \quad \zeta \in \partial S$$

But the weak extremum principle is valid for equation (3.132), hence

$$\tilde{u}(z) \equiv 0, \quad z \in S,$$

i.e., by virtue of (3.131),

$$u(z) \equiv 0, \quad z \in S,$$

which proves uniqueness of the solution of Problem 3.5.2 and, therefore, of Theorem 3.5.6 in the case of Problem 3.5.2 is proved.

In the case of Problem 3.5.5, when $m = 1$, unique solvability is shown in V. Evsin [5]. For $m \geq 1$ the uniqueness of the solution immidiatly follows from the generalized Zaremba-Giraud principle proved in Section 3.1, while the existence of the solution we prove just as in [5] (for the case $a = 0$ see also G. Jaiani [22]) for the case $m = 1$, taking into account that the existence of the m-th order derivative $\frac{\partial^m u}{\partial y^m}$ inside of $\mathbb{R}^2_+$. It follows from the Picard theorem (see, e.g., I. Vekua [35], p. 39, and also I. Vekua [36]). □

Remark 3.5.7 *If there exist solutions of equation (1.1) satisfying BCs of the Problems 3.5.2 and 3.5.4, respectively, when ς is the Jordan arc, f and φ are the continuous functions, then these solutions are unique.*

Proof. Proof is based on consideration of equation (3.132) similarly to the above mentioned proof of uniqueness of the solution of Problem 3.5.2. □

Remark 3.5.8 *Let $b \in [1, +\infty[$, ς be the Jordan arc, f and φ be the continuous functions. If there exists a solution of equation (1.1) satisfying BCs of Problem 3.5.5, then it is unique.*

Proof. If $b > 1$, for the difference u of two possible solutions we have

$$\lim_{y\to 0+} y^{b-1}u = \lim_{y\to 0+} \frac{(-1)^m}{(b-1,m)} y^{b+m-1}\frac{\partial^m u}{\partial y^m} = 0.$$

On the other hand, $u(\zeta) = 0,\ \ \zeta \in \bar{\varsigma}$, i.e.,

$$y^{b-1}u(\zeta) = 0, \quad \zeta \in \bar{\varsigma}.$$

Thus

$$\lim_{z\to\zeta} y^{b-1}u(z) = 0,$$

and the question we have reduced to the uniqueness of the solution of Problem 3.5.3.

If $b = 1$, then

$$\lim_{y\to 0+} \left(\ln\frac{Ae}{y}\right)^{-1} u = \lim_{y\to 0+} \frac{(-1)^m}{(1,m-1)} y^m \frac{\partial^m u}{\partial y^m} = 0$$

and the question we have reduced to the uniqueness of the solution of Problem 3.5.2. □

3.6 Boundary value problems with discontinuous data. Behaviour of the solutions at points of discontinuity of boundary data

Let f be piece-wise continuous function on ∂S, $\overset{*}{I} = \bigcup\limits_{k=1}^{n}\{\xi_k\}$, $\xi_k \in \bar{I}$, $k = \overline{1,n}$, be a set of its points of discontinuity of the first kind, $\overset{(+)}{f}(\xi_k)$ and $\overset{(-)}{f}(\xi_k)$ be limits of f when $\zeta \in \partial S \backslash \overset{*}{I}$ tends to ξ_k in the negative and positive, correspondingly, directions of the circuit of the domain S. Let $\overset{(+)}{\varphi}(\xi_n) \in]0,\pi]$ and $\overset{(-)}{\varphi}(\xi_n) \in]0,\pi]$ be the angles between the x-axis and the smooth Jordan arc ς, correspondingly, at the endpoints ξ_n and ξ_1 of the segment $\bar{I}$ (see [17], [19] and [16], pp. 31-35).

Problem 3.6.1. *Let $b \in]-\infty, 1[$. Find the function $u \in T^0(1, S, \overset{*}{I})$, satisfying BC*

$$\lim_{z\to\zeta} u(z) = f(\zeta), \ \ z \in S, \ \ \zeta \in \partial S \backslash \overset{*}{I}.$$

Problem 3.6.2. *Let $b = 1$. Find the function $u \in T^0\Big(\big(\ln\frac{Ae}{y}\big)^{-1}, S, \overset{*}{I}\Big)$, satisfying BC*

$$\lim_{z\to\zeta} \left(\ln\frac{Ae}{z}y\right)^{-1} u(z) = f(\zeta), \ \ z \in S, \ \ \zeta \in \partial S \backslash \overset{*}{I}. \tag{3.133}$$

Problem 3.6.3. [11] *Let $b\in]1,+\infty[$. Find the function $u\in T^0(y^{b-1},S,\overset{*}{I})$, satisfying BC*

$$\lim_{z\to\zeta} y^{b-1}u(z)=f(\zeta),\;\; z\in S,\;\; \zeta\in\partial S\backslash\overset{*}{I}.$$

Theorem 3.6.4 *There exists such a solution of Problem 3.6.1 which permits the representation*

$$\overset{(1)}{u}(z)=U_1(z)+\sum_{k=1}^{n}\frac{h_k}{\alpha_k(b)}\int\limits_{0}^{\arg(z-\xi_k)}e^{a\tau}\sin^{-b}\tau\,d\tau \tag{3.134}$$

where

$$h_k=\overset{(+)}{f}(\xi_k)-\overset{(-)}{f}(\xi_k),\quad k=\overline{1,n};\;\; \alpha_k(b)=-\Lambda(a,b),\;\; k=\overline{2,n-1};$$

$$\alpha_1(b)=-\int\limits_{0}^{\overset{(-)}{\varphi}}a^{a\tau}\sin^{-b}\tau\,d\tau,\;\; \alpha_n(b)=-\int\limits_{\overset{(+)}{\varphi}}^{\pi}a^{a\tau}\sin^{-b}\tau\,d\tau,$$

$U_1(z)\in T^0(1,S)$ is the solution of Problem 3.5.1 in the case of the following considered on the boundary continuous function

$$f(\zeta)-\sum_{k=1}^{n}\frac{h_k}{\alpha_k(b)}\int\limits_{0}^{\arg(\zeta-\xi_k)}e^{a\tau}\sin^{-b}\tau\,d\tau,\;\; \zeta\in\partial S. \tag{3.135}$$

By approaching the point $z\in S$ to the point of discontinuity $(\xi_k,0)$ of the function f along the different ways lying in S, the solution $\overset{(1)}{u}(z)$ tends to any values between $\overset{(-)}{f}(\xi_k)$ and $\overset{(+)}{f}(\xi_k)$, depending on the way of approaching characterised by the angle between the tangent to the way (curve) at this point and x-axis.

If either $a\in\mathbb{R}^1$, $b\in]0,1[$ or $a=0$, $b=0$, the solution of Problem 3.6.1 i.e., the representation (3.134) is unique.

If either $a\in\mathbb{R}^1$, $b\in]-\infty,0[$ or $a\neq 0$ $b=0$, the solution of Problem 3.6.1 is unique in the class of functions having the same limits of indeterminacy as $z\to\xi_k\in\overset{*}{I}$ (compare with K. Miranda [29], p. 108, Chapter IV, Section 29).

Theorem 3.6.5 *Under the hypotheses of Problem 3.5.2 concerning the arc $\bar{\varsigma}$*[12] *and considered on the boundary continuous function (3.137) (see below), there exists the solution of Problem 3.6.2 which permits the representation*

$$\overset{(2)}{u}(z)=U_2(z)+\sum_{k=1}^{n}\frac{h_k}{\beta_k}\int\limits_{\frac{\pi}{2}}^{\arg(z-\xi_k)}e^{a\tau}\sin^{-1}\tau\,d\tau, \tag{3.136}$$

[11] The problems 3.6.1-3.6.3 when the boundary data have the first type discontinuity points on ς is to be investigated.

[12] Theorem 3.6.5 when $\overset{(\pm)}{\varphi}\neq\frac{\pi}{2}$ remains valid, provided Problem 2.5.2 is solvable for $\overset{(\pm)}{\varphi}\neq\frac{\pi}{2}$.

where

$$\beta_k = -(1+e^{a\pi}), \quad k=\overline{2,n-1},$$

$$\beta_1\Big|_{\overset{(-)}{\varphi}(\xi_1)=\frac{\pi}{2}} = (e^{a\overset{(-)}{\varphi}(\xi_1)}\cos\overset{(-)}{\varphi}(\xi_1)-1)\Big|_{\overset{(-)}{\varphi}(\xi_1)=\frac{\pi}{2}} = -1,$$

$$\beta_n\Big|_{\overset{(+)}{\varphi}(\xi_n)=\frac{\pi}{2}} = (-e^{a\overset{(+)}{\varphi}(\xi_n)}\cos\overset{(+)}{\varphi}(\xi_n)-e^{a\pi})\Big|_{\overset{(+)}{\varphi}=\frac{\pi}{2}} = -e^{a\pi},$$

$U_2(z) \in T^0\left(\left(\ln\frac{Ae}{y}\right)^{-1}, S\right)$ *is the solution of Problem 3.5.2 in the case of the prescribed on the boundary continuous function*

$$f(\zeta) - \sum_{k=1}^{n}\frac{h_k}{\beta_k}\lim_{z\to\zeta\not\equiv\xi_k}\left(\ln\frac{Ae}{y}\right)^{-1}\int\limits_{\frac{\pi}{2}}^{\arg(z-\xi_k)} e^{a\tau}\sin^{-1}\tau\, d\tau, \quad \zeta\in\partial S. \tag{3.137}$$

By approaching the point $z \in S$ to the points of discontinuity $(\xi_k, 0)$ of the function f along the different ways lying in S, the function $\left(\ln\frac{Ae}{y}\right)^{-1}\overset{(2)}{u}(z)$ tends to any value between $\overset{(-)}{f}(\xi_k)$ and $\overset{(+)}{f}(\xi_k)$, depending on the way of approaching characterised by the angle between the tangent to the way (curve) at this point and x-axis.

The solution of Problem 3.6.2 is unique in the class of functions having the same limits of indeterminacy with the weight $\left(\ln\frac{Ae}{y}\right)^{-1}$ as $z \to \xi_k \in \overset{*}{I}$.

Theorem 3.6.6 *There exists the solution of Problem 3.6.3 which permits the representation*

$$\overset{(3)}{u}(z) = y^{1-b}U_3(z) + y^{1-b}\sum_{k=1}^{n}\frac{h_k}{\alpha_k(2-b)}\int\limits_{0}^{\arg(z-\zeta_k)} e^{a\tau}\sin^{b-2}\tau\, d\tau, \tag{3.138}$$

where $y^{1-b}U_3(z) \in T^0(y^{b-1}, S)$ is the solution of Problem 3.5.3 in the case of prescribed on the boundary continuous function

$$f(\zeta) - \sum_{k=1}^{n}\frac{h_k}{\alpha_k(2-b)}\int\limits_{0}^{\arg(z-\zeta_k)} e^{a\tau}\sin^{b-2}\tau\, d\tau, \quad \zeta\in\partial S.$$

By approaching the point $z \in S$ to the points of discontinuity $(\xi_k, 0)$ of the function f along the different ways lying in S, the function $y^{b-1}\overset{(3)}{u}(z)$ tends to any value between $\overset{(-)}{f}(\xi_k)$ and $\overset{(+)}{f}(\xi_k)$, depending on the way of approaching characterised by the angle between the tangent to the way (curve) at this point and x-axis.

If either $a \in \mathbb{R}^1$, $b \in]1,2[$ or $a = 0$, $b = 2$, the solution of Problem 3.6.3, i.e., the representation (3.138), is unique.

If either $a \in \mathbb{R}^1$, $b \in]2, +\infty[$ or $a \neq 0$ $b = 2$, the solution of Problem 3.6.3 is unique in the class of functions having the same limits of indeterminacy with the weight y^{1-b} as $z \to \xi_k \in \overset{*}{I}$.

Proof of Theorem 3.6.4 In view of (3.9) the function

$$\overset{(1)}{u}_k(z) = \frac{h_k}{\alpha_k(b)} \int\limits_0^{\arg(z-\xi_k)} e^{a\tau} \sin^{-b} \tau \, d\tau, \tag{3.139}$$

is a solution of (1.1). It is bounded and continuous in $\bar{S} \backslash \{(\xi_k, 0)\}$. Moreover,

$$\overset{(1)}{u}_k(x,0) = \tag{3.140}$$
$$\begin{cases} \dfrac{h_k}{\alpha_k(b)} \displaystyle\int\limits_0^{\pi} e^{a\tau} \sin^{-b} \tau \, d\tau = \dfrac{h_k}{\alpha_k(b)} \Lambda(a,b) = -h_k, & x \in (\xi_1, \xi_k), \quad k = \overline{2, n}, \\ 0, & x \in (\xi_k, \xi_n), \quad k = \overline{1, n-1}. \end{cases}$$

If $z \to \xi_k$ along the way whose tangent at the point ξ_k makes with the x-axis an angle φ $\Big($Clearly, at the points ξ_k, $k = \overline{2, n-1}$, the angle $\varphi \in [0, \pi]$, at the point ξ_1 the angle $\varphi \in \big[0, \overset{(-)}{\varphi}\big]$, and the point ξ_n the angle $\varphi \in \big[\overset{(+)}{\varphi}, \pi\big[\Big)$, then (3.139) tends to the limit

$$\frac{h_k}{\alpha_k(b)} \int\limits_0^{\varphi} e^{a\tau} \sin^{-b} \tau \, d\tau,$$

By crossing the point ξ_k in negative direction along the curve ∂S, the function $\overset{(1)}{u}_k(\zeta)$, by virtue of (3.140), has the jump

$$0 - (-h_k) = h_k, \quad \text{when} \quad k = \overline{2, n-1};$$

$$0 - \frac{h_k}{\alpha_k(b)} \int\limits_0^{\overset{(-)}{\varphi}} e^{a\tau} \sin^{-b} \tau \, d\tau = h_1, \ \ \text{when} \ \ k = 1,$$

and

$$\frac{h_n}{\alpha_n(b)} \Big[\int\limits_0^{\overset{(+)}{\varphi}} e^{a\tau} \sin^{-b} \tau \, d\tau - \int\limits_0^{\pi} e^{a\tau} \sin^{-b} \tau \, d\tau \Big] = h_n, \ \ \text{when} \ \ k = n.$$

Hence, the function (3.135) remains continuous by crossing each point ξ_k since from the function $f(\zeta)$ which has the jump h_k by crossing ξ_k we subtract the sum of the continuous function

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} \overset{(1)}{u}_j(\zeta) \tag{3.141}$$

and the function $\overset{(1)}{u}_k(\zeta)$ having the same jump h_k which has $f(\zeta)$ at point ξ_k. Thus, the function $U_1(z)$ as the solution of Problem 3.5.1, when on the boundary the continuous function (3.135) is prescribed, exists and is unique. Consequently, (3.134) tends to $f(\zeta)$ as $z \to \zeta \neq \xi_k$, is bounded and satisfies equation (1.1) in S. Let us analyse its behaviour as $z \to \xi_k \in \overset{*}{I}$. Let $z \to \xi_k$ along the way whose tangent at the point ξ_k makes with the x-axis the angle φ. Then from the expression (3.134) it follows that along the above-mentioned way $\overset{(1)}{u}(\zeta)$ tends to

$$\overset{(1)}{u}_\varphi(\xi_k) = \tilde{U}_1(\xi_k) + \frac{h_k}{\alpha_k(b)} \int\limits_0^\varphi e^{a\tau} \sin^{-b}\tau \, d\tau, \tag{3.142}$$

where $\tilde{U}_1(\xi_k)$ is the limit of the sum $U_1(z)$ and (3.141) which does not depend on the way of approaching z to ξ_k. In particular, by approaching ξ_k along the curve ∂S in the negative direction of going around the domain we get

$$\overset{(+)}{f}(\xi_k) = \tilde{U}_1(\xi_k) + \overset{(+)}{\overset{(1)}{u}}_k(\xi_k), \tag{3.143}$$

where

$$\overset{(+)}{\overset{(1)}{u}}_k(\xi_k) = \begin{cases} 0, \quad k = \overline{1, n-1}; \\ \dfrac{h_n}{\alpha_n(b)} \displaystyle\int\limits_0^{\overset{(+)}{\varphi}} e^{a\tau} \sin^{-b}\tau \, d\tau, \quad k = n. \end{cases}$$

Substituting the expression of $\tilde{U}_1(\xi_k)$ determined from (3.143) into (3.142) we obtain

$$\overset{(1)}{u}_\varphi(\xi_k) = \overset{(+)}{f}(\xi_k) + \begin{cases} \dfrac{h_k}{\alpha_k(b)} \displaystyle\int\limits_0^{\varphi} e^{a\tau} \sin^{-b}\tau \, d\tau, \quad k = \overline{1, n-1}; \\ \dfrac{h_n}{\alpha_n(b)} \displaystyle\int\limits_{\overset{(+)}{\varphi}}^{\varphi} e^{a\tau} \sin^{-b}\tau \, d\tau, \quad k = n. \end{cases}$$

The right-hand part of the last equality is the continuous function of φ on the segments $[0, \pi]$, $\big[0, \overset{(-)}{\varphi}\big]$, $\big[\overset{(+)}{\varphi}, \pi\big]$ for $k = \overline{2, n-1}$, $k = 1$, and $k = n$, respectively, and is taking at their ends values $\overset{(+)}{f}(\xi_k)$ and $\overset{(-)}{f}(\xi_k)$. Therefore, according to the second Bolzano-Cauchy theorem (see [6], p. 171), it takes all the values between $\overset{(+)}{f}(\xi_k)$ and $\overset{(-)}{f}(\xi_k)$ on the above segments, depending on the angle φ.

If $a \in \mathbb{R}^1$ and $b \in]0, 1[$, the uniqueness of the representation (3.134) follows from the maximum principle 3.1.3.

If $a = b = 0$ the uniqueness of the representation (3.134) is proved in [26] (see p. 212).

Let either $a \in \mathbb{R}^1$ and $b \in]-\infty, 0[$ or $a \neq 0$, $b = 0$. If we consider the difference of two possible solutions of Problem 3.6.1 with the same limits of indeterminacy as $z \to \xi_k \in \overset{*}{I}$, it is easily seen that this difference on the entire boundary except the points $\xi_k \in \overset{*}{I}$, where the limits along any way are zero, since both the possible solutions have the same limits for the same φ. Taking zero as values of the above difference at the points $\xi_k \in \overset{*}{I}$, we get the solution of equation (1.1) which vanishes on the boundary and is continuous on $\bar{S}$. Then, according to the weak extremum principle, the above difference of two possible solutions equals 0. $\square$

Proof of Theorem 3.6.5 The function

$$\overset{(2)}{u}_k(z) = \frac{h_k}{\beta_k} \int\limits_{\frac{\pi}{2}}^{\arg(z-\xi_k)} e^{a\tau} \sin^{-1}\tau \, d\tau, \tag{3.144}$$

represents the solution of equation (1.1) in S.

The function

$$\Big(\ln \frac{Ae}{y}\Big)^{-1} \overset{(2)}{u}_k(z) \tag{3.145}$$

is bounded and continuous everywhere in $\bar{S} \backslash \{(\xi_k, 0)\}$. Indeed, its continuity on $S \bigcup \varsigma$ is clear. On the set $I \backslash \{\xi_k\}$ (3.145) will become continuous, assuming for its values there the limits

$$\begin{aligned} &\frac{h_k}{\beta_k} \lim_{z\to\xi\neq\xi_k} \Big(\ln \frac{Ae}{y}\Big)^{-1} \int\limits_{\frac{\pi}{2}}^{\arg\,(z-\xi_k)} e^{a\tau} \sin^{-1}\tau \, d\tau \\ &= -\frac{h_k}{\beta_k} \lim_{z\to\xi\neq\xi_k} e^{a\cdot\arg(z-\xi_k)} \cos\arg(z-\xi_k) = \frac{h_k}{\beta_k} \begin{cases} e^{a\pi}, & \xi < \xi_k; \\ -1, & \xi > \xi_k. \end{cases} \end{aligned} \tag{3.146}$$

Using Theorem 2.1.1, it is easy to check (3.146).

If $z \to \xi_k$ along the way whose tangent at the point ξ_k makes with the x-axis an angle φ, then according to Remark 2.1.5 the function (3.145) tends to

$$-\frac{h_k}{\beta_k} e^{a\varphi} \cos\varphi, \tag{3.147}$$

which is bounded with respect to φ (hence, (3.145) is bounded). By crossing the points ξ_k along ∂S in the negative direction of going around the domain S, by virtue of (3.146), the function (3.145) has the jump

$$\frac{h_k}{\beta_k}(-1 - e^{a\pi}) = h_k \ \text{ if } \ k = \overline{2, n-1};$$

$$\frac{h_1}{\beta_1}\big(-1 + e^{a\overset{(-)}{\varphi}} \cos\overset{(-)}{\varphi}\big) = h_1 \ \text{ if } \ k = 1, \ \text{ because of } \ \overset{(-)}{\varphi} = \frac{\pi}{2};$$

$$\frac{h_n}{\beta_n}\big(-e^{a\overset{(+)}{\varphi}} \cos\overset{(+)}{\varphi} - e^{a\pi}\big) = h_n \ \text{ if } \ k = n, \ \text{ because of } \ \overset{(+)}{\varphi} = \frac{\pi}{2}.$$

Hence, the function (3.137) remains continuous by crossing each point ξ_k since from the function $f(\zeta)$ which has the jump h_k by crossing the point ξ_k we subtract the sum of the continuous function

$$\left(\ln\frac{Ae}{y}\right)^{-1}\sum_{\substack{j=1\\ j\neq k}}^{n}\overset{(2)}{u}_j(\zeta) \tag{3.148}$$

and the function (3.145) having the same jump h_k. Thus, by virtue of the hypotheses of the theorem 3.6.5 concerning ς and the function (3.137), there exists the function $U_2(z)$. So, taking into account the properties of the functions (3.144), (3.145), we conclude that the expression (3.136) meets the BC (3.133) as $z \to \xi \neq \xi_k$ and belongs to the class $T^0\Big(\big(\ln\frac{Ae}{y}\big)^{-1}, S, \overset{*}{I}\Big)$. Let us analyse the behaviour of the function (3.145) as $z \to \xi_k \in \overset{*}{I}$. Let $z \to \xi_k$ along the way whose tangent at the point ξ_k makes with the x-axis an angle φ. According to (3.136), (3.147), the function

$$\left(\ln\frac{Ae}{y}\right)^{-1}\overset{(2)}{u}(z)$$

will tend to the limit

$$\overset{(2)}{u}_{\varphi}(\xi_k) = \tilde{U}_2(\xi_k) + \frac{h_k}{\beta_k}e^{a\varphi}\cos\varphi, \tag{3.149}$$

where $\tilde{U}_2(\xi_k)$ is the limit of the sum of the function $\big(\ln\frac{Ae}{y}\big)^{-1}U_2(z)$ and (3.148) which is independent of the way of approaching of z to the point ξ_k. In particular, approaching to the point ξ_k along the curve ∂S in the negative direction, we obtain

$$\overset{(+)}{f}(\xi_k) = \tilde{U}_2(\xi_k) + \overset{(+)}{\overset{(2)}{u}}_k(\xi_k),$$

where

$$\overset{(+)}{\overset{(2)}{u}}(\xi_k) = \begin{cases} -\dfrac{h_k}{\beta_k}, \quad k = \overline{1, n-1}; \\ -\dfrac{h_n}{\beta_n}e^{a\overset{(+)}{\varphi}}\cos\overset{(+)}{\varphi} = 0, \quad k = n. \end{cases}$$

So, (3.149) we rewrite in the form

$$\overset{(2)}{u}(\xi_k) = \overset{(+)}{f}(\xi_k) + \begin{cases} \dfrac{h_k}{\beta_k}(1 - e^{a\varphi}\cos\varphi), & k = 1, , n-1; \\ -\dfrac{h_k}{\beta_k}e^{a\varphi}\cos\varphi, & k = n. \end{cases}$$

□

Next, we argue similarly to the proof of Theorem 3.6.4. Going over to the proof of the uniqueness, we consider the difference u of two possible solutions and verify that on the entire boundary ∂S the function

$$v = \left(\ln \frac{Ae}{y}\right)^{-1} u$$

vanishes. But it satisfies equation (3.132) in S. So that, by virtue of the well-known weak extremum principle for the second order elliptic equations $v \equiv 0$ in S, whence $u \equiv 0$ in S.

Proof of Theorem 3.6.6 after introducing the knew unknown function $v = y^{b-1}u$, we reduce to Theorem 3.6.5, according to the correspondence principle (3.7). □

Remark 3.6.7 *The theorems 3.6.4-3.6.6 remain valid for $\mathbb{R}^2_+$ if for the uniqueness of the solutions, we assume in addition at infinity either (3.48); or*

$$y^{b-1} = O(1), \quad r \to +\infty,$$

when either $a \in \mathbb{R}^1$, $b \in]2, +\infty[$ or $a = 0$, $b = 2$;

$$y^{b-1}u = o(1), \quad r \to +\infty,$$

when either $a \in \mathbb{R}^1$, $b \in]1, 2[$, or $a \neq 0$, $b = 2$; and fulfilment of (3.55) with (3.56) when $a \in \mathbb{R}^1$, $b = 1$, respectively.

In these cases (3.134), (3.136), (3.138) coincide with (3.35)-(3.37), respectively.

let us prove the following lemma in advance.

Lemma 3.6.8 *The following equalities hold:*

$$\Lambda^{-1}(a,b) y^{1-b} \int\limits_{-\infty}^{+\infty} \Big[\int\limits_{0}^{\arg(\xi-\xi_k)} e^{a\tau} \sin^{-b}\tau \, d\tau \Big] e^{a\theta} \rho^{b-2} d\xi$$
$$= \int\limits_{0}^{\arg(\xi-\xi_k)} e^{a\tau} \sin^{-b}\tau \, d\tau, \quad b \in]-\infty, 1[; \tag{3.150}$$

$$(1+e^{a\pi})^{-1} \text{ p.v. } \overset{*}{R}. \int\limits_{-\infty}^{+\infty} \Big[\lim_{z \to \xi \neq \xi_k} (\ln y^{-1})^{-1} \int\limits_{\frac{\pi}{2}}^{\arg(z-\xi_k)} e^{a\tau} \sin^{-1}\tau \, d\tau \Big] e^{a\theta} \rho^{-1} d\xi$$
$$= \int\limits_{\frac{\pi}{2}}^{\arg(z-\xi_k)} e^{a\tau} \sin^{-1}\tau \, d\tau, \tag{3.151}$$

where $k = \overline{2, n-1}$ and the generalized principle value of the integral is defined by the equality

$$\text{p.v.}\overset{*}{R}\int_{-\infty}^{+\infty} = \lim_{R\to+\infty}\int_{-R}^{\overset{*}{R}(R)},$$

here

$$\overset{*}{R} = x + y\cot\overset{*}{\delta}(\delta), \quad \delta = \operatorname{arccot}\frac{x+R}{y}, \tag{3.152}$$

and the function $\overset{*}{\delta}(\delta)$ is given implicitly by the equality

$$\int_{\overset{*}{\delta}}^{\frac{\pi}{2}} e^{-a\tau}\sin^{-1}\tau\, d\tau = \int_{\delta}^{\frac{\pi}{2}} e^{a\tau}\sin^{-1}\tau\, d\tau \tag{3.153}$$

Proof. As

$$\arg(\xi-\xi_k) = \begin{cases} \pi, & \xi \in]-\infty, \xi_k[; \\ 0, & \xi \in]\xi_k, +\infty[, \end{cases}$$

then

$$\Lambda^{-1}(a,b)y^{1-b}\int_{-\infty}^{+\infty}\Big[\int_0^{\arg(\xi-\xi_k)} e^{a\tau}\sin^{-b}\tau\, d\tau\Big]e^{a\theta}\rho^{b-2}d\xi = y^{1-b}\int_{-\infty}^{\xi_k} e^{a\theta}\rho^{b-2}d\xi.$$

whence, after substitution $\xi = x - y\cot\tau$ in the right-hand side, we get (3.150).

It will be observed that the equality (3.150) immediately follows from there, that

$$\int_0^{arg(z-\xi_k)} e^{a\tau}sin^{-b}\tau d\tau, \quad b<1,$$

is such a solution of equation (1.1) which on the boundary $y=0$ takes piece-wise constant values

$$\int_0^{arg(\xi-\xi_k)} e^{a\tau}sin^{-b}\tau d\tau.$$

But such a solution we can respresent by the formula (3.35) which in our case coincides with the right-hand side of the equality (3.150).

Before passing to derivation of the equality (3.151), let us clarify nature of the functions $\overset{*}{\delta} = \overset{*}{\delta}(\delta)$ and $\overset{*}{R} = \overset{*}{R}(R)$ defined by the equalities (3.153) and (3.152), respectively.

From the second equality of (3.152) it is obvious that $\delta = 0$ by $R = +\infty$. If we consider sufficiently big values of R, namely, $R \geq -x$, for the fixed x, y, then $\delta \in [0, \frac{\pi}{2}]$. The right-hand side of (3.153) continuously and strongly monotonically increases (since the integrand is strongly positive) assuming all the values from 0 to $+\infty$, when δ varies from $\frac{\pi}{2}$ to 0. The left-hand side of (3.153) behaves analogously with respect to $\overset{*}{\delta}$. Consequently, to each $\delta \in [0, \frac{\pi}{2}]$, according to (3.153), corresponds one $\overset{*}{\delta} \in [0, \frac{\pi}{2}]$ and conversely. Thus, the equality (3.153) defines the strongly monotonic function $\overset{*}{\delta} = \overset{*}{\delta}(\delta)$, in addition $\overset{*}{\delta}(0) = 0$. But from the first equality of (3.152), we have

$$\lim_{\overset{*}{\delta}\to 0+} \overset{*}{R} = +\infty.$$

So,

$$\lim_{R\to+\infty} \overset{*}{R}(R) = +\infty.$$

According to the definition of $p.v.\overset{*}{R}$, after successive application of (3.146), substitution $\xi = x - y\cot\tau$, and (3.152), (3.153), the left-hand side of (3.151) takes the form

$$(1+e^{a\pi})^{-1} \lim_{R\to+\infty} \left(e^{a\pi} \int_{-R}^{\xi_k} e^{a\theta}\rho^{-1} d\xi - \int_{\xi_k}^{\overset{*}{R}} e^{a\theta}\rho^{-1} d\xi \right)$$

$$= (1+e^{a\pi})^{-1} \lim_{R\to+\infty} \left[e^{a\pi} \int_{\operatorname{arccot}\frac{\mathrm{x+R}}{\mathrm{y}}}^{\arg(z-\xi_k)} e^{a\tau}\sin^{-1}\tau\, d\tau - \int_{\arg(z-\xi_k)}^{\operatorname{arccot}\frac{\mathrm{x}-\overset{*}{\mathrm{R}}}{\mathrm{y}}} e^{a\tau}\sin^{-1}\tau\, d\tau \right]$$

$$= \int_{\frac{\pi}{2}}^{\arg(z-\xi_k)} e^{a\tau}\sin^{-1}\tau\, d\tau$$

$$+(1+e^{a\pi})^{-1} \lim_{R\to+\infty} \left(e^{a\tau} \int_{\operatorname{arccot}\frac{x+R}{y}}^{\frac{\pi}{2}} e^{a\tau}\sin^{-1}\tau\, d\tau + \int_{\operatorname{arccot}\frac{x-\overset{*}{R}}{y}}^{\frac{\pi}{2}} e^{a\tau}\sin^{-1}\tau\, d\tau \right)$$

$$= \int_{\frac{\pi}{2}}^{\arg(z-\xi_k)} e^{a\tau}\sin^{-1}\tau\, d\tau$$

$$+(1+e^{a\pi})^{-1} \lim_{\delta\to 0+} \left(e^{a\pi} \int_{\delta}^{\frac{\pi}{2}} e^{a\tau}\sin^{-1}\tau\, d\tau + \int_{\pi-\overset{*}{\delta}}^{\frac{\pi}{2}} e^{a\tau}\sin^{-1}\tau\, d\tau \right)$$

$$= \int_{\frac{\pi}{2}}^{\arg(z-\xi_k)} e^{a\tau}\sin^{-1}\tau\, d\tau$$

$$+(1+e^{a\pi})^{-1}\lim_{\delta\to 0+}\left(e^{a\pi}\int\limits_{\delta}^{\frac{\pi}{2}} e^{a\tau}\sin^{-1}\tau\,d\tau - e^{a\pi}\int\limits_{\overset{*}{\delta}}^{\frac{\pi}{2}} e^{a\tau}\sin^{-1}\tau\,d\tau\right)$$

$$= \int\limits_{\frac{\pi}{2}}^{\arg(z-\xi_k)} e^{a\tau}\sin^{-1}\tau\,d\tau.$$

So, (3.151) is proved as well, and the proof of Lemma 3.6.8 is complete.

□

Proof of the Remark 3.6.7. By the proof of theorems 3.6.4-3.6.6 we have not used finiteness of the domain except of the existence and uniqueness theorems in the case of the continuous boundary function. Therefore, assuming the function f bounded and applying the existence and uniqueness theorems in the case of the continuous boundary function for $\mathbb{R}^2_+$ the theorems 3.6.4-3.6.6 remain valid also for $\mathbb{R}^2_+$, if the sums in the corresponding expressions we consider from 2 to $n-1$ since in the case under consideration the points ξ_1 and ξ_2 are absent.

Thus, the formulas (3.135) and (3.136) get the forms

$$\begin{aligned}\overset{(1)}{u}_{\Pi}(z) &= \frac{y^{1-b}}{\Lambda(a,b)}\int\limits_{-\infty}^{+\infty}\Big[f(\xi)-\sum_{k=2}^{n-1}\frac{h_k}{\alpha_k(b)}\int\limits_{0}^{\arg(\xi-\xi_k)} e^{a\tau}\sin^{-b}\tau\,d\tau\Big]e^{a\theta}\rho^{b-2}d\xi\\ &+ \sum_{k=2}^{n-1}\frac{h_k}{\alpha_k(b)}\int\limits_{0}^{\arg(z-\xi_k)} e^{a\tau}\sin^{-b}\tau\,d\tau \qquad (3.154)\end{aligned}$$

and

$$\begin{aligned}\overset{(2)}{u}_{\Pi}(z) &= (1+e^{a\tau})^{-1}\text{p.v.}\overset{*}{R}\int\limits_{-\infty}^{+\infty}\Big[f(\xi)\\ &- \sum_{k=2}^{n-1}\frac{h_k}{\beta_k}\lim_{z\to\xi\neq\xi_k}(\ln y^{-1})^{-1}\int\limits_{0}^{\arg(z-\xi_k)} e^{a\tau}\sin^{-1}\tau\,d\tau\Big]e^{a\theta}\rho^{-1}d\xi\\ &+ \sum_{k=2}^{n-1}\frac{h_k}{\beta_k}\int\limits_{0}^{\arg(z-\xi_k)} e^{a\tau}\sin^{-1}\tau\,d\tau, \qquad (3.155)\end{aligned}$$

respectively.

Let us note that we have replaced BC (3.133) by BC (3.33) and applied the formula (3.36) interpretting it in the sense p.v.$\overset{*}{R}$ (relying on Theorem 2.1.7, it can be directly verified that expression (3.36) with the density (3.137), where $Ae \equiv 1$, $\zeta \equiv \xi$, and the sum is taken from 2 to $n-1$, solves Problem 3.2.1 with BC (3.33) when $\overset{*}{I} = \varnothing$ and the boundary function has the form (3.137), $Ae \equiv 1$, $\zeta \equiv \xi$).

By virtue of (3.149), (3.151), it is easily seen that $\overset{(1)}{u}_{\Pi}(z)$ coincides with (3.35), while $\overset{(2)}{u}_{\Pi}(z)$ coincides with (3.36). Of course, we could arrive at this conclusion also in reverse order.

Uniqueness follows from the maximum principles 3.1.1, 3.1.3, 3.1.4.

The case $b > 1$, as it was done many times earlier, we reduce to the case $b < 1$. □

3.7 Approximate solutions of boundary value problems

Below we analyse a question of application of the constructed in Section 3.2 solutions in quadratures of BVPs in the half-plane to solving approximately the corresponding BVPs in the finite domains. The conditions on the domain S, the coefficients of equation (1.1), and the boundary data which guarantee finding of approximate solutions with the preassigned accuracy are established (see [18] and [16], pp. 36-42).

Let $\bar{I}$ contain the segment $[-\gamma, +\gamma]$, $\gamma > 0$, and S contain a half-disk K_R with the radius R.

Let us consider the following BCs:

$$\lim_{z\to\zeta} u(z) = 0, \quad z \in S, \quad \zeta \in \varsigma; \tag{3.156}$$

$$\lim_{z\to x_0} u(z) = f(x_0), \quad x_0 \in \bar{I}\backslash\overset{*}{I}, \quad b \in]-\infty, 1[; \tag{3.157}$$

$$\lim_{z\to x_0} \left(\ln\frac{Ae}{y}\right)^{-1} u(z) = f(x_0), \quad z \in S, \quad x_0 \in \bar{I}\backslash\overset{*}{I}, \quad b = 1; \tag{3.158}$$

$$\lim_{z\to x_0} y^{b-1} u(z) = f(x_0), \quad z \in S, \quad x_0 \in \bar{I}\backslash\overset{*}{I}, \quad b \in]1, +\infty[; \tag{3.159}$$

$$\lim_{z\to x_0} y^{b+m-1}\frac{\partial^m u(z)}{\partial y^m} = f(x_0), \; z \in S, \; x_0 \in \bar{I}\backslash\overset{*}{I}, \; b \in]0, +\infty[, \; m \in \mathbb{N}; \tag{3.160}$$

where

$$f(x_0) := \begin{cases} 0, & x_0 \in \bar{I}\backslash[-\gamma, +\gamma]; \\ f(x_0), & x_0 \in [-\gamma, +\gamma]\backslash\overset{*}{I}. \end{cases}$$

Let

$$M := \sup_{x\in[-\gamma,+\gamma]\backslash\overset{*}{I}} |f(x)| < +\infty.$$

Theorem 3.7.1 *Let $b \in]-\infty, 1[$ and*

$$R > \max\left\{\frac{2\gamma e^{a\overset{*}{a}} M}{\varepsilon\big(1-\overset{*}{\varepsilon}\big)^{\frac{1}{2}}\Lambda(a,b)}, \frac{2\gamma}{\overset{*}{\varepsilon}}\right\}, \tag{3.161}$$

where $\varepsilon > 0$ *is a preassigned exactness of the approximate solution and* $\overset{*}{\varepsilon} \in]0,1[$ *is a fixed arbitrary number,*

$$\overset{*}{a} = \begin{cases} \pi, & a > 0; \\ 0, & a \le 0, \end{cases}$$

then

$$u_1 = \Lambda^{-1}(a,b) y^{1-b} \int\limits_{-\gamma}^{+\gamma} f(\xi) e^{a\theta} \rho^{b-2} d\xi \tag{3.162}$$

represents a solution of the class $T^0(1, S, \overset{*}{I})$ *which satisfies BC (3.157) and the condition*

$$|u_1(\zeta)| < \varepsilon, \;\; \zeta \in \varsigma \tag{3.163}$$

instead of BC (3.156).

If $\overset{*}{I} = \varnothing$ *and* $u_1^T \in T^0(1,S)$ *is the exact solution of BVP (1.1), (3.156), (3.157) (i.e., of a particular case of Problem 3.5.1), then the estimate*

$$\max_{z \in \bar{S}} |u_1(z) - u_1^T(z)| < \varepsilon \tag{3.164}$$

holds.

Theorem 3.7.2 *Let* $b = 1$ *and*

$$R > 2\gamma \max\left\{ \frac{e^{a\overset{*}{a}} M}{\varepsilon \big(1 - \overset{*}{\varepsilon}\big)^{1/2} (1 + e^{a\pi})}, \frac{1}{\overset{*}{\varepsilon}} \right\}, \tag{3.165}$$

then

$$u_2 = (1 + e^{a\pi})^{-1} \int\limits_{-\gamma}^{+\gamma} f(\xi) e^{a\theta} \rho^{-1} d\xi \tag{3.166}$$

represents a solution of the class $T^0\Big(\big(\ln \frac{Ae}{y}\big)^{-1}, S, \overset{*}{I}\Big)$ *which satisfies BC (3.158) and the condition (3.163) instead of BC (3.156).*

If $\overset{*}{I} = \varnothing$ *and* $u_2^T \in T^0\Big(\big(\ln \frac{Ae}{y}\big)^{-1}, S\Big)$ *is the exact solution of BVP (1.1), (3.156), (3.158) (i.e., of particular cases of Problem 3.5.2 and Problem 3.5.4), then the estimate*

$$\max_{z \in \bar{S}} \left(\ln \frac{Ae}{y} \right)^{-1} |u_2(z) - u_2^T(z)| < \varepsilon \tag{3.167}$$

holds.

Theorem 3.7.3 *Let* $b \in]1, +\infty[$ *and*

$$R > \max\left\{ \big(1 - \overset{*}{\varepsilon}\big)^{-1/2} \left[\frac{2\gamma e^{a\overset{*}{a}} M}{\varepsilon \Lambda(a, 2-b)} \right]^{\frac{1}{b}}, \frac{2\gamma}{\overset{*}{\varepsilon}} \right\}. \tag{3.168}$$

then

$$u_3 = \Lambda^{-1}(a, 2-b) \int\limits_{-\gamma}^{+\gamma} f(\xi) e^{a\theta} \rho^{-b} d\xi \tag{3.169}$$

represents a solution of the class $T^0\left(y^{b-1}, S, \overset{*}{I}\right)$ *which satisfies the BC (3.159) and the condition (3.163) instead of BC (3.156) and the condition*

$$y^{b-1}|u_3(z)| < MA^{b-1}, \ \ z \in S.$$

If $\overset{*}{I} = \varnothing$ *and* $u_3^T \in T^0\big(y^{b-1}, S\big)$ *is the exact solution of BVP (1.1), (3.156), (3.159) (i.e., of a particular case of Problem 3.5.3), then the estimate*

$$\max_{z \in \bar{S}} \left(\frac{y}{A}\right)^{b-1} |u_3(z) - u_3^T(z)| < \varepsilon \tag{3.170}$$

holds.

Theorem 3.7.4 *Let* $b \in]0, +\infty[$ *and*

$$R > \max\left\{ \big(1 - \overset{*}{\varepsilon}\big)^{-1/2} \left[\frac{2\gamma b e^{a\overset{*}{a}} M}{\varepsilon (a^2 + b^2)(b, m-1)\Lambda(a, -b)}\right]^{\frac{1}{b}}, \frac{2\gamma}{\overset{*}{\varepsilon}} \right\}. \tag{3.171}$$

then

$$u_4 = \frac{(-1)^m b}{(a^2 + b^2)(b, m-1)\Lambda(a, -b)} \int\limits_{-\gamma}^{+\gamma} f(\xi) e^{a\theta} \rho^{-b} d\xi \tag{3.172}$$

represents a solution of the classes

$$T^0\big(1, S, \overset{*}{I}\big) \cap T^m\big(y^{b+m-1}, S, \overset{*}{I}\big) \ \ \text{and} \ \ T^m\big(y^{b+m-1}, S, \overset{*}{I}\big),$$

when $b \in]0, 1[$ *and* $b \in]1, +\infty[$*, respectively, meeting BC (3.160) and the condition (3.163).*

If $\overset{*}{I} = \varnothing$ *and*

$$u_4^T \in \begin{cases} T^0\big(1, S\big) \cap T^m\big(y^{b+m-1}, S\big), & b \in]0, 1[; \\ T^m\big(y^{b+m-1}, S\big), & b \in]1, +\infty[, \end{cases}$$

is the exact solution of BVP (1.1), (3.156), (3.160) (i.e., of a particular case of Problem 3.5.5 when $b \in]0, 1[$*) and the following estimates*

$$\max_{z \in \bar{S}} |u_4(z) - u_3^T(z)| < \varepsilon, \ \ b \in]0, 1[; \tag{3.173}$$

$$\max_{z \in \bar{S}} \left(\ln \frac{Ae}{y}\right)^{-1} |u_4(z) - u_3^T(z)| < \varepsilon, \ \ b = 1; \tag{3.174}$$

$$\max_{z \in \bar{S}} \left(\frac{y}{A}\right)^{b-1} |u_4(z) - u_3^T(z)| < \varepsilon, \ \ b \in]1, +\infty[, \tag{3.175}$$

hold.

Proof of the Theorem 3.7.1 We extend continuously the function f outside the segment $\bar{I}$ assuming it equal to zero there. Then the exact solution of the Dirichlet problem in $\mathbb{R}^2_+$, according to Theorem 3.2.3, will have the form (3.162) (see the formula (3.35)). Hence, u_1 will satisfy BC (3.157). Now, we choose R in such a way that the solution u_1 be less than preasigned arbitrarily small $\varepsilon > 0$ on the half-circle C_R. Taking into account

$$(1-\overset{*}{\varepsilon})^{\frac{1}{2}} r < \rho, \quad 0 < \overset{*}{\varepsilon} < 1. \tag{3.176}$$

when

$$r > \frac{2\gamma}{\overset{*}{\varepsilon}}, \ \ z \in \mathbb{R}^2_+ \bigcup \mathbb{R}^1, \ \ |\xi| < \gamma,$$

from (3.162), because of

$$\frac{y^{1-b}}{\rho^{2-b}} = \frac{\rho^{1-b} sin^{1-b}\theta}{\rho^{2-b}} \leq \frac{1}{\rho},$$

we obtain

$$|u_1(z)| \leq \frac{e^{a\overset{*}{a}} M}{\Lambda(a,b)} \int\limits_{-\gamma}^{+\gamma} \frac{d\xi}{\rho} \leq \frac{2\gamma e^{a\overset{*}{a}} M}{(1-\overset{*}{\varepsilon})^{\frac{1}{2}} \Lambda(a,b)} \frac{1}{r} \tag{3.177}$$

in $\mathbb{R}^2_+ \backslash K_{\frac{2\gamma}{\overset{*}{\varepsilon}}}$. If we choose R in such a way that (3.161) be fulfilled, then since $r > R$, by virtue of (3.177),

$$|u_1(z)| < \frac{2\gamma e^{a\overset{*}{a}} M}{(1-\overset{*}{\varepsilon})^{\frac{1}{2}} \Lambda(a,b)} \frac{1}{R} < \varepsilon, \ \ z \in R^2_+ \backslash K_R.$$

In particular,

$$|u_1(\zeta)| < \varepsilon, \ \ \zeta \in C_R,$$

as far as ς lies in $R^2_+ \backslash K_R$. So, we may assume that u_1 satisfies up to ε the homogeneous BC on ς.

If $\overset{*}{I} = \varnothing$, then

$$u_1(\xi,0) - u_1^T(\xi,0) = 0, \ \ \xi \in \bar{I},$$

and, in view of (3.163), according to the extremum principle,

$$|u_1(z) - u_1^T(z)| < \varepsilon, \ \ z \in \bar{S},$$

since the last is true on ∂S. The obtained inequality is equally matched to (3.164). $\square$

We prove the theorems 3.7.2-3.7.4 analogously.

Proof of Theorem 3.7.2. The formula (3.166) follows from the formula (3.36). Similarly to the proof of Theorem 3.7.1 we conclude that if R meets the condition (3.165), when $\overset{*}{I} = \varnothing$, then

$$|u_2(\zeta) - u_2^T(\zeta)| < \varepsilon, \ \ \zeta \in \varsigma,$$

$$\lim_{z\to x_0}\left(\ln\frac{Ae}{y}\right)^{-1}[u_2(z)-u_2^T(z)]=0,\quad z\in S,\quad x_0\in\bar I,$$

where $u_2(z)$ has the form (3.166).

Let

$$v_2(z):=\left(\ln\frac{Ae}{y}\right)^{-1}u_2(z),\quad v_2^T(z):=\left(\ln\frac{Ae}{y}\right)^{-1}u_2^T(z).$$

The functions v_2 and v_2^T will satisfy equation (1.1) and the inequality

$$|v_2(\zeta)-v_2^T(\zeta)|<\varepsilon,\ \ \zeta\in\partial S,$$

because of

$$|v_2(\zeta)-v_2^T(\zeta)|<\varepsilon\left(\ln\frac{Ae}{\eta}\right)^{-1}\le\varepsilon,\ \ \zeta\in\varsigma,$$

$$v_2(\xi,0)-v_2^T(\xi,0)=0,\ \ \xi\in\bar I.$$

Therefore, according to the weak extremum principle for equation (3.132), we have the inequality

$$|v_2(z)-v_2^T(z)|<\varepsilon,\ \ z\in\bar S,$$

since the last is true on ∂S. The obtained inequality is equally matched to (3.167). $\square$

Proof of the Theorem 3.7.3. The formula (3.169) follows from the formula (3.37). It is easily seen that if R meets the condition (3.168), when $\overset{*}{I}=\varnothing$, then

$$|u_3(\zeta)-u_3^T(\zeta)|<\varepsilon,\ \ \zeta\in\varsigma,$$

$$\lim_{z\to x_0}y^{b-1}[u_3(z)-u_3^T(z)]=0,\ \ z\in S,\ \ x_0\in\bar I,$$

where $u_3(z)$ has the form (3.169).

Let

$$v_3(z):=\left(\frac{y}{A}\right)^{b-1}u_3(z),\quad v_3^T(z):=\left(\frac{y}{A}\right)^{b-1}u_3^T(z).$$

By the correspondence principle (3.4) the functions v_3 and v_3^T will satisfy the equation

$$E^{(a,2-b)}=0.$$

According to the weak extremum principle

$$|v_3(z)-v_3^T(z)|<\varepsilon,\ \ z\in S,$$

because of

$$|v_3(\zeta)-v_3^T(\zeta)|<\varepsilon\left(\frac{\eta}{A}\right)^{b-1}\le\varepsilon,\ \ \zeta\in\varsigma;$$

$$v_3(\xi,0)-v_3^T(\xi,0)=0,\ \ \xi\in\bar I,$$

i.e., holds

$$\left(\frac{y}{A}\right)^{b-1}|u_3(z)-u_3^T(z)|<\varepsilon,\quad z\in\bar S,$$

which is equally matched to (3.170). □

Proof of the Theorem 3.7.4. The formula (3.172) follows from the formula (3.66). It is easily seen that if R meets the condition (3.171), when $\overset{*}{I}=\varnothing$, then

$$|u_4(\zeta)-u_4^T(\zeta)|<\varepsilon,\quad \zeta\in\varsigma,$$

$$\lim_{z\to x_0} y^{b+m-1}\frac{\partial^m[u_4(\zeta)-u_4^T(\zeta)]}{\partial y^m}=0,\quad z\in S,\quad x_0\in\bar I,$$

where $u_4(z)$ has the form (3.172).

Let $b\in]0,1[$. According to the weak extremum and the Zaremba-Giraud principles (see Section 3.1) the function $u_4-u_4^T$ may attain its extremal values only on ς, i.e.,

$$|u_4(z)-u_4^T(z)|<\varepsilon,\quad z\in S,$$

which equally matches to (3.173).

Let $b=1$. Then

$$\lim_{z\to x_0}\left(\ln\frac{Ae}{y}\right)^{-1}u_4(z)=\lim_{z\to x_0}\frac{\frac{\partial^m u_4(z)}{\partial y^m}}{(-1)^m(1,m-1)y^{-m}}=\frac{f(x_0)}{(-1)^m(1,m-1)},$$
$$z\in S,\quad x_0\in\bar I.$$

Whence,

$$\lim_{z\to x_0}\left(\ln\frac{Ae}{y}\right)^{-1}(-1)^m(1,m-1)u_4(z)=f(x_0),\quad z\in S,\quad x_0\in\bar I.$$

Because of

$$|u_4(\zeta)|<\varepsilon,\quad \zeta\in\varsigma,\tag{3.178}$$

we have

$$|v_4(\zeta)|<(1,m-1)\varepsilon,\quad \zeta\in\varsigma,\tag{3.179}$$

$$\lim_{z\to x_0}\left(\ln\frac{Ae}{y}\right)^{-1}v_4(z)=f(x_0),\quad z\in S,\quad x_0\in\bar I,\tag{3.180}$$

where

$$v_4(z)=(-1)^m(1,m-1)u_4(z).\tag{3.181}$$

By virtue of (3.180), (3.181), and Theorem 3.7.2, evidently,

$$\max_{z\in\bar S}\left(\ln\frac{Ae}{y}\right)^{-1}|v_4(z)-v_4^T(z)|<(1,m-1)\varepsilon,\tag{3.182}$$

where
$$v_4^T(z) = (-1)^m(1, m-1)u_4^T(z). \tag{3.183}$$
In view of (3.181), (3.183), from (3.182) it follows (3.174).

The case $b > 1$ we consider analogously.

$$\lim_{z\to x_0} y^{b-1}u_4(z) = \lim_{z\to x_0} \frac{\frac{\partial^m u_4(z)}{\partial y^m}}{(-1)^m(b-1, m)y^{1-b-m}} = \frac{f(x_0)}{(-1)^m(b-1, m)},$$

$$z \in S, \ \ x_0 \in \bar{I}.$$

Hence,
$$\lim_{z\to x_0} y^{b-1}\overset{*}{v}_4(z) = f(x_0), \ \ z \in S, \ \ x_0 \in \bar{I}, \tag{3.184}$$
where
$$\overset{*}{v}_4(z) = (-1)^m(b-1, m)u_4(z). \tag{3.185}$$
By (3.178), we get
$$|\overset{*}{v}_4(\zeta)| < (b-1, m)\varepsilon, \ \ \zeta \in \varsigma. \tag{3.186}$$
By virtue of (3.184), (3.186), and Theorem 3.7.3, evidently,

$$\max_{z\in\bar{S}} \Big(\frac{y}{A}\Big)^{b-1} |\overset{*}{v}_4(z) - \overset{*}{v}{}_4^T(z)| < (b-1, m)\varepsilon, \tag{3.187}$$

where
$$\overset{*}{v}{}_4^T(z) = (-1)^m(b-1, m)u_4^T(z). \tag{3.188}$$
In view of (3.185) and (3.188), from (3.187) it follows (3.175). □

Remark 3.7.5 *Substituting*
$$\xi = x - y \operatorname{ctg}\theta$$
into the formulas (3.35)-(3.37), (3.66), (3.162), (3.166), (3.169), (3.172), and taking into account

$$\theta = \operatorname{arcctg}\frac{x-\xi}{y}, \qquad \rho^2 = (x-\xi)^2 + y^2,$$

$$x - \xi = \rho\cos\theta, \quad\ y = \rho\sin\theta,$$
$$d\xi = y\sin^{-2}\theta d\theta = \rho\sin^{-1}\theta d\theta,$$

we obtain more suitable for the numerical computation formulas:

$$\frac{y^{1-b}}{\Lambda(a,b)}\int_{-\infty}^{\infty} f(\xi)e^{a\theta}\rho^{b-2}d\xi = \frac{1}{\Lambda(a,b)}\int_0^{\pi} f(x - y\operatorname{ctg}\theta)e^{a\theta}\sin^{-b}\theta d\theta, \quad b < 1;$$

$$\frac{1}{1+e^{\pi}}\int_{-\infty}^{\infty} f(\xi)e^{a\theta}\rho^{-1}d\xi = \frac{1}{1+e^{a\pi}}\int_0^{\pi} f(x - y\operatorname{ctg}\theta)e^{a\theta}\sin^{-1}\theta d\theta, \quad b = 1,$$

where

$$f = O(\sin^{\alpha}\theta), \quad \theta \to 0+; \quad f = O(\sin^{\alpha}(\pi-\theta)), \quad \theta \to \pi-, \quad \alpha > 0,$$

because of

$$|\xi|^{-\alpha} = (x - y\mathrm{ctg}\theta)^{-\alpha} = (x\sin\theta - y\cos\theta)^{-\alpha}\sin^{\alpha}\theta;$$

$$\frac{1}{\Lambda(a,2-b)}\int\limits_{-\infty}^{\infty} f(\xi)e^{a\theta}\rho^{-b}d\xi$$

$$= \frac{y^{1-b}}{\Lambda(a,2-b)}\int\limits_{0}^{\pi} f(x - y\mathrm{ctg}\theta)e^{a\theta}\sin^{b-2}\theta d\theta, \quad b > 1;$$

$$M^{-1}(a,b,m)\int\limits_{-\infty}^{+\infty} f(\xi)e^{a\theta}\rho^{-b}d\xi$$

$$= M^{-1}(a,b,m)y^{1-b}\int\limits_{0}^{\pi} f(x - y\mathrm{ctg}\theta)e^{a\theta}\sin^{b-2}\theta d\theta, \quad (a,b) \in i_{2,m};$$

$$u_1 = \Lambda^{-1}(a,b)y^{1-b}\int\limits_{-\gamma}^{+\gamma} f(\xi)e^{a\theta}\rho^{b-2}d\xi$$

$$= \frac{1}{\Lambda(a,b)}\int\limits_{\mathrm{arcctg}\frac{x+\gamma}{y}}^{\mathrm{arcctg}\frac{x-\gamma}{y}} f(x - y\mathrm{ctg}\theta)e^{a\theta}\sin^{-b}\theta d\theta, \quad b < 1;$$

$$u_2 = (1 + e^{a\pi})^{-1}\int\limits_{-\gamma}^{+\gamma} f(\xi)e^{a\theta}\rho^{-1}d\xi$$

$$= \frac{1}{1 + e^{a\pi}}\int\limits_{\mathrm{arcctg}\frac{x+\gamma}{y}}^{\mathrm{arcctg}\frac{x-\gamma}{y}} f(x - y\mathrm{ctg}\theta)e^{a\theta}\sin^{-1}\theta d\theta, \quad b = 1;$$

$$u_3 = \Lambda^{-1}(a,2-b)\int\limits_{-\gamma}^{+\gamma} f(\xi)e^{a\theta}\rho^{-b}d\xi$$

$$= \frac{y^{1-b}}{\Lambda(a,2-b)}\int\limits_{\mathrm{arcctg}\frac{x+\gamma}{y}}^{\mathrm{arcctg}\frac{x-\gamma}{y}} f(x - y\mathrm{ctg}\theta)e^{a\theta}\sin^{b-2}\theta d\theta, \quad b > 1;$$

$$u_4 = \frac{(-1)^m b}{(a^2+b^2)(b,m-1)\Lambda(a,-b)}\int\limits_{-\gamma}^{+\gamma} f(\xi)e^{a\theta}\rho^{-b}d\xi$$

$$= \frac{(-1)^m b y^{1-b}}{(a^2+b^2)(b,m-1)\Lambda(a,-b)} \int\limits_{\operatorname{arcctg}\frac{x+\gamma}{y}}^{\operatorname{arcctg}\frac{x-\gamma}{y}} f(x - y\operatorname{ctg}\theta) e^{a\theta} \sin^{b-2}\theta d\theta,\ b \in]0,1[,$$

respectively.

3.8 Applications to other degenerate partial differential equations

This section is devoted to application of the previous results on the one hand to the axially symmetric solutions of the equation (see G. Jaiani [16], pp. 43-45, and [21], pp. 56-59)

$$\left(x_2^2 + \cdots + x_p^2\right)^{\frac{1}{2}} \left(u_{x_1x_1} + \cdots + u_{x_px_p}\right) + a u_{x_1} = 0 \tag{3.189}$$

and on the other hand to solutions to the equation (see [16], p. 24)

$$F^{(a,b,c)}u := y^2 \Delta u + a y u_x + b y u_y + c u = 0, \tag{3.190}$$

where a, b, and c are, in general, complex numbers.

In $\mathbb{R}^p$, $p \geq 3$, axially symmetric with respect to x_1 solutions

$$u \in C^2\Big(\mathbb{R}^p \backslash \Big(\mathbb{R}^1 \times \underbrace{\{0\} \times \cdots \times \{0\}}_{(p-1)-times}\Big)\Big)$$

of equation (3.189) will be solutions to equation (1.1) with (1.4).

From Theorem 3.3.8 it follows the following assertions:

1. the expression (3.66) with (1.4) and $p \in \mathbb{N}\backslash\{1,2\}$,

$$m \in \begin{cases} \mathbb{N}^0, & p > 3, \\ \mathbb{N}, & p = 3, \end{cases} \qquad m > 3 - p$$

represents a unique solution of

Problem 3.8.1. *Find in*

$$\mathbb{R}^p \backslash \Big(\mathbb{R}^1 \times \underbrace{\{0\} \times \cdots \{0\}}_{(p-1)-times}\Big)$$

the axially symmetric with respect to x_1 solution

$$u \in C^2\Big(\mathbb{R}^p \backslash \Big(\mathbb{R}^1 \times \underbrace{\{0\} \times \cdots \{0\}}_{(p-1)-times}\Big)\Big)$$

of equation (3.189) satisfying the following conditions:
(i) on the axis of symmetry x_1

$$\lim_{(x_2^2+\cdots+x_p^2)^{1/2} \to 0} (x_2^2 + \cdots + x_p^2)^{\frac{p+m-3}{2}} \frac{\partial^m u(x_1, \ldots, x_p)}{\partial \sqrt{x_2^2 + \cdots + x_p^2}^{\,m}} = f(x_1), \quad x_1 \in \mathbb{R}^1,$$

where f is a continuous function;
if $p=3$ we additionally demand

$$f(x_1)=O(|x_1|^{-\alpha}),\quad |x_1|\to+\infty,\quad \alpha>0, \tag{3.191}$$

and

$$\lim_{(x_2^2+x_3^2)^{\frac{1}{2}}\to 0}\int\limits_{-\infty}^{+\infty}(x_2^2+x_3^2)^{\frac{1}{2}}u(x_1,x_2,x_3)\frac{\partial u(x_1,x_2,x_3)}{\partial\sqrt{x_2^2+x_3^2}}dx_1=0, \tag{3.192}$$

when

$$\lim_{(x_2^2+x_3^2)^{\frac{1}{2}}\to 0}\left[\frac{1}{2}\ln(x_2^2+x_3^2)\right]^{-1}u(x_1,x_2,x_3)=0,\quad x_1\in\mathbb{R}^1; \tag{3.193}$$

(ii) at infinity for $p\geq 4$ we assume

$$u(x_1,\dots,x_p)=\begin{cases} O\Big((x_2^2+\cdots+x_p^2)^{\frac{3-p}{2}}\Big),\ \ x_1^2+\cdots+x_p^2\to+\infty,\\ \text{when either}\ \ a\in\mathbb{R}^1,\ \ p\in\mathbb{N}\backslash\{1,2,3,4\},\ \ \text{or}\ \ a=0,\ \ p=4;\\ o\Big((x_2^2+\cdots+x_p^2)^{\frac{3-p}{2}}\Big),\ \ x_1^2+\cdots+x_p^2\to+\infty,\\ \text{when}\ \ a\neq 0,\ \ p=4,\end{cases}$$

while for $p=3$ we demand

$$u(x_1,x_2,x_3)=O((x_1^2+x_2^2+x_3^2)^{-1}),\quad x_1^2+x_2^2+x_3^2\to+\infty, \tag{3.194}$$

$$\frac{\partial u(x_1,x_2,x_3)}{\partial x_1},\ \ \frac{\partial u(x_1,x_2,x_3)}{\partial\sqrt{x_2^2+x_3^2}}=O((x_1^2+x_2^2+x_3^2)^{-2}),\quad x_1^2+x_2^2+x_3^2\to+\infty, \tag{3.195}$$

2. The expression (3.67) with (1.4) and $p=3$, $m=0$ represents a unique solution of

Problem 3.8.2. *Find in*

$$\mathbb{R}^3\backslash(\mathbb{R}^1\times\{0\}\times\{0\})$$

the axially symmetric with respect to axis x_1 solution $u\in C^2(\mathbb{R}^3\backslash(\mathbb{R}^1\times\{0\}\times\{0\}))$ to the equation

$$(x_2^2+x_3^2)^{\frac{1}{2}}(u_{x_1x_1}+u_{x_2x_2}+u_{x_3x_3})+au_{x_1}=0,$$

satisfying the following conditions:
(i) on the axis of symmetry x_1

$$\lim_{(x_2^2+x_3^2)^{\frac{1}{2}}\to 0}\left[\frac{1}{2}\ln(x_2^2+x_3^2)\right]^{-1}u(x_1,x_2,x_3)=f(x_1),\quad x_1\in\mathbb{R}^1,$$

where f is a continuous function, satisfying (3.191), besides (3.192) is fulfilled for (3.193);
(ii) at infinity (3.194) and (3.195) are fulfilled.

From the identity

$$y^{\frac{b-b^{\pm}}{2}-1}F^{(a,b,c)}\Big(y^{\frac{b^{\pm}-b}{2}}u\Big) \equiv E^{(a,b^{\pm})}u,$$

where

$$b^{\pm} = 1 \pm \sqrt{(1-b)^2-4c}, \tag{3.196}$$

it follows that

$$y^{\frac{b^{\pm}-b}{2}}u^{(a,b^{\pm})} = u^{(a,b,c)}, \tag{3.197}$$

where $u^{(a,b,c)} \in C^2$ is a solution to equation (3.190).

Equality (3.197) associates to each $u^{(a,b,c)}$ a pair of solutions $u^{(a,b^{\pm})}$ to equation

$$E^{(a,b^{\pm})}u = 0$$

and vice versa.

The equality (3.197) makes possible all the results obtained concerning (1.1) to reformulate for the case of equation (3.190). E.g., excluding the case of the negative $(1-b)^2-4c$, we reduce the following problem 3.8.3 to the problem 3.2.1

Problem 3.8.3. *In $\mathbb{R}^2_+$ find $u \in C^2(\mathbb{R}^2_+)$ satisfying equation (3.190) and one of the following BCs*

$$\lim_{z\to x_0} y^{\frac{b-b^{\pm}}{2}}u(z) = f(x_0), \;\; \text{when} \;\; \mathrm{Re}b^{\pm} \in]-\infty,1[;$$
$$\lim_{z\to x_0}\Big(\ln\frac{1}{y}\Big)y^{\frac{b-1}{2}}u(z) = f(x_0), \;\; \text{when} \;\; b^{\pm}=1;$$
$$\lim_{z\to x_0} y^{\frac{b+b^{\pm}}{2}-1}u(z) = f(x_0), \;\; \text{when} \;\; \mathrm{Re}b^{\pm} \in]1,+\infty[,$$

where $z \in \mathbb{R}^2_+$, $x_0 \in \mathbb{R}^1\backslash \overset{*}{I}$, and f meets conditions of the problem 3.2.1.

So, from the Theorem 3.2.3 we get

Theorem 3.8.4 *A solution of Problem 3.8.3 has the form*

$$u(z) = \begin{cases} \dfrac{y^{1-\frac{b+b^{\pm}}{2}}}{\Lambda(a,b^{\pm})}\displaystyle\int_{-\infty}^{+\infty} f(\xi)e^{a\theta}\rho^{b^{\pm}-2}d\xi, & \mathrm{Re}b^{\pm} \in]-\infty,1[;\\ y^{\frac{1-b}{2}}(1+e^{a\pi})^{-1}\displaystyle\int_{-\infty}^{+\infty} f(\xi)e^{a\theta}\rho^{-1}d\xi, & b^{\pm}=1;\\ \dfrac{y^{\frac{b^{\pm}-b}{2}}}{\Lambda(a,2-b^{\pm})}\displaystyle\int_{-\infty}^{+\infty} f(\xi)e^{a\theta}\rho^{-b^{\pm}}d\xi, & \mathrm{Re}b^{\pm} \in]1,\infty[, \end{cases}$$

where when $b^{\pm}=1$ the function f meets conditions of the Theorem 3.2.3 and setting BVPs along with "b" depends on "c" as well.

Remark 3.8.5 *From (3.196) we have $b^+ = 2-b^-$. Therefore,* $\mathrm{Re}b^- > 1$ *for* $\mathrm{Re}b^+ < 1$ *and vice versa.*

Remark 3.8.6 *For $c = 0$ equation (3.190) coincides with (1.1). For $c \neq 0$, as it follows from Theorem 3.8.4, only the weighted boundary value problems may be well-posed. It means that in the last case all the solutions of equation (3.190) are unbounded as $y \to 0+$.*

3.9 The canonical form equation with order and type degenerations

In the present section using the barrier method (see e.g. [1], pp. 187-194, [32], pp. 15-20, and also [2], [3]) the Keldysh [25] theorem for the second-order elliptic in $\mathbb{R}^2_+$ equation of the canonical form with the characteristic parabolic degeneration[13] is generalized for the case of the elliptic equation of the second-order canonical form with order and type degeneration see G. Jaiani [23]. The criteria under which the Dirichlet or Keldysh problems are well-posed are given in a one-sided neighborhood of the degeneration segment, enabling one to write the criteria in a single form. Moreover , some cases are pointed out in which it is even necessary to give a criterion in the neighborhood because it is impossible to establish it on the segment of degeneracy of the equation. In this section we follow the above Paper of G. Jaiani [23].

Let us consider the equation

$$L(u) := y^m \frac{\partial^2 u}{\partial x^2} + y^n \frac{\partial^2 u}{\partial y^2} + a(x,y)\frac{\partial u}{\partial x}$$

$$+b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = 0, \qquad m,n = \text{const} \geq 0, \tag{3.198}$$

in a domain Ω bounded by a sufficiently smooth are ς lying in the upper half-plane $y \geq 0$ and by a segment $\overline{AB}$ of the x-axis;

$$a, b, c, \in \mathcal{A}(\bar{\Omega}), \quad c \leq 0 \ \text{ in } \ \bar{\Omega},^{14} \tag{3.199}$$

where $\mathcal{A}(\bar{\Omega})$ is the class of functions analytic on $\bar{\Omega}$ with respect to x, y.

Let us examine two boundary value problems:

Problem 3.9.1. (Dirichlet Problem) *Find $u \in C^2(\Omega) \cap C(\bar{\Omega})$ in Ω from prescribed continuous values of $L(u)$ in Ω and of u on the boundary $\partial\Omega$.*

Problem 3.9.2. (Keldysh Problem) *Find bounded $u \in C^2(\Omega) \cap C(\Omega \cup \varsigma)$ in Ω from prescribed continuous values of $L(u)$ in Ω and of u only on the part ς of the boundary $\partial\Omega$.*

---

[13] which we obtain from (3.198) for $m = 0$.

[14] Theorem 3.9.3 remains true when analyticity in (3.199) is replaced by Lipschitz continuity (see footnotes of the present section and take into account that integral representation by means of Green's function of a solution of the Dirichlet problem for the non-degenerate elliptic equation is valid also in the case of Lipschitz continuous coefficients).

$C(\bar{\Omega})$ is a set of functions continuous on closure of Ω. $C^2(\Omega)$ is a set of functions with continuous derivatives of orders ≤ 2 in Ω.

Let

$$I_\delta := \{(x,y) \in \Omega :\ 0 < y < \delta,\ \ \delta = \text{const } > 0\}.$$

Theorem 3.9.3 *If either $n < 1$ or $n \geq 1$* [15] *and*

$$b(x,y) < y^{n-1} \quad \text{on} \quad \bar{I}_\delta,^{16} \tag{3.200}$$

the Dirichlet problem is well-posed while the Keldysh problem has an infinite number of solutions.
If $n \geq 1$ and

$$b(x,y) \geq y^{n-1} \quad \text{in} \quad I_\delta,^{17} \tag{3.201}$$

furthermore

$$a(x,y) = O(y^m), \qquad y \to 0+ \tag{3.202}$$

(O is the Landau symbol), the Keldysh problem is well-posed while the Dirichlet problem, in general, has no solutions.

Proof. We look for desired solutions as follows (see, e.g. Bitsadze [1], p. 189; Smirnov [32], p. 16).

A solution of the Keldysh problem with continuous data on ς we construct as follows. Let $f(x,y) \in C\overline{\Omega})$. We construct a sequence of increasing domains such that $\Omega_{h_n} \in \Omega$, $n \in \mathbb{N}$ with a smooth boundary. For all the points of Ω_{h_n} where h_n is sufficiently small and of boundary S_{h_n} of Ω_{h_n} we have $y \geq h_n > 0$. The boundary S_{h_n} of the domain Ω_{h_n} coincides with ς for $y > h_n$ denote it ς_n and outside of some neighborhood of the endpoints $\{P_i\} \in \varsigma_n$ go along the straight lines $y = h_n$ parallel to the x-axis ($h_n \to 0$ *as* $n \to \infty$). Let $u_{h_n}(x,y)$ be a solution of the Dirichlet problem for equation (3.198) obtaining values $f(x,y)$ on the boundary S_{h_n}. Such solutions exist since in $\overline{\Omega}_{h_n}$ equation (3.198) is not degenerate one. In view of $c \leq 0$ in $\overline{\Omega}_{h_*}$ we have

$$\text{Uniformly } \ |u_{h_n}| < M, \ \text{ where } \ M = \max_{(x,y)\in\overline{\Omega}} |f(x,y)|. \tag{3.203}$$

Let h_* be an arbitrary small fixed value of h_n starting from the value $h_1 < h_*$ for all the values $h_{n+1} < h_n < h_*$ the family $\{u_{h_n}\}$ is fully definite.

Let $G_*(x,y,\xi,\eta)$ be the Green function of the Dirichlet problem for equation (3.198) in Ω_{h_*}. Then,

$$u_{h_n}(x,y) = \int\limits_{S_{h_*}} u_{h_n}(s) \frac{\partial G_*(x,y,\xi,\eta)}{\partial \nu} ds, \quad h_n < h_*, \tag{3.204}$$

where ν is a normal to S_{h_*}. Therefore, taking into account (3.203), it follows equicontinuity and uniform boundedness of the set of functions $\{u_{h_n}\}$ on Ω_{h_*}.

[15] If $1 < n < 2$, $b(x,0) \leq 0$ then the Dirichlet problem is well-posed (see Remark 3.9.5).

[16] Closure is essential, since we need $b(x,y) - y^{n-1} < 0$ on closing of the neigborhood, containing a part of the line $y = 0$.

[17] There does not exist such $(x_0, 0)$ that $b(x_0, 0) = 0$ (see Remark 3.9.10).

According to the Arzelá Theorem $\{u_{h_n}\}$ will be compact inside Ω, i.e., we may select subsequence $u_{h_{n_i}}(x,y)$ uniformly converging to some function

$$u(x,y) = \lim_{n_i\to\infty} u_{h_{n_i}}(x,y), \tag{3.205}$$

which, by virtue of (3.199), will be a unique solution of (3.198) in Ω which will take the value $f(x,y)$ on ς.

The above-constructed solution (3.205) of equation (3.198) will serve as a unique solution of the Keldish problem, provided we find the function W (see bellow), while it will serve as a unique solution of the Dirichlet problem, provided we find the s.c. barrier function (see bellow).

In Bitsadze [1] it is shown for equation (3.198) that

— if for any point $(x_0,0)$, $x_0 \in \overline{AB}$, there exists barrier $v \in C^2(\omega^\delta_{x_0})$, where

$$\omega^\delta_{x_0} := \{(x,y)\in\Omega:\ y>0,\ (x-x_0)^2+y^2<\delta,\ \ \delta = \text{const } >0\},$$

such that

$$v \in C(\overline{\omega^\delta_{x_0}}),$$

$$v(x_0,0)=0,$$

$$v>0 \quad \text{in} \quad \overline{\omega^\delta_{x_0}} \setminus \{(x_0,0)\},$$

$$L(v) < \eta = \text{const } < 0 \quad \text{in} \quad \omega^\delta_{x_0},$$

then the Dirichlet problem is well-posed, since constructed by the above way solution tends to prescribed values of the function f of the segment $\overline{AB}$.

— if there exists $W \in C^2(\Omega)$ such that:

$$W>0 \ \text{ in } \ \ \Omega\cup\sigma,$$

$$\lim_{y\to 0+} W(x,y) = +\infty$$

uniformly with respect to x,

$$L(W) < 0 \ \text{ inside } \ \ \Omega,$$

then the Keldysh problem is well-posed,

Indeed, let $u(x,y)$ be a solution of equation (3.198) which vanishes on ς, by virtue of $L(\varepsilon W \pm u) < 0$ inside of Ω, the function $\varepsilon W \pm u$ may not have a negative minimum in Ω and, since its limits on the boundary $\partial\Omega$ are positive, inside of Ω we have $\varepsilon W \pm u > 0$, i.e.,$|u| \le \varepsilon W$, whence, because of arbitrariness of $\varepsilon > 0$, $u \equiv 0$ in $\overline{\Omega}$.

Let us show that by (3.200) the function

$$v(x,y) = (-\ln y)^{-1} + (x-x_0)^2$$

may serve as a barrier function.

Indeed, taking into account (3.198), we have

$$\begin{aligned} L(v) &= 2y^m + 2y^{n-2}(-\ln y)^{-3} - y^{n-2}(-\ln y)^{-2} + 2a\cdot(x-x_0) \\ &+ b\cdot y^{-1}(-\ln y)^{-2} + cv = [b(x,y) - y^{n-1}]y^{-1}(-\ln y)^{-2} \\ &+ 2y^{n-2}(-\ln y)^{-3} + 2y^m + 2a\cdot(x-x_0) + cv \\ &< \eta = \text{const } < 0 \quad \text{in} \quad \omega^{\delta}_{x_0}, \end{aligned} \tag{3.206}$$

since the sign of $L(v)$ when $y \to 0+$ is defined by the first term of (3.206)

$$[b(x,y) - y^{n-1}]y^{-1}(-\ln y)^{-2}, \tag{3.207}$$ [18]

and for $n \geq 1$, in view of (3.200), there does not exist such $(x_0, 0)$ that $b(x_0, 0) = 0$ (see Remark 3.9.9) and, therefore,

$$\lim_{y\to 0+} L(v) = -\infty. \tag{3.208}$$

If $0 \leq n < 1$, we rewrite the first term of (3.206) as

$$[y^{1-n}b(x,y) - 1]y^{n-2}(-\ln y)^{-2}. \tag{3.209}$$

Because of (3.199)

$$\lim_{y\to 0+}[y^{1-n}b(x,y) - 1] = -1.$$ [19]

Therefore, (3.208) holds in this case too.

It is easily seen that the other properties of the barrier are also fulfilled.

To prove the second part of the theorem, let us consider the function

$$W(x,y) = -\ln y - (x-\alpha)^l + k,$$

where $x - \alpha > 1$, $\alpha, k = \text{const}$, $l > 2$ is an integer.

Obviously,

---

[18] If $n \geq 2$, only (3.207) tends to infinity (namly to $-\infty$), since

$$\lim_{y\to 0+} 2y^{n-2}(-\ln y)^{-3} = 0.$$

If $1 < n < 2$, then

$$\lim_{y\to 0+} \frac{\big[b(x,y) - y^{n-1}\big]y^{-1}(-\ln y)^{-2}}{2y^{n-2}(-\ln y)^{-3}} = \lim_{y\to 0+} \frac{1}{2}\big[b(x,y) - y^{n-1}\big]y^{1-n}(-\ln y)$$

$$= \begin{cases} -\infty & \text{for} \quad 1 < n < 2 \\ \lim\limits_{y\to 0+} \frac{1}{2}\big[b(x,y)y^{1-n} - 1\big](-\ln y) = -\infty & \text{for} \quad 0 \leq n < 1. \end{cases}$$

[19] Evidently, for negativeness of the limit it will be sufficient if

$$\lim_{y\to 0+} y^{1-n}b(x,y) = \gamma = \text{const } < 1.$$

it is true, because of $\gamma = 0$ for continuous $b(x,y)$ in $\overline{\Omega}$ and all the more for Lipschitz continuous and analytic function $b(x,y)$ in $\overline{\Omega}$. So, in our case $\gamma = 0$.

$$
\begin{aligned}
L(W) &= -l(l-1)(x-\alpha)^{l-2}y^m + y^{n-2} - la\cdot(x-\alpha)^{l-1} - \frac{b}{y} + cW \\
&= \frac{y^{n-1}-b(x,y)}{y} - \frac{1}{3}l[y^m(l-1)+3a\cdot(x-\alpha)](x-\alpha)^{l-2} \\
&- \frac{2}{3}l(l-1)(x-\alpha)^{l-2}y^m + cW.
\end{aligned} \tag{3.210}
$$

In view of (3.202), l can be chosen so that

$$
l-1 > 3\max_{\Omega}(x-\alpha)\sup_{\Omega}\frac{|a|}{y^m} \geq \frac{3|a|(x-\alpha)}{y^m} \quad \text{in} \quad \Omega. \tag{3.211}
$$

On the other hand, by virtue of (3.201),

$$
\frac{y^{n-1}-b(x,y)}{y} \leq 0; \ \text{ in } \ I_\delta.
$$

Hence, taking into account that $1 < (x-\alpha)^{l-2}$ and

$$
-\frac{2}{3}l(l-1)y^m > -\frac{2}{3}l(l-1)(x-\alpha)^{l-2}y^m \quad \text{in} \quad \Omega,
$$

from (3.210) we have

$$
\begin{aligned}
L(W) &< -\frac{2}{3}l(l-1)(x-\alpha)^{l-2}y^m + cW \\
&\leq -\frac{2}{3}l(l-1)y^m < 0; \ \text{ in } \ I_\delta,
\end{aligned} \tag{3.212}
$$

since $W > 0$ in $\Omega\cup\sigma$ for suitably chosen k. It is clear that there exist $A, l = \text{const}$ such that

$$
\frac{y^{n-1}-b(x,y)}{y} < A; \ \text{ and } \ l(l-1) > \frac{3A}{y^m}; \ \text{ in } \ \overline{\Omega\setminus I_\delta}.
$$

Further,

$$
L(W) < A - \frac{2}{3}l(l-1)y^m + cW < -\frac{1}{3}l(l-1)y^m < 0; \ \text{ in } \ \overline{\Omega\setminus I_\delta}. \tag{3.213}
$$

from (3.212) and (3.213) there follows

$$
L(W) < 0 \quad \text{in} \quad \Omega.
$$

The fulfillment of the other properties of the function W is obvious. □

The following remarks should be considered as tasks for subsequent discussion on the topic.

Remark 3.9.4 *Condition (3.202) is not necessary. If $a(x, y) \geq 0$ in I_δ then*

$$L(W) < -l(l-1)(x-\alpha)^{l-2}y^m < -l(l-1)y^m < 0; \quad \text{in} \quad I_\delta,$$

and, by virtue of (3.213), which is valid since (3.211) holds for $\overline{\Omega \setminus I_\delta}$, the theorem remains true without restriction (3.202). On the other hand, if (3.201) is fulfilled in Ω and $c < 0$ in $\bar{\Omega}$ or $b(x, y) > y^{n-1}$ in Ω and $c \leq 0$ in $\bar{\Omega}$, then

$$W^* = -\ln y + k$$

can serve as the Keldysh function, since

$$L(W^*) = \frac{y^{n-1} - b(x, y)}{y} + cW^* < 0 \quad \text{in} \quad \Omega,$$

and condition (3.202) is again unnecessary.

Remark 3.9.5 *When $1 < n < 2$, $b(x, 0) \leq 0$, the sign of $L(v)$ (see (3.206)) is defined by (3.207). Since $b \in \mathcal{A}(\bar{\Omega})$,*

$$[b(x, y) - y^{n-1}]y^{-1}\ln^{-2} y = [b(x, 0) + \frac{\partial b(x, 0)}{\partial y}y + O(y^2)]y^{-1}(\ln y)^{-2}$$

$$-y^{n-2}\ln^{-2} y \leq \left[\frac{\partial b(x, 0)}{\partial y} - O(y)\right]\ln^{-2} y - y^{n-2}\ln^{-2} y,$$

where the first term tends to zero and the second one tends to - ∞. Therefore, (3.208) is fulfilled and the Dirichlet problem is well-posed. The same remains true if $b(x, y)$ is not analytic but it is Lipschitz continuous i.e., $b(x, y) - b(x, 0) \leq \text{const}\, y$ in $\overline{\Omega}$, since in some neighborhood of $y = 0$ we will have

$$b(x, y)y^{-1}\ln^{-2} y = \Big[b(x, y) - b(x, 0)\Big]y^{-1}\ln^{-2} y + b(x, 0)y^{-1}\ln^{-2} y$$

$$\leq \text{const } \ln^{-2} y + b(x, 0)y^{-1}ln^{-2}y \leq \text{const } \ln^{-2} y,$$

by virtue of

$$b(x, y) - b(x, 0) < \text{const}\, y \quad \text{in} \quad \overline{\Omega},$$

and

$$\lim_{y \to 0_+} b(x, y)y^{-1}\ln^{-2} y \leq \text{const} \lim_{y \to 0_+} \ln^{-2} y = 0.$$

So, in this case the Keldysh criterion of well-posedness of the Dirichlet problem is valid also for equation (3.198).

Remark 3.9.6 *Because of the continuity on $\bar{\Omega}$ of both sides of (3.201) for $n \geq 1$, (3.201) holds also in $\bar{I}_\delta$.*

Remark 3.9.7 *Because of (3.199) $b(x, y) \not\equiv y^{n-1}$ in $\bar{\Omega}$ when $\{n\} \neq 0$ ($\{n\}$ is a fractional part of n) since $y^{n-1}, \{n\} \neq 0$, is neither analytic nor Lipschitz continuous on $\bar{\Omega}$.*

Remark 3.9.8 *For $0 \le n < 1$, because of the boundedness (see (3.199)) of $b(x,y)$ and $\lim\limits_{y\to 0+} y^{n-1} = +\infty$. (3.200) is always fulfilled in I_δ and (3.201) cannot take place in I_δ. In that case from (3.200) in I_δ there follows*

$$y^{1-n}b(x,y) - 1 < 0 \ \text{ in } \ \bar{I}_\delta,$$

and the well-posedness of the Dirichlet problem is clear (see the proof of the Theorem 3.9.3). Hence, we can embrace the case $0 \le n < 1$ with condition (3.200). But because of the clearness of the question (for $0 \le n < 1$ the Dirichlet problem is always well-posed), this case is considered separately.

Remark 3.9.9 *When (3.200) holds we have either $n = 1,\ b(x,0) < 1$* or *$n > 1,\ b(x,0) < 0$ and vice versa. Indeed, when $n = 1$, (3.200) obviously implies $b(x,0) < 1, x \in \overline{AB}$, and from the latter there follows (3.200) since $[1 - b(x,y)] \in C(\bar{\Omega})$ (see (3.199)) and has to preserve its sign in closure of some $I_{\delta^*} \subset I_\delta,\ \delta^* < \delta$. If $n > 1$ from (3.200) there follows $b(x,0) < 0$ and from the latter as above $b(x,y) < 0$ in some I_{δ^*} and, therefore, there obviously follows (3.200), since $b(x,y) < 0 \le y^{n-1}$ in $\bar{I}_{\delta^*}$.*

Remark 3.9.10 *When (3.201) holds we have either $n = 1, b(x,0) \ge 1$ or $1 < n < 2,\ b(x,0) > 0$ (see here also the next paragraph) or $n \ge 2,\ b(x,0) \ge 0$ for $x \in \overline{AB}$. The reverse motion is not true, in general, but if $n = 1,\ b(x,0) > 1$ or $n > 1,\ b(x,0) > 0,\ x \in \overline{AB}$, then*

$$b(x,y) > y^{n-1} \text{ in } \ I_\delta. \tag{3.214}$$

In the latter case, there exist b_0 and δ such that $b(x,y) \ge b_0 = \text{const} > 0$ in $\bar{I}_\delta$. Hence, (3.214) will be fulfilled if $\delta = b_0^{\frac{1}{n-1}}$. The arguments of the proof of the second part of Theorem 3.9.3 concerning W, are correct also for $n = 1,\ b(x,0) > 1$; and for $1 < n < 2,\ b(x,0) > 0$, since in both the cases we can choose δ in such a way that $b(x,y) - 1 > 0$ in I_δ respectively, where $\delta := min\{\delta^*, \tilde{\delta}\},\ \ \delta^* := b_0^{\frac{1}{n-1}},\ \ b_0 := min b(x,y) > 0$ on $\bar{I}_{\tilde{\delta}}$ (such a $\tilde{\delta}$ exists since $b(x,0) > 0$ preserves positive sign in some right neighbourhood of the segment $\overline{AB}$), $b(x,y) \ge b_0 = (\delta^*)^{n-1} \ge \delta^{n-1} \ge y^{n-1}$.

For $1 < n < 2$ condition (3.201) does not exclude the existence of such $x_0 \in \overline{AB}$ where $b(x_0,0) = 0$. But in that case, because of (3.199),

$$b(x_0,y) = b(x_0,0) + \frac{\partial b(x_0,0)}{\partial y}y + \frac{1}{2}\frac{\partial^2 b(x_0,0)}{\partial y^2}y^2 + \cdots =$$

$$= y\left[\frac{\partial b(x_0,0)}{\partial y} + \frac{1}{2}\frac{\partial^2 b(x_0,0)}{\partial y^2}y + \cdots\right] = y\varkappa(x_0,y)^{20}, \ \ 0 \le y < \delta,$$

[20] The same is true if $b(x,y)$ is Lipschitz continuous in $\overline{\Omega}$ since from

$$|b(x,y) - b(x,0)| \le \text{const}\, y \quad \text{in} \quad \overline{\Omega}$$

it follows that

$$|b(x_0,y)| \le \text{const}\, y,$$

with $\varkappa(x_0, y)$ *bounded for* $0 \le y < \delta$ *and, in view of (3.201), we have*

$$y\varkappa(x_0, y) \ge y^{n-1}, \ \ 0 < y < \delta,$$

i.e.,

$$\varkappa(x_0, y) \ge y^{n-2}, \ \ 0 < y < \delta,$$

which means the unboundedness of $\varkappa$. *This is a contradiction. Therefore, if* $1 < n < 2$, *then* $b(x_0, 0) \ne 0$ *and* $b(x, 0) > 0$ *for all* $x \in \overline{AB}$. *The other cases are obvious.*

Example 3.9.11 *Consider the following four exercises*

$$n = 1 \quad \text{and} \quad b(x, y) = \pm 1 \pm y.$$

Remark 3.9.12 *For* $m = 0$ *from Theorem 3.9.3 there follows the Keldysh Theorem [25].*

For reader's convenience we state the Keldysh Theorem for equation (3.198) with $m = 0$ here

Theorem 3.9.13 *If*
either 1) $n < 1$,
or 2) $n = 1$, *and* $b(x, 0) < 1$,
or 3) $1 < n < 2$, *and* $b(x, 0) \le 0$,
or 4) $n \ge 2$, *and* $b(x, 0) < 0$,
then the Dirichlet problem is uniquely solvable, while the Keldysh problem has an infinite number of bounded solutions.
If
either 1) $n = 1$, *and* $b(x, 0) \ge 1$;
or 2) $1 < n < 2$, *and* $b(x, 0) > 0$;
or 3) $n \ge 2$, *and* $b(x, 0) \ge 0$,
the Dirichlet problem is not solvable, in general, while the Keldysh problem is uniquely solvable.

Remark 3.9.14 *Let us consider in* Ω *two equations: one with order and type degeneration*

$$y^m \frac{\partial^2 u}{\partial x^2} + y^n \frac{\partial^2 u}{\partial y^2} + b(x, y) \frac{\partial u}{\partial y} = 0 \tag{3.215}$$

and an other one with characteristic type degeneration

$$\frac{\partial^2 u}{\partial x^2} + y^n \frac{\partial^2 u}{\partial y^2} + b(x, y) \frac{\partial u}{\partial y} = 0, \tag{3.216}$$

where

$$b(x, y) = b_0 y^{[n]-1}, \ \ b_0 = \text{const}\,, \ \ m \ge [n] - 1, \ \ n \ge 2, \tag{3.217}$$

and $[n]$ *is the integral part of* n.

---

i.e.,

$$\varkappa(x_0, y) = \frac{b(x_0, y)}{y}.$$

In both cases $b(x,0)=0$. Hence, in view of the Keldysh Theorem 3.9.13 [25], the Keldysh problem is well-posed for (3.216) by any b_0. Similarly, it is expected the well-posedness of the Keldysh problem for (3.215). Now, let us apply the Theorem 3.9.3 and check the fulfillment of (3.201). The condition (3.214) will be fulfilled for (3.201) iff

$$b_0 \geq y^{\{n\}} \ \text{ in } \ I_\delta. \tag{3.218}$$

The latter will be fulfilled iff

$$b_0 \geq 1 \ \text{ when } \ \{n\}=0, \tag{3.219}$$

$$b_0 > 0 \ \text{ when } \ \{n\}>0. \tag{3.220}$$

(Indeed, if n is an integer, (3.218) and (3.219) coincide. When n is not an integer, for any $b_0>0$ we can find the neighborhood I_δ, $\delta=b_0^{\frac{1}{\{n\}}}$ where (3.218) will be fulfilled). In these cases the well-posedness of the Keldysh problem follows from our Theorem 3.9.3.

If

$$\begin{aligned} &b_0 < 1 \ \text{ when } \ \{n\}=0,\\ &b_0 \leq 0 \ \text{ when } \ \{n\}>0, \end{aligned} \tag{3.221}$$

(3.200) is fulfilled in I_δ but in $\bar{I}_\delta$ the inequality cannot be strong and we are not able to use our theorem. However, after dividing both sides of (3.215) by $y^{[n]-1}$ in Ω, we obtain the equation

$$y^{m-[n]+1}\frac{\partial^2 u}{\partial x^2}+y^{\{n\}+1}\frac{\partial^2 u}{\partial y^2}+b_0\frac{\partial u}{\partial y}=0,$$

and now we can apply our theorem, which by (3.221) asserts the well-posedness of Dirichlet problem.

Thus, for both equations (3.215) and (3.216) $b(x,0)=0$. Nevertheless the Keldysh problem is well-posed for (3.216) for any b_0; the Keldysh problem is well-posed for (3.215) for some b_0 (see (3.219), (3.220)), and the Dirichlet problem is well-posed for other b_0 (see (3.221)). It means that for equation (3.216) with type degeneration the well-posedness of admissible problems depends on values of $b(x,y)$ on the line of degeneracy of the equation, while for equation (3.215) with order and type degeneration, the well-posedness of admissible problems essentially depends on the behavior of $b(x,y)$ in a neighborhood not on the segment of degeneracy of the equation. In other words, the Keldysh Theorem in the classical formulation for equation with characteristic type degeneration can not be valid for equations with order and type degeneration.

Therefore, when $m>0, n>2, b(x,0)=0$, the well-posedness of the boundary value problems for (3.198), even under assumptions (3.199), essentially depends on additional properties of $b(x,y)$ in the neighborhood (see (3.200),(3.201)) of line of degeneracy of (3.198), i.e., it is necessary to give the criteria in the neighborhood because it is impossible to establish them on the segment of degeneracy of the equation.

Remark 3.9.15 *Theorem 3.9.14 remains true if analyticity in (3.199) is replaced by Hölder continuity on* $\overline{\Omega}$ $a, b, c \in C^{(0,\lambda)}(\overline{\Omega})$ *which garanties existence of the Green function* k *and of the representation*

$$u(x, y) = \int\limits_{\Gamma} k(x, y, s) f(s) ds$$

of a regular solution (see A. Bitsadze [1], [2] K. Miramda [29] in the part of $\bar{\Omega}$*, where equation (3.198) is elliptic and in addition* $b(x, y)$ *is Lipshitz continuous on* $\bar{\Omega}$ *(concerning the last see footnotes of this section).*

Remark 3.9.16 *If* $n = 0$*, from (3.198) we get a degenerate elliptic equation with non-characteristic parabolic degeneration and as it was expected from Theorem 3.9.3 it follows that the Dirichlet problem is well-posed, because of* $n < 1$ *for any* $m \geq 0$*.*

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Chapter 4

Singular Generalized Analytic Functions

In this chapter the generalized Cauchy-Riemann system of the first order partial differential equations generated by the second order partial differential equation of the general form in the plane has been considered. The solutions $u(x, y)$, $v(x, y)$ of the above system, called conjugate generalized harmonic functions, satisfy first, second or third order partial differential equations, depending on coefficients of the generator equation. The question of solving of boundary value problems for one conjugate generalized harmonic function by means of solutions of corresponding boundary value problems for another one has been investigated. As an example of a generator, the degenerate equation

$$y\left(\varphi_{xx} + \varphi_{yy}\right) + a\varphi_x + b\varphi_y = 0,\ a, b = \text{const}\,,$$

has been treated.

4.1 Introduction

The Laplace equation

$$\varphi_{xx} + \varphi_{yy} = 0$$

after introduction of notations

$$u := \varphi_y,\ v := \varphi_x \tag{4.1}$$

leads to the Cauchy-Riemann system

$$u_x = v_y,\ v_x = -u_y$$

By given u (or v) one can define v (u) up to an additive constant. The necessary and sufficient condition ensuring it

$$u_{xx} = -u_{yy}\ \left(v_{yy} = -v_{xx}\right)$$

is fulfilled if u $(v) \in C^2$. It is also known that from solution of the Dirichlet problem with respect to one of conjugate functions u and v one can obtain the solution of the Neumann problem for another one. The present chapter deals with similar questions when the generator of a system of first order partial differential equations is the general second order partial differential equation of two independent variables [5]:

$$L\varphi := A(x,y)\varphi_{xx} + 2B(x,y)\varphi_{xy} + C(x,y)\varphi_{yy} + a(x,y)\varphi_x + b(x,y)\varphi_y = 0. \quad (4.2)$$

Taking into account (4.1), we have the following system (compare with [7])

$$u_x = v_y, \quad (4.3)$$

$$Av_x + 2Bv_y(= 2Bu_x) + Cu_y + av + bu = 0. \quad (4.4)$$

Definition 4.1.1 *The functions u and v satisfying the system (4.3), (4.4) will be called conjugate generalized harmonic functions in the sense of the system (4.3), (4.4), provided equation (4.2) is elliptic.*

In the case of harmonic functions u and v the combination

$$w(z) = u(x,y) + iv(x,y), \quad z = x + iy,$$

leads to an analytic function of the complex variable z provided $u, v \in C^1$. In this chapter considering, in particular, the EPD equation as a generating equation we construct singular analytic functions. In contrast to the complex analytic functions, when both the real and imaginary parts are harmonic functions, i.e., they satisfy the same second order equation in case of singular generalized analytic functions the imaginary part satisfies EPD equation, while the real part fulfills a third order equation. Moreover, we solve (in general) weighted BVPs for singular generalized analytic functions, when on the straight line of singularity the m-th order derivative either of the imaginary or of the real part with the corresponding weight is prescribed. in the particular case, when $a = b = 0$, we get the classical Schwartz formula for the analytic function in the half-plane.

4.2 Equations for conjugate generalized harmonic functions

The characteristic form of the system (4.3), (4.4) has the form

$$\begin{vmatrix} 1 & -\lambda \\ C\lambda & A + 2B\lambda \end{vmatrix} = A + 2B\lambda + C\lambda^2.$$

Therefore, the equation (4.2) and the system (4.3), (4.4) are of the same type or degeneration.

Let $A, B, C, a, b, \in C^2(\Omega)$, domain $\Omega \subset R^2$.

Case 4.2.1. $A \equiv 0$, $a \equiv 0$ *in* Ω.

For u from (4.4) we obtain the first order equation

$$L_1 u := 2Bu_x + Cu_y + bu = 0 \quad \text{in } \Omega. \quad (4.5)$$

Case 4.2.2. $A \equiv 0$, $a \not\equiv 0$ *in* Ω.

From (4.4) we immediately have v expressed in terms of u:

$$v = -a^{-1} L_1 u \quad \text{ in } \ \Omega_a, \tag{4.6}$$

where

$$\Omega_a := \{(x,y) \in \Omega : a(x,y) \neq 0\},$$

After differentiation of (4.6) with respect to y, and substitution of the obtained expression into (4.3), we get

$$\begin{aligned} 2Bu_{xy} + Cu_{yy} + (2B_y + a - 2a^{-1}a_y B)\, u_x \\ + (C_y + b - a^{-1}a_y C)\, u_y + (b_y - a^{-1}ba_y)\, u = 0 \quad \text{ in } \ \Omega_a. \end{aligned} \tag{4.7}$$

On the remained subset $\Omega \backslash \Omega_a$ we have (4.5).

Case 4.2.3. $A \not\equiv 0$, $(aA^{-1})_y \equiv 0$ *in* Ω.

From (4.4) we have

$$v_x = -A^{-1} L_1 u - aA^{-1} v \quad \text{ in } \ \Omega_A, \tag{4.8}$$

where

$$\Omega_A := \{(x,y) \in \Omega : A(x,y) \neq 0\}$$

The expression

$$v_x dx + v_y dy,$$

where v_x and v_y are given by (4.8) and (4.3), correspondingly, will be the total differential if and only if (iff):

$$\begin{aligned} u_{xx} = {} & A_y A^{-2} L_1 u \\ & -A^{-1}(2B_y u_x + 2Bu_{xy} + C_y u_y + Cu_{yy} + b_y u + bu_y) \\ & -(aA^{-1})_y\, v - aA^{-1} v_y \quad \text{ in } \ \Omega_A. \end{aligned} \tag{4.9}$$

Further,

$$\begin{aligned} u_{xx} = {} & A^{-1}[A_y A^{-1}(2Bu_x + Cu_y + bu) \\ & -2B_y u_x - 2Bu_{xy} - Cu_{yy} \\ & -(C_y + b)\, u_y - b_y u] - aA^{-1} u_x \quad \text{ in } \ \Omega_A, \end{aligned}$$

since in case under consideration

$$\left(aA^{-1}\right)_y \equiv 0 \ \text{ and } \ v_y = u_x.$$

Therefore,

$$\begin{aligned} Au_{xx} + 2Bu_{xy} + Cu_{yy} + (a + 2B_y - 2A^{-1}A_y B)\, u_x \\ + (b + C_y - A^{-1}A_y C)\, u_y \\ + (b_y - bA^{-1}A_y)\, u = 0 \quad \text{ in } \ \Omega_A. \end{aligned} \tag{4.10}$$

In $(\Omega \backslash \Omega_A) \cap \Omega_a$ we have (4.7); in $(\Omega \backslash \Omega_a) \cap (\Omega \backslash \Omega_A)$, i.e., in $\Omega \backslash (\Omega_A \cup \Omega_a)$ we have (4.5).

Case 4.2.4. $A \not\equiv 0,\ \left(aA^{-1}\right)_y \not\equiv 0$ *in* Ω.

In view (4.3), replacing in (4.9) v_y by u_x, and determining v from the obtained expression, we have

$$\begin{aligned} v = &-\left[A\left(aA^{-1}\right)_y\right]^{-1}\left[Au_{xx} + 2Bu_{xy} + Cu_{yy}\right. \\ &+ \left(a + 2B_y - 2A^{-1}A_yB\right)u_x + \left(b + C_y - A^{-1}A_yC\right)u_y \\ &\left.+ \left(b_y - bA^{-1}A_y\right)u\right] \quad \text{in } \Omega_{(aA^{-1})_y} \cap \Omega_A, \end{aligned} \tag{4.11}$$

where

$$\Omega_{(aA^{-1})_y} := \left\{(x, y) \in \Omega : \left(aA^{-1}\right)_y \neq 0\right\}.$$

In order to exclude v from the condition (4.11), we have to substitute its expression given by (4.11) [assuming $u \in C^3(\Omega)$] into (4.3) which should be fulfilled by u and v:

$$\begin{aligned} &Au_{xxy} + 2Bu_{xyy} + Cu_{yyy} + \left(AK + A_y\right)u_{xx} \\ &+ \left(a + 4B_y - 2A^{-1}A_yB + 2BK\right)u_{xy} + \left(b + 2C_y - A^{-1}A_yC + CK\right)u_{yy} \\ &+ \left[A\left(aA^{-1}\right)_y + \left(a + 2B_y - 2A^{-1}A_yB\right)K - 2\left(A^{-1}A_yB\right)_y + 2B_{yy} + a_y\right]u_x \\ &+ \left[\left(b + C_y - A^{-1}A_yC\right)K - \left(A^{-1}A_yC\right)_y - bA^{-1}A_y + C_{yy} + 2b_y\right]u_y \\ &+ \left[\left(b_y - bA^{-1}A_y\right)K - \left(bA^{-1}A_y\right)_y + b_{yy}\right]u = 0, \quad \text{in } \Omega_{(aA^{-1})_y} \cap \Omega_A, \end{aligned} \tag{4.12}$$

where

$$K := A\left(A^{-1}a\right)_y\left\{\left[A\left(aA^{-1}\right)_y\right]^{-1}\right\}_y.$$

In $\left(\Omega\backslash\Omega_{(aA^{-1})_y}\right) \cap \Omega_A$ we have (4.10); in $\left[\Omega\backslash\left(\Omega_{(aA^{-1})_y} \cup \Omega_A\right)\right] \cap \Omega_a$ we have (4.7); in $\Omega\backslash\left(\Omega_{(aA^{-1})_y} \cup \Omega_A \cup \Omega_a\right)$ we have (4.5).

Because of symmetry

$$u \longleftrightarrow v,\ A \longleftrightarrow C,\ a \longleftrightarrow b,\ x \longleftrightarrow y, \tag{4.13}$$

the equation for v will have the following forms.

Case 4.2.5. $C \equiv 0,\ b \equiv 0,$ *in* Ω.

$$L_2 v := 2Bv_y + Av_x + av = 0 \quad \text{in } \Omega. \tag{4.14}$$

Case 4.2.6. $C \equiv 0,\ b \not\equiv 0,$ *in* Ω.

$$\begin{aligned} &2Bv_{yx} + Av_{xx} + \left(2B_x + b - 2b^{-1}b_xB\right)v_y \\ &\quad + \left(A_x + a - b^{-1}b_xA\right)v_x + \left(a_x - ab^{-1}b_x\right)v = 0 \quad \text{in } \Omega_b, \end{aligned} \tag{4.15}$$

where

$$\Omega_b := \left\{(x, y) \in \Omega : b(x, y) \neq 0\right\}.$$

On the remained subset we have (4.14).

Case 4.2.7. *$C \not\equiv 0$, $(bC^{-1})_x \equiv 0$ in Ω.*

$$\begin{aligned} &Cv_{yy} + 2Bv_{yx} + Av_{xx} + (b + 2B_x - 2BC^{-1}C_x)\, v_y \\ &+ (a + A_x - AC^{-1}C_x)\, v_x + (a_x - aC^{-1}C_x)\, v = 0 \quad \text{ in } \Omega_C, \end{aligned} \tag{4.16}$$

where

$$\Omega_C := \{(x, y) \in \Omega : C(x, y) \neq 0\}.$$

In $(\Omega \backslash \Omega_C) \cap \Omega_b$ we have (4.15); in $\Omega\, (\Omega_C \cup \Omega_b)$ we have (4.14).

Case 4.2.8. *$C \not\equiv 0$, $(bC^{-1})_x \not\equiv 0$ in Ω.*

$$\begin{aligned} &Cv_{yyx} + 2Bv_{yxx} + Av_{xxx} + (CM + C_x)\, v_{yy} \\ &+ \big(b + 4B_x + 2BC^{-1}C_x + 2BM\big)\, v_{yx} + \big(a + 2A_x - AC^{-1}C_x + AM\big)\, v_{xx} \\ &+ \big[C\left(bC^{-1}\right)_x + \big(b + 2B_x - 2BC^{-1}C_x\big)\, M\ - 2\left(BC^{-1}C_x\right)_x + 2B_{xx} + b_x\big]\, v_y \\ &+ \big[\big(a + A_x - AC^{-1}C_x\big)\, M - \big(AC^{-1}C_x\big)_x\ - aC^{-1}C_x + A_{xx} + 2a_x\big]\, v_x \\ &+ \big[\big(a_x - aC^{-1}C_x\big)\, M - \big(aC^{-1}C_x\big)_x + a_{xx}\big]\, v = 0 \quad \text{ in } \Omega_{(bC^{-1})_x} \cap \Omega_C, \end{aligned} \tag{4.17}$$

where

$$M := C\left(bC^{-1}\right)_x \cdot \left\{\left[C\left(bC^{-1}\right)_x\right]^{-1}\right\}_x, \ \Omega_{(bC^{-1})_x} := \left\{(x, y) \in \Omega : \left(bC^{-1}\right)_x \neq 0\right\}.$$

In $\left[\Omega \backslash \Omega_{(bC^{-1})_x}\right] \cap \Omega_C$ we have (4.16); in $\left[\Omega \backslash \left(\Omega_{(bC^{-1})_x} \cup \Omega_C\right)\right] \cap \Omega_b$ we have (4.15); in $\Omega \backslash \left[\Omega_{(bC^{-1})_x} \cup \Omega_C \cup \Omega_b\right]$ we have (4.14).

Corollary 4.2.9 *If A, B, C, a, b are independent of x then v will be satisfying the generator equation (4.2);*
If A, B, C, a, b are independent of y then u will be satisfying the generator equation (4.2);
If A, B, C, a, b are constants then both the conjugate functions will be satisfying the generator equation (4.2).

Proof. Let A, B, C, a, b be independent of x. Then Case 4.2.8 is excluded since $(bC^{-1})_x \equiv 0$; the equations (4.15), (4.16) will coincide with (4.2); after differentiation with respect to x from (4.14) we also obtain (4.2). The second part of the corollary will be proved by analogy with preceding. □

Remark 4.2.10 *In Cases 4.2.1, 4.2.5 conjugate functions satisfy first order equations; in Cases 4.2.4 and 4.2.8 they satisfy the third order equations which are of composite type provided that (4.2) is elliptic; in Cases 4.2.2, 4.2.3 and 4.2.6, 4.2.7 they satisfy the second order equations with main parts coinciding with the main part of (4.2), and therefore, all the above equations are of the same type.*

Remark 4.2.11 *The equations (4.12) and (4.17) are the pure differential equations of lowest order for $u \in C^3$ and $v \in C^3$, correspondingly, in appropriate cases. But they are also satisfying the following loaded integro-differential equations of the second order:*

$$\int_{x_0}^{x}\left[\left(A^{-1}L_1u\right)_y(t,y)+\left(A^{-1}L_1u\right)(t,y)\int_{x_0}^{t}\left(aA^{-1}\right)_y(\xi,y)d\xi\right]e^{\int_{x_0}^{t}\left(aA^{-1}\right)(\xi,y)d\xi}dt$$
$$+\left[\int_{y_0}^{y}u_x\left(x_0,\tau\right)d\tau+C_0-\int_{x_0}^{x}\left(A^{-1}L_1u\right)(t,y)e^{\int_{x_0}^{t}\left(aA^{-1}\right)(\xi,y)d\xi}dt\right]$$
$$\times\int_{x_0}^{x}\left(aA^{-1}\right)_y(t,y)dt+e^{\int_{x_0}^{t}\left(aA^{-1}\right)(\xi,y)d\xi}u_x(x,y)-u_x\left(x_0,y\right)=0,$$
$$C_0=v\left(x_0,y_0\right); \tag{4.18}$$

$$\int_{y_0}^{y}\left[\left(C^{-1}L_2v\right)_x(x,\tau)+\left(C^{-1}L_2v\right)(x,\tau)\int_{y_0}^{\tau}\left(bC^{-1}\right)_x(x,\eta)d\eta\right]e^{\int_{y_0}^{\tau}\left(bC^{-1}\right)(x,\eta)d\eta}d\tau$$
$$+\left[\int_{x_0}^{x}v_y\left(t,y_0\right)dt+C_*-\int_{y_0}^{y}\left(C^{-1}L_2v\right)(x,\tau)e^{\int_{y_0}^{\tau}\left(bC^{-1}\right)(x,\tau)d\eta}d\tau\right]$$
$$\times\int_{y_0}^{y}\left(bC^{-1}\right)_x(x,\tau)d\tau+e^{\int_{y_0}^{y}\left(bC^{-1}\right)(x,\eta)d\eta}v_y(x,y)-v_y\left(x,y_0\right)=0,\ C_*=u\left(x_0,y_0\right).$$

Proof. From (4.4) we have

$$v_x+aA^{-1}v+A^{-1}L_1u=0.$$

Obviously,

$$v(x,y)=e^{-\int_{x_0}^{x}\left(aA^{-1}\right)(t,y)dt}\left[l(y)-\int_{x_0}^{x}\left(A^{-1}L_1u\right)(t,y)e^{\int_{x_0}^{t}\left(aA^{-1}\right)(\xi,y)d\xi}dt\right]. \tag{4.19}$$

In order to determine $l(y)$ we have to substitute (4.19) into (4.3):

$$u_x=v_y=e^{-\int_{x_0}^{x}\left(aA^{-1}\right)(\xi,y)d\xi}\left\{l'(y)-\int_{x_0}^{x}\left[\left(A^{-1}L_1u\right)(t,y)\right.\right.$$
$$\left.\left.+\left(A^{-1}L_1u\right)(t,y)\int_{x_0}^{t}\left(aA^{-1}\right)_y(\xi,y)d\xi\right]e^{\int_{x_0}^{t}\left(aA^{-1}\right)(\xi,y)d\xi}dt\right\}$$

$$-\left[l(y)-\int\limits_{x_0}^{x}\left(A^{-1}L_1u\right)(t,y)e^{\int\limits_{x_0}^{t}\left(aA^{-1}\right)(\xi,y)d\xi}dt\right]$$

$$\times\int\limits_{x_0}^{x}\left(aA^{-1}\right)_y(\xi,y)d\xi e^{-\int\limits_{x_0}^{t}\left(aA^{-1}\right)(\xi,y)d\xi}. \tag{4.20}$$

Since $l(y)$ is independent of x, substituting $x=x_0$ into (4.20), we obtain

$$u_x(x_0,y)=l'(y). \tag{4.21}$$

Hence,

$$l(y)=\int\limits_{y_0}^{y}u_x(x_0,\tau)\,d\tau+C_0. \tag{4.22}$$

Substituting (4.21), (4.22) into (4.20), we get (4.18)

From (4.19) and (4.22) there follows

$$v(x_0,y_0)=l(y_0)=C_0.$$

□

4.3 Construction of conjugate functions in terms of each other

If u is given, then in Case 4.2.1 from (4.3) there follows

$$v(x,y)=\int\limits_{y_0}^{y}u_x(x,\tau)d\tau+v_0(x)\ \text{ in }\ \Omega \tag{4.23}$$

and, therefore, the following statement is valid.

Statement 4.3.1. *In Case 4.2.1 v is defined up to an arbitrary function $v_0(x)$.*

Statement 4.3.2. *In Cases 4.2.2 and 4.2.4 v is uniquely defined by the formulas (4.6) and (4.11), correspondingly.*

In Case 4.2.3, substituting (4.23) into (4.4), for $v_0(x)$ we have the following equation:

$$\begin{aligned}A(x,y_0)v_0'(x)+a(x,y_0)v_0(x)+2B(x,y_0)u_x(x,y_0)\\+C(x,y_0)u_y(x,y_0)+b(x,y_0)u(x,y_0)=0,\end{aligned}$$

taking into account that v_0 is independent of y, and setting there $y=y_0$. Further,

$$\begin{aligned}&v_0'(x)+(aA^{-1})(x)v_0(x)+2(A^{-1}B)(x,y_0)u_x(x,y_0)\\&\quad+(A^{-1}C)(x,y_0)u_y(x,y_0)+(bA^{-1})(x,y_0)u(x,y_0)=0\ \text{ in }\ \Omega_A\end{aligned} \tag{4.24}$$

since in this case

$$\left(aA^{-1}\right)_y \equiv 0, \tag{4.25}$$

and, therefore, aA^{-1} depends only on x. Obviously, the general solution of (4.24) has the form

$$\begin{gathered} v_0(x) = e^{-\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi} \Bigg\{ C_0 - \int\limits_{x_0}^{x} \left[2\left(A^{-1}B\right)(t,y_0)\, u_x(t,y_0)\right. \\ \left. + \left(A^{-1}C\right)(t,y_0)\, u_y(t,y_0) + \left(bA^{-1}\right)(t,y_0)\, u(t,y_0)\right] \\ \times e^{\int\limits_{x_0}^{t}\left(aA^{-1}\right)(\xi)d\xi} dt \Bigg\}, \quad C_0 = \text{const}\,, \text{ in } \ \Omega_A. \end{gathered} \tag{4.26}$$

After substitution of (4.26) into (4.23), which is valid in all the cases, we have

$$\begin{gathered} v(x,y) = e^{-\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi} \Bigg\{ C_0 + \int\limits_{y_0}^{y} e^{\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi} u_x(x,\tau)d\tau \\ - \int\limits_{x_0}^{x} e^{\int\limits_{x_0}^{t}\left(aA^{-1}\right)(\xi)d\xi} A^{-1}(t,y_0) \\ \times \left[2B(t,y_0)\, u_x(t,y_0) + C(t,y_0)\, u_y(t,y_0) + b(t,y_0)\, u(t,y_0)\right] dt \Bigg\} \ \text{ in } \ \Omega_A. \end{gathered} \tag{4.27}$$

On the other hand the integral

$$\begin{gathered} \int\limits_{(x_0,y_0)}^{(x,y)} \Bigg\{ e^{\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi} u_x(x,y)dy - e^{\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi} A^{-1}(x,y) \\ \times \left[2B(x,y)u_x(x,y) + C(x,y)u_y(x,y) + b(x,y)u(x,y)\right] dx \Bigg\} \end{gathered} \tag{4.28}$$

is independent of a curve of integration lying in Ω_A if

$$\begin{aligned} & e^{\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi} \left[\left(aA^{-1}\right)(x,y)u_x(x,y) + u_{xx}\right] = -e^{\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi} \\ & \quad \times \Big[2\left(A^{-1}B\right)_y(x,y)u_x(x,y) + 2\left(A^{-1}B\right)(x,y)u_{xy}(x,y) \\ & \quad + \left(A^{-1}C\right)_y(x,y)u_y(x,y) + \left(A^{-1}C\right)(x,y)u_{yy}(x,y) \\ & \quad + \left(bA^{-1}\right)_y(x,y)u(x,y) + \left(bA^{-1}\right)(x,y)u_y(x,y)\Big]. \end{aligned} \tag{4.29}$$

Hence, after multiplying both sides of (4.29) by

$$A \cdot e^{-\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi},$$

we obtain (4.10) which is really fulfilled in Case 4.2.3. Now, taking as the curve of integration a piecewise linear curve connecting points (x_0, y_0), (x, y_0), (x, y), the integral (4.28) will be coincided with the integral

$$\int\limits_{y_0}^{y} e^{\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi} u_x(x,\tau)d\tau - \int\limits_{x_0}^{x} e^{\int\limits_{x_0}^{t}\left(aA^{-1}\right)(\xi)d\xi} A^{-1}(t,y_0)$$

$$\times \left[2B\left(t,y_0\right)u_x\left(t,y_0\right)+C\left(t,y_0\right)u_y\left(t,y_0\right)+b\left(t,y_0\right)u\left(t,y_0\right)\right]dt.$$

Therefore, we can rewrite (4.27) as follows

$$v(x,y)=e^{-\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi}\left\langle C_0+\int\limits_{(x_0,y_0)}^{(x,y)}e^{\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi}\left\{u_x(x,y)dy-A^{-1}(x,y)\right.\right.$$

$$\left.\left.\times\left[2B(x,y)u_x(x,y)+C(x,y)u_y(x,y)+b(x,y)u(x,y)\right]dx\right\}\right\rangle \text{ in } \Omega_A. \qquad (4.30)$$

If $u=0$, then

$$v(x,y)=C_0e^{-\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi},$$

and this last pair (u,v) should satisfy the system (4.3), (4.4). It is easily seen that this pair fulfills (4.4). Substituting them into (4.3), we have

$$-C_0e^{-\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi}\int\limits_{x_0}^{x}\left(aA^{-1}\right)_y d\xi=0.$$

Hence, it should be realized

$$C_0\int\limits_{x_0}^{x}\left(aA^{-1}\right)_y d\xi=0 \text{ in } \Omega_A.$$

But, by virtue of (4.25),

$$\int\limits_{x_0}^{x}\left(aA^{-1}\right)_y d\xi=0 \text{ in } \Omega_A.$$

Consequently, C_0 can be an arbitrary constant.

Statement 4.3.3. *In Case 4.2.3 v has been defined in Ω_A in terms of u up to the additive term*

$$C_0e^{-\int\limits_{x_0}^{x}\left(aA^{-1}\right)(\xi)d\xi},$$

where C_0 is an arbitrary constant.

Because of symmetry (4.13) three following statements hold:

Statement 4.3.4. *In Case 4.2.5, by given v,*

$$u(x,y)=\int\limits_{x_0}^{x}v_y(t,y)dt+u_0(y) \text{ in } \Omega,$$

where $u_0(y)$ is an arbitrary function.

Statement 4.3.5. *In Cases 4.2.6 and 4.2.8 u is uniquely defined in terms of v by the formulas*

$$u = -b^{-1} L_2 v \ \text{ in } \ \Omega_b,$$

and

$$\begin{aligned} u = -\left[C\left(bC^{-1}\right)_y\right]^{-1} & \left[Cv_{yy} + 2Bv_{yx} + Av_{xx}\right. \\ & + \left(b + 2B_x - 2BC^{-1}C_x\right) v_y + \left(a + A_x - AC^{-1}C_x\right) v_x \\ & \left. + \left(a_x - aC^{-1}C_x\right) v\right] \ \text{ in } \ \Omega_{\left(bC^{-1}\right)_x} \cap \Omega_C, \end{aligned}$$

correspondingly.

Statement 4.3.6. *In Case 4.2.7 u has been defined in Ω_C in terms of v up to the additive term*

$$C_0 e^{-\int\limits_{y_0}^{y} \left(bA^{-1}\right)(\eta) d\eta},$$

where C is an arbitrary constant. It has been given by the formula

$$u(x,y) = e^{-\int\limits_{y_0}^{y} \left(bC^{-1}\right)(\eta) d\eta} \left\langle C_0 + \int\limits_{(x_0,y_0)}^{(x,y)} e^{\int\limits_{y_0}^{y} \left(bC^{-1}\right)(\eta) d\eta} \left\{ v_y(x,y)dx - C^{-1}(x,y) \right.\right.$$

$$\left.\left. \times \left[2B(x,y)v_y(x,y) + A(x,y)v_x(x,y) + a(x,y)v(x,y)\right] dy \right\} \right\rangle \ \text{ in } \ \Omega_C. \tag{4.31}$$

Remark 4.3.7 *By given solution φ of (4.2) the corresponding [in view of (4.1)] solution (u,v) of the system (4.3), (4.4) is uniquely defined. By given (u,v) the corresponding [in view of (4.1)] solution φ of (4.2) is defined up to an additive constant C_0:*

$$\varphi(x,y) = \int\limits_{(x_0,y_0)}^{(x,y)} v dx + u dy + C_0. \tag{4.32}$$

Equation (4.3) guarantees independence of this integral on a form of the curve of integration. Equation (4.4) guarantees that (4.32) is the solution of (4.2).

In fact, from (4.32) there follows (4.1), and after substitution of (4.1) into (4.4) we obtain (4.2).

Remark 4.3.8 *If we consider the more general than (4.2) equation*

$$L\varphi + c\varphi = 0 \tag{4.33}$$

then (4.1) does not lead to the system of type (4.3), (4.4) since $c\varphi$ in (4.33) can not be expressed in terms of u and v without integration. But if in the domain Ω there exists a positive regular solution ψ of (4.33), then substituting into (4.33) the product

$$\varphi = \chi \cdot \psi,$$

where φ is a solution of (4.33), we will have

$$\begin{aligned} A\chi_{xx} + 2B\chi_{xy} + C\chi_{yy} & \\ + \left[a + 2A(\ln\psi)_x + 2B(\ln\psi)_y\right] \chi_x & \\ + \left[b + 2B(\ln\psi)_x + 2C(\ln\psi)_y\right] \chi_y & = 0. \end{aligned}$$

Hence, χ will be a solution of an equation of type (4.2). If (4.2) is of the canonical form and elliptic, then its positive regular solution (see [10], [11]) always exists, in general, locally.

Remark 4.3.9 *Summarizing the Statements 4.3.1 - 4.3.6, we arrive at the following conclusion: if u and v are conjugate functions in sense of the system (4.3), (4.4), then u is defined in terms of v up to the addend*

$$u_*(y) = \begin{cases} u_0(y) & \text{in Case 4.2.5;} \\ 0 & \text{in Cases 4.2.6, 4.2.8;} \\ C_0 e^{-\int\limits_{x_0}^{x} \left(bC^{-1}\right)(\eta)d\eta} & \text{in Case 4.2.7.} \end{cases}$$

and v is defined in terms of u up to the addend

$$v_*(y) = \begin{cases} v_0(y) & \text{in Case 4.2.1;} \\ 0 & \text{in Cases 4.2.2, 4.2.4;} \\ C_0 e^{-\int\limits_{x_0}^{x} \left(aA^{-1}\right)(\xi)d\xi} & \text{in Case 4.2.3.} \end{cases}$$

Remark 4.3.10 *Let R be the operator corresponding to the equations (4.5), (4.7), (4.10), (4.12) and I be the operator corresponding to the equations (4.14) - (4.17). Then in Ω*

$$Ru = 0, \tag{4.34}$$

$$Iv = 0. \tag{4.35}$$

If in the half-plane $y > 0$ denoted as R^2_+ for the equation (4.34) the boundary value problem (BVP) with the boundary condition (BC)

$$\lim_{y \to 0+} \gamma_{m+1}(y) \frac{\partial^m u}{\partial y^m} = f(x), \tag{4.36}$$

where $\gamma_{m+1}(y)$ is a certain weight function, is uniquely solvable under some restrictions, then under some restrictions on u the BVP for the equation (4.35) with BC

$$\lim_{y \to 0+} \gamma_{m+1}(y) \frac{\partial^{m+1} v}{\partial y^{m+1}} = f'(x) \tag{4.37}$$

will be solvable up to the addend $v_*(x)$.

In fact, after differentiation of (4.3) m times with respect to y, we have

$$\frac{\partial^{m+1} u}{\partial y^m \partial x} = \frac{\partial^{m+1} v}{\partial y^{m+1}}. \tag{4.38}$$

Further, after differentiation of (4.36) with respect x, in view (4.38) we have

$$f'(x) = \lim_{y \to 0+} \gamma_{m+1}(y) \frac{\partial^{m+1} u}{\partial y^m \partial x} = \lim_{y \to 0+} \gamma_{m+1}(y) \frac{\partial^{m+1} v}{\partial y^{m+1}} \tag{4.39}$$

If in R^2_+ for the equation (4.35) the BVP with with BC (4.37) is uniquely solvable under same restrictions, under same restrictions on v for the equation (4.34) the BVP with BC (4.36) will be solvable up to the addend $u_*(y)$; here BC (4.36) will be fulfilled up to an additive constant.

In fact, from (4.38) and (4.37) there follows (4.39). After integration of the latter, we will have

$$\lim_{y\to 0+}\gamma_{m+1}(y)\frac{\partial^m u}{\partial y^m} = f(x) + \text{const}\,.$$

4.4 A system generated by EPD equation

Let $A \equiv C \equiv y$, $B \equiv 0$, $a, b = \text{const}\,.$ Then the equation (4.2) will have the form

$$y\varphi_{xx} + y\varphi_{yy} + a\varphi_x + b\varphi_y = 0, \tag{4.40}$$

and the system (4.3), (4.4) generated by it will have the form

$$u_x = u_y, \;\; yv_x + yu_y + av + bu = 0. \tag{4.41}$$

Since in R^2_+ $A \equiv y \neq 0$ then $\left(aA^{-1}\right)_y = -ay^{-2} \neq 0$ if $a \neq 0$, and from (4.12) there follows that

$$y\left(u_{xxy} + u_{yyy}\right) + 2u_{xx} + au_{xy} + (b+2)u_{yy} = 0; \tag{4.42}$$

if $a = 0$ then $\left(aA^{-1}\right)_y = 0$, and (4.10) gives

$$y\left(u_{xx} + u_{yy}\right) + bu_y - by^{-1}u = 0, \tag{4.43}$$

i.e.,

$$y^2\left(u_{xx} + u_{yy}\right) + byu_y - bu = 0. \tag{4.44}$$

After differentiation of (4.44) with respect to y and division by y, we obtain (4.42), where $a = 0$. Therefore, for any a (4.42) is fulfilled but the class of regular solutions of (4.42) when $a = 0$ is wider than the class of regular solutions of (4.43). Hence, these two equations are not equivalent.

Since in R^2_+ $C \equiv y \neq 0$, then $\left(bC^{-1}\right)_x = b\left(y^{-1}\right)_x = 0$, and from (4.16) there follows

$$E^{(a,b)}v := y\left(v_{xx} + v_{yy}\right) + av_x + bv_y = 0. \tag{4.45}$$

Remark 4.4.1 *Let us consider (see Section 3.8)*

$$F^{(a,b,c)}u := y^2\left(u_{xx} + u_{yy}\right) + ayu_x + byu_y + cu = 0, \;\; a, b, c = \text{const}\,. \tag{4.46}$$

From identity

$$y^{\frac{b-b^{\pm}}{2}-1}F^{(a,b,c)}\left(y^{\frac{b^{\pm}-b}{2}}u\right) \equiv E^{(a,b\pm)}u,$$

where

$$b^{\pm} := 1 \pm \sqrt{(1-b)^2 - 4c}, \tag{4.47}$$

there follows

$$y^{\frac{b^{\pm}-b}{2}} u^{(a,b\pm)} = u^{(a,b,c)}. \tag{4.48}$$

The last relation represents one to one correspondence between solution $u^{(a,b,c)}$ of the equation (4.46) and solutions $u^{(a,b\pm)}$ of the equation

$$E^{(a,b\pm)} u = 0.$$

If $c = 0$ then (4.46), after division by y, when $y \neq 0$, coincides with (4.45). In this case from (4.47) it is obvious that either $b^{\pm} = b$ or $b^{\pm} = 2 - b$. If $c \neq 0$, then $b^{\pm} \neq b, 2 - b$.

Thus, if $a \neq 0$, then third order equation (4.42) and second second order equation (4.45) are conjugate in sense of the system (4.41). Hence, in view of Remark 4.3.10 from the solutions of BVP for second order equation (4.45) we can receive solution of the corresponding BVP for the third order equation (4.42) which is of composite type since the equation (for composite type equations see [6], [8])

$$y\left(\xi^2\eta + \eta^3\right) = 0, \ \text{ i.e., } \ y\left[\frac{\eta}{\xi} + \left(\frac{\eta}{\xi}\right)^3\right] = 0,$$

when $y > 0$, has both real $\left(\frac{\eta}{\xi} = 0\right)$, and imaginar $\left(\frac{\eta}{\xi} = \pm i\right)$ solutions. If $y = 0$, the order of (4.42) degenerates.

Let

$$\gamma_m(y) = \begin{cases} 1 & \text{when } b < 1 - m; \\ -(\ln y)^{-1} & \text{when } b = 1 - m; \\ y^{b+m-1} & \text{when } b > 1 - m, \end{cases} \tag{4.49}$$

where $m \in \{0, 1, 2, \ldots\}$.

Let remind that

$$T^m\left(\gamma_m(y)\right) := \left\{ v \in C^2\left(R_+^2\right) : \gamma_m(y)\frac{\partial^m v}{\partial y^m} \in C\left(R_\varepsilon^2\right) \right\},$$

$$R_\varepsilon^2 := \{(x,y) : x \in R^1, \ 0 \le y \le \varepsilon < 1\};$$

$$T_n^m\left(\gamma_m(y)\right) := \left\{ v \in T^m\left(\gamma_m(y)\right) : v \in C\left(R_+^2 \cup R^1\right), u(x,0) \in \underset{*}{C}^n\left(R^1\right), n \ge 1 \right\};$$

$$\underset{\circ}{T}_n^m\left(\gamma_m(y)\right) := \left\{ v \in T_n^m\left(\gamma_m(y)\right) : \lim_{|x| \to +\infty} v(x,0) = 0 \right\},$$

$$T_0^m := \underset{\circ}{T}_0^m := T^m;$$

$\underset{*}{C}^n$ be class of bounded functions from C^m with bounded derivatives;

$$f(\xi) \in \begin{cases} \underset{*}{C}^{-m} & \text{when } b < 1 - m \text{ or } b = 1 - m, \ m > 0; \\ \underset{*}{C}^{-1} & \text{when } b = 1 - m, \ m = 0; \\ C & \text{and } f(\xi) = O\left(|\xi|^{-\alpha}\right), \ |\xi| \to +\infty, \ \alpha > 1 - b, \\ & \text{if } b \in]-\infty, 1], \ \text{when } b > 1 - m, \end{cases}$$

where $\underset{*}{C}^{-m}$ be class of continuous functions with bounded antiderivatives of the order
$\leq m$; $\underset{\circ}{C}^{-m} \subset \underset{*}{C}^{-m}$ be class of functions with the m-th order antiderivative vanishing at infinity;

$$\alpha_m := \begin{cases} 0 & \text{when } b \leq 1-m, \\ -m & \text{when } b > 1-m. \end{cases}$$

Theorem 4.4.2 *(see Section 3.3 and also [2], [3] §2.1, and [4], §1). - The BVP for (4.45) with BC*

$$\lim_{y \to 0+} \gamma_m(y) \frac{\partial^m v}{\partial y^m} = f(x), \ \ x \in R^1,$$

is always solvable in $T^m_{m+\alpha_m}(\gamma_m(y))$. *If* $b \leq 1-m,\ m=0$*; or* $b > 1-m$*, the solution is unique; if* $b \leq 1-m,\ m>0$*, it is defined up to an additive constant under some restrictions at infinity (e. g., boundedness if* $b<1-m$ *and* $b<0$ *or* $u = O\left(y^{1-b}\right)$*,* $r := x^2+y^2 \to +\infty$ *if* $b > 1-m$ *and* $b > 2$*; also Problem 4.4.5). If* $b < 1-m,\ m>0$*, and* $f \in \underset{\circ}{C}^{-m}$*, the solution is unique in* $\underset{\circ}{T}{}^m_m(\gamma_m(y))$ *under the above restrictions at infinity. The solution has the form*

$$v(x,y) = \begin{cases} M_m^{-1}(a, 2-b, 0) y^{1-b} \displaystyle\int_{-\infty}^{+\infty} \underset{*}{f}^{(-m)}(\xi) e^{\alpha\theta} \rho^{b-2} d\xi & \text{when } b < 1-m, \\ d_m^{-1}(a) y^m \displaystyle\int_{-\infty}^{+\infty} \underset{*}{f}^{(-m)}(\xi) e^{\alpha\theta} \rho^{-m-1} d\xi & \text{when } b = 1-m, \\ M_0^{-1}(a,b,m) \displaystyle\int_{-\infty}^{+\infty} f(\xi) e^{\alpha\theta} \rho^{-b} d\xi & \text{when } b > 1-m, \end{cases} \tag{4.50}$$

where

$$M_k(a,b,m) := y^{b+m-1} \int_{-\infty}^{+\infty} (\xi - x)^k \frac{\partial^m e^{\alpha\theta} \rho^{-b}}{\partial y^m} d\xi,$$

$$d_m(a) := \begin{cases} (m+1) M_m(a, m+3, 0) + a M_{m+1}(a, m+3, 0) & \text{for } \ m > 0; \\ 1 + e^{a\pi} \ \ \text{for } \ m = 0, \end{cases}$$

$$\rho := (x-\xi)^2 + y^2, \ \ \theta = \operatorname{arcctg} \frac{x-\xi}{y} \in [0, \pi].$$

Problem 4.4.3. *Let* $b < -m$. *Find* $u \in C^3\left(R^2_+\right)$ *satisfying the equation (4.42) and BC*

$$\lim_{y \to 0+} \frac{\partial^m u}{\partial y^m} = f(x), \ x \in R^1, \ f \in \underset{*}{C}^1\left(R^1\right) \cap \underset{*}{C}^{-m}\left(R^1\right), \tag{4.51}$$

when conjugate function (in sense of the system (4.41))

$$v \in T^{m+1}_{m+1}(1), \text{ and } v = O(1), \ r \to +\infty.$$

Problem 4.4.4. *Let $b=-m$. Find $u \in C^3\left(R_+^2\right)$ satisfying the equation (4.42) and BC*

$$\lim_{y\to 0+}(-\ln y)^{-1}\frac{\partial^m u}{\partial y^m}=f(x),\ x\in R^1,\ f\in C^1\left(R^1\right)\cap \underset{*}{C}^{-m}\left(R^1\right), \tag{4.52}$$

when conjugate function (in sense of the system (4.41)

$$v\in T_{m+1}^{m+1}\left((-\ln y)^{-1}\right),\ v=O(1),\ r\to+\infty.$$

Problem 4.4.5. *Let $b>-m$. Find $u \in C^3\left(R_+^2\right)$ satisfying the equation (4.42) and BC*

$$\lim_{y\to 0} y^{b+m}\frac{\partial^m u}{\partial y^m}=f(x),\ x\in R^1,\ f\in \underset{*}{C}^1\left(R^1\right), \tag{4.53}$$

where

$$f(\xi)=O\left(|\xi|^{-\alpha}\right),\ |\xi|\to+\infty,\ \alpha>-b\ \text{ if }\ b\in]-\infty,0],$$

when conjugate function (in sense of the system (4.41))

$$\begin{array}{lll} v=O\left(y^{1-b}\right), & r\to+\infty, & \text{for}\quad b\in]2,+\infty[;\\ v=o\left(y^{1-b}\right), & r\to+\infty, & \text{for}\quad b\in]1,2];\\ v=O\left(r^{-1}\right), & v_x,v_y=\left(r^{-2}\right), & r\to+\infty, \end{array}$$

$$\lim_{y\to 0+} y\int_{-\infty}^{+\infty} v\cdot v_y dx=0\ \text{ when} \lim_{y\to 0+}(\ln y)^{-1}v=0,\ x\in R^1,\ \text{for}\ \ b=1;$$

$$v\in C\left(R_+^2\cup R^1\right),\ v=o(1),\ r\to+\infty,\ \text{for}\ \ b\in]0,1[;$$

$$v=\overset{*}{I}+O\left(r^{-1}\right),\ v_x=\overset{*}{I}_x+O\left(r^{-2}\right),\ v_y=\overset{*}{I}_y+O\left(r^{-2}\right),\ \ r\to+\infty,$$

$$\overset{*}{I}:=M_0^{-1}(a,b,m+1)\int_{-\infty}^{+\infty} f'(\xi)e^{\alpha\theta}\rho^{-b}d\xi,$$

$$\lim_{y\to 0+} y^b\int_{-\infty}^{+\infty} v\cdot v_y dx=0 \text{ when} \lim_{y\to 0+} y^{b+m}\frac{\partial^{m+1}v}{\partial y^{m+1}}=0,\ x\in R^1,\ \text{for}\ b\in]-\infty,0].$$

Theorem 4.4.6 *All the solutions of the Problems 4.4.3 - 4.4.5, correspondingly, have the following form*

$$u(x,y)=\begin{cases} M_{m+1}^{-1}(a,2-b,0)y^{-b}\\ \qquad\times\displaystyle\int_{-\infty}^{+\infty} \underset{*}{f}^{(-m)}(\xi)(\xi-x)e^{\alpha\theta}\rho^{b-2}d\xi+C_*y^{-b},\ b<-m;\\ d_{m+1}^{-1}(a)y^m\displaystyle\int_{-\infty}^{+\infty} \underset{*}{f}^{(-m)}(\xi)(\xi-x)e^{\alpha\theta}\rho^{-m-2}d\xi+C_*y^m,\\ \qquad b=-m;\\ M_0^{-1}(a,b,m+1)\displaystyle\int_{-\infty}^{+\infty} f(\xi)\frac{\partial e^{\alpha\theta}\rho^{-b}}{\partial y}d\xi,\ \ b>-m, \end{cases} \tag{4.54}$$

where $C_* = \text{const}$, $\underset{*}{f}^{(-m)}$ *means bounded one among antiderivatives of the m-th order:*

$$f^{(-m)}(\xi) := \int\limits_{\xi_0}^{\xi} \frac{(\xi-\tau)^{m-1}}{(m-1)!} f(\tau)d\tau + \sum_{k=0}^{m-1} c_k \xi^k, \ \ \xi \in R^1, \ \ c_k = \text{const}\,.$$

Proof. According to Remark 4.3.10, in order to solve Problems 4.4.3 - 4.4.5, we have to solve BVP for (4.45) with BC (4.37). Therefore, by virtue of Theorem 4.4.2 (see (4.49), (4.50), where we have to replace m and f by $m+1$ and f', correspondingly),

$$v(x,y) = \begin{cases} M_{m+1}^{-1}(a, 2-b, 0) y^{1-b} \int\limits_{-\infty}^{+\infty} [f'(\xi)]_*^{(-m-1)} e^{\alpha\theta}\rho^{b-2} d\xi, \quad b < -m; \\ d_{m+1}^{-1}(a) y^{m+1} \int\limits_{-\infty}^{+\infty} [f'(\xi)]_*^{(-m-1)} e^{\alpha\theta}\rho^{-m-2} d\xi, \ b = -m; \\ M_0^{-1}(a, b, m+1) \int\limits_{-\infty}^{+\infty} f'(\xi) e^{\alpha\theta}\rho^{-b} d\xi, \ \ b > -m, \end{cases} \tag{4.55}$$

where $[\cdot]_*^{-m-1}$ means bounded antiderivative of the $m+1$ order.

If $b < -m$, in view of the statement 4.3.6, taking into account (4.55), we have

$$u(x,y) = y^{-b} y_0^b \Bigg\{ C_0 + \int\limits_{(x_0,y_0)}^{(x,y)} y^b y_0^{-b} \Big[M_{m+1}^{-1}(a, 2-b, 0)$$

$$\times \int\limits_{-\infty}^{+\infty} \underset{*}{f}^{(-m)}(\xi) \frac{\partial y^{1-b} e^{\alpha\theta}\rho^{b-2}}{\partial y} d\xi dx - y^{-1} M_{m+1}^{-1}(a, 2-b, 0) y^{1-b}$$

$$\times \int\limits_{-\infty}^{+\infty} \underset{*}{f}^{(-m)}(\xi) \left(y \frac{\partial e^{\alpha\theta}\rho^{b-2}}{\partial x} + a e^{\alpha\theta}\rho^{b-2} \right) d\xi dy \Big] \Bigg\}$$

since

$$[f'(\xi)]_*^{(-m-1)} = \begin{cases} \left\{ [f'(\xi)]_*^{(-1)} \right\}^{(-m)} = \left\{ [f(\xi) + \text{const}\,]_*^{(-1)} \right\}^{(-m+1)} \\ = \left\{ [f(\xi)]_*^{(-1)} \right\}^{(-m+1)} \\ = \left\{ \underset{*}{f}^{(-1)}(\xi) \right\}^{(-m+1)} \\ = \cdots = \underset{*}{f}^{(-m)}(\xi) \ \text{ when } m > 0, \\ [f'(\xi)]_*^{(-1)} = f(\xi) + \text{const} \ \ \text{ when } m = 0, \end{cases}$$

and the last constant we take equal to zero in order to fulfill (4.51) exactly (see also the end of the Remark 4.3.10). In case $m > 0$, we also have to take equal to zero the above constant and all other constants, but the last, arising by integration. Otherwise the boundedness of the next antiderivative will be violated. The last restriction is connected with the question of uniqueness of representation (4.50). If we did not care for the question of uniqueness, we could take $[f'(\xi)]^{(-m-1)}$ instead of $\overset{*}{f}{}^{-m}(\xi)$ but then we had to take the above constant equal to zero in order to fulfill (4.51) exactly.

Evidently,

$$\frac{\partial y^{1-b}e^{\alpha\theta}\rho^{b-2}}{\partial y} = \frac{\partial(\xi - x)y^{-b}e^{\alpha\theta}\rho^{b-2}}{\partial x},$$

$$y\frac{\partial e^{\alpha\theta}\rho^{b-2}}{\partial x} + ae^{\alpha\theta}\rho^{b-2} = -\frac{\partial(\xi - x)e^{\alpha\theta}\rho^{b-2}}{\partial y}.$$

Hence,

$$u(x,y) = y^{-b}\left\{y_0^b C_0 + M_{m+1}^{-1}(a, 2-b, 0)\int\limits_{-\infty}^{+\infty} \overset{*}{f}{}^{(-m)}(\xi)\right.$$

$$\left.\times \int\limits_{(x_0,y_0)}^{(x,y)} \left[\frac{\partial(\xi - x)e^{\alpha\theta}\rho^{b-2}}{\partial x}dx + \frac{\partial(\xi - x)e^{\alpha\theta}\rho^{b-2}}{\partial y}dy\right] d\xi\right\}$$

$$= y^{-b}\left\{y_0^b C_0 + M_{m+1}^{-1}(a, 2-b, 0)\int\limits_{-\infty}^{+\infty} \overset{*}{f}{}^{(-m)}(\xi)\left[(\xi - x)e^{\alpha\theta}\rho^{b-2}\right]\Big|_{(x_0,y_0)}^{(x,y)} d\xi\right\}$$

$$= y^{-b}\left[C_* + M_{m+1}^{-1}(a, 2-b, 0)\int\limits_{-\infty}^{+\infty} \overset{*}{f}{}^{(-m)}(\xi)(\xi - x)e^{\alpha\theta}\rho^{b-2}d\xi\right], \tag{4.56}$$

where

$$C_* := y_0^b C_0 - M_{m+1}^{-1}(a, 2-b, 0)$$

$$\times \int\limits_{-\infty}^{+\infty} \overset{*}{f}{}^{(-m)}(\xi)\left[(\xi - x)e^{\alpha\theta}\rho^{b-2}\right]\Big|_{(x_0,y_0)} d\xi = \text{const}.$$

It is easily seen directly that (4.56) satisfies (4.51).

Case $b = -m$ can be considered in analogous way.

If $b > -m$, after integration by parts from (4.55) we obtain

$$v(x,y) = -M_0^{-1}(a, b, m+1)\int\limits_{-\infty}^{+\infty} f(\xi)\frac{\partial e^{\alpha\theta}\rho^{-b}}{\partial \xi}d\xi$$

$$= M_0^{-1}(a, b, m+1)\int\limits_{-\infty}^{+\infty} f(\xi)\frac{\partial e^{\alpha\theta}\rho^{-b}}{\partial x}d\xi.$$

Then, in view of (4.31),

$$u(x,y) = y^{-b}y_0^b \left\{ C_0 + \int\limits_{(x_0,y_0)}^{(x,y)} y^b y_0^{-b} \left[M_0^{-1}(a,b,m+1) \right.\right.$$
$$\times \int\limits_{-\infty}^{+\infty} f(\xi)\frac{\partial^2 e^{\alpha\theta}\rho^{-b}}{\partial x \partial y} d\xi dx - y^{-1}M_0^{-1}(a,b,m+1)$$
$$\left.\left. \times \int\limits_{-\infty}^{+\infty} f(\xi)\left(y\frac{\partial^2 e^{\alpha\theta}\rho^{-b}}{\partial x^2} + a\frac{\partial e^{\alpha\theta}\rho^{-b}}{\partial x}\right) d\xi dy \right]\right\}.$$

But

$$y^{b-1}\left(y\frac{\partial^2 e^{\alpha\theta}\rho^{-b}}{\partial x^2} + a\frac{\partial e^{\alpha\theta}\rho^{-b}}{\partial x}\right) = -\frac{\partial y^b \dfrac{\partial e^{\alpha\theta}\rho^{-b}}{\partial y}}{\partial y}.$$

Hence,

$$u(x,y) = y^{-b}\left\{ y_0^b C_0 + M_0^{-1}(a,b,m+1)\int\limits_{-\infty}^{+\infty} f(\xi) \right.$$
$$\left. \times \int\limits_{(x_0,y_0)}^{(x,y)} \left[\frac{\partial y^b \dfrac{\partial e^{\alpha\theta}\rho^{-b}}{\partial y}}{\partial x} dx + \frac{\partial y^b \dfrac{\partial e^{\alpha\theta}\rho^{-b}}{\partial y}}{\partial y} dy \right]\right\} d\xi$$
$$= y^{-b}\left[C_* + M_0^{-1}(a,b,m+1)\int\limits_{-\infty}^{+\infty} f(\xi) y^b \frac{\partial e^{\alpha\theta}\rho^{-b}}{\partial y} d\xi \right], \tag{4.57}$$

where

$$C_* := y_0^b C_0 - M_0^{-1}(a,b,m+1)\int\limits_{-\infty}^{+\infty} f(\xi)\left[y^b \frac{\partial e^{\alpha\theta}\rho^{-b}}{\partial y}\right]\Bigg|_{(x_0,y_0)}^{(x,y)} d\xi = \text{const}\,.$$

It is easily seen immediately that (4.57) satisfies (4.53), if $C_* = 0$. □

Remark 4.4.7 *Since in case under consideration u satisfies also (4.18), from Theorem 4.4.6 there follows that (4.54) (4.54) is the solution of BVP for loaded integro-differential equation of second order (4.18), where $A \equiv C \equiv y$, $B \equiv 0$, $a, b = \text{const}$, with the appropriate BVC out of (4.51) - (4.53).*

Remark 4.4.8 *Let us now consider the case $a = 0$. In the case of the Dirichlet problem for $v \equiv u^{(0,b)}$ the weight is $\gamma_0(y)$ (see (4.49), Section 3.2 and also [3]). Let us find the weight for the Neumann problem for the equation (4.45), where $a = 0$. The conjugate function u satisfies (4.44), i.e., $u \equiv u^{(0,b,-b)}$. On the one hand, in view of (4.48),*

$$u^{(0,b,-b)} = y^{-b}u^{(0,-b)}. \tag{4.58}$$

On the other hand, by virtue of (4.3), conjugate in sense of (4.41) functions $u^{(0,b,-b)}$ *and* $u^{(0,b)}$ *satisfy the following relation*

$$u_x^{(0,b,-b)} = u_y^{(0,b)}. \tag{4.59}$$

Evidently, from (4.58) we have

$$u_x^{(0,b,-b)} = y^{-b} u_x^{(0,-b)}.$$

Taking into account the latter, from (4.59) there follows

$$u_x^{(0,-b)} = y^b u_y^{(0,b)}. \tag{4.60}$$

Since in R_+^2 *BVP with BC*

$$\lim_{y \to 0+} \gamma_0(y) u^{(0,-b)}(x,y) = f(x), \quad x \in R^1, \tag{4.61}$$

where

$$\gamma_0(y) = \begin{cases} 1 & \text{when } b > -1; \\ -(\ln y)^{-1} & \text{when } b = -1; \\ y^{-b-1} & \text{when } b < -1, \end{cases}$$

is correct, obviously,

$$\lim_{y \to 0+} \gamma_0(y) u_x^{(0,-b)}(x,y) = f'(x), \quad x \in R^1. \tag{4.62}$$

Finally, from (4.60), (4.62) there follows

$$\lim_{y \to 0+} y^b \gamma_0(y) u_y^{(0,b)}(x,y) = f'(x), \quad x \in R^1,$$

where

$$y^b \gamma_0(y) = \begin{cases} y^{-1} & \text{when } b < -1; \\ -(y \ln y)^{-1} & \text{when } b = -1; \\ y^b & \text{when } b > -1, \end{cases}$$

is the weight function for the Neumann problem. The solution of the Neumann problem can be constructed in the following way: Under conditions of Theorem 4.4.2 when $m = 0$*, the unique solution* $u^{(0,-b)}$ *of the BVP with BC (4.61) is given by (4.50), where* $m = 0$*. Further, from (4.58) we find* $u^{(0,b,-b)}$ *and than in usual way by means of (4.30) we find its conjugate function, which is just desired solution.*

4.5 BVPs for singular generalized analytic functions

Introducing the notation

$$w := u + iv, \quad \partial_z := \frac{1}{2}\left(\partial_x - i\partial_y\right), \quad \partial_{\bar z} := \frac{1}{2}\left(\partial_x + i\partial_y\right), \quad z := x + iy,$$

and taking into account that, by virtue of (4.3),

$$\partial_{\bar z} w + \partial_z \bar w = 0,$$

we can rewrite the system (4.3), (4.4) in the following form

$$\alpha \partial_z w + \beta \partial_{\bar z} w - \beta \partial_z \bar w - \bar\alpha \partial_{\bar z} \bar w + \gamma w - \bar\gamma \bar w = 0$$

where

$$\alpha := \frac{A-C}{2} + iB, \quad \beta := \frac{A+C}{2}, \quad \gamma := \frac{a+ib}{2}.$$

If

$$AC - B^2 = \beta^2 - (\mathrm{Re}\alpha)^2 - (\mathrm{Im}\alpha)^2 > 0,\,^1$$

then w is called a generalized analytic [11] or pseudo-analytic [1] function.

In particular, the system (4.41) can be rewritten in the following form

$$(z-\bar z)\,\partial_{\bar z} w(z,\bar z) + \mathrm{Re}\,[(ia-b)w(z,\bar z)] = 0, \quad a, b = \mathrm{const}\,. \tag{4.63}$$

Hence, the solution w of the equation (4.63) is generalized analytic function when $y \neq 0$, and when $y = 0$, the equation (4.63) degenerates in algebraic one. Let $a \neq 0$.

Problem 4.5.1. *Find $w(z) \in C^2(\mathbb{R}^2_+)$ fulfilling in $\mathbb{R}^2_+$ the equation (4.63) when either*

$$\lim_{y\to 0+} \gamma_{m+1}(y) \frac{\partial^m \mathrm{Re}w}{\partial y^m} = f(x) \tag{4.64}$$

or

$$\lim_{y\to 0+} \gamma_m(y) \frac{\partial^m \mathrm{Im}w}{\partial y^m} = f(x) \tag{4.65}$$

under conditions of Theorems 4.4.6 and 4.4.2 respectively, and under following additional conditions in case of boundary condition (4.65):

$$f(\xi) = O\left(|\xi|^{-\alpha}\right), \quad |\xi| \to +\infty, \quad \alpha > b, \quad \text{if } \ m = 0,\ 0 \le b < 1,$$
$$\text{when } \ b > 1-m;$$

$$f^{(-m)}(\xi) = O\left(|\xi|^{-\alpha}\right),\ |\xi| \to +\infty,\ \alpha > 1-m, \quad \text{if } m = 0, 1,$$
$$\text{when } b = 1-m;$$

and when $1-m < b \le 0$, the arbitrary constants in the expression of $f^{(-1)}(\xi)$ arising by integration should be taken equal to zero.

Theorem 4.5.2 *All the solutions of Problem 4.5.1 have the following forms:*

$$w(z) = D^{-1} y^{-b} \int\limits_{-\infty}^{+\infty}{}_{*} f^{(-m)}(\xi) e^{a\cdot \arg(\xi - z)} \frac{|\xi - z|^b}{\xi - z} d\xi + C_* y^{-b}, \tag{4.66}$$

where

$$D := M_{m+1}(a, 2-b, 0) \quad \textit{when} \ \ b < -m, \tag{4.67}$$

---

[1] $B = \mathrm{Im}\,\alpha, \quad A = \beta + \mathrm{Re}\,\alpha, \quad C = \beta - \mathrm{Re}\,\alpha.$

$$D := d_{m+1}(a) \quad when \;\; b = -m, \tag{4.68}$$

$$D := M_m(a, 2-b, 0) \quad when \;\; b < 1-m, \tag{4.69}$$

$$D := d_m(a) \quad when \;\; b = 1-m, \tag{4.70}$$

C_* is an arbitrary real constant;

$$w(z) = 2i \int_{-\infty}^{+\infty} F(\xi) \partial_z \left(e^{a \cdot \arg(z-\xi)} |\xi - z|^{-b} \right) d\xi, \tag{4.71}$$

where

$$F(\xi) := M_0^{-1}(a, b, m+1) f(\xi) \quad when \;\; b > -m, \tag{4.72}$$

$$F(\xi) := M_0^{-1}(a, b, m) f^{(-1)}(\xi) \quad when \;\; b < 1-m. \tag{4.73}$$

(4.67), (4.68), (4.72) and (4.69), (4.70), (4.73) correspond to the boundary conditions (4.64) and (4.65), respectively.

Proof. In view of the well-known Picard Theorem (see, e.g., [9], [10]), solutions $u, v \in C^2\left(R_+^2\right)$ of the system (4.41) are analytic functions with respect to the real variables x, y since v satisfies (4.45), and u satisfies the non-homogeneous equation

$$u_{xx} + u_{yy} + ay^{-1}u_x + (b+1)y^{-1}u_y = -y^{-1}v_x$$

with analytic in R_+^2 with respect to x, y coefficients and the right hand side.

In case of the boundary condition (4.64), according to the Theorem 4.4.6, the solution w of the Problem 4.5.2 can be constructed by means of (4.54), (4.55) as $u + iv$, taking into account that

$$\theta = \arg(\xi - z), \quad \rho = |\xi - z|,$$

$$\xi - z + iy = \frac{(\xi - z)^2 + y^2}{\xi - z} = \frac{\rho^2}{\xi - z},$$

$$\frac{\partial}{\partial y}\left(e^{\alpha\theta}\rho^{-b}\right) + i\frac{\partial}{\partial x}\left(e^{\alpha\theta}\rho^{-b}\right) = 2i\partial_z\left(e^{\alpha\theta}\rho^{-b}\right).$$

In case of boundary condition (4.65), at first we find v which will have the form (4.50). Further, in the similar way as proof of Theorem 4.4.6, we find conjugate u. Therefore, it will have the form (4.54), where in coefficients of integrals and in conditions with respect to b the non-negative integer m should be replaced by $m - 1$. At last, we calculate $u + iv$. □

Remark 4.5.3 *If $m \geq 1$, and $b < 0$, is cases (4.67) - (4.70), the representation (4.66), because of $\underset{*}{f}^{(-m)}(\xi)$, besides of $C_* y^{-b}$, contains as well the following arbitrariness:*

$$CD^{-1} \int_{-\infty}^{+\infty} e^{a \cdot \operatorname{arcctg}(-t)} \left(1 + t^2\right)^{\frac{b}{2}} \frac{dt}{t - i}$$

$$= CD^{-1} \int_{-\infty}^{+\infty} e^{a\cdot \operatorname{arcctg}(-t)}(t+i)\left(1+t^2\right)^{\frac{b}{2}-1} dt$$

$$= CD^{-1}\left[M_1(a,2-b,0)+iM_0(a,2-b,0)\right]$$

$$= CD^{-1}\left[-\frac{a}{b}M_0(a,2-b,0)+iM_0(a,2-b,0)\right]$$

$$= CD^{-1}\left(i-\frac{a}{b}\right)M_0(a,2-b,0),$$

where C is an arbitrary real constant.

In the case (4.73), if $b>0$, because of $f^{(-1)}(\xi)$, the representation (4.71) contains the following arbitrariness

$$2iC\int_{-\infty}^{+\infty}\partial_z\left[e^{a\cdot\arg(z-\xi)}|\xi-z|^{-b}\right]d\xi$$

$$= iC\left\{-e^{a\cdot\arg(z-\xi)}|\xi-z|^{-b}\Big|_{-\infty}^{+\infty} - i\int_{-\infty}^{+\infty}\frac{\partial}{\partial y}\left[e^{a\cdot\arg(z-\xi)}|\xi-z|^{-b}\right]d\xi\right\}$$

$$= C\cdot M_0(a,b,1)y^{-b},$$

where C is an arbitrary real constant.

Let now $a=0$, then u satisfies the second order equation (4.44). In this case the boundary value problem can be set and solved in the similar way, taking into account that, by virtue of Remark 4.4.8, now

$$\gamma_0(y)=\begin{cases} 1 & \text{if } b<1-m \text{ and } m=2k, \text{ or} b=-2k\geq 1-m;\\ y^{-1} & \text{if } b<-m \text{ and } m=2k+1;\\ (-y\ln y)^{-1} & \text{if } b=-m \text{ and } m=2k+1;\\ y^{b+m-1} & \text{if } -m<b<1-m \text{ and } m=2k+1,\\ & \text{or } b>1-m,\ m\geq 0 \text{ and } b\neq -2k>1-m;\\ (-\ln y)^{-1} & \text{if } b=1-m \text{ and } m=2k, \end{cases}$$

$$k\in\{0,1,\ldots\}.$$

4.6 Some general remarks

In Section 4.2 some sets have been introduced.

Since functions indicated in indices of sets

$$\Omega_a,\ \Omega_A,\ \Omega_{(aA^{-1})_y},\ \Omega_b,\ \Omega_C,\ \Omega_{(bC^{-1})_x}$$

(which are subsets of a domain Ω and whose closures are supports of the above functions) are continuous (even more $a,b,A,C\in C^2(\Omega)$, $(aA^{-1})_y\in C^1(\Omega_A)$, $(bC^{-1})^x\in C^1(\Omega_C)$), the above sets are open and locally simply connected i.e., for any point from each of the above sets there exists simply connected domain contained in the above set and containing point under consideration. On the other hand the above sets can be multiply connected domains and even unions of domains without joint points.

In Section 4.3 in all the cases we have formulas for construction of v by means of u and vice versa of u by means of v either only by differentiation [see (4.6), (4.11), and Statement 4.4.5] or by differentiation and quadratures [see (4.23), (4.30), and Statements 4.4.4, 4.4.6].

Here should be emphasized that the formulas (4.6), (4.11) mentioned in Statement 4.4.2 (or formulas of Statement 4.4.5), where v (correspondingly u) is determined by u (correspondingly by v) only by differentiation without integration, are valid in indicated sets without any additional restrictions. The formulas (4.23), (4.30), (4.31) and the formula of Statement 4.4.4, containing integration, are valid locally, in general, in indicated sets. The above formulas will be valid globally in indicated sets if we demand simply connectness of corresponding sets (such are they in Sections 4.4 and 4.5). Moreover, in (4.23) and in Statement 4.4.4, Ω should be convex parallel to axis y and x, correspondingly (i.e., whenever the set contains two points lying on a line parallel to the axis, it contains the segment connecting the above points). In formulas (4.30), (4.31) there have been taken into the consideration the following reasonings: if a path of the integration lies in two-dimensional domain and the integrand depends only on one variable, then the curvilinear integral of the second kind (with respect to the above variable) is equal to the integral along the projection of the above path on the corresponding axis, where the integrand can be defined by means of parallel (to another axis) transfer.

Thus, we have the above-described chain of the first order partial differential equations system (4.3), (4.4), its generator partial differential equation of the second order (4.2), and the conjugate first (4.5), (4.14), second (4.7), (4.10), (4.15), (4.16) and third (4.12), (4.17) order partial differential equations. Now, if we are able to solve certain boundary value problems for one of them, then we can solve the corresponding (not the same) boundary value problems for other ones. This is illustrated in Sections 4.4 and 4.5 in the case when the generator equation is a second order elliptic differential equation (4.40) (which arises in the theory of elastic cusped plates) with the order degeneration. So, in Section 4.4, by means of solutions of the boundary value problem when on the boundary the m-th order derivative of the solution of the second order equation (4.45) with the suitable weight is given, the corresponding boundary value problem for the third order partial differential equation (4.42) has been solved; by means of the weight function for Dirichlet problem for the second order degenerate equation (4.45), the weight functions of Neumann problem for conjugate second order degenerate equation (4.44) have been constructed and the way how to solve the above weighted Neumann problem has been shown. In Section 4.5, for a singular generalized analytic function [i.e., for solution of the first order degenerate (singular) complex partial differential equation (4.63) generated by the degenerate equation (4.40)], the weighted Riemann-Hilbert type problems have been solved.

References to Chapter 4

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Chapter 5

Weighted Boundary Value Problems for Higher Order Partial Differential Equations

In this chapter, based on the results of Chapter 3, we investigate (in general weighted) Dirichlet and Riquier BVPs for higher order degenerate PDEs. In particular, the above-mentioned equations are obtained by iterating elliptic EPD operators with different constant coefficients. We give two ways of constructing solutions to weighted, in general, BVPs for degenerate higher (even) order equations. Here we employ the results of Sections 3.2 and 3.3 concerning weighted, in general, BVPs for second order degenerate EPD equations.

5.1 The iterated EPD equation

The $2n$ order equation

$$\left(\prod_{j=0}^{n-1} E^{(a_j,b_j)}\right)\varphi(x,y)=0,\ \ n\in\mathbb{N}\backslash\{1\}, \tag{5.1}$$

where a_j, b_j, $j=0,\ldots,n-1$, are (in general) complex constants, will be called the iterated EPD equation (see, G. Jaiani [2], [4]-[7] and [8], pp. 46-57).

The principal part of the equation (5.1) has the form

$$y^n\Delta^n u=y^n\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)^n u=y^n\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\frac{\partial^{2n}}{\partial x^{2(n-k)}\partial y^{2k}}.$$

Let us consider corresponding $2n$ order form with respect to real constants λ_1, λ_2:

$$K\left(\lambda_1,\lambda_2\right)=y^n\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\lambda_1^{2(n-k)}\lambda_2^{2k}=y^2\left(\lambda_1^2+\lambda_2^2\right)^n.$$

Since the conical manifold

$$K\left(\lambda_1,\lambda_2\right)=y^2\left(\lambda_1^2+\lambda_2^2\right)^n=0$$

for $y \neq 0$, has not real points except the point $\lambda_1 = 0$, $\lambda_2 = 0$, equation (5.1) is of elliptic type for $y \neq 0$ according to classification (see [1], p.10). When $y = 0$ equation (5.1) has an order degeneration from the order $2n$ to n. Let us also consider the following particular case of equation (5.1) (see, G. Jaiani [2], [3], [5], [6] and [7], pp.59-95)

$$E_n^b \varphi := \left(\prod_{j=0}^{n-1} E^{(a_j,b)} \right) \varphi(x,y) = 0, \quad n \in \mathbb{N} \backslash \{1\}. \tag{5.2}$$

It is easily seen that operator E_n^b is independent of the order of factors $E^{(a_j,b)}$. The following corresponding principle

$$\varphi^{(b)}(x,y) = y^{1-b} \varphi^{(2-b)}(x,y) \tag{5.3}$$

is valid. The principle (5.3) to each solution $\varphi^{(b)}(z)$ corresponds solution $\varphi^{(2-b)}(z)$ and vice versa. The above corresponding principal follows from the identity

$$E_n^b[\varphi(z)] = y^{1-b} E_n^{2-b}[y^{1-b}\varphi(z)] \tag{5.4}$$

where $\varphi \in C^{2n}\left(\mathbb{R}_+^2\right)$. The identity (5.4) evidently is true for $n = 1$ [see (5.3)]. Let it be true for $n = m-1 \geq 1$, then

$$\begin{aligned} E_m^b \varphi = E^{(a_{m-1},b)} \left(E_{m-1}^b \varphi \right) &= E^{(a_{m-1},b)} = y^{1-b} E_{m-1}^{2-b} \left(y^{b-1} \varphi \right) \\ &= y^{b-1} E^{(a_{m-1},2-b)} \left[E_{m-1}^{2-b} \left(y^{b-1} \varphi \right) \right] y^{1-b} E_m^{2-b} \left(y^{b-1} \varphi \right), \end{aligned}$$

i.e., (5.4) is true for $n = m$ too. Thus, according to the method of mathematical induction (5.4) is true for an arbitrary $n \in \mathbb{N}$.

5.2 The first BVP in the half-plane

In this section we assume that

$$a_j \neq a_k, \quad j \neq k, \ \ j,k = 0, \ldots, n-1. \tag{5.5}$$

For the sake of brevity let $z := (x,y)$ and $x_0 := (x_0, 0)$.

Problem 5.2.1. *In $\mathbb{R}_+^2$ find a function $\varphi \in C^{2n}\left(\mathbb{R}_+^2\right)$ satisfying equation (5.2) in $\mathbb{R}_+^2$ and either BCs*

$$\lim_{z \to x_0} \frac{\partial^k \varphi(z)}{\partial y^k} = f_k\left(x_0\right), \ \ k = 0, \ldots, n-1, \ \ \mathrm{Re} b < 2-n, \tag{5.6}$$

or BCs

$$\lim_{z \to x_0} \frac{\partial^k y^{b-1} \varphi(z)}{\partial y^k} = f_k\left(x_0\right), \ \ k = 0, \ldots, n-1, \ \ \mathrm{Re} b > n, \tag{5.7}$$

where $z \in \mathbb{R}_+^2$, $x_0 \in \mathbb{R}^1$, and $f_k^{(j-k)} \in \underset{*}{C}\left(\mathbb{R}^1\right)$, $j = 0, \ldots, n-2$, $k = 0, \ldots, n-1$; while $f_k^{(n-1-k)} \in \underset{*}{C}\left(\mathbb{R}^1 \backslash \overset{*}{I}\right)$, $k = 0, \ldots, n-1$; $\overset{*}{I}$ is a union of discontinuity points of the first kind of the last functions; subscript "" means the subset of bounded functions of the corresponding class of functions.*

Theorem 5.2.2 *A solution of the Problem 5.2.1 has the following form*

$$\overset{((1)}{\varphi}(z)=\frac{y^{1-b}}{D(b)}\sum_{j=0}^{n-1}\int\limits_{-\infty}^{+\infty}D_j(b,\xi)e^{a_j\theta}\rho^{b-2}d\xi,\ \ \text{when}\ \ \mathrm{Re}b<2-n,\tag{5.8}$$

and

$$\overset{((2)}{\varphi}(z)=\frac{1}{D(2-b)}\sum_{j=0}^{n-1}\int\limits_{-\infty}^{+\infty}D_j(2-b,\xi)e^{a_j\theta}\rho^{-b}d\xi,\ \ \text{when}\ \ \mathrm{Re}b>n,\tag{5.9}$$

$$D_j\left(b,\xi\right):=$$

$$\begin{vmatrix} \Lambda_0\left(a_0,b\right) & \dots & \Lambda_0\left(a_{j-1},b\right) & f_0\left(\xi\right) & \Lambda_0\left(a_{j+1},b\right) & \dots & \Lambda_0\left(a_{n-1},b\right)\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \Lambda_{n-1}\left(a_0,b\right) & \dots & \Lambda_{n-1}\left(a_{j-1},b\right) & f_{n-1}^{(1-n)}\left(\xi\right) & \Lambda_{n-1}\left(a_{j+1},b\right) & \dots, & \Lambda_{n-1}\left(a_{n-1},b\right) \end{vmatrix}.$$

Proof. We look for solution of the Problem 5.2.1 under BC (5.6) in the form of the following sum

$$\varphi(z)=\sum_{j=0}^{n-1}\psi_j(z),\tag{5.10}$$

where functions $\psi_j(z)\in C^2\left(\mathbb{R}^2_+\right)$, $j=0,\dots,n-1$; are solutions of the following BVPs

$$E^{(a_j,b)}\psi_j(z)=0,\ \ z\in\mathbb{R}^2_+,\ \ j=0,\dots,n-1,$$

$$\lim_{z\to x_0}\frac{\partial^j\psi_j(z)}{\partial y^j}=F_j\left(x_0\right),\ \ z\in\mathbb{R}^2_+,\ \ x_0\in\begin{cases}\mathbb{R}^1, & j=0,\dots,n-2,\\ \mathbb{R}^1\backslash\overset{*}{I}, & j=n-1,\end{cases}$$

provided that functions F_j satisfy the conditions of Problem 5.2.1 with respect to f_j. At the same time functions F_j, $j=0,...,n-1$, should be chosen in such a way that

$$\lim_{z\to x_0}\sum_{j=0}^{n-1}\frac{\partial^k\psi_j(z)}{\partial y^k}=f_k\left(x_0\right),\ \ z\in\mathbb{R}^2_+,\ \ x_0\in\begin{cases}\mathbb{R}^1, & k=0,\dots,n-2,\\ \mathbb{R}^1\backslash\overset{*}{I}, & k=n-1.\end{cases}\tag{5.11}$$

If Re$b<1-j$ and $a_j\neq 0$ for odd j (it could be assumed with out loss of generality, since on the one hand, in view of (5.5), only one of the constants a_j can be equal to zero; on the other hand operator E^b_n is independent of the order of cofactor operator $E^{(a_j,b)}$ and, therefore, we always can give the even index j to the coefficient a_j which is equal to zero), then, by virtue of Theorem 3.3.8 (see formula (3.63)) and Remark 3.3.10

$$\psi_j(z)=\frac{y^{1-b}}{\Lambda_j\left(a_j,b\right)}\int\limits_{-\infty}^{+\infty}F_j^{(-j)}(\xi)e^{a_j\theta}\rho^{b-2}d\xi,\ j=0,\dots,n-1.\tag{5.12}$$

Substituting (5.12) into (5.11) and let $z \to x$, we get

$$\sum_{j=0}^{n-1} \frac{\Lambda_k(a_j,b)}{\Lambda_j(a_j,b)} \underset{*}{F_j}^{(k-j)}(x) = f_k(x),\ x \in \left\{ \begin{array}{l} \mathbb{R}^1,\ \ k=0,\dots,n-2, \\ \mathbb{R}^1 \backslash \overset{*}{I},\ k=n-1. \end{array} \right. , \tag{5.13}$$

whence, after differentiation of the k-th equation $(n-k-1)$-times we obtain

$$\sum_{j=0}^{n-1} \frac{\Lambda_k(a_j,b)}{\Lambda_j(a_j,b)} F_j^{(n-j-1)}(x) = f_k^{(n-k-1)}(x),\ k=0,\dots,n-1. \tag{5.14}$$

Solving the system of algebraic equations (5.14) with respect to functions $F_j^{(n-j-1)}(x)$, we get

$$F_j^{(n-j-1)}(x) = \tilde{D}^{-1}(b) \sum_{j=0}^{n-1} A_{kj} f_k^{(n-k-1)}(x), \tag{5.15}$$

where

$$\tilde{D}(b) := \tilde{D}(a_0,\dots,a_{n-1},b) := D(a_0,\dots,a_{n-1},b) \left[\prod_{j=0}^{n-1} \Lambda_j(a_j,b) \right]^{-1} \tag{5.16}$$

is the determinant of the system (5.14). Here

$$D(b) := D(a_0,\dots,a_{n-1},b) = \begin{vmatrix} \Lambda_0(a_0,b), & \dots, & \Lambda_0(a_{n-1},b), \\ \vdots & \vdots & \vdots \\ \Lambda_{n-1}(a_0,b) & \dots, & \Lambda_{n-1}(a_{n-1},b) \end{vmatrix},$$

and A_{kj} is the cofactor of its element

$$\frac{\Lambda_k(a_j,b)}{\Lambda_j(a_j,b)}$$

of the determinant $\tilde{D}(b)$.

By virtue of (2.64), and well-known properties of determinants we have

$$D(b) := D\left(a_0, \dots, a_{n-1}, b\right) = (-1)^{n-1} \frac{\prod\limits_{j=0}^{n-1} \Lambda\left(a_j, b\right)}{\prod\limits_{k=0}^{n-2} \left[\prod\limits_{j=0}^{k} (b+j)\right]}$$

$$\times \begin{vmatrix} 1 & \dots \\ a_0 & \dots \\ \overset{(2)}{c}{}_0 - a_0^2 & \dots \\ . & \\ . & \\ . & \\ \begin{cases} \sum\limits_{j=0}^{\frac{k-2}{2}} \overset{(k)}{c}{}_{2j} a_0^{2j} - a_0^k, & k \in \mathbb{N}_2 \\ \sum\limits_{j=0}^{\frac{k-3}{2}} \overset{(k)}{c}{}_{2j+1} a_0^{2j+1} + a_0^k, & k \in \mathbb{N}_1, \quad \sum\limits_{j=0}^{-1} (\cdot) := 0 \end{cases} & \dots \\ . & \\ . & \\ . & \\ \begin{cases} \sum\limits_{j=0}^{\frac{n-3}{2}} \overset{(n-1)}{c}_{2j} a_0^{2j} - a_0^{n-1}, & n-1 \in \mathbb{N}_2 \\ \sum\limits_{j=0}^{\frac{n-4}{2}} \overset{(n-1)}{c}_{2j+1} a_0^{2j+1} + a_0^{n-1}, & n-1 \in \mathbb{N}_1 \end{cases} & \dots \end{vmatrix}$$

(j-th column we can get from the first one replacing a_0 by a_{j-1}, $j = \overline{2, n-1}$)

$$= (-1)^{n-1} \frac{\prod\limits_{j=0}^{n-1} \Lambda\left(a_j, b\right)}{\prod\limits_{k=0}^{n-2} (b+j)^{n-1-j}} \begin{vmatrix} 1 & & 1 \\ a_0 & \dots & a_{n-1} \\ a_0^2 & \dots & a_{n-1}^2 \\ \vdots & \dots & \vdots \\ a_0^{n-1} & & a_{n-1}^{n-1} \end{vmatrix} \times \begin{cases} (-1)^{\frac{n-1}{2}}, \ (n-1) \in \mathbb{N}_2; \\ (-1)^{\frac{n-1}{2}}, \ (n-1) \in \mathbb{N}_1 \end{cases}$$

(taking into account that the last one is the Wandermond determinant)

$$= (-1)^{\frac{\tilde{n}}{2}} \frac{\prod\limits_{j=0}^{n-1} \Lambda\left(a_j, b\right)}{\prod\limits_{j=0}^{n-2} (b+j)^{n-1-j}} \prod_{0 \le k < j \le n-1} \left(a_j - a_k\right),$$

where $\tilde{n}$ is the greatest even number which is not grater than n. Hence, by virtue of (2.59),

$$D(b) \neq 0$$

and, moreover, by virtue of (5.16), (2.59),

$$\tilde{D}(b) \neq 0,$$

when (5.5) is fulfilled and $a_j, b \in \mathbb{R}^1$. If a_j and b are complex numbers, then we have to assume that (2.55), (2.56) are fulfilled. Since,

$$\begin{aligned}&A_{n-1\,n-1} = \tilde{D}\left(a_0, \ldots, a_{n-2}, b\right),\\ &A_{n-1j} = \tilde{D}\left(a_0, \ldots, a_{j-1}, a_{n-1}, a_{j+1}, \ldots, a_{n-2}, b\right),\ j = 0, \ldots, n-2,\end{aligned}$$

under same restrictions

$$A_{n-1j} \neq 0,\ j = 0, \ldots, n-1. \tag{5.17}$$

On the one hand after integration by parts $(n-j-1)$-times

$$\begin{aligned}&\int\limits_{\xi_0}^{\xi} \frac{(\xi-\tau)^{n-2}}{(n-2)!} F_j^{(n-j-1)}(\tau) d\tau\\ &= \sum_{l-1}^{n-j-2} (-1)^{l+1} \frac{(\xi-\xi_0)^{n-2-l}}{(n-2-l)!} F_j^{(n-j-2-1)}(\xi_0) + \int\limits_{\xi_0}^{\xi} \frac{(\xi-\tau)^{j-1}}{(j-1)!} F_j(\tau) d\tau\\ &\qquad = \underset{*}{F_j}^{(-j)}(\xi) + \sum_{l=0}^{n-2} \underset{j}{\tilde{c}_l}\, \xi^l,\ j = 0, \ldots, n-1,\ \sum_{l=0}^{-1}(\cdot) := 0. \end{aligned} \tag{5.18}$$

On the other hand taking into account equalities (5.15), (5.17),

$$\begin{aligned}&\int\limits_{\xi_0}^{\xi} \frac{(\xi-\tau)^{n-2}}{(n-2)!} F_j^{(n-j-1)}(\tau) d\tau\\ &= \tilde{D}^{-1}(b) \sum_{k=0}^{n-1} A_{kj} \int\limits_{\xi_0}^{\xi} \frac{(\xi-\tau)^{n-2}}{(n-2)!} f_k^{(n-j-1)}(\tau) d\tau\\ &= \tilde{D}^{-1}(b) \sum_{k=0}^{n-1} A_{kj} \left[\underset{*}{f_k}^{(-k)}(\xi) + \sum_{l=0}^{n-2} \underset{k}{\approx c_l}\, \xi^l \right] = \tilde{D}^{-1}(b) \left\{ \sum_{k=0}^{n-2} A_{kj} \underset{*}{f_k}^{(-k)}(\xi) \right.\\ &\left. + A_{n-1\,j} \left[\underset{*}{f_{n-1}}^{(-n+1)}(\xi) + A_{n-1\,j}^{-1} \sum_{k=0}^{n-1} A_{kj} \sum_{l=0}^{n-2} \underset{k}{\approx c_l} \xi^l \right] \right\}\\ &= D^{-1}(b) \left\{ \sum_{k=0}^{n-2} A_{kj} \underset{*}{f_k}^{(-k)}(\xi) + A_{n-1\,j} \left[\underset{*}{f_{n-1}}^{(-n+1)}(\xi) + \sum_{l=0}^{n-2} \overset{*}{c}_l \xi^l \right] \right\}, \end{aligned} \tag{5.19}$$

$$\underset{j}{\tilde{c}_l}, \underset{k}{\approx c_l}, \overset{*}{c}_l = \text{const}\,.$$

From the equality of the left hand sides of (5.18), (5.19) it follows that

$$\underset{*}{F_j}^{(-j)}(\xi) = \tilde{D}^{-1}(b) \left\{ \sum_{k=0}^{n-2} A_{kj} \underset{*}{f_k}^{(-k)}(\xi) \right.$$

$$+A_{n-1\,j}\left[\underset{*}{f}{}_{n-1}^{(-n+1)}(\xi)+\sum_{l=0}^{n-2}\left(\overset{*}{c}_l-\underset{j}{\tilde{c}_l}A_{n-1\,j}^{-1}\tilde{D}(b)\right)\xi^l\right]\Bigg\}$$

$$=\tilde{D}^{-1}(b)\sum_{k=0}^{n-2}A_{kj}\underset{*}{f_k}{}^{(-k)}(\xi).$$

Therefore, by virtue of (5.16), we obtain

$$\underset{*}{F_j}{}^{(-j)}(\xi)=\Lambda_j\left(a_j,b\right)\frac{D_j(b,\xi)}{D(b)}. \tag{5.20}$$

Substituting (5.20) into (5.12), in view of (5.10), we get (5.8).

From (5.20) we have

$$F_j(x)=\Lambda_j\left(a_j,b\right)D^{-1}(b)D_j^{(j)}(b,x),\quad j=0,\dots,n-1.$$

Hence, under the restrictions of the theorem we conclude that functions $F_j(x)$, $j=0,...,n-1$, satisfy conditions of the Problem 5.2.1 with respect to f_j.

If $f_k(x)\equiv 0$, $k=0,...,n-1$, then the solution of the problem under consideration can be represented as (5.8), where function $f_0(x)\equiv 0$, while functions $f_j^{(-j)}(x)$, $j=0,...,n-1$ are arbitrary polynomials of order $(j-1)$. It is easy to see that this solution is trivial one (s. [7], p. 63, 64).

Solution of Problem 5.2.1 with boundary condition (5.7) according to the correspondence principle (5.3), can be reduced to the Problem 5.2.1 under boundary condition (5.6). □

Remark 5.2.3 *Let*

$$\mathrm{Re}b>n+p_k,\ p_k\in\mathbb{N}^0,\ k=0,\dots,n-1.$$

It is easily check that (5.9) is a solution of equation (5.2) which satisfies the following BCs (s. [7], pp. 66-69)

$$\lim_{z\to x_0}\frac{\dfrac{\partial^{p_k}}{\partial y^{p_k}}\left(y^{k-b+1}\dfrac{\partial^k y^{b-1}\varphi}{\partial y^k}\right)}{\dfrac{\partial^{p_k}y^{k-b+1}}{\partial y^{p_k}}}=f_k\left(x_0\right),\ z\in\mathbb{R}_+^2,\ (x_0)\in\mathbb{R}^1,\ k=0,\dots,n-1.$$

Remark 5.2.4 *A solution of Problem 5.2.1 under the BCs (5.6) is unique if*

$$a_j,\ b\in\mathbb{R}^1,\ \overset{*}{I}=\varnothing,$$

$$\varphi,\ E_i^b\varphi\in\underset{*}{C}\left(\mathbb{R}_+^2\bigcup\mathbb{R}^1\right),\ j=1,\dots,n-1; \tag{5.21}$$

$$\lim_{z\to x_0}y^{j-k-l}\frac{\partial^{p+q-l}\Delta^{j-k}\varphi}{\partial x^p\partial y^{q-l}}=0, \tag{5.22}$$

$j-k-l>0,\ p+q=k,\ j=1,\dots,n-1,\ k=0,\dots,j,\ \ l=0,\dots,q,\ p,q=0,\dots,k;$

$$\frac{\partial^j\varphi}{\partial x^s\partial y^{l-s}}\in C\left(\mathbb{R}_+^2\bigcup\mathbb{R}^1\right), \tag{5.23}$$

$s=0,\dots,j,\ \ j=1,\dots,n-1.$

Proof. It can be proved that

$$E_j^b \equiv \sum_{k=0}^{j} \sum_{p,q=0,\ldots,k}^{p+q=k} \sum_{l=0}^{q} \overset{j}{a}{}_{kl}^{pq} y^{j-k-l} \frac{\partial^{p+q-l}\Delta^{j-k}}{\partial x^p \partial y^{q-l}}$$

$$= \sum_{k=0}^{j} \sum_{p,q=0,\ldots,k}^{p+q=k} \sum_{l=0}^{q} \sum_{r=0}^{j-k} \underset{r}{\overset{j}{b}}{}_{kl}^{pq} y^{j-k-l} \frac{\partial^{2j-l-k}}{\partial x^{p+2r} \partial y^{q-l+2(j-k-r)}}, \tag{5.24}$$

where $\overset{j}{a}{}_{kl}^{pq}$, $\underset{r}{\overset{j}{b}}{}_{kl}^{pq}$ are certain constants, in particular $\overset{j}{a}{}_{kl}^{pq} = \underset{r}{\overset{j}{b}}{}_{kl}^{pq} = 0$ for $j-k-l<0$ and $\overset{j}{a}{}_{00}^{00} = 1$, $j=1,\ldots,n-1$.

If φ satisfies BCs (5.6), then, according to (5.23),

$$\lim_{z\to x_0} \sum_{k=0}^{j} \sum_{\substack{p,q=0,\ldots,k\\ q\geq j-k}}^{p+q=k} \overset{j}{a}{}_{k\,j-k}^{pq} \frac{\partial^{2k-j}\Delta^{j-k}\varphi}{\partial x^p \partial y^{q-j+k}}$$

$$= \lim_{z\to x_0} \sum_{k=0}^{j} \sum_{\substack{p,q=0,\ldots,k\\ q\geq j-k}}^{p+q=k} \sum_{r=0}^{j-k} \underset{r}{\overset{j}{b}}{}_{k\,j-k}^{pq} \frac{\partial^{j}\varphi}{\partial x^{p+2r} \partial y^{j-p+2r}}$$

$$= \sum_{k=0}^{j} \sum_{\substack{p,q=0,\ldots,k\\ q\geq j-k}}^{p+q=k} \sum_{r=0}^{j-k} \underset{r}{\overset{j}{b}}{}_{k\,j-k}^{pq} f_{j-p-2r}^{(p+2r)}(x_0), \quad j=1,\ldots,n-1. \tag{5.25}$$

If now, φ is a difference of two possible solutions (i.e., $f_j=0$, $x\in\mathbb{R}^1$, $j=1,\ldots,n-1$, for this difference), which satisfy conditions (5.21)-(5.23), then by virtue of (5.6), (5.24), (5.22), (5.25), we have

$$\lim_{z\to x_0}\varphi = 0, \quad \lim_{z\to x_0} E_j^b\varphi = 0, \qquad j=1,\ldots,n-1.$$

Hence, $\varphi(x,y)\equiv 0$ how it is proved below in Section 5.3 for the case $n=2$. □

Remark 5.2.5 *A solution of Problem 5.2.1 under the BC (5.7) is unique if*

$$a_j,\ b\in\mathbb{R}^1,\ \overset{*}{I}=\varnothing,$$

$$y^{b-1}\varphi,\ y^{b-1}E_i^b\varphi \in \underset{*}{C}\left(\mathbb{R}_+^2\bigcup\mathbb{R}^1\right),\ j=1,\ldots,n-1;$$

$$\lim_{z\to x_0} y^{j-k-l}\frac{\partial^{k-l}\Delta^{j-k}\left(y^{b-1}\varphi\right)}{\partial x^p\partial y^{q-l}} = 0,$$

$j-k-l>0,\ p+q=k,\ j=1,\ldots,n-1,\ k=0,\ldots,j,\ \ l=0,\ldots,q,\ p,q=0,\ldots,k;$

$$\frac{\partial^j y^{b-1}\varphi}{\partial x^s\partial y^{j-s}} \in C\left(\mathbb{R}_+^2\bigcup\mathbb{R}^1\right),\ s=0,\ldots,j,\ j=1,\ldots,n-1.$$

Remark 5.2.6 *Solution (5.8) satisfies conditions (5.21)-(5.23).*

Proof. By virtue of (5.7), when $\alpha+\beta=j$ we have

$$\lim_{z\to x_0} y^{j-k-l}\frac{\partial^{k-l}\Delta^{j-k}\overset{(1)}{\varphi}}{\partial x^p\partial y^{q-l}} = \lim_{z\to x_0}\sum_{r=0}^{j-k}\left(\begin{array}{c} j-k \\ r \end{array}\right) y^{j-k-l}\frac{\partial^{2j-k-l}\overset{(1)}{\varphi}}{\partial x^{p+2r}\partial y^{q-l+2(j-k-r)}}$$

$$= D^{-1}(b)\sum_{r=0}^{j-k}\left[\left(\begin{array}{c} j-k \\ r \end{array}\right)\sum_{\eta=0}^{n-1}\lim_{z\to x_0} y^{j-k-l}\right.$$

$$\left.\times\int\limits_{\infty}^{\infty} D_\eta^{(j)}(b,\xi)\frac{\partial^{j-k-l}y^{1-b-\beta}(\xi-x)^\beta e^{a_\eta\theta}\rho^{b-2}}{\partial x^{p+2r-\alpha}\partial y^{q-l+2(j-k-r)-\beta}}d\xi\right]$$

$$= D^{-1}(b)\sum_{r=0}^{j-k}\left[\left(\begin{array}{c} j-k \\ r \end{array}\right)\sum_{\eta=0}^{n-1} D_\eta^{(j)}(b,x_0)\, y^{j-k-l+1}\right.$$

$$\left.\times\int\limits_{\infty}^{\infty}\left.\frac{\partial^{j-k-l}y^{1-b-\beta}(\xi-x)^\beta e^{a_\eta\theta}\rho^{b-2}}{\partial x^{p+2r-\alpha}\partial y^{q-l+2(j-k-r)-\beta}}\right|_{\xi=x+yt} dt\right] = 0.$$

So, (5.22) takes place. Since (5.24) is true for $\overset{(1)}{\varphi}$, from (5.22)-(5.24) it follows validity (5.11) for $\overset{(1)}{\varphi}$ (boundedness becomes clear after substitution $\xi = x + yt$). $\square$

5.3 The generalized Riquier problem in the half-plane

The method used in the previous subsection excludes consideration of the case when either $a_i = a_j$ for $i \neq j$ or $b_i \neq b_j$ for $i \neq j$.

For the sake of simplicity we consider the fourth order equation

$$E^{(a_1,b_1)}\circ E^{(a_0,b_0)}\varphi_0 = 0 \tag{5.26}$$

and apply another method of solution of basic BVPs which allows investigation of the general case for (5.1).

Let the constants $a_1, b_1 \in \mathbb{R}^1$, $j = 1,\ldots,n-1$.

Let us introduce the following classes of functions.

$K^{m_0,m_1}(\overset{0}{\gamma}(y),\overset{1}{\gamma}(y))$, $m_\delta\in N^0$, $\delta=0,1$, is the class of functions φ_0 satisfying the conditions:

$$\varphi_0\in C^4(\mathbb{R}_2^+),\quad E^{(a_1,b_1)}\circ E^{(a_0,b_0)}\varphi_0 = 0,$$

$$\overset{\delta}{\gamma}(y)\frac{\partial^{m_\delta}\varphi_\delta}{\partial y^{m_\delta}}\in C(\mathbb{R}_\varepsilon^2),\quad \delta=0,1,$$

where

$$\varphi_1(x_1, y) := E^{(a_0,b_0)}\varphi_0(x, y),$$

$$\mathbb{R}^2_\varepsilon := \{(x, y) : \ x \in \mathbb{R}^1, \ 0 \le y \le \varepsilon = \text{const } < 1\},$$

$K^{m_0,m_1}_{n_0,n_1}(\overset{0}{\gamma}(y), \overset{1}{\gamma}(y))$, $n_\delta \in N$, is the class of functions φ_0 satisfying the conditions:

(i) $\varphi_0 \in K^{m_0,m_1}(\overset{0}{\gamma}(y), \overset{1}{\gamma}(y))$;

(ii) $\varphi_\delta \in C(\mathbb{R}^2_+ \cup \mathbb{R}^1)$, $\delta = 0, 1$, $\underset{|x|\to+\infty}{lim} \varphi_1(x, 0) = 0$;

(iii) $\varphi_\delta \in \underset{*}{C}^{n_\delta}(\mathbb{R}^1)$, $\delta = 0, 1$.

$K^{m_0,m_1}_{0,0} := \overset{*}{K}^{m_0,m_1}$.

$K^{m_0,m_1}_{,n_1}(\overset{0}{\gamma}(y), \overset{1}{\gamma}(y))$ (or $K^{m_0,m_1}_{n_0,}(\overset{0}{\gamma}(y), \overset{1}{\gamma}(y))$) is the class of functions φ_0 satisfying all the conditions of the class

$$K^{m_0,m_1}_{n_0,n_1}$$

except of the second and third conditions for $\delta = 0$ ($\delta = 1$).

$K^{m_0,n_1}_{0,n_1} \equiv K^{m_0,m_1}_{,n_1}$.

$K^{m_0,m_1}_{n_0,0} \equiv K^{m_0,m_1}_{n_0,}$.

$K^{m_0,m_1}_{,0} \equiv K^{m_0,m_1}$.

$K^{m_0,m_1}_{0,} \equiv K^{m_0,m_1}$.

$\underset{0}{K}{}^{m_0,m_1}_{n_0,n_1}(\overset{0}{\gamma}(y), \overset{1}{\gamma}(y)) \subset K^{m_0,m_1}_{n_0,n_1}(\overset{0}{\gamma}(y), \overset{1}{\gamma}(y))$, $n_0, n_1 \in \mathbb{N}^0$ is the class of functions φ_0 satisfying the conditions

$$\underset{|x|\to+\infty}{lim} \varphi_1(x, 0) = 0 \quad \text{for} \quad n_0 \neq 0.$$

Bellow (see [2], [4], [5], also [7], pp. 69-95 and [8], pp. 46-57) the following BVPs are solved.

Problem 5.3.1. *Let* $b_1 \in \left]-\infty, 2 - m_0\right[$ *and* $(a_\delta, b_\delta) \in i_{1,m_\delta}$, $\delta = 0, 1$. *Find a function* $\varphi_0 \in K^{m_0,m_1}_{m_0,m_1}(1, 1)$ *which satisfies the following boundary conditions*

$$\lim_{z\to x_0} \frac{\partial^{m_\delta}\varphi_\delta}{\partial y^{m_\delta}} = f_\delta\left(x_0\right), \ z \in \mathbb{R}^2_+, \ x_0 \in \mathbb{R}^1, \ \delta = 0, 1, \tag{5.27}$$

where

$$f_\delta \in \underset{*}{C}^{(-m_\delta-\delta)}, \ \delta = 0, 1, \ f_1 \in \underset{*}{C}^{-m^0-m_1-1}, \ \varphi_1 := E^{(a_0,b_0)}\varphi_0,$$

and the conditions

$$\varphi_\delta(x, y) = \begin{cases} O(1), \ r \to \infty \ \text{ when either } \ a_\delta \in \mathbb{R}^1, b_\delta \in]-\infty, 0[\\ \qquad\qquad \text{or } \ a_\delta = 0, \ b_\delta = 0; \\ o(1), \ r \to \infty \ \text{ when either } \ a_\delta \in \mathbb{R}^1, b_\delta \in]0, 1[\\ \qquad\qquad \text{or } \ a_\delta \neq 0, \ b_\delta = 0, \end{cases} \tag{5.28}$$

$\delta = 0, 1$.

Problem 5.3.2. *Let* $b_1 \in]-\infty, 2-m_0[$ *and* $(a_0, b_0) \in i_{3,m_0}$, $(a_1, b_1) \in i_{1,m_1}$. *Find a function*

$$\varphi_0 \in K^{m_0,m_1}_{m_0,m_1}\left(\left(\ln\frac{1}{y}\right)^{-1}, 1\right)$$

which satisfies the following BCs:

(i) (5.27) *for* $\delta = 1$;

(ii)

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{\partial^{m_0}\varphi_0}{\partial y^{m_0}} = f_0(x_0),\ z \in \mathbb{R}^2_+,\ x_0 \in \mathbb{R}^1, \tag{5.29}$$

where

$$f_1 \in \underset{*}{C}^{(-m_1-1)};\ f_1 \in \underset{*}{C}^{m_0-m_1-2}$$

$$\left(\text{or}\ \underset{0}{f_1}^{(m_0-m_1-1)}(\xi) = O\left(|\xi|^{-\alpha}\right),\ |\xi| \to +\infty,\ \alpha > 0\right); \tag{5.30}$$

$$f_0 \in \begin{cases} \underset{*}{C}^{(-m_0)}, & m_0 > 0, \\ \underset{*}{C}^{(-1)}, & m_0 = 0, \end{cases}$$

$$\big(\text{or if}\ \ f_0 \in C\ \ \text{and}\ \ f_0(\xi) = O\left(|\xi|^{-\alpha}\right),\ |\xi| \to +\infty,\ \alpha > 0, for\ \ m_0 = 0\big); \tag{5.31}$$

and if $m_0 > 0$, $m_1 \geq 0$, *then* φ_0 *satisfies conditions (5.28); while when* $m_0 = 0$, $m_1 \geq 0$ *it satisfies (3.55), (3.56), and (5.28)* $\delta = 1$.

Problem 5.3.3. *Let* $m_0 < m_1 + 1$ *and* $(a_0, b_0) \in i_{1,m_0}$, $(a_1, b_1) \in i_{3,m_1}$. *Find a function*

$$\varphi_0 \in K^{m_0,m_1}_{m_0,m_1}\left(1, \left(\ln\frac{1}{y}\right)^{-1}\right)$$

which satisfies the following BCs:

(i) (5.27) *for* $\delta = 0$;

(ii)

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{\partial^{m_1}\varphi_1}{\partial y^{m_1}} = f_1(x_0),\ z \in \mathbb{R}^2_+,\ x_0 \in \mathbb{R}^1, \tag{5.32}$$

where $f_\delta \in \underset{*}{C}^{(-m_\delta-\delta)}$; $\delta = 0, 1$, *and if* $m_0 \geq 0$, $m_1 > 0$, *then* φ_0 *satisfies conditions (5.28); while when* $m_0 \geq 0$, $m_1 = 0$ *it satisfies condition (5.28) for* $\delta = 0$ *and* φ_1 *satisfies (3.55), (3.56).*

Problem 5.3.4. *Let* $m_0 < m_1 + 1$ *and* $(a_\delta, b_\delta) \in i_{3,m_\delta}$, $\delta = 0, 1$. *Find a function*

$$\varphi_0 \in K^{m_0,m_1}_{m_0,m_1}\left(\left(\ln\frac{1}{y}\right)^{-1}, \left(\ln\frac{1}{y}\right)^{-1}\right)$$

which satisfies boundary conditions (5.29), (5.32) where $f_\delta \in \underset{*}{C}^{(-m_\delta-\delta)}$, $\delta = 0, 1$, *when* $m_0 > 0$ *(if* $m_0 = 0$, *either (5.31) is valid or* $f_0 \in \underset{*}{C}^{(-1)}$, $f_1 \in \underset{*}{C}^{(-m-2)}$ *or*

(5.30) is fulfilled for $m_0 = 0$); and if $m_\delta > 0$, $\delta = 0,1$, then φ_0 satisfies condition (5.28), if $m_\delta = 0$, $\delta = 0,1$, satisfy (3.55), (3.56) φ_δ, $\delta = 0,1$, if $m_0 = 0$, $m_1 > 0$ φ_0 satisfies (3.55), (3.56) for φ_0 and φ_1 satisfies (5.28) when $\delta = 1$, while if $m_0 > 0$, $m_1 = 0$, then φ_1 satisfies conditions ((3.55), (3.56) and φ_0 satisfies condition (5.28) for $\delta = 0$.

Problem 5.3.5. *Let $b_0 = b_1 = b$; moreover $b > 1 - m_0$ and either $a_0 \neq 0$, or*

$$a_0 = 0, \;\; b \neq 0, -2, \ldots, -2\left(m_0 - \left[\frac{m_0}{2}\right] - 1\right), \; m_0 \in \mathbb{N}\backslash\{1\}.$$

In addition $b > 1 - m_1$ and either $a_1 \neq 0$, or

$$a_1 = 0, \;\; b \neq 0, -2, \ldots, -2\left(m_1 - \left[\frac{m_1}{2}\right] - 1\right), \; m_1 \in \mathbb{N}\backslash\{1\}.$$

Find function $\varphi_0 \in K^{m_0,m_1}\left(y^{b+m_0-1}, y^{b+m_1-1}\right)$ which satisfies BCs

$$\lim_{z \to x_0} y^{b+m_\delta-1} \frac{\partial^{m_\delta}\varphi_\delta}{\partial y^{m_\delta}} = f_\delta\left(x_0\right), \; z \in \mathbb{R}^2_+, \; x_0 \in \mathbb{R}^1, \; \delta = 0,1, \tag{5.33}$$

where $f_\delta \in \underset{*}{C}^{(-\delta)}$, $\delta = 0,1$; if $b \in]-\infty, 1]$, then

$$f_0(\xi), \underset{*}{f_1}^{(-1)}(\xi) = O\left(|\xi|^{-\alpha}\right), \; (\xi) \to +\infty, \; \alpha > 1 - b,$$

and conditions (3.53)-(3.59) for φ_0 and φ_1, where u, a are replaced by φ_0, a_0 and φ_1, a_1, respectively.

Theorem 5.3.6 *The solutions of Problems 5.3.1-5.3.5 have the following forms*

$$\overset{(1)}{\varphi}_0 = \Lambda^{-1}_{m_0}\left(a_0, b_0\right) y^{1-b_0} \int_{-\infty}^{+\infty} \left\{ \underset{*}{f_0}^{(-m_0)}(\xi) + \left[H_{m_0}\left(\frac{\pi}{2}, a_0, a_1 - a_0, b_0, b_1 - b_0\right) \right.\right.$$
$$\left.\left. -\Lambda_{m_0}\left(a_0, b_0\right) \Omega\left(\theta, \frac{\pi}{2}, a_1 - a_0, b_1 - b_0\right)\right] \frac{\underset{*}{f_1}^{(-m_1-1)}(\xi)}{\Lambda_{m_1}\left(a_1, b_1\right)} \right\} e^{a_0\theta} \rho^{b_0-2} d\xi, \tag{5.34}$$

$$\overset{(2)}{\varphi}_0 = d^{-1}_{m_0}\left(a_0\right) y^{m_0} \int_{-\infty}^{+\infty} \left\{ \underset{*}{f_0}^{(-m_0)}(\xi) \right.$$
$$+ \left[a_0 H_{m_0+1}\left(\frac{\pi}{2}, a_0, a_1 - a_0, -m_0 - 1, b_1 + m_0 - 1\right) \right.$$
$$+ \left(m_0 + 1\right) H_{m_0}\left(\frac{\pi}{2}, a_0, a_1 - a_0, -m_0 - 1, b_1 + m_0 - 1\right)$$
$$\left. +\Lambda_{m_0+1}\left(a_1, b_1 - 2\right) - d_{m_0}\left(a_0\right) \Omega\left(\theta, \frac{\pi}{2}, a_1 - a_0, b_1 - m_0 - 1\right)\right]$$
$$\left. \times \Lambda^{-1}_{m_1}\left(a_1, b_1\right) \underset{*}{f_1}^{(-m_1-1)}(\xi) \right\} e^{a_0\theta} \rho^{-m_0-2} d\xi, \tag{5.35}$$

$$\begin{aligned}
&\overset{(3)}{\varphi}_0 = \Lambda_{m_0}^{-1}(a_0,b_0)\, y^{1-b_0} \\
&\times \int\limits_{-\infty}^{+\infty} \left\{ \underset{*}{f_0}^{(-m_0)}(\xi) + \left[H_{m_0}\left(\frac{\pi}{2}, a_0, a_1-a_0, b_0, 1-m_1-b_0\right) \right.\right. \\
&\left. -\Lambda_{m_0}(a_0,b_0)\,\Omega\left(\theta, \frac{\pi}{2}, a_1-a_0, 1-m_1-b_0\right)\right] \\
&\left. \times d_{m_1}^{-1} \underset{*}{f_1}^{(-m_1-1)}(\xi) \right\} e^{a_0\theta}\rho^{b_0-2}\, d\xi, \qquad (5.36) \\
&\overset{(4)}{\varphi}_0 = d_{m_0}^{-1}(a_0)\, d_{m_1}^{-1}(a_1)\, y^{m_0} \int\limits_{-\infty}^{+\infty} \left\{ d_{m_1}(a_1)\, \underset{*}{f_0}^{(-m_0)}(\xi) \right. \\
&- \left[d_{m_0}(a_0)\,\Omega\left(\theta, \frac{\pi}{2}, a_1-a_0, m_0-m_1\right) \right. \\
&-a_0 H_{m_0+1}\left(\frac{\pi}{2}, a_0, a_1-a_0, -m_0-1, m_0-m_1\right) - \Lambda_{m_0+1}(a_1, -m_1-1) \\
&\left. -(m_0+1)\, H_{m_0}\left(\frac{\pi}{2}, a_0, a_1-a_0, -m_0-1, m_0-m_1\right)\right] \\
&\left. \times \underset{*}{f_1}^{(-m_1-1)}(\xi) \right\} e^{a_0\theta}\rho^{-m_0-1}\, d\xi, \qquad (5.37)
\end{aligned}$$

$$\overset{(5)}{\varphi}(x,y) = \begin{cases}
M^{-1}(a_0,b,m_0) \int\limits_{-\infty}^{+\infty} \left[f_0(\xi) \right. \\
\left. + \dfrac{M(a_1,b,m_0)}{(a_1-a_0)\, M(a_1,b,m_1)} \underset{*}{f_1}^{(-1)}(\xi) \right] e^{a_0\theta}\rho^{-b}\, d\xi \\
-(a_1-a_0)^{-1} M^{-1}(a_1,b,m_1) \int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-1)}(\xi) e^{a_0\theta}\rho^{-b}\, d\xi, \ a_1 \neq a_0; \\
M^{-1}(a_0,b,m_0) \int\limits_{-\infty}^{+\infty} \left[f_0(\xi) + \dfrac{M(a,b,m_0)}{M(a,b,m_1)} \underset{*}{f_1}^{(-1)}(\xi) \right] e^{a_0\theta}\rho^{-b}\, d\xi \\
-M^{-1}(a,b,m_1) \int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-1)}(\xi)\, \theta e^{a\theta}\rho^{-b}\, d\xi, \ a_1 = a_0 = a,
\end{cases} \qquad (5.38)$$

respectively, where

$$\begin{aligned}
&H_k(\theta_0, a_0, a_1-a_0, \gamma, b_1-b_0) \\
&:= \int\limits_{-\infty}^{+\infty} t^k e^{a_0 \operatorname{arcctg}(-t)} \Omega\left(\operatorname{arcctg}(-t), \theta_0, a_1-a_0, b_1-b_0\right) \left(1+t^2\right)^{\frac{\gamma}{2}-1} dt. \qquad (5.39)
\end{aligned}$$

Problems 5.3.1-5.3.4 for $m_0 = 0$ and Problem 5.3.5 are uniquely solvable.

Solutions of Problems 5.3.1-5.3.4 for $m_0 > 0$ *are defined up to an additive constant.*

If $f_\delta \in \underset{0}{C}^{(-m_\delta-\delta)}$, $\delta = 0, 1,$ *then Problems 5.3.1-5.3.4 are uniquely solvable in the classes*

$$\underset{0}{K}{}^{m_0,m_1}_{m_0,m_1}(1,1),\quad \underset{0}{K}{}^{m_0,m_1}_{m_0,m_1}\left(\left(\ln\frac{1}{y}\right)^{-1}, 1\right),$$

$$\underset{0}{K}{}^{m_0,m_1}_{m_0,m_1}\left(1, \left(\ln\frac{1}{y}\right)^{-1}\right),\quad \underset{0}{K}{}^{m_0,m_1}_{m_0,m_1}\left(\left(\ln\frac{1}{y}\right)^{-1}, \left(\ln\frac{1}{y}\right)^{-1}\right),$$

respectively (when we are looking for solutions in the above mentioned classes in the expressions (5.34)-(5.37) of solutions stars should be replaced by zeros).

Remark 5.3.7 *If* $m_0 = 0$, *then in solutions (5.34) and (5.36) of Problem 5.3.1 and Problem 5.3.3, respectively, the terms which correspond to the constant* C *in the expression of* $\underset{0}{f_1}^{(-m_1-1)}(\xi)$ *are equal to zero, since, by virtue of (5.39),*

$$y^{1-b_0}\int\limits_{-\infty}^{+\infty}\Big[H_0\left(\frac{\pi}{2}, a_0, a_1-a_0, b_0, b_1-b_0\right)$$
$$-\Lambda\left(a_0, b_0\right)\Omega\left(\theta, \frac{\pi}{2}, a_1-a_0, b_1-b_0\right)\Big]\, Ce^{a_0\theta}\rho^{b_0-2}d\xi = 0$$

and this assertion remains also for $b_1 = 1 - m_1$.

If $m_0 = 0$, *then in solutions (5.35) and (5.37) of Problem 5.3.1 and Problem 5.3.4, respectively, under the assumption of Theorem 5.3.6, either*

$$\underset{*}{f_1}^{(-m_1-1)}(\xi) = O\left(|\xi|^{-\alpha}\right),\ |\xi| \to +\infty,\ \alpha > 0, \tag{5.40}$$

or exists $\underset{*}{f_1}^{(-m_1-2)}(\xi)$. *But then* $\underset{*}{f_1}^{(-m_1-1)}(\xi)$ *may not contain the arbitrary additive constant, otherwise, either (5.40) will be violated or* $\underset{*}{f_1}^{(-m_1-2)}(\xi)$ *will be unbounded.*

In (5.38) the sum of the terms corresponding to the arbitrary additive constants in $\underset{*}{f_1}^{(-1)}$ *is equal to zero, since, because of equalities*

$$\int\limits_{-\infty}^{+\infty} e^{a\theta}\rho^{-b}d\xi = y^{1-b}M(a,b,0),\ z \in \mathbb{R}^2_+,$$

$$\int\limits_{-\infty}^{+\infty} \theta e^{a\theta}\rho^{-b}d\xi = y^{1-b}M(a,b,1,0),\ z \in \mathbb{R}^2_+,$$

$$M\left(a_\delta, b, m_0\right) = (-1)^{m_0}\left(b-1, m_0\right) M\left(a_\delta, b, 0\right),\ \delta = 0, 1,$$

$$M_0\left(a, b, 1, m_0\right) = (-1)^{m_0}\left(b-1, m_0\right) M_0\left(a_\delta, b, 1, 0\right).$$

The expression

$$M^{-1}(a_0,b,m_0)\int\limits_{-\infty}^{+\infty} CM^{-1}(a_1,b,m_1)\left\{\begin{array}{c}\dfrac{M(a_1,b,m_0)}{a_1-a_0}\\ M_0(a,b,1,m_0)\end{array}\right\}e^{a_0\theta}\rho^{-b}d\xi$$

$$-M^{-1}(a_1,b,m_1)\int\limits_{-\infty}^{+\infty} C\left\{\begin{array}{c}(a_1-a_0)^{-1}\\ \theta\end{array}\right\}e^{a_1\theta}\rho^{-b}d\xi$$

$$=\frac{Cy^{1-b}}{M(a_1,b,m_1)}\left[M^{-1}(a_0,b,m_0)\left\{\begin{array}{l}(a_1-a_0)^{-1}M(a_0,b,0)\,M(a_1,b,m_0)\\ M_0(a,b,1,m_0)\,M(a,b,0)\end{array}\right\}\right.$$

$$\left.-\left\{\begin{array}{l}(a_1-a_0)^{-1}M(a_1,b,0)\\ M_0(a,b,1,0)\end{array}\right\}\right]=0,\ \ b>1,$$

where in the braces the upper and lower expressions correspond to the cases $a_1\neq a_0$ and $a_1=a_0=a$, respectively.

Remark 5.3.8 *If in the formulas (5.34)-(5.38) f_0 and f_1 are piece-wise continuous functions satisfying all the hypotheses of problems 4.3.1-4.3.5, except everywhere continuity, then the expressions (5.34)-(5.38) will satisfy equation (5.26) in $\mathbb{R}^2_+$ and the pairs of BCs (5.27); (5.27) for $\delta=1$, (5.29); (5.27) for $\delta=0$, (5.32); (5.29), (5.32); (5.33), respectively, at the points of continuity of the functions f_0 and f_1, even if $a_\delta,\ b_\delta,\ \delta=0,1$, be complex numbers and $\operatorname{Re} b_\delta,\ \delta=0,1$, satisfy the same conditions which were satisfied by the real constants $b_\delta,\ \delta=0,1$ (except of the cases $b_0=1-m_0$ and $b_1=1-m_1$, when $b_\delta,\ \delta=0,1$, are always supposed real ones). Naturally, we exclude the complex values of a_δ and $b_\delta,\ \delta=0,1$, when the dinominators in (5.34)-(5.38) vanish.*

Proof of Theorem 5.3.6. Since

$$E^{(a_0,b_0)}\varphi_0=\varphi_1, \tag{5.41}$$

we equivalently reduce Problems 5.3.1 - 5.3.5, to the pairs of BVPs like Problem 3.3.1 - 3.3.5 for the homogeneous equation

$$E^{(a_1,b_1)}\varphi_1=0 \tag{5.42}$$

and for the non-homogeneous equation (5.41). Here the function φ_1 in the right-hand side of equation (5.41) is a solution of the certain BVP for equation (5.42). In that cases, when in the Theorem 5.3.6 is claimed uniquely solvability of Problems 5.3.1 - 5.3.5 there are assumed that the conditions with respect to φ_1 and φ_0 of uniquely solvability of the corresponding Problems 3.3.1 - 3.3.5 are fulfilled.

Thus, if we consider a difference of two possible solutions of Problems 5.3.1 - 5.3.5, first we get certain BVP of the form of Problems 3.3.1 - 3.3.5 for equation (5.42) with homogeneous BCs under assumptions of uniquely solvability of the BVP, i.e.,

$$\varphi_1(x,y)\equiv 0,\quad z\in\mathbb{R}^2_+,$$

and then we get a certain BVP of the form of Problems 3.3.1 - 3.3.5 for equation (5.41) which is converted in to the homogeneous one with homogeneous BCs under assumptions of uniquely solvability of the BVP, i.e.,

$$\varphi_0(x,y) \equiv 0, \quad z \in \mathbb{R}^2_+.$$

□

Now, we pass to the constructing of solutions. Problem 5.3.1 is equivalently reduced to the following pair of BVPs.

Problem 5.3.9. *Let* $(a_1, b_1) \in i_{1,m_1}$. *Find a function*

$$\varphi_1 \in T^{m_1}_m(1)$$

satisfying equation (5.42) and conditions (5.27), (5.28) for $\delta = 1$.

Problem 5.3.10. *Let* $b_1 \in]-\infty, 2-m_0[$ *and* $(a_0, b_0) \in i_{1,m_0}$. *Find a function*

$$\varphi_0 \in T^{m_1}_{m_1}(1)$$

satisfying the non-homogeneous (instead of homogeneous according to the definition of the class $T^{m_1}_{m_1}(1)$*) equation (5.41) and conditions (5.27), (5.28) for* $\delta = 0$.

By virtue of (3.62), all the solutions of Problem 5.3.9 have the form

$$\varphi_1 = \Lambda^{-1}_{m_1}(a_1, b_1)\, y^{1-b_1} \int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-m_1)}(\xi) e^{a_1\theta} \rho^{b_1-2} d\xi. \tag{5.43}$$

Let us find a particular solution of equation (5.41) with the right-hand side of the form (5.43). We look for it in the integral form

$$\varphi_{p1}(x,y) = \Lambda^{-1}_{m_1}(a_1, b_1) \int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-m_1)}(\xi)\omega_1(x-\xi, y) d\xi, \ z \in \mathbb{R}^2_+, \tag{5.44}$$

where the function $\omega_1(x-\xi, y)$ is to be determined.

Assuming (5.44) twice differentiable under sign of integral, from (5.41) we obtain

$$E^{(a_0,b_0)}\omega_1 = y^{1-b_1} e^{a_1\theta} \rho^{b_1-2}.$$

In the polar coordinate system the last equation we rewrite in the form

$$\rho^2 \frac{\partial^2\omega_1}{\partial\rho^2} + \rho\frac{\partial\omega_1}{\partial\rho} + \frac{\partial^2\omega_1}{\partial\theta^2} + a_0\rho\,\mathrm{ctg}\theta\frac{\partial\omega_1}{\partial\rho} - a_0\frac{\partial\omega_1}{\partial\theta} + b_0\rho\frac{\partial\omega_1}{\partial\rho} + b_0\mathrm{ctg}\theta\frac{\partial\omega_1}{\partial\theta} = e^{a_1\theta}\sin^{-b_1}\theta.$$

Whence, particular solutions depending only on θ satisfy the following equation

$$\frac{\partial^2\omega_1}{\partial\theta^2} + (b_0\mathrm{ctg}\theta - a_0)\frac{\partial\omega_1}{\partial\theta} = e^{a_1\theta}\sin^{-b_1}\theta$$

and have the form

$$\omega_1 = \int\limits_{\theta_0}^{\theta} e^{a_0 t} \sin^{-b_0} t \left[C_1 + \int\limits_{\theta_0}^{t} e^{(a_1 - a_0)\tau} \sin^{(b_0 - b_1)} \tau d\tau \right] dt + C_2, \tag{5.45}$$

where $C_1, C_2, \theta_0 = \text{const}$, $\theta_0 =]0, \pi[$. Let

$$C_1 = C_2 = 0, \ \theta_0 = \frac{\pi}{2}.$$

$\Big($If

$$b_1 = b_2 = b, \ \theta_0 = \frac{\pi}{2}, \ C_2 = 0, \ C_1 = \begin{cases} (a_1 - a_0)^{-1} e^{(a_1 - a_0)\frac{\pi}{2}}, \ a_1 \neq a_0, \\ \frac{\pi}{2}, \ a_1 = a_0, \end{cases}$$

then

$$\omega_1(x - \xi, y) = \begin{cases} (a_1 - a_0)^{-1} \int\limits_{\frac{\pi}{2}}^{\theta} e^{a_1\theta} \sin^{-b}\theta d\theta, \ a_1 \neq a_0, \\ \int\limits_{\frac{\pi}{2}}^{\theta} e^{a\theta} \sin^{-b}\theta d\theta, \ a_1 = a_0 = a. \Big) \end{cases} \tag{5.46}$$

Substituting (5.45) into (5.44), after integration by parts, regarding

$$\underset{*}{f_1}^{(-m_1-1)}(\xi) = -O\left(|\xi|^{-\alpha}\right), \ |\xi| \to +\infty, \ \alpha > 0, \tag{5.47}$$

by virtue of (3.13)-(3.15) we arrive at the desired particular solution

$$\varphi_{p1}(x, y) = -\Lambda_{m_1}^{-1}(a_1, b_1)$$
$$\times \int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-m_1-1)}(\xi) e^{a_0\theta} \sin^{-b_0}\theta \Omega\left(\theta, \frac{\pi}{2}, a_1 - a_0, b_1 - b_0\right) \frac{\partial\theta}{\partial\xi} d\xi$$

$$= -\frac{y^{1-b_0}}{\Lambda_{m_1}(a_1, b_1)} \int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-m_1-1)}(\xi) e^{a_0\theta} \rho^{b_0-2} \Omega\left(\theta, \frac{\pi}{2}, a_1 - a_0, b_1 - b_0\right) d\xi. \tag{5.48}$$

Now, supposing $f_1 \in \underset{*}{C}^{-m_1-1}$ (without restriction (5.47)) it is easily seen that (5.48) is a particular solution of equation (5.41), when φ_1 has the form (5.43). Indeed,

$$E^{(a_0, b_0)}\varphi_{p1}(x, y) = -\Lambda_{m_1}^{-1}(a_1, b_1)$$
$$\times \int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-m_1-1)}(\xi) \left\{\rho^{-2} e^{a_1\theta} \sin^{1-b_1}\theta \left[a_1 \sin\theta + (2 - b_1)\cos\theta\right]\right\} d\xi$$
$$= -\Lambda_{m_1}^{-1}(a_1, b_1) \int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-m_1-1)}(\xi) \frac{\partial}{\partial\xi}\left[\rho^{-1} e^{a_1\theta} \sin^{1-b_1}\theta\right] d\xi$$

$$= \Lambda_{m_1}^{-1}(a_1, b_1) \int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-m_1)} \rho^{-1} e^{a_1\theta} \sin^{1-b_1}\theta d\xi$$

$$= \Lambda_{m_1}^{-1}(a_1, b_1)\, y^{1-b_1} \int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-m_1)}(\xi) e^{a_1\theta} \rho^{b_1-2} d\xi.$$

Further, in view of $b_1 < 2 - m_0$,

$$\lim_{z\to x_0} \frac{\partial^{m_0}\varphi_{p1}}{\partial y^{m_0}} = -H_{m_0}(\theta_0, a_0, a_1 - a_0, b_0, b_1 - b_0)\, \Lambda_{m_1}^{-1}(a_1, b_1)\, \underset{*}{f_1}^{(m_0-m_1-1)}(x_0).$$

The solution of Problem 5.3.10 will be the sum of the particular solution (5.48) and the solution of the equation

$$E^{(a_0,b)} \overset{0}{\varphi}(x, y) = 0, \ z \in \mathbb{R}^2_+$$

satisfying the BCs condition

$$\lim_{z\to x_0} \frac{\partial^{m_0} \overset{0}{\varphi}}{\partial y^{m_0}} = f_0(x_0) - \lim_{z\to x_0} \frac{\partial^{m_0}\varphi_{p1}}{\partial y^{m_0}}.$$

Whence, having constructed all the solution $\overset{0}{\varphi}$ according to the formula (3.63) and summing it with (5.48) we obtain all the solutions of Problem 5.3.1.

We solve similarly other problems. E.g., by solving Problem 5.3.5 we need to find a particular solution of equation (5.41), when the right-hand side has the form

$$M^{-1}(a_1, b, m_1) \int\limits_{-\infty}^{+\infty} f_1(\xi) e^{a_1\theta} \rho^{-b} d\xi.$$

We are looking for a particular solution in the form

$$\varphi_{p5} = M^{-1}(a_1, b, m_1)\, y^{1-b} \int\limits_{-\infty}^{+\infty} f_1(\xi)\omega_5(x-\xi, y) d\xi. \tag{5.49}$$

In this case for the function ω_5 we get the equation

$$E^{(a_0,b)} y^{1-b}\omega_5(x-\xi, y) = e^{a_1\theta}\rho^{-b}.$$

Hence, by virtue of the identity (3.4), we have

$$E^{(a_0,2-b)}\omega_5(x-\xi, y) = y^{b-1} e^{a_1\theta}\rho^{-b}.$$

But the particular solution of this equation we have already found and it has the form (5.46), provided b is replaced by $2-b$. Substituting it into (5.49), after simple transformations, we obtain

$$\varphi_{p5} = \begin{cases} -\dfrac{M^{-1}(a_1, b, m_1)}{a_1 - a_0} \displaystyle\int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-1)}(\xi) e^{a_1\theta}\rho^{-b} d\xi, \ a_1 \neq a_0; \\ -M^{-1}(a, b, m_1) \displaystyle\int\limits_{-\infty}^{+\infty} \underset{*}{f_1}^{(-1)}(\xi)\theta e^{a_1\theta}\rho^{-b} d\xi, \ a_1 = a_0 = a. \end{cases}$$

Therefore,

$$\lim_{z\to x_0} y^{b+m_0-1}\frac{\partial^{m_0}\varphi_{p5}}{\partial y^{m_0}}$$
$$=\begin{cases} -\left(a_1-a_0\right)^{-1}M^{-1}\left(a_1,b,m_1\right)M\left(a_1,b,m_0\right)\underset{*}{f_1}^{(-1)}\left(x_0\right), \ a_1\neq a_0; \\ -M^{-1}\left(a,b,m_1\right)M_0\left(a,b,1,m_0\right)\underset{*}{f_1}^{(-1)}\left(x_0\right), \ a_1=a_0=a. \end{cases}$$

Then acting as above by solving of Problem 5.3.1 we arrive at (5.38).

The solutions of Problems 5.3.1 - 5.3.5 make possible to solve a number of BVPs of the type of the first BVP, in particular, the first BVP proper. It should be noted that such a passage is possible only in the case of degeneration of the order of the equation. The equation under consideration in the present chapter is such one.

Problem 5.3.11. *Let $b_0<-m_1$, $f_0\in \underset{*}{C}^{-m_0}\bigcap\underset{*}{C}^{m_1-m_0+1}$. Find a solution of Problem 5.3.1 satisfying BC*

$$\lim_{z\to x_0}\frac{\partial^{m_1+1}\varphi_0}{\partial y^{m_1+1}}=\tilde f_1\left(x_0\right), \ z\in\mathbb{R}^2_+, \ x_0\in\mathbb{R}^1, \ \tilde f_1\in\underset{*}{C}^{-m_1-1}\bigcap\underset{*}{C}^{m_0-m_1+1}, \quad (5.50)$$

(instead of BC (5.27) for $\delta=1$) and the following conditions

$$\lim_{z\to x_0} y\frac{\partial^{m_1}\Delta\varphi_0}{\partial y^{m_1}}=0, \ z\in\mathbb{R}^2_+, \ x_0\in\mathbb{R}^1, \quad (5.51)$$

$$\lim_{z\to x_0}\frac{\partial^{m_1+1}\varphi_0}{\partial x^{1+\delta}\partial y^{m_1-\delta}}=\Lambda^{-1}_{m_0}\left(a_0,b_0\right)\left\{\Lambda_{m_1-\delta}\left(a_0,b_0\right)\underset{*}{f_0}^{(m_1-m_0+1)}\left(x_0\right)\right.$$
$$\left.+\frac{H^{m_0,m_1-\delta}}{H^{m_0,m_1+1}}\left[\Lambda_{m_0}\left(a_0,b_0\right)\tilde f_1\left(x_0\right)-\Lambda_{m_1+1}\left(a_0,b_0\right)\underset{*}{f_0}^{(m_1-m_0+1)}\left(x_0\right)\right]\right\}, \quad \delta=0,1, \quad (5.52)$$

where

$$H^{m,n}:=H_m\left(\frac{\pi}{2},a_0,a_1-a_0,b_0,b_1-b_0\right)\Lambda_n\left(a_0,b_0\right)$$
$$-\Lambda_m\left(a_0,b_0\right)H_n\left(\frac{\pi}{2},a_0,a_1-a_0,b_0,b_1-b_0\right). \quad (5.53)$$

Problem 5.3.12. *Let $b_0\in[-m_1,1-m_1[$, $f_0\in\underset{*}{C}^{m_1-m_0+1}$. Find a solution of Problem 5.3.1 satisfying BC*

$$\lim_{z\to x_0}\frac{\partial^{m_1}}{\partial y^{m_1}}\left(y\frac{\partial^2}{\partial y^2}+b_0\frac{\partial}{\partial y}\right)\varphi_0=\tilde f_1\left(x_0\right), \ \tilde f_1\in\underset{*}{C}^{-m_1-1}\bigcap\underset{*}{C}^{m_0-m_1-1} \quad (5.54)$$

(instead of BC (5.27) for $\delta=1$) and the following conditions

$$\lim_{z\to x_0} y\frac{\partial^{m_1+2}\varphi_0}{\partial x^2\partial y^{m_1}}=0, \ z\in\mathbb{R}^2_+, \ x_0\in\mathbb{R}^1, \quad (5.55)$$

$$\lim_{z\to x_0}\frac{\partial^{m_1+1}\varphi_0}{\partial x^{1+\delta}\partial y^{m_1-\delta}}=\Lambda_{m_0}^{-1}(a_0,b_0)\left\langle\Lambda_{m_1-\delta}^{-1}(a_0,b_0)\underset{*}{f_0}^{(m_1-m_0+1)}(x_0)\right.$$
$$+\left(\overset{*}{H}{}^{m_0,m_1}\right)^{-1}H^{m_0,m_1-\delta}\left\{\Lambda_{m_0}(a_0,b_0)\tilde{f}_1(x_0)\right.$$
$$\left.\left.+\left[m_1\Lambda_{m_1-1}(a_0,b_0)+a_0\Lambda_{m_1}(a_0,b_0)\right]\underset{*}{f_0}^{(m_1-m_0+1)}(x_0)\right\}\right\rangle, \tag{5.56}$$

$$\overset{*}{H}{}^{m_0,m_1}:=\Lambda_{m_0}(a_0,b_0)\left[\Lambda_{m_1}(a_1,b_1)+m_1H_{m_1-1}\left(\frac{\pi}{2},a_0,a_1-a_0,b_0,b_1-b_0\right)\right.$$
$$\left.+a_0H_{m_1}\left(\frac{\pi}{2},a_0,a_1-a_0,b_0,b_1-b_0\right)\right]$$
$$-H_{m_0}\left(\frac{\pi}{2},a_0,a_1-a_0,b_0,b_1-b_0\right)\left[m_1\Lambda_{m_1-1}(a_0,b_0)+a_0\Lambda_{m_1}(a_0,b_0)\right],$$
$$m_1H_{m_1-1}(.,.,.,.,.)\big|_{m_1=0}\equiv 0,\ m_1\Lambda_{m_1-1}(.,.)\big|_{m_1=0}\equiv 0.$$

Problem 5.3.13. *Let $m_1<m_0-1,\ \ m_0\in\mathbb{N}\setminus\{1\},\ \ f_0\in \underset{*}{C}^{m_1-m_0+1},\ \ \tilde{f}_1$ satisfy the condition (5.30), f_0 satisfy the condition (5.31). Find a solution of Problem 5.3.2 satisfying BC (5.50) (instead of BC (5.27) for $\delta=1$) and the following conditions*

$$\lim_{z\to x_0}\frac{\partial^{m_1+1}\varphi_0}{\partial x^{1+\delta}\partial y^{m_1-\delta}}=d_{m_0}^{-1}(a_0)\left\{\Lambda_{m_1-\delta}(a_0,1-m_0)\underset{*}{f_0}^{(m_1-m_0+1)}(x_0)\right.$$
$$+\frac{\tilde{H}^{m_0+1,m_1}}{\tilde{H}^{m_0+1,m_1+1}}\left[d_{m_0}(a_0)\tilde{f}_1(x_0)\right.$$
$$\left.\left.-\Lambda_{m_1+1}(a_0,1-m_0)\underset{*}{f_0}^{(m_1-m_0+1)}(x_0)\right]\right\}, \tag{5.57}$$

where

$$\tilde{H}^{m.n_1}:=\left[a_0H_m\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,b_1+m-1\right)\right.$$
$$\left.+mH_{m-1}\left(\frac{\pi}{2},a_0,a_1-a_0,-m,b_1+m-2\right)+\Lambda_{m+1}(a_1,b_1-2)\right]\Lambda_n(a_0,2-m)$$
$$-d_{m-1}(a_0)H_n\left(\frac{\pi}{2},a_0,a_1-a_0,2-m,b_1-2+m\right).$$

Problem 5.3.14. *Let $b_0<1-m_1,\ \ f_0\in\underset{*}{C}^{m_1-m_0+1}$. Find a solution of Problem 5.3.3 satisfying the following BC*

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{\partial^{m_1}}{\partial y^{m_1}}\left(y\frac{\partial^2}{\partial y^2}+b_0\frac{\partial}{\partial y}\right)\varphi_0=\tilde{f}_1(x_0), \tag{5.58}$$

where $\tilde{f}_1$ satisfies the conditions of Problem 5.3.3 with respect of f_1 (instead of BC (5.32)) and the following conditions

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}y\frac{\partial^{m_1+2}\varphi_0}{\partial x^2\partial y^{m_1}}=0,\ z\in\mathbb{R}^2_+,\ x_0\in\mathbb{R}^1, \tag{5.59}$$

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{\partial^{m_1+1}\varphi_0}{\partial x^{1+\delta}\partial y^{m_1-\delta}}=0,\ z\in\mathbb{R}^2_+,\ x_0\in\mathbb{R}^1,\ \delta=0,1. \tag{5.60}$$

Problem 5.3.15. *Let* $m_1 = m_0 = m$, $f_0 \in \underset{*}{C}^1$ *and* $f_0^{(0)}$, $\tilde{f}_1$ *satisfy conditions of Problem 5.3.4 with respect to* f_1. *Find a solution of Problem 5.3.4 satisfying BC (5.58) (instead of BC (5.32)) and the conditions (5.59) and*

$$\left(\ln\frac{1}{y}\right)^{-1} \frac{\partial^{m+1}\varphi_0}{\partial x^{1+\delta}\partial y^{m-\delta}} \begin{cases} \in C\left(R_\varepsilon^2\right), & \delta = 0; \\ = o(1), \ z \to x_0, & \delta = 1. \end{cases} \tag{5.61}$$

Problem 5.3.16. *Let* $f_0 \in \underset{*}{C}^1$ *and* $f_0^{(1)}$, $\tilde{f}_1$ *satisfy the conditions of Problem 5.3.5 with respect to* f_1. *Find a solution of Problem 5.3.5 satisfying the following BC*

$$\lim_{z\to x_0} y^{b+m_1-1} \frac{\partial^{m_1}}{\partial y^{m_1}} \left(y\frac{\partial^2}{\partial y^2} + b\frac{\partial}{\partial y}\right)\varphi_0 = \tilde{f}_1\left(x_0\right) \tag{5.62}$$

(instead of BC (5.33) for $\delta = 1$*) and the following conditions*

$$\lim_{z\to x_0} y^{b+m_1} \frac{\partial^{m_1+2}\varphi_0}{\partial x^2\partial y^{m_1}} = 0, \tag{5.63}$$

$$\lim_{z\to x_0} y^{b+m_1-1} \frac{\partial^{m_1+1}\varphi_0}{\partial x^2\partial y^{m_1-1}} = 0, \tag{5.64}$$

$$\lim_{z\to x_0} y^{b+m_1-1} \frac{\partial^{m_1+1}\varphi_0}{\partial x\partial y^{m_1}}$$

$$= \begin{cases} M^{-1}\left(a_0,b,m_0\right) M\left(a_0,b,m_1\right) f_0'\left(x_0\right) + \tilde{M}_0\left(a_0,a_1,b,m_0,m_1\right) \\ \times \dfrac{M\left(a_0,b,m_0\right)\tilde{f}_1\left(x_0\right) + a_0 M\left(a_0,b,m_1\right) f_0'\left(x_0\right)}{a_1 M\left(a_1,b,m_1\right) M\left(a_0,b,m_0\right) - a_0 M\left(a_1,b,m_0\right) M\left(a_0,b,m_1\right)}, \\ \ a_1 \neq a_0; \\ M^{-1}\left(a,b,m_0\right) M\left(a,b,m_1\right) f_0'\left(x_0\right) + \tilde{M}_1\left(a,a,b,m_0,m_1\right) \\ \times \dfrac{\tilde{f}_1\left(x_0\right) + aM\left(a,b,m_1\right) M^{-1}\left(a,b,m_0\right) f_0'\left(x_0\right)}{M\left(a,b,m_1\right) - a_0\tilde{M}_1\left(a,a,b,m_0,m_1\right)}, \ a_1 = a_0 = a, \end{cases} \tag{5.65}$$

where $z \in \mathbb{R}_+^2$, $x_0 \in \mathbb{R}^1$,

$$\begin{aligned} &\tilde{M}_j(a_0,a_1,b,m_0,m_1) \\ &:= M_0(a_1,b,j,m_0)M(a_0,b,m_1)M^{-1}(a_0,b,m_0) - M_0(a_1,b,j,m_1). \end{aligned}$$

Remark 5.3.17 *If* $m_1 = 0$, *then the conditions (5.52), (5.56), (5.57), (5.60), (5.61) for* $\delta = 1$ *and the condition (5.64) in Problems 5.3.11 - 5.3.16 are absent. If* $m_1 = m_0 = 0$, *then the conditions (5.52), (5.56) may be replaced by the following equivalent condition*

$$\frac{\partial\varphi}{\partial x} \in C\left(R_\varepsilon^2\right).$$

If $m_1 = m_0 =: m$, *then the condition (5.65) may be replaced by the equivalent condition*

$$y^{b+m-1}\frac{\partial^{m+1}\varphi_0}{\partial x\partial y^m} \in C^2\left(R_\varepsilon^2\right).$$

Theorem 5.3.18 *All the solutions of the Problems 5.3.11 and 5.3.12 we get from (5.34) substituting there*

$$f_1(x) = \frac{\Lambda_{m_1}(a_1, b_1)\left[\Lambda_{m_0}(a_0, b_0)\tilde{f}_1(x) - \Lambda_{m_1+1}(a_0, b_0)\underset{*}{f_0}^{(m_1-m_0+1)}(x)\right]}{H^{m_0, m_1+1}} \tag{5.66}$$

and

$$f_1(x) = \frac{\Lambda_{m_0}(a_0, b_0)\tilde{f}_1(x)\left[m_1\Lambda_{m_1-1}(a_0, b_0) - a_0\Lambda_{m_1}(a_0, b_0)\underset{*}{f_0}^{(m_1-m_0+1)}(x)\right]}{\Lambda_{m_1}^{-1}(a_1, b_1)\overset{*}{H}^{m_0, m_1+1}}, \tag{5.67}$$

respectively.
All the solutions of Problem 5.3.16 we get from (5.35) substituting there

$$f_1(x) = \frac{d_{m_0}(a_0)\tilde{f}_1(x) + \Lambda_{m_1+1}(a_0, 1-m_0)\underset{*}{f_0}^{(m_1-m_0+1)}(x)}{\Lambda_{m_1}^{-1}(a_1, b_1)\tilde{H}^{m_0+1, m_1+1}}. \tag{5.68}$$

All the solutions of Problem 5.3.14 we get from (5.36), where

$$f_1(x) = \tilde{f}_1(x). \tag{5.69}$$

All the solutions of Problem 5.3.15 we get from (5.37), where $m_1 = m_0 =: m$ *and*

$$f_1(x) = a_0\underset{*}{f}'(x) + \tilde{f}_1(x). \tag{5.70}$$

All the solutions of Problem 5.3.15 has the form (5.38), where

$$f_1(x) = \begin{cases} \dfrac{(a_1 - a_0)M(a_1, b, m_1)\left[M(a_0, b, m_0)\tilde{f}_1(x_0) + a_0M(a_0, b, m_1)f_0'(x_0)\right]}{a_1M(a_1, b, m_1)M(a_0, b, m_0) - a_0M(a_1, b, m_0)M(a_0, b, m_1)}, \\ a_1 \neq a_0; \\ \dfrac{\tilde{f}_1(x_0) + aM(a, b, m_1)M^{-1}(a, b, m_0)f_0'(x_0)}{1 - a_0M^{-1}(a, b, m_1)\tilde{M}_1(a, a, b, m_0, m_1)}, \ a_1 = a_0 = a. \end{cases} \tag{5.71}$$

Solutions of Problem 5.3.6 and for $m_0 = 0$ *of Problems 5.3.11, 5.3.12, 5.3.14, 5.3.15 are uniquely determined.*

Solutions of Problem 5.3.10 and for $m_0 > 0$ *of Problems 5.3.11, 5.3.12, 5.3.14, 5.3.15 are determined up to an additive constant.*

Remark 5.3.19 *The classes of uniqueness of solutions of Problem 5.3.10 and for* $m_0 > 0$ *of Problems 5.3.11, 5.3.12, 5.3.14, 5.3.15 easily follow from Theorem 5.3.6.*

Remark 5.3.20 *If $m_0 = m_1 + 1$, as it could be easily foreseen, from (5.53) it follows that*

$$H^{m_0,m_1+1} = 0.$$

If $m_1 = m_0 = 0$, then

$$H^{m_0,m_1+1} \neq 0. \tag{5.72}$$

While for other values of $m_1,\ m_0$, using (2.65) and the equality

$$H_k\left(\frac{\pi}{2}, a_0, a_1 - a_0, \gamma, b_1 - b_0\right) = -\frac{1}{\gamma + k - 1}\left[a_0 H_{k-1}\left(\frac{\pi}{2}, a_0, a_1 - a_0, \gamma, b_1 - b_0\right)\right.$$

$$\left.+(k-1)H_{k-2}\left(\frac{\pi}{2}, a_0, a_1 - a_0, \gamma, b_1 - b_0\right)\right],$$

$$\gamma < \left\{\begin{array}{l} 1 - k \ \ when \ \ b_1 - b_0 \le 1; \\ 2 - k + b_0 - b_1 \ \ when \ \ b_1 - b_0 > 0, \end{array}\right. \quad k \ge 2,$$

we easily obtain conditions on the coefficients for fulfilment of (5.72).

Similarly, can be investigated the nominators of the expressions (5.67), (5.68), (5.71).

Proof of Theorem 5.3.18. Conditions (5.51), (5.52), (5.55)-(5.57), (5.59)-(5.61), (5.63)-(5.65) lead the question of investigation of the uniqueness of solutions of Problems 5.3.11-5.3.16 to the question of investigation of the uniqueness of solutions of Problems 5.3.1-5.3.5. E.g., in the case of Problem 5.3.15, because of (5.59) and (5.61) we have

$$\lim_{z \to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{\partial^{m_1}\varphi_1}{\partial y^{m_1}} = a_0 \underset{*}{f}'(x_0) + \tilde{f}_1(x),\ z \in \mathbb{R}^2_+,\ x_0 \in \mathbb{R}^1.$$

Therefore, the difference of two possible solutions of Problem 5.3.15 is the solution of Problem 5.3.4 with the homogeneous BCs.

It is easy to prove that under admissible conditions for f_0 of Problems 5.3.11-5.3.16 the solution $\overset{(1)}{\varphi}_0$ of Problem 5.3.1 satisfies the conditions (5.51), (5.52), (5.55), and (5.56); the solution $\overset{(2)}{\varphi}_0$ of Problem 5.3.2 satisfies the conditions (5.51) and (5.57); the solution $\overset{(3)}{\varphi}_0$ of Problem 5.3.3 satisfies the conditions (5.59), (5.60); the solution $\overset{(4)}{\varphi}_0$ of Problem 5.3.4 satisfies the conditions (5.59), (5.61); the solution $\overset{(5)}{\varphi}_0$ of Problem 5.3.5 satisfies the conditions (5.63)-(5.65).

E.g. we can easily follow the following calculations

$$\lim_{z \to x_0} y \frac{\partial^{m_1+2}\overset{(1)}{\varphi}_0}{\partial x^2 \partial y^{m_1}}$$

$$= \Lambda^{-1}_{m_0}(a_0, b_0) \lim_{z \to x_0} y \frac{\partial}{\partial y} y^{2-b_0-m_1} \int_{-\infty}^{+\infty} \left\{ \underset{*}{f_0}^{(m_1-m_0+1)}(\xi)(\xi - x)^{m_1-1}\right.$$

$$+\left[H_{m_0}\left(\frac{\pi}{2}, a_0, a_1 - a_0, b_0, b_1 - b_0\right) - \Lambda_{m_0}(a_0, b_0)\,\Omega\left(\theta, \frac{\pi}{2}, a_1 - a_0, b_1 - b_0\right)\right]$$

$$\times\Lambda_{m_1}^{-1}(a_1,b_1)\,f_1(\xi)(\xi-x)^{m_1-1}\Big\}\,e^{a_0\theta}\rho^{b_0-2}d\xi$$

$$=\lim_{z\to x_0}\int_{-\infty}^{+\infty}\Bigg[\Lambda_{m_0}^{-1}(a_0,b_0)\,\underset{*}{f_0}^{(m_1-m_0+1)}(x+yt)$$

$$+\frac{H_{m_0}\left(\frac{\pi}{2},a_0,a_1-a_0,b_0,b_1-b_0\right)}{\Lambda_{m_0}(a_0,b_0)\,\Lambda_{m_1}(a_1,b_1)}f_1(x+yt)\Bigg]$$

$$\times\left\{y^2\frac{\partial}{\partial y}\left[y^{2-b_0-m_1}(\xi-x)^{m_1-1}e^{a_0\theta}\rho^{b_0-2}\right]\right\}_{|\xi=x+yt}dt$$

$$-\Lambda_{m_1}^{-1}(a_1,b_1)\lim_{z\to x_0}\int_{-\infty}^{+\infty}f_1(x+yt)\left\{y^2\frac{\partial}{\partial y}\left[y^{2-b_0-m_1}\Omega\left(\theta,\frac{\pi}{2},a_1-a_0,b_1-b_0\right)\right.\right.$$

$$\left.\left.\times(\xi-x)^{m_1-1}e^{a\theta}\rho^{b_0-2}\right]\right\}_{|\xi=x+yt}dt\,\Bigg[\Lambda_{m_0}^{-1}(a_0,b_0)\,\underset{*}{f_0}^{(m_1-m_0+1)}(x_0)$$

$$+\frac{H_{m_0}\left(\frac{\pi}{2},a_0,a_1-a_0,b_0,b_1-b_0\right)}{\Lambda_{m_0}(a_0,b_0)\,\Lambda_{m_1}(a_1,b_1)}f_1(x_0)\Bigg]$$

$$\times\lim_{z\to x_0}\int_{-\infty}^{+\infty}\left\{y^2\frac{\partial}{\partial y}\left[y^{2-b_0-m_1}(\xi-x)^{m_1-1}e^{a\theta}\rho^{b_0-2}\right]\right\}_{|\xi=x+yt}dt$$

$$-\Lambda_{m_1}^{-1}(a_1,b_1)\,f_1(x_0)\lim_{z\to x_0}\int_{-\infty}^{+\infty}\left\{y^2\frac{\partial}{\partial y}\left[y^{2-b_0-m_1}\Omega\left(\theta,\frac{\pi}{2},a_1-a_0,b_1-b_0\right)\right.\right.$$

$$\left.\left.\times(\xi-x)^{m_1-1}e^{a\theta}\rho^{b_0-2}\right]\right\}_{|\xi=x+yt}dt=0,$$

when $b_0<2-m_1,\ m_1\in\mathbb{N}$, since, because of (3.123),

$$\int_{-\infty}^{+\infty}\left\{y^2\frac{\partial}{\partial y}\left[y^{2-b_0-m_1}(\xi-x)^{m_1-1}e^{a\theta}\rho^{b_0-2}\right]\right\}_{|\xi=x+yt}dt=0\ when\ b_0<2-m_1;$$

and

$$\int_{-\infty}^{+\infty}\left\{y^2\frac{\partial}{\partial y}\left[y^{2-b_0-m_1}\Omega\left(\theta,\frac{\pi}{2},a_1-a_0,b_1-b_0\right)(\xi-x)^{m_1-1}e^{a\theta}\rho^{b_0-2}\right]\right\}_{|\xi=x+yt}dt$$

$$=y\frac{\partial}{\partial y}\left[y^{2-b_0-m_1}\int_{-\infty}^{+\infty}\Omega\left(\theta,\frac{\pi}{2},a_1-a_0,b_1-b_0\right)(\xi-x)^{m_1-1}e^{a\theta}\rho^{b_0-2}d\xi\right]$$

$$=y\frac{\partial}{\partial y}H_{m_1-1}\left(\frac{\pi}{2},a_1-a_0,b_0,b_1-b_0\right)$$

when $b_0<2-m_1,\ b_1<3-m_1,\ m_1\in\mathbb{N}$.

According to the similar arguments we get

$$\lim_{z\to x_0}y\frac{\partial^{m_1+2}\overset{(1)}{\varphi}_0}{\partial y^{m_1+2}}=0\ \ when\quad b_0<-m_1,\quad b_1<1-m_1,\quad z\in\mathbb{R}^2_+,\quad x_0\in\mathbb{R}^1.$$

If $m_1=m_0=m$, then

$$\lim_{z\to x_0}\left(\ln\frac{1}{y}\right)^{-1}\frac{\partial^{m+1}\overset{(4)}{\varphi}_0}{\partial x\partial y^m}=\lim_{z\to x_0} y\left\langle d_m^{-1}(a_0)d_m^{-1}(a_1)\int\limits_{-\infty}^{+\infty}\left\{d_m(a_1)\underset{*}{f_0'}(\xi)(\xi-x)^m\right.\right.$$

$$-\left[d_m(a_0)\Omega\left(\theta;\frac{\pi}{2},a_1-a_0,0\right)-a_0H_{m+1}\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,0\right)\right.$$

$$\left.\left.-\Lambda_{m+1}(a_1,-m-1)-(m+1)H_m\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,0\right)\right]f_1(\xi)(\xi-x)^m\right\}$$

$$\left.\times e^{a_0\theta}\rho^{-m-3}\left[a_0(x-\xi)-(m+1)y\right]d\xi+d_m(a_0)\int\limits_{-\infty}^{+\infty}f_1(\xi)(\xi-x)^{m+1}e^{a_1\theta}\rho^{-m-3}d\xi\right\rangle$$

$$=-d_m^{-1}(a_0)d_m^{-1}(a_1)\left\{d_m(a_1)\left[a_0\Lambda_{m+1}(a_0,-m-1)+(m+1)\Lambda_m(a_0,-m-1)\right]\underset{*}{f_0'}(x_0)\right.$$

$$-d_m(a_0)\left[a_0H_{m+1}\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,0\right)\right.$$

$$\left.+(m+1)H_m\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,0\right)\right]f_1(x_0)$$

$$+\left[a_0H_{m+1}\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,0\right)+\Lambda_{m+1}(a_1,-m-1)\right.$$

$$\left.+(m+1)H_m\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,0\right)\right]\left[a_0\Lambda_{m+1}(a_0,-m-1)\right.$$

$$\left.+(m+1)\Lambda_m(a_0,-m-1)\right]f_1(x_0)-d_m(a_0)\Lambda_{m+1}(a_1,-m-1)f_1(x_0)\}$$

$$=-d_m^{-1}(a_0)d_m^{-1}(a_1)\left\{d_m(a_0)d_m(a_1)\underset{*}{f_0'}(x_0)\right.$$

$$-d_m(a_0)\left[a_0H_{m+1}\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,0\right)\right.$$

$$\left.+(m+1)H_m\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,0\right)\right]f_1(x_0)$$

$$+d_m(a_0)\left[a_0H_{m+1}\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,0\right)+\Lambda_{m+1}(a_1,-m-1)\right.$$

$$\left.+(m+1)H_m\left(\frac{\pi}{2},a_0,a_1-a_0,-m-1,0\right)\right]f_1(x_0)$$

$$-d_m(a_0)\Lambda_{m+1}(a_1,-m-1)f_1(x_0)\}=-\underset{*}{f_0'}(x_0)\ \ for\quad z\in\mathbb{R}^2_+,\quad x_0\in\mathbb{R}^1.$$

If $a_1\neq a_0$ then

$$\lim_{z\to x_0} y^{b+m_1}\frac{\partial^{m_1+2}\overset{(5)}{\varphi}_0}{\partial x^2\partial y^{m_1}}$$

$$= \lim_{z\to x_0} y^{b+m_1}\left\{M_0^{-1}(a_0,b,m_0)\int\limits_{-\infty}^{+\infty}\left[\underset{*}{f_0'}(\xi)+\frac{M(a_1,b,m_0)}{(a_1-a_0)M(a_1,b,m_1)}f_1(\xi)\right]\right.$$

$$\left.\times\frac{\partial^{m_1+1}}{\partial x\partial y^{m_1}}\left(e^{a_0\theta}\rho^{-b}\right)d\xi-\frac{(a_1-a_0)^{-1}}{M(a_1,b,m_1)}\int\limits_{-\infty}^{+\infty}f_1(\xi)\frac{\partial^{m_1+1}e^{a_0\theta}\rho^{-b}}{\partial x\partial y^{m_1}}d\xi\right\}$$

$$=M_0^{-1}(a_0,b,m_0)\left[\underset{*}{f_0'}(x_0)+\frac{M(a_1,b,m_0)f_1(x_0)}{(a_1-a_0)M(a_1,b,m_1)}\right]\int\limits_{-\infty}^{+\infty}\left(y^{b+m_1+1}\frac{\partial^{m_1+1}e^{a_0\theta}\rho^{-b}}{\partial x\partial y^{m_1}}\right)\Bigg|_{\xi=x+yt}dt$$

$$-(a_1-a_0)^{-1}M^{-1}(a_1,b,m_1)f_1(x_0)\int\limits_{-\infty}^{+\infty}\left(y^{b+m_1+1}\frac{\partial^{m_1+1}e^{a_0\theta}\rho^{-b}}{\partial x\partial y^{m_1}}\right)\Bigg|_{\xi=x+yt}dt$$

$$=M_0^{-1}(a_0,b,m_0)\left[\underset{*}{f_0'}(x_0)+\frac{M(a_1,b,m_0)f_1(x_0)}{(a_1-a_0)M(a_1,b,m_1)}\right]y\frac{\partial}{\partial x}\int\limits_{-\infty}^{+\infty}y^{b+m_1+1}\frac{\partial^{m_1}e^{a_0\theta}\rho^{-b}}{\partial y^{m_1}}d\xi$$

$$-(a_1-a_0)^{-1}M^{-1}(a_1,b,m_1)f_1(x_0)y\frac{\partial}{\partial x}\int\limits_{-\infty}^{+\infty}\left(y^{b+m_1+1}\frac{\partial^{m_1+}e^{a_0\theta}\rho^{-b}}{\partial x\partial y^{m_1}}\right)\Bigg|_{\xi=x+yt}d\xi$$

$$=M^{-1}(a_0,b,m_0)\left[\underset{*}{f_0'}(x_0)+\frac{M(a_1,b,m_0)}{(a_1-a_0)M(a_1,b,m_1)}f_1(x_0)\right]y\frac{\partial}{\partial x}M(a_0,b,m_1)$$

$$-(a_1-a_0)^{-1}M^{-1}(a_1,b,m_1)f_1(x_0)y\frac{\partial}{\partial x}M(a_1,b,m_1)=0\ \ when\ \ z\in\mathbb{R}^2_+,\ x_0\in\mathbb{R}^1$$

□

Now, taking into account (5.51), (5.52), (5.55)-(5.57), (5.59)-(5.61), (5.63)-(5.65), and comparing in pairs (5.27) for $\delta=1$ with (5.50); (5.27) for $\delta=1$ with (5.54); (5.32) with (5.58); (5.33) for $\delta=1$ with (5.64) it is easy to obtain (5.66)-(5.71).

Remark 5.3.21 *After constructing explicitly solutions of Problems 5.3.11-5.3.16 without taking into account the conditions (5.51), (5.52), (5.55)-(5.57), (5.59)-(5.61), (5.63)-(5.65) we directly check that they are solutions for the complex constants a_δ, b_δ, $\delta=0,1$ (see also Remark 5.3.8) as well. So that the above conditions we need only for the analysing the question of the uniqueness of the solutions.*

Remark 5.3.22 *In the particular case when $m_1=m_0=0$, $b_1=b_0=b<0$, $a_1=ic$, $a_0=ic$ with the real constant $c\neq 0$ the operator E_2^b is real one and the real solution (5.34) of Problem 5.3.11 with regard to (5.66) after some simple* *transformations will get the form*

$$\varphi(x,y)=y^{1-b}\int\limits_{-\infty}^{+\infty}\left\{\frac{cA(c,b)f_0(\xi)+bB(c,b)\underset{*}{\widetilde{f}_1}^{(-1)}(\xi)}{c[A^2(c,b)+B(c,b)]}\cos(c\theta)\right.$$
$$\left.+\frac{cB(c,b)f_0(\xi)-bA(c,b)\underset{*}{\widetilde{f}_1}^{(-1)}(\xi)}{c[A^2(c,b)+B(c,b)]}\sin(c\theta)\right\}\rho^{b-2}d\xi,\quad z\in\mathbb{R}^2_+.\quad (5.73)$$

As we have shown (see Section 2.1, the formulas (2.55), (2.56) and the corresponding consequent ones)

$$A^2(c,b)+B^2(c,b)=2^{2b}(1-b)^{-2}\pi^2\tilde{B}^{-2}\left(\frac{2+c+b}{2},\frac{2-c-b}{2}\right)>0$$

for $b<\pm c$.

If $b=-1$ *and* $c=1$, *then the operator* $y^{-2}E_2^{-1}$ *will be biharmonic one and from (5.73) we have the following well-known formula*

$$\varphi(x,y)=\frac{2y^3}{\pi}\int\limits_{-\infty}^{+\infty}f_0(\xi)\rho^{-4}d\xi+\frac{y^2}{\pi}\int\limits_{-\infty}^{+\infty}\tilde{f}_1^{(-1)}(\xi)\rho^{-2}d\xi$$

for the half-plane.

Remark 5.3.23 *If* $m_1=m_0=0,\ b_1=b_0=b$, *then the solution of Problem 5.3.12 has the form*

$$\varphi(x,y)$$
$$=\begin{cases}\dfrac{y^{1-b}}{a_1-a_0}\displaystyle\int\limits_{-\infty}^{+\infty}\left[\frac{a_1f_0(\xi)+\underset{*}{\widetilde{f}_1}^{(-1)}(\xi)}{\Lambda(a_0,b)}e^{a_0\theta}-\frac{a_0f_0(\xi)+\underset{*}{\widetilde{f}_1}^{(-1)}(\xi)}{\Lambda(a_1,b)}e^{a_1\theta}\right]\rho^{b-2}d\xi\\ \qquad\qquad \text{when }\ a_1\neq a_0;\\[2ex] \dfrac{y^{1-b}}{\Lambda(a,b)}\\ \times\displaystyle\int\limits_{-\infty}^{+\infty}\left\{\frac{[\Lambda(a,b)+aM_0(a,2-b,1,0)]f_0(\xi)+M_0(a,2-b,1,0)\underset{*}{\widetilde{f}_1}^{(-1)}(\xi)}{\Lambda(a,b)}\right.\\ \left.-\,[af_0(\xi)+\underset{*}{\widetilde{f}_1}^{(-1)}(\xi)]\theta\right\}e^{a\theta}\rho^{b-2}d\xi\ \text{ when }\ a_1=a_0=a\end{cases}\quad (5.74)$$

$\varphi\in K^{0,0}$ *(1,1) is unique and satisfies the conditions (5.28) and*

$$\frac{\partial\varphi}{\partial x}\in C(\mathbb{R}^2_\varepsilon),\quad \lim_{z\to x_0}\frac{\partial^2\varphi}{\partial x^2}=0,\quad z\in\mathbb{R}^2_+,\quad x_0\in\mathbb{R}^1.$$

Remark 5.3.24 *If $b > 0$, $a_i = ic$, $a_0 = ic$, where real number $c \neq 0$, then we easily reduce the solution of Problem 5.3.16 to the following form*

$$\begin{aligned}
\varphi(x,y) &= \frac{b}{c(b^2-c^2)[A^2(c,-b)+\overset{*}{A}{}^2(c,-b)]}\int\limits_{-\infty}^{+\infty}\left\{\left[\frac{(-1)^{m_0}cA(c,-b)}{(b,m_0-1)}f_0(\xi)\right.\right.\\
&+ \left.\frac{(-1)^{m_1}B(c,-b)}{(b,m_1-1)}\ \underset{*}{\tilde{f}}{}_1^{(-1)}(\xi)\right]\cos(c\theta)+\left[\frac{(-1)^{m_0}cB(c,-b)}{(b,m_0-1)}f_0(\xi)\right.\\
&+ \left.\left.\frac{(-1)^{m_1}\,A(c,-b)}{(b,m_1-1)}\ \underset{*}{\tilde{f}}{}_1^{(-1)}(\xi)\right]\sin(c\theta)\right\}\rho^{-b}d\xi.
\end{aligned} \tag{5.75}$$

Evidently,

$$A^2(c,-b)+B^2(c,-b)=2^{-2b}(1+b)^{-2}\pi^2\tilde{B}^{-2}\left(\frac{2+c+b}{2},\frac{2-c+b}{2}\right)>0$$

for $b > \pm c$.

5.4 On a way of constructing solutions of BVPs for higher order equations

In this section the simple way of constructing solutions of boundary value problems for higher order equations by means of solutions of boundary value problems for equations of less order is pointed out (see, G. Jaiani [9]).

Suppose that a domain $\Omega \in \mathbb{R}^2$ and its boundary is $\partial\Omega$. We seek a function φ satisfying

$$F\varphi := \Big(\prod_{j=0}^{n-1}E_j\Big)\varphi = 0 \ \text{ in } \ \Omega, \tag{5.76}$$

$$B_j\varphi = f_j, \ \ j=\overline{0,n-1}, \ \text{ on } \ \partial\Omega, \tag{5.77}$$

where E_j, $j=\overline{0,n-1}$, are second order elliptic operators which can degenerate on the part of the boundary or on the whole one; B_j, $j=\overline{0,n-1}$, are differential operators of certain order (zero order is also admitted), in general, containing the weight functions and f_j, $j=\overline{0,n-1}$, are given functions. Suppose that the operator F remains unchangeable by an arbitrary rearrangement of operators E_j, $j=\overline{0,n-1}$.

Let us assume that the problems

$$L_k\varphi_k = 0 \ \text{ in } \ \Omega, \quad B_k^*\varphi_k=\psi_k, \ \ k=\overline{0,n-1}, \ \text{ on } \ \partial\Omega, \tag{5.78}$$

where B_k^*, $k=\overline{0,n-1}$, are certain differential operators, ψ_k, $k=\overline{0,n-1}$, are given functions, are solvable in classical sence and the solutions have the form

$$\varphi_k=\Psi_k(\psi_k), \ \ k=\overline{0,n-1}, \tag{5.79}$$

where the operator Ψ_k, $k=\overline{0,n-1}$, are such that one can define

$$B_j\Psi_k(\psi_k), \ \ k,j=\overline{0,n-1}, \ \text{ on } \ \partial\Omega.$$

Theorem 5.4.1 *Suppose that the system*

$$\sum_{k=0}^{n-1} B_j \Psi_k(\psi_k) = f_j, \quad j = \overline{0, n-1}, \quad \text{on } \partial\Omega \tag{5.80}$$

is solvable with respect to ψ_k, $k = \overline{0, n-1}$. *Then the problem (5.76), (5.77) is solvable in the classical sense and the solution has the form*

$$\varphi = \sum_{k=0}^{n-1} \Psi_k(\psi_k). \tag{5.81}$$

Proof. Let us seek the solution of the problem (5.76), (5.77) in the form

$$\varphi = \sum_{k=0}^{n-1} \varphi_k, \tag{5.82}$$

where φ_k, $k = \overline{0, n-1}$, are the solutions of the problems (5.78). Since E_j, $j = \overline{0, n-1}$, in (5.76) are rearrangable, (5.82), obviously, is solution of (5.76) and, by virtue of (5.79), has the form (5.81). On the other hand, in view of (5.80), such φ satisfies (5.77).

It is clear that a similar proposition holds true also for more general cases in the sense of type and order of operators E_j, $j = \overline{0, n-1}$, and a number of independent variables under various initial, boundary and mixed conditions.

By application of this proposition there are two principal moments: suitable choice of boundary operators B_k^*, $k = \overline{0, n-1}$, and solvability of the system (5.80). □

This method has been applied to investigation of equation (5.2) (see Section 5.2 and also [9]).

5.5 Some general comments and problems to be solved

In this chapter problems set for the iterated EPD equation in the half-plane are completely investigated. All the solutions are constructed in explicit form, in quadratures.

The analogues BVPs are to be investigated for the iterated EPD equation in the finite domain as it was done for the single EPD equation.

The methods of approximate and numerical solution of the posed BVPs are to be developed.

Efficiency of the explicit solutions constructed in Section 5.2 and Section 5.3 for numerical solution of BVPs should be studied.

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