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ON THE PULLBACK STABILITY OF A QUOTIENT MAP WITH RESPECT TO A CLOSURE OPERATOR

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ABSTRACT. There are well-known characterizations of the hereditary quotient maps in the category of topological spaces, (that is, of quotient maps stable under pullback along embeddings), as well as of universal quotient maps (that is, of quotient maps stable under pullback). These are precisely the so-called pseudo-open maps, as shown by Arhangel'skii, and the bi-quotient maps of Michael, as shown by Day and Kelly, respectively. In this paper hereditary and stable quotient maps are characterized in the broader context given by a category equipped with a closure operator. To this end, we derive explicit formulae and conditions for the closure in the codomain of such a quotient map in terms of the closure in its domain.

1. Introduction

In the category $\mathcal{T}op$ of topological spaces, a quotient map is just an epimorphism $f : X \to Y$ for which $B \subseteq Y$ is closed whenever $p^{-1}(B)$ is closed. A pseudo-open map is an epimorphism f such that for each $B \subseteq Y$ the closure of B in Y coincides with the image by f of the closure of $f^{-1}(B)$ in X. Without any changes these concepts make sense in an arbitrary category equipped with a Dikranjan-Giuli closure operator [10]. Hence there is a natural notion of *c-quotient* map [8] and of c-pseudo-open map, which is known under the name of *c-final* morphism (cf. [6] and [14]). Explicitly, a morphism $f : X \to Y$ is c-final if $c_Y(n) \cong f(c_X(f^{-1}(n)))$ for all subobjects n of Y.

Our starting point in this paper is to consider the expression on the right of the last formula as an operator F of n, and to characterize c-quotient maps in terms of this operator (Theorem 3.1) which, in some sense, gives a measure of the distance between the concepts of c-quotient and c-final morphism. In Section 4 we use this characterization to show that the notion of c-quotient ascends along both c-closed or c-open monomorphism, that is, if in the pullback diagram

$$\begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "p'"'] & V \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{1}$$

p' is a c-closed or c-open monomorphism, then with f also f' is a c-quotient morphism. This result follows the same pattern as other facts proved in [11] and [14] about properties

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ascending along special morphisms. In Section 5 we characterize hereditary c-quotients as the c-final morphisms, provided that we deal with morphisms whose codomain has a subobject lattice which is a Boolean algebra. Our main result concerns the characterization of *universal c-quotient* maps, that is, of those c-quotient maps preserved by any pullback. In $\mathcal{T}op$, with the usual Kuratowski closure, they were characterized by Michael [16] and Day and Kelly [9] in terms of filters and open sets. Here we present another characterization in terms of an inequality involving the closures in the domain and in the codomain of the map in question (Theorem 6.3). The point of this inequality is that it still gives at least a sufficient criterion for universality of a c-quotient in a fairly general categorical context which is described in Theorem 6.10.

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2. Preliminaries

Throughout this paper, we shall be working in a finitely complete category $\mathcal{X}$ with a proper factorization system $(\mathcal{E}, \mathcal{M})$ for morphisms (see, for example, [1, 13, 17]). We shall further assume that $\mathcal{X}$ is $\mathcal{M}$*-complete*, that is, $\mathcal{M}$ is pullback-stable and the intersection of a (possibly large) family of morphisms in $\mathcal{M}$ with common codomain exists in $\mathcal{X}$ and belongs to $\mathcal{M}$. By a *subobject* of an object X we mean an $\mathcal{M}$-morphism with codomain in X. We use sub$X$ to indifferently denote the class of all subobjects of X or the usual class of equivalence classes of isomorphic $\mathcal{M}$-morphisms with codomain in $\mathcal{X}$. The $\mathcal{M}$-completeness of $\mathcal{X}$ implies that, for every object $X \in \mathcal{X}$, sub$X$ is a complete (pre)ordered class.

2.1. Every morphism $f : X \to Y$ gives the image-preimage adjunction

$$f(-) \dashv f^{-1}(-) : \mathrm{sub}Y \longrightarrow \mathrm{sub}X\,.$$

Thus, we have that:
(a) $m \leq f^{-1}(f(m))$ and $f(f^{-1}(n)) \leq n$, for all $f : X \to Y$, $m \in \mathrm{sub}X$ and $n \in \mathrm{sub}Y$;
(b) $f^{-1}\left(\bigwedge_{i\in I} n_i\right) \cong \bigwedge_{i\in I} f^{-1}(n_i)$ and $f\left(\bigvee_{i\in I} m_i\right) \cong \bigvee_{i\in I} f(m_i)$.
Moreover, we have the following properties (cf. [14]):
(c) $m \cong f^{-1}(f(m))$, if f is monic and $\mathcal{E}$ is stable under pullbacks;
(d) $f(f^{-1}(n)) \cong n$, if and only if every pullback of f along an $\mathcal{M}$-morphism belongs to $\mathcal{E}$.

2.2. Let the diagram (1) in Introduction be a pullback diagram. Then, for every $m \in \mathrm{sub}X$, by image-preimage adjunction, $f'({p'}^{-1}(m)) \leq p^{-1}(f(m))$; we say that (1) satisfies the *Beck-Chevalley Property (BCP)* if, moreover, we have $f'({p'}^{-1}(m)) \cong p^{-1}(f(m))$. We shall make use of the following result (cf. [15, 7]):

Every pullback in $\mathcal{X}$ satisfies (BCP) if and only if $\mathcal{E}$ is stable under pullback.

2.3. A *closure operator* c of $\mathcal{X}$ with respect to $(\mathcal{E},\mathcal{M})$ is given by a family of functions $c_X : \mathrm{sub}X \to \mathrm{sub}X$ ($X \in \mathcal{X}$) such that, for $m, n \in \mathrm{sub}X$, $m \le c_X(m)$, $c_X(m) \le c_X(n)$ if $m \le n$, and all $f : X \to Y$ in $\mathcal{X}$ are c-continuous, that is, $f(c_X(m)) \le c_Y(f(m))$. A subobject $m \in \mathrm{sub}X$ is *c-closed* if $m \cong c_X(m)$ and the operator c is called *idempotent* if $c_X(m)$ is c-closed for all $m \in \mathrm{sub}X$, $X \in \mathcal{X}$. A morphism $f : X \to Y$ is said to be *c-closed* if, for each $m \in \mathrm{sub}X$, $f(c_X(m)) \cong c_Y(f(m))$, and it is said to be *c-open* if, for each $n \in \mathrm{sub}Y$, $f^{-1}(c_Y(n)) \cong c_X(f^{-1}(n))$. A morphism $f : X \to Y$ is said to be a *c-quotient* provided that it belongs to $\mathcal{E}$ and, for each $n \in \mathrm{sub}Y$, the c-closedness of $f^{-1}(n)$ implies the c-closedness of n. It is said to be *c-final* if, for each $n \in \mathrm{sub}Y$, $c_Y(n) \cong f(c_X(f^{-1}(n)))$.

2.4. Recall that a commutative diagram as in (1) in Introduction is said to be a $\mathcal{E}$-weak pullback whenever the canonical morphism $e : U \to X \times_Y V$ belongs to $\mathcal{E}$. The (BCP) property is enjoyed by all pullbacks in $\mathcal{X}$ iff it is enjoyed by all $\mathcal{E}$-weak pullbacks (see [15, 7, 14]). All results on pullbacks presented in sections 3 and 4 are also true for $\mathcal{E}$-weak pullbacks. This simply follows from the fact that all proofs work if, instead of pullbacks, we have $\mathcal{E}$-weak pullbacks.

3. A characterization of c-quotients

In this section we give a characterization of c-quotient which does not involve the notion of c-closed subobject. Furthermore, it sheds light on the difference between c-quotient morphisms and c-final morphisms.

Given a morphism $f : X \to Y$ in $\mathcal{X}$, we have the function

$$F : \mathrm{sub}Y \longrightarrow \mathrm{sub}Y$$

which assigns to each $n \in \mathrm{sub}Y$ the $\mathcal{M}$-subobject $f(c_X(f^{-1}(n)))$.

F is clearly extensive and monotone. Moreover, a morphism $f : X \to Y$ is c-final exactly whenever $c_Y \le F$, that is, $c_Y(n) \le F(n)$ for each $n \in \mathrm{sub}Y$. We call F the *c-final function associated to f*. We are going to show that c-quotient morphisms can also be characterized by means of the corresponding c-final function.

We obtain an ascending ordinal chain of functions from $\mathrm{sub}Y$ into $\mathrm{sub}Y$,

$$(F^\alpha)_{\alpha \in Ord},$$

by putting

$$F^0 = Id_{\mathrm{sub}Y},\ F^\alpha = FF^{\alpha-1},\ F^\beta = \bigvee_{\gamma<\beta} F^\gamma$$

for every successor ordinal α and for every limit ordinal β. We denote by F^∞ the function which, to each $n \in \mathrm{sub}Y$, assigns $\bigvee_{\alpha \in Ord} F^\alpha(n)$.

3.1. Theorem. *Let $\mathcal{X}$ be $\mathcal{M}$-wellpowered. Then a morphism $f : X \to Y$ is a c-quotient if and only if $c_Y \le F^\infty$.*

PROOF. Let us assume that $c_Y \leq F^\infty$. By definition of F^α, it is clear that, for each ordinal α, $F^\alpha(1_Y) \leq f(1_X)$. Thus, we have that $1_Y \cong c_Y(1_Y) \leq F^\infty(1_Y) \leq f(1_X)$; consequently, $f(1_X)$ is an isomorphism, so that $f \in \mathcal{E}$. Let $n \in \mathrm{sub}Y$ be such that $f^{-1}(n)$ is c-closed, that is, $c_X(f^{-1}(n)) \cong f^{-1}(n)$. Then, we have that $F(n) = f(c_X(f^{-1}(n)) \cong f(f^{-1}(n)) \leq n$. From this, it immediately follows that $F^\alpha(n) \leq n$ for each α. Thus $c_Y(n) \leq n$, that is, n is c-closed.

Conversely, let $f : X \to Y$ be a c-quotient. Since $\mathcal{X}$ is $\mathcal{M}$-wellpowered, the chain $(F^\gamma)_{\gamma \in Ord}$ is stationary, that is, there is some ordinal α such that $F^\beta \cong F^\alpha$ for all $\beta \geq \alpha$. Let $n \in \mathrm{sub}Y$; we are going to prove that $c_Y(n) \leq F^\alpha(n)$. For that, we show that $F^\alpha(n)$ is c-closed.

In fact, we have that

$$\begin{aligned} f^{-1}(F^\alpha(n)) &\cong f^{-1}(F^{\alpha+1}(n)) \\ &= f^{-1}(f(c_X(f^{-1}(F^\alpha(n))))) \\ &\geq c_X(f^{-1}(F^\alpha(n))), \end{aligned}$$

from what follows that $f^{-1}(F^\alpha(n))$ is c-closed. Thus, since f is a c-quotient, $F^\alpha(n)$ is c-closed. Now, using the fact that $n \leq F^\alpha(n)$, we obtain that $c_Y(n) \leq F^\alpha(n)$. ■

3.2. REMARK. Assume that $\mathcal{X}$ is $\mathcal{M}$-wellpowered and that $\mathcal{E}$ is pullback stable along $\mathcal{M}$-morphisms. Then, as a consequence of the above theorem, we have that: If the closure operator c is idempotent, then a morphism $f : X \to Y$ is c-quotient if and only if $c_Y \cong F^\infty$, if and only if there is some $\alpha \in Ord$ such that $c_Y \cong F^\alpha$. In fact, let c be idempotent and let $f : X \to Y$ belong to $\mathcal{E}$. Then, for each $n \in \mathrm{sub}Y$ and each $\alpha \in Ord$, we have that $F(n) = f(c_X(f^{-1}(n)) \leq f(f^{-1}(c_Y(n)) \cong c_Y(n)$ (by (c) of 2.1) and, assuming $F^\alpha(n) \leq c_Y(n)$, it follows that $F^{\alpha+1}(n) \cong F(F^\alpha(n)) \leq f(c_X(f^{-1}(c_Y(n)))) \leq f(f^{-1}(c_Y^2(n)) \cong c_Y^2(n) \cong c_Y(n)$. Thus, it it is clear that $F^\alpha(n) \leq c_Y(n)$ for all $\alpha \in Ord$.

4. Pullbacks of c-quotients

From now on, we assume that $\mathcal{E}$ is stable under pullback and so, by 2.2, that every pullback in $\mathcal{X}$ satisfies (BCP).

4.1. Let

$$\begin{tikzcd} U \arrow[r, "f'"] \arrow[d, "p'"'] & V \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2}$$

be a pullback diagram. In [14] it was proved that:

If f is c-final and p' c-initial then f' is c-final and p c-initial.

We are going to study the parallel situation corresponding to replacing c-final by c-quotient and c-initial by c-closed or c-open monomorphism.

4.2. First, we need the following definition:

By a chain of subobjects of X, we mean a family $(n_i)_{i\in I}$ of subobjects of X such that $(I,\leq)$ is a linear ordered set and $n_i \leq n_j$ whenever $i \leq j$. Given a morphism $g : X \to Y$, we say that *$\mathcal{M}$-unions of chains of subobjects of Y are preserved by the inverse image under g* provided that, for each chain $(n_i)_{i\in I}$ of subobjects of Y, it holds that

$$g^{-1}(\bigvee_{i\in I} n_i) \cong \bigvee_{i\in I} g^{-1}(n_i)\,.$$

We point out that the preservation by inverse images under monomorphisms of $\mathcal{M}$-unions of chains is fulfilled by numerous categories; for instance, all monotopological categories over Set for $\mathcal{M}$ the class of embeddings, the categories of vector spaces, groups and rings, for $\mathcal{M}$ the injective homomorphisms.

Let us consider the pullback diagram (2), where p' and p are monomorphisms. Then, we prove the following two assertions:

4.3. Theorem. *If f is a c-quotient and p' c-closed, then f' is a c-quotient; furthermore, p is c-closed if c is idempotent.*

Proof. Let $n \in \mathrm{sub}V$ be such that ${f'}^{-1}(n)$ is a c-closed subobject of U. We show that n is c-closed, thus f' is a c-quotient. Since p' is a c-closed morphism, we obtain that $p'({f'}^{-1}(n)) \cong p'(c_U({f'}^{-1}(n)) \cong c_X(p'({f'}^{-1}(n)))$, that is, $p'({f'}^{-1}(n))$ is c-closed.

Thus, since, by (BCP), $p'({f'}^{-1}(n)) \cong f^{-1}(p(n))$, we conclude that $f^{-1}(p(n))$ is c-closed. As f is a c-quotient, it follows that $p(n)$ is c-closed, i.e.,

$$c_Y(p(n)) \cong p(n)\,. \tag{3}$$

Then,

$$\begin{aligned} c_V(n) &\leq p^{-1}(p(c_V(n))), && \text{by 2.1(a)}\\ &\leq p^{-1}(c_Y(p(n))), && \text{by } c\text{-continuity}\\ &\cong p^{-1}(p(n)), && \text{from (3)}\\ &\cong n, && \text{by 2.1(c)} \end{aligned}$$

Therefore, n is c-closed.

Now, if c is idempotent, in order to show that p is c-closed, it suffices to show that p maps c-closed subobjects to c-closed subobjects (see [14]). Let n be a c-closed subobject of V. The idempotency of c implies that, then, ${f'}^{-1}(n)$ is also c-closed, since

$$c_U({f'}^{-1}(n)) \cong c_U({f'}^{-1}(c_V(n))) \leq {f'}^{-1}(c_V^2((n))) \cong {f'}^{-1}(c_V(n)) \cong {f'}^{-1}(n)\,.$$

Consequently, $p'({f'}^{-1}(n))$ is c-closed and, from the (BCP) property, the same happens to $f^{-1}(p(n))$. So, the fact that f is c-quotient implies that $p(n)$ is c-closed. ∎

4.4. Theorem. *Let $\mathcal{X}$ be $\mathcal{M}$-wellpowered and let $\mathcal{M}$-unions of chains of subobjects of Y be preserved by inverse images under monomorphisms. If f is a c-quotient and p' is c-open, then f' is a c-quotient; furthermore, p is c-open if c is idempotent.*

PROOF. We are going to make use of the characterization of c-quotients given in Section 2. Let F' be the c-final function associated to f'. First we show that, for each $n \in \mathrm{sub}Y$ and each ordinal β, we have that

$$p^{-1}(F^{\beta}(n)) \cong F'^{\beta}(p^{-1}(n)). \tag{4}$$

For $\beta = 0$, it is clear. Assuming the property for all ordinals less then β, we have: For $\beta = \alpha + 1$,

$$\begin{aligned} p^{-1}(F^{\beta}(n)) &= p^{-1}(f(c_X(f^{-1}(F^{\alpha}(n))))) \\ &\cong f'(p'^{-1}(c_X(f^{-1}(F^{\alpha}(n))))), && \text{by (BCP)} \\ &\cong f'(c_U(p'^{-1}(f^{-1}(F^{\alpha}(n))))), && \text{since } p' \text{ is } c\text{-open} \\ &\cong f'(c_U(f'^{-1}(p^{-1}(F^{\alpha}(n))))), && \text{by (BCP)} \\ &\cong f'(c_U(f'^{-1}(F'^{\alpha}(p^{-1}(n))))), && \text{by inductive hypothesis} \\ &= F'^{\beta}(p^{-1}(n)) \end{aligned}$$

For β a limit ordinal,

$$\begin{aligned} p^{-1}(F^{\beta}(n)) &= p^{-1}(\textstyle\bigvee_{\gamma<\beta} F^{\gamma}(n)) \\ &\cong \textstyle\bigvee_{\gamma<\beta} p^{-1}(F^{\gamma}(n)), && \text{by the preservation of } \mathcal{M}\text{-unions} \\ & && \text{of chains by inverse image} \\ &\cong \textstyle\bigvee_{\gamma<\beta} F'^{\beta}(p^{-1}(n)), && \text{by inductive hypothesis} \end{aligned}$$

Let now $m \in \mathrm{sub}V$; we want to prove that, for some $\alpha \in Ord$, $c_V(m) \leq F'^{\alpha}(m)$. We have that

$$\begin{aligned} c_V(m) &\leq p^{-1}(p(c_V(m))), && \text{by 2.1(a)} \\ &\leq p^{-1}(c_Y(p(m))), && \text{by } c\text{-continuity} \\ &\leq p^{-1}(F^{\alpha}(p(m))), && \text{for some } \alpha \in Ord, \text{ since} \\ & && f \text{ is } c\text{-quotient and } \mathcal{X} \text{ is } \mathcal{M}\text{-} \\ & && \text{wellpowered,} \\ &\cong F'^{\alpha}(p^{-1}(p(m))), && \text{by (4)} \\ &\cong F'^{\alpha}(m), && \text{by 2.1(c).} \end{aligned}$$

Therefore, F' is a c-quotient.

Let us now show that, assuming that c is idempotent, p is c-open. Since $\mathcal{X}$ is $\mathcal{M}$-wellpowered, and f and f' are c-quotients, there is some ordinal α such that $c_Y \cong F^{\alpha}$ and $c_V \cong F'^{\alpha}$ (see Remark 3.2). Thus, it follows that:

$$\begin{aligned} p^{-1}(c_Y(n)) &\cong p^{-1}(F^{\alpha}(n)) \\ &\cong F'^{\alpha}(p^{-1}(n)), \text{ by (4)} \\ &\cong c_V(p^{-1}(n)). \end{aligned}$$

■

4.5. Remark. The idempotency of c is necessary for p to be c-closed in Theorem 4.3. The same happens to the c-openness of p in Theorem 4.4. To show that, let us consider the topological category $\mathcal{SG}ph$ of spatial graphs, that is, the category of graphs (X, E) such that the relation E is reflexive, and maps which preserve the edges. Let $\mathcal{M}$ be the class of embeddings of subgraphs and let c be the up-closure, that is, $c = \uparrow$ with $\uparrow_X (M) = \{x \in X \mid (\exists a \in M) a \to x\}$, for each $M \subseteq X$. It is well-known that the up-closure is not idempotent in $\mathcal{SG}ph$ (cf. [11]). In the diagram (2), let $U = V = X$ be the spatial graph $0 \to 1 \to 2$ and let Y be the spatial graph obtained from X by joining the edge $0 \to 2$. Let the underlying map of each one of the morphisms f, f', p and p' be the identity. This way, the diagram (2) is a pullback, f is $\uparrow$-quotient, p' is simultaneously $\uparrow$-closed and $\uparrow$-open, but p is not $\uparrow$-closed since $p(\uparrow_V (\{0\}) = \{0,1\} \neq \{0,1,2\} = \uparrow_Y (p(\{0,1\}))$; and p is not $\uparrow$-open because $p^{-1}(\uparrow_Y (\{0\}) = \{0,1,2\} \neq \{0,1\} = \uparrow_V (p^{-1}(\{0\}))$.

5. Hereditary c-quotients

From now on, we assume that the closure operator c is *hereditary*, i.e., for all $m : M \to Y$, $y : Y \to X$ in $\mathcal{M}$, it holds that $y \wedge c_X(y \cdot m) \cong y \cdot c_Y(m)$, or, equivalently, $c_Y(m) \cong y^{-1}(c_X(y \cdot m))$.

5.1. It is clear that the heredity of c is equivalent to every morphism in $\mathcal{M}$ being c-initial (cf. [14]).

5.2. A *hereditary c-quotient* is a morphism whose pullback along any $\mathcal{M}$-morphism is a c-quotient. Clearly any hereditary c-quotient is a c-quotient. It is well-known that in $\mathcal{T}op$, for the Kuratowski closure k, the hereditary k-quotients coincide with the k-final morphisms. In this section, we are going to show that this is a particular instance of a general property. Indeed, next we prove that, under convenient conditions, the hereditary c-quotients actually coincide with c-final morphisms.

5.3. Theorem. *Let subY be a Boolean algebra. Then a morphism with codomain in Y is a hereditary c-quotient if and only if it is c-final.*

Proof. From 4.1 and 5.1, it immediately follows that every c-final morphism is a hereditary c-quotient.

Conversely, let $f : X \to Y$ be a hereditary c-quotient and let $n : N \to Y$ be a subobject of Y. We want to prove that $c_Y(n) \cong f(c_X(f^{-1}(n)))$. Let $m : M \to Y$ be the complement of $f(c_X(f^{-1}(n)))$ and put $r = m \vee n : R \to Y$. Let $n_r : N \to R$ be the $\mathcal{M}$-morphism for which $r \cdot n_r = n$ and consider the pullback

$$\begin{tikzcd} f^{-1}(R) \arrow[r, "g"] \arrow[d, "f^{-1}(r)"'] & R \arrow[d, "r"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

By hypothesis, g is a c-quotient; we show that $g^{-1}(n_r)$ is c-closed, so that n_r is c-closed. To this end, since c is hereditary, it suffices to show that $f^{-1}(r) \wedge c_X(f^{-1}(r) \cdot g^{-1}(n_r)) \cong f^{-1}(r) \cdot g^{-1}(n_r))$, that is, that $f^{-1}(r) \wedge c_X(f^{-1}(n)) \cong f^{-1}(n)$. In fact, this isomorphism occurs since, on the one hand, it is clear that $f^{-1}(n) \leq f^{-1}(r) \wedge c_X(f^{-1}(n))$; and, on the other hand, we have that

$$\begin{aligned} f^{-1}(r) \wedge c_X(f^{-1}(n)) &\leq f^{-1}(r) \wedge f^{-1}(f(c_X(f^{-1}(n)))), \quad \text{by 2.1(a)} \\ &\cong f^{-1}(r \wedge f(c_X(f^{-1}(n)))), \quad \text{by 2.1(b)} \\ &\cong f^{-1}((m \vee n) \wedge f(c_X(f^{-1}(n)))) \\ &\cong f^{-1}(m \wedge f(c_X(f^{-1}(n))) \vee (n \wedge f(c_X(f^{-1}(n))))) \\ &\cong f^{-1}(0_X \vee n), \quad \text{by 2.1(d)} \\ &\cong f^{-1}(n)\,. \end{aligned}$$

Therefore, $c_R(n_r) \cong n_r$ and so, $r \cdot c_R(n_r) \cong n$. Thus, since the heredity of c assures that $r \wedge c_Y(n) \cong r \cdot c_R(n_r)$, we have that $r \wedge c_Y(n) \cong n$ and, consequently, that $m \wedge c_Y(n) \leq n$. Then, we obtain that

$$c_Y(n) \wedge m \leq n \wedge m \leq f(c_X(f^{-1}(n))) \wedge m \cong 0_Y\,, \tag{5}$$

and that

$$c_Y(n) \vee m \cong f(f^{-1}(c_Y(n))) \vee m \geq f(c_X(f^{-1}(n))) \vee m \cong 1_Y\,. \tag{6}$$

Since m is the complement of $f(c_X(f^{-1}(n)))$, it follows from (5) and (6) that $c_Y(n) \cong f(c_X(f^{-1}(n)))$. ∎

6. On universal c-quotients

6.1. A morphism $f : X \to Y$ is said to be a *universal c-quotient* provided that every pullback of f along an arbitrary morphism is a c-quotient.

6.2. A quotient map in the category of topological spaces is just a k-quotient, with k the Kuratowski closure operator. The universal quotients in $\mathcal{T}op$ were characterized in [16] and [9] as being just those continuous maps $f : X \to Y$ which fulfill one of the following equivalent conditions:

(a) For each $y \in Y$ and each open coverage $(G_I)_{i \in I}$ of $f^{-1}(y)$, there is some finite subset J of I such that $y \in \operatorname{int}(\bigcup_{j \in J} f(G_j))$.

(b) For each filter base $\mathcal{B}$ in Y, if $y \in Y$ adheres to $\mathcal{B}$, then some $x \in f^{-1}(y)$ adheres to $f^{-1}(\mathcal{B})$.

These two conditions do not give clear information about the behaviour of f with respect to the closure operator. However several properties on the category of topological spaces are particular cases of properties in a much general setting, namely a category equipped with a closure operator. So, one natural question is: how to characterize universal quotients in $\mathcal{T}op$ by means of the closure operator? The answer is given by the following theorem.

6.3. Theorem. *In $\mathcal{T}op$, a morphism $f : X \to Y$ is a universal c-quotient, for c the Kuratowski closure operator, if and only if, for each family $(n_i)_{i\in I}$ of subobjects of Y, the condition*

$$\text{(GF)} \qquad \bigwedge_{\substack{J\subseteq I\\ J\text{finite}}} c_Y(\bigwedge_{j\in J} n_j) \leq f(\bigwedge_{i\in I} c_X(f^{-1}(n_i)))$$

holds.

Proof. Along the proof, we denote the Kuratowski closure of a subset F of a space by $\overline{F}$. The condition (b) stated in 6.2 is clearly equivalent to the following condition:

(c) For each filter $\mathcal{F}$ in Y, $\cap_{F\in\mathcal{F}}\overline{F} \subseteq f(\cap_{F\in\mathcal{F}}\overline{f^{-1}(F)})$.

We show that it is equivalent to (GF). Assume (GF) and let $\mathcal{F}$ be a filter in Y. Putting $\mathcal{F} = \{F_i : i \in I\}$ and denoting by $\mathcal{J}$ the set of all finite subsets of I, we have that

$$\cap_{J\in\mathcal{J}}\, \overline{\cap_{j\in J}F_i} \subseteq f(\cap_{i\in I}\overline{f^{-1}(F_i)})\,. \tag{7}$$

But, for all finite subsets J of I, $\cap_{j\in J}F_i \in \mathcal{F}$; then $\{\overline{\cap_{j\in J}F_i},\, J \in \mathcal{J}\} = \{\overline{F_i},\, i \in I\}$ and (c) follows.

Conversely, assume (c). Let $(N_i)_{i\in I}$ be a family of subsets of Y. If, for some finite subset J of I, $\cap_{j\in J}N_i = \emptyset$, then (GF) is trivially verified. Otherwise, let $\mathcal{F}$ be the filter generated by $(N_i)_{i\in I}$. From (c), we have that

$$\cap_{F\in\mathcal{F}}\,\overline{F} \subseteq f(\cap_{F\in\mathcal{F}}\overline{f^{-1}(F)})\,. \tag{8}$$

But

$$\cap_{F\in\mathcal{F}}\,\overline{F} = \cap_{J\in\mathcal{J}}\overline{\cap_{j\in J}N_i} \tag{9}$$

and

$$f(\cap_{F\in\mathcal{F}}\overline{f^{-1}(F)}) = f(\cap_{J\in\mathcal{J}}\overline{f^{-1}(\cap_{j\in J}N_j}) = f(\cap_{J\in\mathcal{J}}\overline{\cap_{j\in J}f^{-1}(N_j)})\,. \tag{10}$$

Since $\{\overline{\cap_{j\in J}f^{-1}(N_j)},\, J\in\mathcal{J}\} \supseteq \{\overline{f^{-1}(N_i)},\, i \in I\}$, we have that

$$\cap_{J\in\mathcal{J}}\,\overline{\cap_{j\in J}f^{-1}(N_j)} \subseteq \cap_{i\in I}\overline{f^{-1}(N_i)}\,. \tag{11}$$

Then, from (8), (9) (10) and (11), we conclude that $\cap_{J\in\mathcal{J}}\overline{\cap_{j\in J}N_i} \subseteq f(\cap_{i\in I}\overline{f^{-1}(N_i)})$. ∎

6.4. Remarks.

1. It is clear that in a category with a closure operator, any morphism f fulfilling the condition (GF) is, in particular, c-final.
2. For the up-closure $\uparrow$ in the category of spatial graphs $\mathcal{SG}ph$, we have that:

   The universal $\uparrow$-quotients coincide with the $\uparrow$-final morphisms.

   To show this, first we recall that, in $\mathcal{SG}ph$, the $\uparrow$-final morphisms are just those surjections $f : X \to Y$ which fulfill the following condition: $y_1 \to y_2 \Leftrightarrow \exists x_1, x_2 \in X\, (x_1 \to x_2$ and $f(x_i) = y_i$, $i = 1, 2)$ (cf. [8]). Let

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "r"'] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a pullback diagram, where $f : X \to Y$ is $\uparrow$-final and s is an arbitrary morphism. We show that g is also $\uparrow$-final. Given (x_1, z_1) and (x_2, z_2) in W, $(x_1, z_1) \to (x_2, z_2)$ is an edge in W iff $x_1 \to x_2$ is an edge in X and $z_1 \to z_2$ is an edge in Z. So, let $z_1 \to z_2$ be an edge in Z; then $s(z_1) \to s(z_2)$ belongs to Y, from what follows that there is some edge $x_1 \to x_2$ in X such that $f(x_i) = s(z_i)$, $i = 1, 2$. Consequently, $(x_1, z_1) \to (x_2, z_2)$ belongs to W with $g(x_1, z_1) = z_1$ and $g(x_2, z_2) = z_2$.

3. The condition (GF) does not give the desired characterization in the general setting. In fact, it fails to be a necessary condition in the category $\mathcal{SG}ph$ equipped with the up-closure. To show that, consider the following example: Let X be the spatial graph with underlying set $\bigcup\limits_{i \in \mathbb{N}} \{x_i, w_i\}$ and with non loop edges $x_i \to w_i$, $i \in \mathbb{N}$, and let Y be the spatial graph with underlying set $\bigcup\limits_{i \in \mathbb{N}} \{y_i\} \cup \{y\}$ and with non loop edges $y_i \to y$, $i \in \mathbb{N}$; let $f : X \to Y$ be defined by $f(x_i) = y_i$ and $f(w_i) = y$. Then, f is clearly $\uparrow$-final, so, from 2., a universal $\uparrow$-quotient. But it does not satisfy condition (GF). To see that, consider the family of all subobjects of Y of the form $N_i = \{y_k \mid k \geq i\}$; then $y \in \bigwedge\limits_{\substack{J \subseteq I \\ J \text{finite}}} \uparrow_Y (\bigwedge\limits_{j \in J} N_j)$ but $y \notin f(\bigwedge_{i \in I} \uparrow_X (f^{-1}(N_i)))$.

6.5. Next, we are going to show that in a suitable setting, which generalizes the one of the category of topological spaces, the condition (GF) is sufficient. For this purpose, we need to use the notion of point.

Let $\mathcal{X}$ have a terminal object 1; then, since $\mathcal{E} \subseteq \mathrm{Epi}(\mathcal{X})$, each morphism with domain 1, being a split monomorphism, belongs to $\mathcal{M}$. The subobjects of an $\mathcal{X}$-object X with domain 1 are said to be *points of* X. The class of all points of X is denoted by $\mathrm{pt}X$. The notion of point, in a sense which generalizes the one used here, was studied in detail in [4] (see also [5]). Let $m : M \to X$ be a subobject of X and let $x \in \mathrm{pt}X$. By $\mathrm{pt}(m)$ we denote the set of all points of X of the form mz, with $z \in \mathrm{pt}M$. We say that the category $\mathcal{X}$ *has enough points* provided that, for each $X \in \mathcal{X}$, $1_X \cong \bigvee \mathrm{pt}X$. This is equivalent to saying that, for each $\mathcal{X}$-object X and each $m \in \mathrm{sub}X$, $m \cong \vee \mathrm{pt}(m)$. In fact, it is clear that the last assumption implies that $\mathcal{X}$ has enough points; conversely, we have that $\bigvee \mathrm{pt}(m) \cong \bigvee_{z \in \mathrm{pt}M}(mz) \cong m(\bigvee_{z \in \mathrm{pt}M} z) \cong m(1_M) \cong m$.

The following three lemmas will be useful to prove 6.10 below.

6.6. Lemma. *In a category with enough points, let m, m_i and n belong to sub$X$ and let $f : X \to Y$ be a morphism. Then:*

(a) $m \leq n \Leftrightarrow pt(m) \subseteq pt(n)$; and, thus, $m \cong n \Leftrightarrow pt(m) = pt(n)$.

(b) $pt(\bigwedge_{i \in I} m_i) = \bigcap_{i \in I} pt(m_i)$.

(c) $pt(m) = \emptyset \Rightarrow m \cong o_X$*; furthermore, we have that* $pt(m) = \emptyset \Leftrightarrow m \cong o_X$ *whenever* $o_X \not\cong 1_X$.

(d) $x \in pt(m) \Rightarrow f(x) \in pt(f(m))$.

PROOF. (a) and (b) are immediate.

(c) $\mathrm{pt}(m) = \emptyset \Rightarrow \bigvee \mathrm{pt}(m) \cong o_X \Leftrightarrow m \cong o_X$. To show that $o_X \not\cong 1_X$ ensures that $\mathrm{pt}(o_X) = \emptyset$, let $z \in \mathrm{pt}(o_X)$. Then $z \leq o_X$, and so, $z \cong o_X$. But then, $O_X \cong 1$ and, consequently, for each $x \in \mathrm{pt}X$, it holds that $x \cong o_X$, from what follows that $1_X \cong \bigvee \mathrm{pt}X \cong o_X$.

(d) Let $m : M \to X$ be a subobject of X and let $z \in \mathrm{pt}M$ and $x \in \mathrm{pt}X$ be such that $x = mz$. Let $f(m) \cdot e$ be the $(\mathcal{E}, \mathcal{M})$-factorization of the morphism fm. Then, we have that $f(x) \cong fx = fmz = f(m)ez$; this means that $f(x) \in \mathrm{pt}(f(m))$. ■

In the following, if sub$X$ is a Boolean algebra and $m \in \mathrm{sub}X$, we write m^* to denote the complement of m.

6.7. LEMMA. *In a category with enough points, let subX and subY be Boolean algebras such that* $o_Y \not\cong 1_Y$*, and let* $f : X \to Y$ *be a morphism. Then, for* $m \in subX$ *and* $n, n_i \in subY$*, we have that:*

(a) $y \in pt(n) \Leftrightarrow y \wedge n^* \cong o_Y \Leftrightarrow y \notin pt(n^*)$.

(b) $pt(\bigvee_{i \in I} n_i) = \bigcup_{i \in I} pt(n_i)$*, for any finite set* I.

(c) If complements are preserved under the inverse image by f*, then:*

(i) $m \wedge f^{-1}(n) \cong o_X \Leftrightarrow f(m) \wedge n \cong o_Y$.

(ii) $pt(f(m)) = \{f(x) : x \in pt(m)\}$.

PROOF. (a) Let $y \in \mathrm{pt}Y$. Then $y \in \mathrm{pt}(n) \Leftrightarrow y \leq n \Leftrightarrow y \wedge n^* \cong o_Y \Leftrightarrow \mathrm{pt}(y \wedge n^*) = \emptyset \Leftrightarrow \mathrm{pt}(y) \wedge \mathrm{pt}(n^*) = \emptyset \Leftrightarrow y \notin \mathrm{pt}(n^*)$, the last but two equivalence being a consequence of 6.6 (c).

(b) Using (a) above and (b) and (c) of 6.6, we conclude that $y \in \mathrm{pt}(n_1 \vee n_2)$ iff $y \notin \mathrm{pt}(n_1^*) \cap \mathrm{pt}(n_2^*)$. Again by (a), we get that the last condition is equivalent to $y \in \mathrm{pt}(n_1)$ or $y \in \mathrm{pt}(n_2)$.

(c) (i) If $m \wedge f^{-1}(n) \cong o_X$, then $m \leq (f^{-1}(n))^* \cong f^{-1}(n^*)$. Consequently, $f(m) \leq n^*$ and, then, $f(m) \wedge n \cong o_Y$. Conversely, if $m \wedge f^{-1}(n) \not\cong o_X$, then, by 6.6(c), there is some $x \in \mathrm{pt}(m \wedge f^{-1}(n))$. Using Lemma 6.6, one concludes that $f(x) \in \mathrm{pt}(f(m)) \cap \mathrm{pt}(n)$ and, then, since $o_Y \not\cong 1_Y$, that $f(m) \wedge n \not\cong o_Y$.

(c) (ii) The inclusion $\{f(x) : x \in pt(m)\} \subseteq \mathrm{pt}(f(m))$ is clear, from 6.6 (d). On the other hand, for $y \in \mathrm{pt}Y$, if $y \in \mathrm{pt}(f(m))$ then $y \wedge f(m) \not\cong o_Y$ and, from (i), $f^{-1}(y) \wedge m \not\cong o_X$. Thus $\mathrm{pt}(f^{-1}(y)) \cap \mathrm{pt}(m) \neq \emptyset$. This means that, for some $x \in \mathrm{pt}X$, $x \in \mathrm{pt}(f^{-1}(y)) \cap \mathrm{pt}(m)$ and, thus, $y \in \{f(x) : x \in \mathrm{pt}(m)\}$. ■

6.8. LEMMA. *For a subobject* $m : M \to X$ *of* X*, the following two first assertions are equivalent and they are equivalent to (c) in a category with enough points.*

(a) $m : M \to X$ *is c-closed.*

(b) $m \cong \bigwedge\{n \in subX : n \text{ c-closed and } m \leq n\}$.

(c) If $x \in pt(X)$ *is such that* $m \leq n \Rightarrow x \in pt(n)$ *for all c-closed subobjects* n *of* X*, then* $x \in pt(m)$.

PROOF. (a)$\Leftrightarrow$ (b): This is straightforward.

(b)$\Leftrightarrow$ (c): This follows from (a) and (b) of Lemma 6.6. ■

6.9. Given a pullback

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "r"'] & W \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{12}$$

let $\mathcal{N}$ be the family of all subobjects of Z of the form $n = r^{-1}(n_1) \vee g^{-1}(n_2)$ for some c-closed subobjects $n_1 \in \mathrm{sub}X$ and $n_2 \in \mathrm{sub}W$. We say that the pullback (12) is *c-initial*, provided that the c-closed subobjects of Z are just those $m : M \to Z$ in sub$Z$ such that $m \cong \bigwedge\{n \in \mathrm{sub}X : n \in \mathcal{N} \text{ and } m \leq n\}$. If $\mathcal{X}$ has enough points, this is equivalent to say that m fulfills the following property: for any $z \in \mathrm{pt}Z$, if $z \in \mathrm{pt}(n)$, for all $n \in \mathcal{N}$ such that $m \leq n$, then $z \in \mathrm{pt}(m)$.

We point out that for $\mathcal{X} = \mathcal{T}op$, $\mathcal{M}$ the class of embeddings and c the Kuratowski closure operator, the c-initiality for pullbacks coincides with the usual initiality of pullbacks in $\mathcal{T}op$.

6.10. THEOREM. *Let $\mathcal{X}$ have enough points and let c be an idempotent closure operator in $\mathcal{X}$ such that any finite union of c-closed subobjects is a c-closed subobject. Moreover, let the diagram (12) be a c-initial pullback, let sub$X$, sub$Y$, sub$Z$ and sub$W$ be Boolean algebras, with $o_Y \not\cong 1_Y$, and let all morphisms in (12) preserve complements under the inverse image. Then, if $f : X \to Y$ fulfills condition (GF), it is a universal c-quotient.*

PROOF. If $o_X \cong 1_X$ or $o_W \cong 1_W$, it is trivially concluded that g is a c-quotient. Thus, besides the assumption $o_Y \not\cong 1_Y$, we also assume, without loss of generality, that $o_X \not\cong 1_X$ and $o_W \not\cong 1_W$. If $f : X \to Y$ fulfills condition (GF), then it is clearly a c-quotient; in particular, it belongs to $\mathcal{E}$ and so $g \in \mathcal{E}$. Let $n : N \to W$ be a subobject of W such that $g^{-1}(n)$ is c-closed. In order to show that n is c-closed, we are going to use (c) of Lemma 6.8: we show that, for $w \in \mathrm{pt}W$, if $w \notin \mathrm{pt}(n)$, then there exists a c-closed subobject u of W such that $n \leq u$ and $w \notin \mathrm{pt}(u)$.

Suppose that $w \notin \mathrm{pt}(n)$. Then, by (d) of Lemma 6.6, for each $z \in \mathrm{pt}(g^{-1}(w))$, we have that $z \notin \mathrm{pt}(g^{-1}(n))$. Consequently, using the c-initiality of the pullback (12), we obtain that, for each $z \in \mathrm{pt}(g^{-1}(w))$, there are c-closed subobjects n_{1z} and n_{2z} of X and W, respectively, such that

$$g^{-1}(n) \leq r^{-1}(n_{1z}) \vee g^{-1}(n_{2z}) \quad \text{and} \quad z \notin \mathrm{pt}(r^{-1}(n_{1z}) \vee g^{-1}(n_{2z}))\,. \tag{13}$$

Put $I = \mathrm{pt}(g^{-1}(w))$ and denote by $\mathcal{J}$ the class of all finite subsets of I. For each $z \in \mathrm{pt}(g^{-1}(w))$, let

$$m_z = (f(n_{1z}^*))^*\ .$$

Then, by (GF),

$$\bigwedge_{J\in\mathcal{J}} c_Y(\bigwedge_{z\in J} m_z) \leq f(\bigwedge_{z\in I} c_X(f^{-1}(m_z)))\,. \tag{14}$$

First, we show that, for each $z \in I$, it holds that $f^{-1}(m_z) \leq n_{1z}$. Indeed,

$$\begin{aligned} m_z = (f(n_{1z}^*))^* \;&\Rightarrow m_z \wedge f(n_{1z}^*) \cong o_Y \\ &\Leftrightarrow f^{-1}(m_z) \wedge n_{1z}^* \cong o_X, \text{ by 6.7(c)(i)} \\ &\Rightarrow f^{-1}(m_z) \leq n_{1z} \end{aligned}$$

Consequently, since n_{1z} is c-closed, we have that $c_X(f^{-1}(m_z)) \leq n_{1z}$.

Now, we are going to show that $s(w) \notin \mathrm{pt}(c_Y(\bigwedge_{z\in J} m_z))$ for some $J \in \mathcal{J}$. For each $z \in \mathrm{pt}(g^{-1}(w))$, since $z \notin \mathrm{pt}(r^{-1}(n_{1z}))$, we have that $z \wedge r^{-1}(n_{1z}) \cong o_Z$; then, by 6.7(c)(i), $r(z) \wedge n_{1z} \cong o_X$. Since $c_X(f^{-1}(m_z)) \leq n_{1z}$, we obtain that $r(z) \wedge c_X(f^{-1}(m_z)) \cong o_X$. Consequently, for all $z \in \mathrm{pt}(g^{-1}(w))$, we have that $r(z) \wedge (\bigwedge_{z\in I} c_X(f^{-1}(m_z))) \cong o_X$. Then, by Lemma 6.7(c)(ii), we conclude that $r(g^{-1}(w)) \wedge (\bigwedge_{z\in I} c_X(f^{-1}(m_z))) \cong o_X$, that is, by (BCP), that $f^{-1}(s(w)) \wedge (\bigwedge_{z\in I} c_X(f^{-1}(m_z))) \cong o_X$. By 6.7(c)(i), it follows that $s(w) \wedge f\,(\bigwedge_{z\in I} c_X(f^{-1}(m_z))) \cong o_Y$ and, consequently, using (14), that $s(w) \wedge (\bigwedge_{J\in\mathcal{J}}(c_Y(\bigwedge_{z\in J} m_z))) \cong o_Y$. This implies that $s(w) \notin \mathrm{pt}\,(c_Y(\bigwedge_{z\in J} m_z))$, for some $J \in \mathcal{J}$, and, thus, for this J, $w \notin \mathrm{pt}\,(s^{-1}(c_Y(\bigwedge_{z\in J} m_z)))$. On the other hand, for each $z \in J$, since $z \notin \mathrm{pt}(g^{-1}(n_{2z}))$, we have, by 6.6(c), that $z \wedge g^{-1}(n_{2z}) \cong o_Z$, what, combined with 6.7(c)(i) and the fact that $g(z) \cong w$, assures that $w \wedge n_{2z} \cong o_W$, that is, that $w \notin \mathrm{pt}(n_{2z})$. Now, using 6.7(b), we get that $w \notin \mathrm{pt}\,(s^{-1}(c_Y(\bigwedge_{z\in J} m_z)) \bigvee(\bigvee_{z\in J} n_{2z}))$.

The subobject $s^{-1}(c_Y(\bigwedge_{z\in J} m_z))$ is c-closed, since c is idempotent. Then, the subobject $s^{-1}(c_Y(\bigwedge_{z\in J} m_z)) \bigvee(\bigvee_{z\in J} n_{2z})$, being the finite union of c-closed subobjects of W, is, by assumption, c-closed. We are going to show that it has n as its subobject, what completes the proof. Let $t \in \mathrm{pt}(n)$. By (13) and using (BCP), we have that, for each $z \in J$, $t \leq s^{-1}(f(n_{1z})) \vee n_{2z}$. If $t \leq n_{2z}$ for some $z \in J$, then it is clear that $t \in \mathrm{pt}\,(s^{-1}(c_Y(\bigwedge_{z\in J} m_z)) \bigvee(\bigvee_{z\in J} n_{2z})$. Otherwise, since t is a point, for each $z \in J$, we get that $t \wedge n_{2z} \cong o_W$, and, by 6.6(d), that $g^{-1}(t) \wedge g^{-1}(n_{2z}) \cong o_X$. Consequently, as $g^{-1}(t) \leq r^{-1}(n_{1z}) \vee g^{-1}(n_{2z})$, and we are working in a Boolean algebra, it follows that $g^{-1}(t) \leq r^{-1}(n_{1z})$. Thus, we have that $r(g^{-1}(t)) \leq n_{1z}$, that is, $f^{-1}(s(t)) \leq n_{1z}$. This implies that $f^{-1}(s(t)) \wedge n_{1z}^* \cong o_X$. Then, using Lemma 6.7(c)(i) twice, we obtain in succession, that $s(t) \wedge f(n_{1z}^*) \cong o_Y$ and $t \wedge s^{-1}\,(f(n_{1z}^*)) \cong o_W$. Hence, $t \leq (s^{-1}\,(f(n_{1z}^*)))^*$, and thus, $t \leq s^{-1}(m_z)$. Consequently, one concludes that $t \leq \bigwedge_{z\in J} s^{-1}(m_z) \leq s^{-1}(c_Y(\bigwedge_{z\in J} m_z))$. Therefore, we have that $t \in \mathrm{pt}\,(s^{-1}(c_Y(\bigwedge_{z\in J} m_z)) \bigvee(\bigvee_{z\in J} n_{2z})$. ∎

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