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$\mathscr{V}$-CAT IS LOCALLY PRESENTABLE OR LOCALLY BOUNDED IF $\mathscr{V}$ IS SO

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ABSTRACT. We show, for a monoidal closed category $\mathscr{V} = (\mathscr{V}_0, \otimes, I)$, that the category $\mathscr{V}$**-Cat** of small $\mathscr{V}$-categories is locally λ-presentable if $\mathscr{V}_0$ is so, and that it is locally λ-bounded if the closed category $\mathscr{V}$ is so, meaning that $\mathscr{V}_0$ is locally λ-bounded and that a side condition involving the monoidal structure is satisfied.

Many important properties of a monoidal category $\mathscr{V}$ are inherited by the category $\mathscr{V}$**-Cat** of small $\mathscr{V}$-categories. For instance, if $\mathscr{V}$ is symmetric monoidal, $\mathscr{V}$-Cat has a canonical symmetric monoidal structure, as was observed already in [4]. Much later [7, Remark 5.2], it was realized that if $\mathscr{V}$ is only *braided* monoidal then $\mathscr{V}$**-Cat** still has a canonical monoidal structure, although it need not have a braiding unless the braiding on $\mathscr{V}$ is in fact a symmetry. Similarly, it is straightforward to show that $\mathscr{V}$**-Cat** is monoidal *closed* when $\mathscr{V}$ is closed and complete, and that $\mathscr{V}$**-Cat** is complete when $\mathscr{V}$ is so. All of these results are essentially routine; the less trivial fact that $\mathscr{V}$**-Cat** is cocomplete when $\mathscr{V}$ is so was first proved in [11].

The properties of $\mathscr{V}$ or $\mathscr{V}$**-Cat** that we consider here are of a less basic nature, being conditions on $\mathscr{V}$ which allow proofs by transfinite induction of the existence of various important adjoints. The best known of these conditions is *local presentability* [5], but there is also the notion of *local boundedness* [8], which is more general than local presentability, but also much more common, and sufficient for the central existence results of [8, Chapter 6], from which follow the basic results of the theory of enriched projective sketches. Recall that to be locally presentable is to be locally λ-presentable for some regular cardinal λ, and similarly that to be locally bounded is to be locally λ-bounded for some λ. It would be one thing to prove that $\mathscr{V}$**-Cat** is locally presentable if $\mathscr{V}$ is so (in the sense that its underlying ordinary category $\mathscr{V}_0$ is so); here we prove the stronger result that $\mathscr{V}$**-Cat** is locally λ-presentable if $\mathscr{V}_0$ is so, so that the passage from $\mathscr{V}$ to $\mathscr{V}$**-Cat** does not require the regular cardinal λ to be changed. When it comes to local boundedness, we prove that $\mathscr{V}$**-Cat** is locally λ-bounded when $\mathscr{V}$ is so "as a closed category", meaning that $\mathscr{V}_0$ is locally λ-bounded and satisfies a side condition involving the monoidal structure. We recall the precise definitions of local λ-presentability and local λ-boundedness in Section 2, but the common aspect is that $\mathscr{V}_0$ is cocomplete and has a small set $\mathscr{G}$ of objects forming in some sense a generator of $\mathscr{V}_0$, with the representables $\mathscr{V}_0(G, -) : \mathscr{V}_0 \to \mathbf{Set}$ preserving certain colimits: λ-filtered colimits in the locally λ-presentable case, and λ-filtered unions

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(with respect to a given factorization system on $\mathscr{V}_0$) in the locally λ-bounded case.

All categories are assumed to have small hom-sets.

1. Unions

For this section we consider a cocomplete category $\mathscr{K}$ with a proper factorization system $(\mathscr{E}, \mathscr{M})$; recall that $(\mathscr{E}, \mathscr{M})$ is *proper* when each $\mathscr{E}$ is an epimorphism and each $\mathscr{M}$ a monomorphism — equivalently, when each $\mathscr{M}$ is a monomorphism and each coretraction is in $\mathscr{M}$. Note that a map $f : A \to B$ lies in $\mathscr{E}$ precisely when each factorization $f = mg$ with $m \in \mathscr{M}$ has m invertible.

A small family $(m_j : A_j \to B)_{j \in J}$ of maps with a common codomain is said to be *jointly in* $\mathscr{E}$ if the induced map $m : \sum_j A_j \to B$ is in $\mathscr{E}$; this is equivalent to saying that there is no proper $\mathscr{M}$-subobject of B through which each m_j factorizes. When moreover each m_j is in $\mathscr{M}$, we say that the family constitutes an $\mathscr{M}$*-union*, or that B is the $\mathscr{M}$-union of the m_j.

More generally, the $\mathscr{M}$-union of a small family $(m_j : A_j \to B)_{j \in J}$ of $\mathscr{M}$-subobjects of B is defined to be the unique $\mathscr{M}$-subobject $n : A \to B$ containing the m_j for which the corresponding $n_j : A_j \to A$ constitute an $\mathscr{M}$-union. We may calculate this $\mathscr{M}$-union by taking the $(\mathscr{E}, \mathscr{M})$-factorization of $m : \sum_j A_j \to B$.

We shall need to speak of *preservation* of $\mathscr{M}$-unions only in the case of representable functors. We say that the representable functor $\mathscr{K}(X, -) : \mathscr{K} \to \mathbf{Set}$ preserves the $\mathscr{M}$-union $(m_j : A_j \to B)_{j \in J}$ if the functions $\mathscr{K}(X, m_j) : \mathscr{K}(X, A_j) \to \mathscr{K}(X, B)$ are jointly surjective; in more concrete terms this says that any map $f : X \to B$ factorizes through some m_j.

Given a small family $(m_j : A_j \to B)_{j \in J}$ we can preorder the set J by setting $j \leq k$ whenever $A_j \leq A_k$ as $\mathscr{M}$-subobjects of B. Then the A_j are the object values of a functor $A : J \to \mathscr{K}$, and we may form $\mathrm{colim} A$ and the induced $h : \mathrm{colim} A \to B$. It is easy to see that the m_j are an $\mathscr{M}$-union if and only if $h \in \mathscr{E}$.

Finally for a regular cardinal λ, the preorder J is said to be λ-filtered if it is so as a category: that is, if for each subset K of J with cardinality less than λ, the A_k with $k \in K$ are all contained in some A_j. By a λ-filtered $\mathscr{M}$-union $(m_j : A_j \to B)_{j \in J}$ we mean one for which J is λ-filtered.

2. Locally presentable and locally bounded categories

In this section we continue to consider a cocomplete category $\mathscr{K}$; from time to time we shall further suppose it to be equipped with a proper factorization system $(\mathscr{E}, \mathscr{M})$.

Let λ be a regular cardinal. An object X of $\mathscr{K}$ is said to be λ-*presentable* [5] if the representable functor $\mathscr{K}(X, -) : \mathscr{K} \to \mathbf{Set}$ preserves λ-filtered colimits, and λ-*bounded* [6] if $\mathscr{K}(X, -)$ preserves λ-filtered $\mathscr{M}$-unions.

A small set $\mathscr{G}$ of objects of $\mathscr{K}$ is said to be a *strong generator* if an arrow $f : A \to B$ is invertible whenever $\mathscr{K}(G, f) : \mathscr{K}(G, A) \to \mathscr{K}(G, B)$ is bijective for each $G \in \mathscr{G}$; while

$\mathscr{G}$ is an $(\mathscr{E},\mathscr{M})$*-generator* if this is true for arrows $f : A \to B$ in $\mathscr{M}$. Clearly $\mathscr{G}$ is an $(\mathscr{E},\mathscr{M})$-generator precisely when, for each $A \in \mathscr{K}$, the family of all maps $G \to A$ with $G \in \mathscr{G}$ is jointly in $\mathscr{E}$; that is, when the evident map $\epsilon_A : \sum_{G\in\mathscr{G}} \mathscr{K}(G, A) \bullet G \to A$ lies in $\mathscr{E}$; here we are writing $X \bullet A$ for the coproduct of X copies of A. When $(\mathscr{E},\mathscr{M})$ is the proper factorization system (strong epimorphisms, monomorphisms), we have the well-known result (see for instance [6, Proposition 2.5.3]) that an $(\mathscr{E},\mathscr{M})$-generator is the same thing as a strong generator. In our cocomplete category $\mathscr{K}$, the pair (strong epimorphisms, monomorphisms) is certainly a proper factorization system if $\mathscr{K}$ admits arbitrary cointersections of strong epimorphisms.

The cocomplete $\mathscr{K}$ is said to be *locally λ-presentable* if it has a strong generator all of whose objects are λ-presentable; it is a consequence that $\mathscr{K}$ is then complete. This and many other facts about locally presentable categories can be found in the books [1, 5, 10].

The cocomplete $\mathscr{K}$ is said to be *locally λ-bounded* with respect to a proper factorization system $(\mathscr{E},\mathscr{M})$ if it has an $(\mathscr{E},\mathscr{M})$-generator all of whose objects are λ-bounded, and if moreover $\mathscr{K}$ admits arbitrary cointersections (even large ones, if need be) of maps in $\mathscr{E}$. The definition of locally λ-bounded category given in [8] included the further assumption of completeness, but once again this is a consequence of the other axioms, as we show in Corollary 2.2 below.

As well as being complete, every locally λ-presentable category is well-powered; it follows that it has a proper factorization system $(\mathscr{E},\mathscr{M})$ in which $\mathscr{M}$ consists of the monomorphisms and $\mathscr{E}$ the strong epimorphisms. For this factorization system, an $(\mathscr{E},\mathscr{M})$-generator is, as we observed above, the same thing as a strong generator. Locally presentable categories are also well-copowered, and so arbitrary $\mathscr{E}$-cointersections exist. Finally, it turns out (see [6, Lemma 2.3.1]) that in a locally λ-presentable category every λ-presentable object is λ-bounded; we deduce that every locally λ-presentable category is locally λ-bounded. The converse, however, is false: see [5, p.104] or [6, p.190] for examples of locally λ-bounded categories that are not locally μ-presentable for any μ.

A cocomplete monoidal closed category is said to be *locally λ-bounded as a closed category* if its underlying ordinary category is locally λ-bounded and, in addition, the functors $A \otimes -$ and $- \otimes A$ map $\mathscr{E}$ into $\mathscr{E}$ for all objects A. The latter condition is clearly equivalent to the condition that $e \otimes e' \in \mathscr{E}$ whenever $e, e' \in \mathscr{E}$, and it turns out to be vacuous if $\mathscr{M}$ consists of all the monomorphisms.

In fact all the examples of closed categories considered in [8] have some factorization system for which they are locally bounded. Algebraic examples, such as the categories **Set**, **Cat**, and **Ab** of sets, categories, and abelian groups are all locally finitely presentable, as is the combinatorial example **SSet**, the category of simplicial sets. The reason for using the weaker notion of local boundedness rather than local presentability is the desire to include such topological examples as the categories **CGTop**, **QTop**, and **Ban** of compactly generated topological spaces, quasi-topological spaces, and Banach spaces, which are not locally presentable, but are locally bounded. The example **QTop** is not $\mathscr{E}$-wellcopowered, which explains why we must explicitly require arbitrary cointersections of maps in $\mathscr{E}$. For the details, and for many further examples, including Lawvere's closed category given by the interval $[0, \infty]$ of the reals, see [8, Chapter 6].

For our promised proof that every locally bounded category is complete we use an (apparently unpublished) $(\mathscr{E},\mathscr{M})$-variant of Freyd's Special Adjoint Functor Theorem, namely:

2.1. PROPOSITION. *Let the cocomplete category $\mathscr{K}$ have the factorization system $(\mathscr{E},\mathscr{M})$ for which $\mathscr{E}$ is contained in the epimorphisms; suppose that $\mathscr{K}$ admits arbitrary cointersections of maps in $\mathscr{E}$, and that $\mathscr{K}$ has an $(\mathscr{E},\mathscr{M})$-generator $\mathscr{G}$. Then every cocontinuous functor $S:\mathscr{K}\to\mathscr{L}$ has a right adjoint.*

PROOF. To provide a right adjoint to S is equally to provide, for each $D\in\mathscr{L}$, a terminal object of the comma category S/D, whose objects are pairs $(C, f : SC\to D)$ and whose maps $(C,f)\to(C',f')$ are maps $x:C\to C'$ with $Sx.f=f'$. The forgetful functor $U:S/D\to\mathscr{K}$ creates colimits (and hence reflects epimorphisms). We get an induced factorization system, still called $(\mathscr{E},\mathscr{M})$, on S/D by taking $x:(C,f)\to(C',f')$ to be in $\mathscr{E}$ or in $\mathscr{M}$ when Ux is so; once again every $\mathscr{E}$ is an epimorphism. Finally, the small set consisting of the (C,f) with $C\in\mathscr{G}$ forms an $(\mathscr{E},\mathscr{M})$-generator for S/D. Thus $(S/D,\mathscr{E},\mathscr{M})$ has just the properties required in the proposition of $(\mathscr{K},\mathscr{E},\mathscr{M})$. So it suffices to prove that the $\mathscr{K}$ of the proposition has a terminal object.

Form in $\mathscr{K}$ the coproduct $H=\sum_{G\in\mathscr{G}}G$, and let $\zeta:H\to K$ be the cointersection of *all* the maps in $\mathscr{E}$ having domain H; of course $\zeta\in\mathscr{E}$ and is an epimorphism. Any two maps $f,g:A\to K$ must coincide: for their coequalizer $h:K\to L$ is in $\mathscr{E}$, so that $h\zeta$ is in $\mathscr{E}$, whence $kh\zeta=\zeta$ for some k by the definition of ζ as the smallest $\mathscr{E}$-quotient, so that in fact $kh=1$ and h is invertible.

To exhibit K as the desired terminal object it remains only to show that, for each $A\in\mathscr{K}$, there is a map $A\to K$. For each $G\in\mathscr{G}$ and $A\in\mathscr{K}$ we have the trivial function $\mathscr{K}(G,A)\to 1$ into the singleton set, so that we have an induced map $t:\sum_{G\in\mathscr{G}}\mathscr{K}(G,A)\bullet G\to\sum_{G\in\mathscr{G}}G$. Form in $\mathscr{C}$ the pushout

$$\begin{tikzcd}
\sum_{G\in\mathscr{G}}\mathscr{K}(G,A)\bullet G \arrow[r, "\epsilon_A"] \arrow[d, "t"'] & A \arrow[d, "r"] \\
\sum_{G\in\mathscr{G}}G \arrow[r, "s"'] & L\ ;
\end{tikzcd}$$

here ϵ_A lies in $\mathscr{E}$ since $\mathscr{G}$ is an $(\mathscr{E},\mathscr{M})$-generator, so that its pushout s also lies in $\mathscr{E}$. By the definition of K, therefore, there is a map $v:L\to K$, and thus a map $vr:A\to K$. ∎

2.2. COROLLARY. *Let the cocomplete category $\mathscr{K}$ have a factorization system $(\mathscr{E},\mathscr{M})$ for which every $\mathscr{E}$ is an epimorphism, and suppose that $\mathscr{K}$ admits arbitrary cointersections of maps in $\mathscr{E}$ and has an $(\mathscr{E},\mathscr{M})$-generator $\mathscr{G}$. Then $\mathscr{K}$ is complete.*

PROOF. For each small category $\mathscr{C}$ we seek a right adjoint to the diagonal $\Delta:\mathscr{K}\to[\mathscr{C},\mathscr{K}]$; and this adjoint exists by the proposition, since $[\mathscr{C},\mathscr{K}]$ has colimits formed pointwise and Δ is cocontinuous. ∎

2.3. Remark. Given a cocomplete category $\mathscr{K}$, to give a factorization system $(\mathscr{E}, \mathscr{M})$ having each $\mathscr{E}$ epimorphic and admitting arbitrary cointersections of maps in $\mathscr{E}$, it suffices by [3, Lemma 3.1] to give a class $\mathscr{E}$ of epimorphisms in $\mathscr{K}$, closed under composition and stable under pushout, for which arbitrary cointersections of maps in $\mathscr{E}$ exist and lie in $\mathscr{E}$.

Before leaving this section, we make a final observation of rather lesser importance. We have discussed what it means for a monoidal closed category to be locally bounded *as a closed category*, but we have not considered local presentability for closed categories. In [9], a monoidal closed category $\mathscr{V}$ was defined to be *locally λ-presentable as a closed category* if its underlying category $\mathscr{V}_0$ was locally λ-presentable and the λ-presentable objects of $\mathscr{V}_0$ were closed under the monoidal structure: that is, the unit I was λ-presentable and $X \otimes Y$ was λ-presentable whenever X and Y were so. The observation we wish to make here is the following:

2.4. Proposition. *If $\mathscr{V}$ is a monoidal closed category and $\mathscr{V}_0$ is locally λ-presentable, then there exists a regular cardinal μ for which $\mathscr{V}$ is locally μ-presentable as a closed category.*

Proof. Observe that the set of λ-presentable objects is (essentially) small, so the set of objects of the form $G \otimes H$ where G and H are λ-presentable is (essentially) small. Thus there exists a regular cardinal μ with the property that I is μ-presentable and $G \otimes H$ is μ-presentable whenever G and H are λ-presentable. But now if A and B are μ-presentable objects, then we may write $A = \mathrm{colim}_i G_i$ and $B = \mathrm{colim}_j H_j$ where the colimits in question are μ-small, and where each G_i and each H_j is λ-presentable. Then

$$\begin{aligned} A \otimes B &= \mathrm{colim}_i G_i \otimes \mathrm{colim}_j H_j \\ &= \mathrm{colim}_{i,j}(G_i \otimes H_j) \end{aligned}$$

and each $G_i \otimes H_j$ is μ-presentable; thus $A \otimes B$ is a μ-small colimit of μ-presentable objects, and thus is itself μ-presentable. This proves that $\mathscr{V}$ is locally μ-presentable as a closed category. ∎

3. $\mathscr{V}$-**Cat** is finitarily monadic over $\mathscr{V}$-**Gph**

For this section we suppose that $\mathscr{V}$ is a monoidal category which is cocomplete, and that the functors $A \otimes -$ and $- \otimes A$ preserve colimits for all objects A of $\mathscr{V}$, as is certainly the case if the monoidal $\mathscr{V}$ is *closed.*

As a preliminary to our investigation of $\mathscr{V}$-**Cat**, we consider the category $\mathscr{V}$-**Gph** of $\mathscr{V}$-graphs and their morphisms. Recall that a $\mathscr{V}$-graph is a pair (X, A), where X is a (small) set, and A is a family $(A(x,y))_{x,y\in X}$ of objects of $\mathscr{V}$. A $\mathscr{V}$-graph morphism from (X, A) to (Y, B) is a pair (f, φ) where $f : X \to Y$ is a function from X to Y, and φ is a family $(\varphi_{x,y} : A(x, y) \to B(fx, fy))_{x,y\in X}$ of morphisms in $\mathscr{V}$. We write $P : \mathscr{V}\text{-}\mathbf{Gph} \to \mathbf{Set}$ for the functor sending a $\mathscr{V}$-graph (X, A) to its set X of objects, and sending (f, φ) to f.

There is an evident forgetful functor $U : \mathscr{V}\text{-}\mathbf{Cat} \to \mathscr{V}\text{-}\mathbf{Gph}$ which is monadic, as was proved in [2] under the hypotheses above, and more generally when $\mathscr{V}$ is a suitable bicategory; and much earlier in [11] when $\mathscr{V}$ is symmetric monoidal closed. In this section we shall show that the monad in question is finitary — meaning that it preserves filtered colimits; in the next, we show that $\mathscr{V}$**-Gph** is locally λ-presentable if $\mathscr{V}$ is so; it will then follow that $\mathscr{V}$**-Cat** is locally λ-presentable if $\mathscr{V}$ is so, by [5, Satz 10.3]. Accordingly we begin by studying colimits in $\mathscr{V}$**-Gph**.

Following [2], we shall analyze $\mathscr{V}$-graphs in terms of the more general $\mathscr{V}$*-matrices.* If X and Y are sets, a $\mathscr{V}$-matrix S from X to Y is a family $(S(y,x))_{(x,y)\in X\times Y}$ of objects of $\mathscr{V}$; thus a $\mathscr{V}$-graph is just a set X equipped with a $\mathscr{V}$-matrix $A : X \to X$. The value of $\mathscr{V}$-matrices is that they can be composed: if $S : X \to Y$ and $T : Y \to Z$ are $\mathscr{V}$-matrices, then their composite $TS : X \to Z$ is defined by

$$(TS)(z,x) = \sum_{y\in Y} T(z,y) \otimes S(y,x).$$

There is now a bicategory $\mathscr{V}$**-Mat** in which the objects are the (small) sets, the 1-cells are the $\mathscr{V}$-matrices, and a 2-cell between $\mathscr{V}$-matrices $S, S' : X \to Y$ is a family $(\sigma_{y,x} : S(y,x) \to S'(y,x))_{(x,y)\in X\times Y}$ of morphisms of $\mathscr{V}$.

For objects X and Y of $\mathscr{V}$**-Mat**, the hom-category $\mathscr{V}$**-Mat**(X,Y) is just $\mathscr{V}^{Y\times X}$, which is cocomplete since $\mathscr{V}$ is so, with colimits formed pointwise from those in $\mathscr{V}$. Furthermore, if $S : Y \to Y'$ and $R : X' \to X$ are arbitrary $\mathscr{V}$-matrices, the functors $\mathscr{V}$**-Mat**(X,S) : $\mathscr{V}$**-Mat**$(X,Y) \to$ $\mathscr{V}$**-Mat**(X,Y') and $\mathscr{V}$**-Mat**(R,Y) : $\mathscr{V}$**-Mat**$(X,Y) \to$ $\mathscr{V}$**-Mat**(X',Y) are cocontinuous; we express this fact by saying that "composition commutes with colimits".

A function $f : X \to Y$ determines $\mathscr{V}$-matrices $f_* : X \to Y$ and $f^* : Y \to X$ with

$$f_*(y,x) = f^*(x,y) = \begin{cases} I & \text{if } fx = y \\ 0 & \text{otherwise} \end{cases}$$

where I denotes the unit object and 0 the initial object of $\mathscr{V}$. The reader will easily construct a natural bijection between 2-cells $f_*A \to B$ and 2-cells $A \to f^*B$, and so deduce that f_* is left adjoint to f^* in the bicategory $\mathscr{V}$**-Mat**. In fact it is also easy to describe explicitly the unit $1_X \to f^*f_*$ and the counit $f_*f^* \to 1_Y$.

We have already observed that a $\mathscr{V}$-graph is an object X of $\mathscr{V}$**-Mat** equipped with a 1-cell $A : X \to X$; a morphism of $\mathscr{V}$-graphs from (X,A) to (Y,B) can be seen as a function $f : X \to Y$ equipped with a 2-cell $\varphi : A \to f^*Bf_*$, as the following calculation shows:

$$\begin{aligned}(f^*Bf_*)(z,x) &= \sum_{y\in Y} f^*(z,y) \otimes (Bf_*)(y,x) \\ &= (Bf_*)(fz,x) \\ &= \sum_{y\in Y} B(fz,y) \otimes f_*(y,x) \\ &= B(fz,fx).\end{aligned}$$

In fact, because of the adjunction $f_* \dashv f^*$ in the bicategory $\mathscr{V}$-**Mat**, there is a bijection (of "mates") between 2-cells $\varphi : A \to f^*Bf_*$ and 2-cells $\widehat{\varphi} : f_*Af^* \to B$; explicitly, we find that

$$(f_*Af^*)(u,v) = \sum_{\substack{fx=u\\ fy=v}} A(x,y),$$

and now for $x \in f^{-1}(u)$ and $y \in f^{-1}(v)$ the (x,y)-component of $\widehat{\varphi}_{u,v} : (f_*Af^*)(u,v) \to B(u,v)$ is $\varphi_{x,y}$.

As shown in [2], colimits in $\mathscr{V}$-**Gph** can be described as follows. Let $\mathscr{J}$ be a small category, and $(X,A) : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Gph}$ a functor; we denote the image of an object j under (X,A) by (X_j, A_j) and the image of a morphism $\theta : j \to k$ by (X_θ, A_θ). Consider the functor $X = P(X,A) : \mathscr{J} \to \mathbf{Set}$, and form its colimit $\bar{X}$ with colimit cone $(q_j : X_j \to \bar{X})_{j\in\mathscr{J}}$. There is a functor $\widetilde{A} : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Mat}(\bar{X},\bar{X})$ sending j to $(q_j)_*A_j(q_j)^*$ and sending a morphism $\theta : j \to k$ to $(q_k)_*\widehat{A}_\theta(q_k)^* : (q_k)_*(X_\theta)_*A_j(X_\theta)^*(q_k)^* \to (q_k)_*A_k(q_k)^*$, where $\widehat{A}_\theta : (X_\theta)_*A_j(X_\theta)^* \to A_k$ is the mate, as above, of $A_\theta : A_j \to (X_\theta)^*A_k(X_\theta)_*$. As we saw above, the colimit of $\widetilde{A}$ is formed pointwise from colimits in $\mathscr{V}$: write $\bar{A} : \bar{X} \to \bar{X}$ for this colimit, with colimit cone $\alpha'_j : (q_j)_*A_j(q_j)^* \to \bar{A}$. Now we have in $\mathscr{V}$-**Mat** a cone $(q_j,\alpha_j) : (X_j,A_j) \to (\bar{X},\bar{A})$, where $\alpha_j : A_j \to (q_j)^*\bar{A}(q_j)_*$ is the 2-cell for which $\widehat{\alpha}_j : (q_j)_*A_j(q_j)^* \to \bar{A}$ is α'_j; and it is shown in [2] that this is a colimit cone for $(X,A) : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Gph}$. (Of course we henceforth drop the name α'_j in favour of $\widehat{\alpha}_j$.)

We need below to consider functors $(X,A) : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Gph}$ and $(X,B) : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Gph}$ with the same $X : \mathscr{J} \to \mathbf{Set}$; accordingly we introduce the category $\mathscr{V}\text{-}\mathbf{Gph}^{(2)}$ defined by the pullback

$$\begin{tikzcd} \mathscr{V}\text{-}\mathbf{Gph}^{(2)} \arrow[r, "Q"] \arrow[d, "R"'] & \mathscr{V}\text{-}\mathbf{Gph} \arrow[d, "P"] \\ \mathscr{V}\text{-}\mathbf{Gph} \arrow[r, "P"'] & \mathbf{Set} \end{tikzcd}$$

in **Cat**; observe that, since $\mathscr{V}$-**Gph** and **Set** are cocomplete and P is cocontinuous, $\mathscr{V}\text{-}\mathbf{Gph}^{(2)}$ is cocomplete and the functors Q and R jointly create colimits. An object of $\mathscr{V}\text{-}\mathbf{Gph}^{(2)}$ is a pair $((X,A),(X,B))$ of $\mathscr{V}$-graphs with the same underlying set X, which we henceforth write as (X,A,B); and a morphism has the form $(f,\alpha,\beta) : (X,A,B) \to (X',A',B')$ where $(f,\alpha) : (X,A) \to (X',A')$ and $(f,\beta) : (X,B) \to (X',B')$ are morphisms in $\mathscr{V}$-**Gph**. To give a pair of functors as in the first sentence of this paragraph is of course to give a single functor from $\mathscr{J}$ to $\mathscr{V}\text{-}\mathbf{Gph}^{(2)}$. In the same way we can define $\mathscr{V}\text{-}\mathbf{Gph}^{(n)}$ with objects $(X,A_1,\dots,A_n)$ by taking the fibred product in **Cat** of n copies of $P : \mathscr{V}\text{-}\mathbf{Gph} \to \mathbf{Set}$, and $\mathscr{V}\text{-}\mathbf{Gph}^{(\mathbb{N})}$ by taking the fibred product of copies indexed by the set $\mathbb{N}$ of natural numbers; and we have the corresponding results about colimits in $\mathscr{V}\text{-}\mathbf{Gph}^{(n)}$ and $\mathscr{V}\text{-}\mathbf{Gph}^{(\mathbb{N})}$.

Consider the functor $S : \mathscr{V}\text{-}\mathbf{Gph}^{(2)} \to \mathscr{V}\text{-}\mathbf{Gph}$ sending (X,A,B) to $(X,A+B)$, where the sum $A+B$ of matrices is of course the coproduct in $\mathscr{V}^{X\times X}$; the value of S on morphisms is given by the evident sum of 2-cells using the distributive law for matrices.

This functor S preserves colimits, for if $(X, A, B) : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Gph}^{(2)}$, it is clear from the description above of colimits in $\mathscr{V}\text{-}\mathbf{Gph}$ that the colimit of $(X, A + B)$ is $(\bar{X}, \bar{A} + \bar{B})$, where $(\bar{X}, \bar{A})$ and $(\bar{X}, \bar{B})$ are the colimits of (X, A) and (X, B). Similarly of course for sums of any size: the form we need below is:

3.1. Lemma. *The functor $S : \mathscr{V}\text{-}\mathbf{Gph}^{(\mathbb{N})} \to \mathscr{V}\text{-}\mathbf{Gph}$ sending $(X, (A_n)_{n\in\mathbb{N}})$ to $(X, \sum_{n\in\mathbb{N}} A_n)$ preserves colimits.* ∎

We also need to consider the functor $M : \mathscr{V}\text{-}\mathbf{Gph}^{(2)} \to \mathscr{V}\text{-}\mathbf{Gph}$ which sends (X, A, B) to (X, AB), where AB denotes as before the matrix product. We must of course define M on morphisms too. Recall that the α of a morphism $(f, \alpha) : (X, A) \to (X', A')$ can be seen as a matrix $\alpha : A \to f^* A' f_*$, but can equally be described by its mate $\widehat{\alpha} : f_* A f^* \to A'$ under the adjunction $f_* \dashv f^*$. But there is of course yet another equivalent form, namely $\bar{\alpha} : f_* A \to A' f_*$. In fact we find that $(f_*A)(x', x) = \sum_{fy=x'} A(y, x)$, that $(A'f_*)(x', x) = A'(x', fx)$, and that $\bar{\alpha}_{x',x}$ has $\alpha_{y,x}$ as its y-component. Now the value of M on $(f, \alpha, \beta) : (X, A, B) \to (X', A', B')$ is $(f, \gamma) : (X, AB) \to (X', A'B')$ where γ is determined in terms of its mate $\bar{\gamma}$ by the pasting composite

$$
\begin{tikzcd}
X \arrow[r, "f_*"] \arrow[d, "AB"'] \arrow[dr, phantom, "\overset{\bar{\gamma}}{\Rightarrow}"] & X' \arrow[d, "A'B'"] \\
X \arrow[r, "f_*"'] & X'
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X \arrow[r, "f_*"] \arrow[d, "B"'] \arrow[dr, phantom, "\overset{\bar{\beta}}{\Rightarrow}"] & X' \arrow[d, "B'"] \\
X \arrow[r, "f_*"] \arrow[d, "A"'] \arrow[dr, phantom, "\overset{\bar{\alpha}}{\Rightarrow}"] & X' \arrow[d, "A'"] \\
X \arrow[r, "f_*"'] & X'.
\end{tikzcd}
$$

This comes, as the reader will easily see, to taking for $\gamma_{z,x} : (AB)(z, x) \to (A'B')(fz, fx)$ the composite

$$\sum_{y\in X} A(z, y)B(y, x) \xrightarrow{\sum \alpha_{z,y}\beta_{y,x}} \sum_{y\in X} A'(fz, fy)B'(fy, fx) \xrightarrow{\kappa} \sum_{y'\in X'} A'(fz, y')B'(y', fx) \ ,$$

where the y-component of κ is the fy-injection into the final sum; we included the less elementary description of γ given above since it makes clearer the functoriality of M. The result we need is:

3.2. Lemma. *The functor $M : \mathscr{V}\text{-}\mathbf{Gph}^{(2)} \to \mathscr{V}\text{-}\mathbf{Gph}$ preserves filtered colimits.*

Proof. Consider a functor $(X, A, B) : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Gph}^{(2)}$ with $\mathscr{J}$ filtered. Using the notation above, we recall that the colimit of $(X, A) : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Gph}$ is $(\bar{X}, \bar{A})$ with colimit cone $(q_j, \alpha_j) : (X_j, A_j) \to (\bar{X}, \bar{A})$, where $q_j : X_j \to \bar{X}$ is the colimit cone for $X : \mathscr{J} \to \mathbf{Set}$ and $\widehat{\alpha}_j : (q_j)_* A_j (q_j)^* \to \bar{A}$ is the colimit cone for the functor $\widetilde{A} : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Mat}(\bar{X}, \bar{X})$ sending j to $\widetilde{A}j = (q_j)_* A_j (q_j)^*$ and sending $\theta : j \to k$ to $\widetilde{A}\theta = (q_k)_* \widehat{A}_\theta (q_k)^*$. Similarly the colimit of (X, B) is $(\bar{X}, \bar{B})$ with colimit cone (q_j, β_j), where $\widehat{\beta}_j : (q_j)_* B_j (q_j)^* \to \widehat{B}$ is the colimit cone for $\widetilde{B} : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Mat}(\bar{X}, \bar{X})$.

The composite of M with the functor (X, A, B) is a functor $(X, C) : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Gph}$ where $C_j = A_jB_j$ and where C_θ for $\theta : j \to k$ is such that $\bar{C}_\theta$ is a pasting composite of $\bar{A}_\theta$ and $\bar{B}_\theta$: see the definition of M on morphisms above. This functor, of course, has the colimit cone $(q_j, \gamma_j) : (X_j, C_j) \to (\bar{X}, \bar{C})$ where $\widehat{\gamma}_j : (q_j)_*A_jB_j(q_j)^* = (q_j)_*C_j(q_j)^* \to \bar{C}$ is the colimit cone for the functor $\widetilde{C} : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Mat}(\bar{X}, \bar{X})$ sending j to $\widetilde{C}_j = (q_j)_*A_jB_j(q_j)^*$.

The functor M, however, sends the colimit $(\bar{X}, \bar{A}, \bar{B})$ of (X, A, B) to $(\bar{X}, \bar{A}\bar{B})$, and sends the colimit cone (q_j, α_j, β_j) of (X, A, B) to the cone $(q_j, \delta_j) : (X_j, A_j, B_j) \to (\bar{X}, \bar{A}\bar{B})$ where δ_j is determined through the pasting equation

$$
\begin{tikzcd}
X_j \arrow[r, "(q_j)_*"] \arrow[dd, "A_jB_j"'] & \bar{X} \arrow[dd, "\bar{A}\bar{B}"] \\
\arrow[r, phantom, "\overset{\bar{\delta}_j}{\Rightarrow}"] & \; \\
X_j \arrow[r, "(q_j)_*"'] & \bar{X}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
X_j \arrow[r, "(q_j)_*"] \arrow[d, "B_j"'] \arrow[dr, phantom, "\overset{\bar{\beta}_j}{\Rightarrow}"] & \bar{X} \arrow[d, "\bar{B}"] \\
X_j \arrow[r, "(q_j)_*"] \arrow[d, "A_j"'] \arrow[dr, phantom, "\overset{\bar{\alpha}_j}{\Rightarrow}"] & \bar{X} \arrow[d, "\bar{A}"] \\
X_j \arrow[r, "(q_j)_*"'] & \bar{X}
\end{tikzcd} .
$$

To say that M preserves the colimit of (X, A, B) is to say that the cone (q_j, δ_j) is a colimit cone, and hence, by the above, to say that the cone

$$\widehat{\delta}_j : (q_j)_*A_jB_j(q_j)^* \longrightarrow \bar{A}\bar{B}$$

is a colimit cone in $\mathscr{V}\text{-}\mathbf{Mat}(\bar{X}, \bar{X})$ over the functor $\widetilde{C}$.

On the other hand, since composition of matrices commutes with colimits, the colimit cones $\widehat{\alpha}_j : \widetilde{A}_j \to \bar{A}$ and $\widehat{\beta}_j : \widetilde{B}_j \to \bar{B}$ give by composition a colimit cone $\widehat{\alpha}_j\widehat{\beta}_k : \widetilde{A}_j\widetilde{B}_k \to \bar{A}\bar{B}$ over the functor $\mathscr{J} \times \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Mat}(\bar{X}, \bar{X})$ sending (j, k) to $\widetilde{A}_j\widetilde{B}_k$ and similarly defined on morphisms. Because $\mathscr{J}$ is filtered, however, the diagonal $\mathscr{J} \to \mathscr{J} \times \mathscr{J}$ is final; so that $\widehat{\alpha}_j\widehat{\beta}_j : \widetilde{A}_j\widetilde{B}_j \to \bar{A}\bar{B}$ is a colimit cone for the functor $\widetilde{A}\widetilde{B} : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Mat}(\bar{X}, \bar{X})$ sending j to $\widetilde{A}_j\widetilde{B}_j = (q_j)_*A_j(q_j)^*(q_j)_*B_j(q_j)_*$.

We have the unit $\eta_j : 1_{X_j} \to (q_j)^*(q_j)_*$ of the adjunction $(q_j)_* \dashv (q_j)^*$, and thus for each j a 2-cell

$$(q_j)_*A_j\eta_jB_j(q_j)^* : (q_j)_*A_jB_j(q_j)^* \to (q_j)_*A_j(q_j)^*(q_j)_*B_j(q_j)^* ,$$

which we may write as $\zeta_j : \widetilde{C}_j \to \widetilde{A}_j\widetilde{B}_j$; a straightforward calculation verifies that these are the components of a natural transformation $\zeta : \widetilde{C} \to \widetilde{A}\widetilde{B} : \mathscr{J} \to \mathscr{V}\text{-}\mathbf{Mat}(\bar{X}, \bar{X})$. Using the adjunction $(q_j)_* \dashv (q_j)^*$ to express the $\widehat{\delta}_j$ in terms of their mates $\bar{\delta}_j$ and hence in terms of α and β, we find that the cone $\widehat{\delta}_j : \widetilde{C}_j \to \bar{A}\bar{B}$ is just the composite of ζ_j with the colimit cone $\widehat{\alpha}_j\widehat{\beta}_j : \widetilde{A}_j\widetilde{B}_j \to \bar{A}\bar{B}$. So the $\widehat{\delta}_j$ constitute a colimit cone if and only if the $\bar{\zeta} : \bar{C} \to \bar{A}\bar{B}$ induced by $\zeta : \widetilde{C} \to \widetilde{A}\widetilde{B}$ is invertible.

Recall our earlier calculation of a matrix composite f_*Af^*. This gives us, for $x, y \in \bar{X}$,

$$\begin{aligned}\widetilde{C}_j(x,y) &= \big((q_j)_*A_jB_j(q_j)^*\big)(x,y)\\ &= \sum_{\substack{\rho,\sigma,\tau\in X_j\\ q_j\rho=x\\ q_j\sigma=y}} A_j(\rho,\tau)B_j(\tau,\sigma)\end{aligned}$$

and

$$\begin{aligned}(\widetilde{A}_j\widetilde{B}_j)(x,y) &= \big((q_j)_*A_j(q_j)^*(q_j)_*B_j(q_j)^*\big)(x,y)\\ &= \sum_{z\in\bar{X}}\sum_{\substack{r,t\in X_j\\ q_jr=x\\ q_jt=z}}\sum_{\substack{p,s\in X_j\\ q_jp=z\\ q_js=y}} A_j(r,t)B_j(p,s)\ ;\end{aligned}$$

and it follows easily from the explicit description of the unit $1_{X_j} \to (q_j)^*(q_j)_*$ that $(\zeta_j)_{x,y} : \widetilde{C}_j(x,y) \to (\widetilde{A}_j\widetilde{B}_j)(x,y)$ is the map whose (ρ,σ,τ)-component is the (z,r,t,p,s)-coprojection where $r = \rho$, $s = \sigma$, $t = p = \tau$, and $z = q_j\tau$.

We complete the proof by constructing an inverse $\bar{\xi} : \bar{A}\bar{B} \to \bar{C}$ of $\bar{\zeta}$, or equally inverses $\bar{\xi}_{x,y} : (\bar{A}\bar{B})(x,y) \to \bar{C}(x,y)$ of $\bar{\zeta}_{x,y}$; here $\bar{\xi}_{x,y}$ is to be the map induced on the colimit by a cone $(\xi_j)_{x,y} : (\widetilde{A}_j\widetilde{B}_j)(x,y) \to \bar{C}(x,y)$. By the formula above for $(\widetilde{A}_j\widetilde{B}_j)(x,y)$, it suffices to give for each (z,r,t,p,s) the appropriate component $(\xi_j)_{x,y,z;r,t,p,s} : A_j(r,t)B_j(p,s) \to \bar{C}(x,y)$. Now since $q_jt = q_jp$, there is by the filteredness of $\mathscr{J}$ some $\theta : j \to k$ with $X_\theta t = X_\theta p = t' \in X_k$, say. Write r' for $X_\theta r$ and s' for $X_\theta s$. We take for $(\xi_j)_{x,y,z;r,t,p,s}$ the composite

$$A_j(r,t)B_j(p,s) \xrightarrow{(A_\theta)_{r,t}(B_\theta)_{p,s}} A_k(r',t')B_k(t',s') \xrightarrow{\ \lambda\ } \widetilde{C}_k(x,y) \xrightarrow{(\widehat{\gamma}_k)_{x,y}} \bar{C}(x,y)\ ,$$

where λ is the appropriate coprojection in the expression above for $\widetilde{C}_j(x,y)$, but now with k in place of j. It is easy to verify, first, that $(\xi_j)_{x,y,z;r,t,p,s}$ is independent of our choice of a $\theta : j \to k$ with $X_\theta t = X_\theta p$, so that $(\xi_j)_{x,y}$ is well-defined; and second that the $(\xi_j)_{x,y} : (\widetilde{A}_j\widetilde{B}_j)(x,y) \to \bar{C}(x,y)$ constitute a cone, thus inducing a map $\bar{\xi}_{x,y} : \bar{A}\bar{B}(x,y) \to \bar{C}(x,y)$ determined by $\bar{\xi}_{x,y}(\widehat{\alpha}_j\widehat{\beta}_j)_{x,y} = (\xi_j)_{x,y}$.

That $\bar{\xi}_{x,y}\bar{\zeta}_{x,y} = 1$ follows easily because, in applying $\bar{\xi}_{x,y}$ on the image of $\bar{\zeta}_{x,y}$ we may, since here $t = p = \tau$, take $\theta : j \to k$ to be 1_j. To say that $\bar{\zeta}_{x,y}\bar{\xi}_{x,y} = 1$ is to say that $\bar{\zeta}_{x,y}\bar{\xi}_{x,y}(\widehat{\alpha}_j\widehat{\beta}_j)_{x,y} = (\widehat{\alpha}_j\widehat{\beta}_j)_{x,y}$ for each j. However $\bar{\zeta}_{x,y}\bar{\xi}_{x,y}(\widehat{\alpha}_j\widehat{\beta}_j)_{x,y} = \bar{\zeta}_{x,y}(\xi_j)_{x,y}$, whose $(z;r,t,p,s)$-component by the above is

$$\bar{\zeta}_{x,y}(\widehat{\gamma}_k)_{x,y}\lambda\left((A_\theta)_{r,t}(B_\theta)_{p,s}\right) = (\widehat{\alpha}_k\widehat{\beta}_k)_{x,y}(\zeta_k)_{x,y}\lambda\left((A_\theta)_{r,t}(B_\theta)_{p,s}\right),$$

and it follows from the explicit description above of $(\zeta_k)_{x,y}$ that $(\zeta_k)_{x,y}\lambda$ is just the coprojection $A_k(r',t')B_k(t',s') \to (\widetilde{A}_k\widetilde{B}_k)(x,y)$, which we shall write as κ_k. If we similarly write κ_j for the coprojection $A_j(r,t)B_j(p,s) \to (\widetilde{A}_j\widetilde{B}_j)(x,y)$, we have $\kappa_k((A_\theta)_{r,t}(B_\theta)_{p,s}) = (\widetilde{A}_\theta\widetilde{B}_\theta)_{x,y}\kappa_j$, so that $(\widehat{\alpha}_k\widehat{\beta}_k)_{x,y}\kappa_k((A_\theta)_{r,t}(B_\theta)_{p,s}) = (\widehat{\alpha}_k\widehat{\beta}_k)_{x,y}(\widetilde{A}_\theta\widetilde{B}_\theta)_{x,y}\kappa_j = (\widehat{\alpha}_j\widehat{\beta}_j)_{x,y}\kappa_j$, which is the $(z;r,t,p,s)$-component of $(\widehat{\alpha}_j\widehat{\beta}_j)_{x,y}$, as desired. So the $\bar{\zeta}_{x,y}$ are indeed invertible, which completes the proof. ∎

We shall now describe the endofunctor T of $\mathscr{V}$**-Gph** underlying the "free $\mathscr{V}$-category" monad. Recall from [2] that T sends a $\mathscr{V}$-graph (X,A) to (X,A') where $A' = \sum_{n\in\mathbb{N}} A^n$ is the free monoid on A in the monoidal category given by $\mathscr{V}$**-Mat**(X,X) with matrix multiplication as its tensor product; and that the unit $(X,A) \to (X,A')$ of the adjunction is $(1,\rho_A)$ where $\rho_A : A \to A'$ is the injection of the summand $A = A^1$ into $\sum A^n$. From this we can calculate the value of T on morphisms, which leads to the following description of T. For each $n \in \mathbb{N}$ there is an endofunctor T_n of $\mathscr{V}$**-Gph** sending (X,A) to (X,A^n); and because $PT_n = P$, these T_n are the components of a functor $T_\mathbb{N} : \mathscr{V}\textbf{-Gph} \to \mathscr{V}\textbf{-Gph}^{(\mathbb{N})}$; whereupon T is the composite $ST_\mathbb{N}$, where $S : \mathscr{V}\textbf{-Gph}^{(\mathbb{N})} \to \mathscr{V}\textbf{-Gph}$ is the functor so denoted in Lemma 3.1. Since S preserves all colimits by Lemma 3.1, T will be finitary (that is, will preserve filtered colimits) if $T_\mathbb{N}$ is so. Since the projections $\mathscr{V}\textbf{-Gph}^{(\mathbb{N})} \to \mathscr{V}\textbf{-Gph}$ jointly create colimits, $T_\mathbb{N}$ will be finitary if each T_n is so. However T_1 is the identity endofunctor 1 of $\mathscr{V}$**-Gph**, while T_2 is the composite $M(1,1)$, where $(1,1) : \mathscr{V}\textbf{-Gph} \to \mathscr{V}\textbf{-Gph}^{(2)}$ is the functor each of whose components is 1; and T_{n+1} for $n \geq 1$ is (isomorphic to) the composite $M(T_n,1)$. Since the projections $\mathscr{V}\textbf{-Gph}^{(2)} \to \mathscr{V}\textbf{-Gph}$ jointly create colimits, it follows inductively from Lemma 3.2 that T_n is finitary for $n \geq 1$.

It remains to consider the endofunctor T_0 of $\mathscr{V}$**-Gph** sending (X,A) to $(X,1_X)$, where 1_X is the identity matrix with $(1_X)_{x,y}$ being I for $x = y$ and 0 otherwise. This is the composite of the forgetful functor $P : \mathscr{V}\textbf{-Gph} \to \textbf{Set}$ and the evident functor $H : \textbf{Set} \to \mathscr{V}\textbf{-Gph}$ sending X to $(X,1_X)$. Since P preserves all colimits, it will suffice to show that H preserves filtered colimits. Suppose then that $X : \mathscr{J} \to \textbf{Set}$ with $\mathscr{J}$ filtered has as before the colimit cone $(q_j : X_j \to \bar{X})$, and consider the colimit of HX; our claim is that the colimit of the $(X_j, 1_{X_j})$ is $(\bar{X}, 1_{\bar{X}})$. By our description of colimits in $\mathscr{V}$**-Gph**, we have to show that $1_{\bar{X}}$ is the colimit in $\mathscr{V}\textbf{-Mat}(\bar{X},\bar{X})$ of the $(q_j)_* 1_{X_j}(q_j)^*$. Since

$$\left((q_j)_* 1_{X_j}(q_j)^*\right)(x,y) = \sum_{\substack{q_j r = x\\ q_j s = y}} (1_{X_j})(r,s) \ ,$$

there is nothing to prove for $x \neq y$, the cone being constant at 0. For $x = y$ the above gives

$$\left((q_j)_* 1_{X_j}(q_j)^*\right)(x,x) = q_j^{-1}(x) \bullet I \ ,$$

the coproduct of $q_j^{-1}(x)$ copies of I; and we are claiming that the colimit in $\mathscr{V}$ of the $q_j^{-1}(x) \bullet I$ is I. However $(\ ) \bullet I : \textbf{Set} \to \mathscr{V}$ preserves colimits, so that it suffices to observe that in **Set** we have $\operatorname{colim}(q_j^{-1}(x)) = 1$. But filtered colimits in **Set** commute with finite limits; and the above is precisely what we get on pulling back the colimit $q_j : X_j \to \bar{X}$ along $x : 1 \to \bar{X}$. This completes the proof of:

3.3. Theorem. *The monad on $\mathscr{V}$-**Gph** whose algebras are $\mathscr{V}$-categories is finitary.*

An equivalent formulation is:

3.4. Corollary. *The forgetful functor $U : \mathscr{V}\text{-}\mathbf{Cat} \to \mathscr{V}\text{-}\mathbf{Gph}$ is finitary.*

4. $\mathscr{V}$-Cat is locally presentable if $\mathscr{V}$ is so

As in Section 3, we continue to suppose that the monoidal category $\mathscr{V}$ is cocomplete and that the functors $A \otimes -$ and $- \otimes A$ preserve colimits, as they surely do when $\mathscr{V}$ is closed. To avoid pathologies in our use of the "strong generator" notion, we further suppose that $\mathscr{V}_0$ admits arbitrary cointersections of strong epimorphisms, which ensures that (strong epimorphisms, monomorphisms) is a factorization system on $\mathscr{V}_0$. This presents no problem, since our main goal is the study of the case where $\mathscr{V}_0$ is locally presentable.

It is convenient to introduce, for each object G of $\mathscr{V}$, the $\mathscr{V}$-graph $(2, \bar{G})$ having $2 = \{0, 1\}$ for its set of objects and having

$$\bar{G}(0,1) = G, \quad \bar{G}(0,0) = \bar{G}(1,1) = \bar{G}(1,0) = 0,$$

where this last 0 is the initial object of $\mathscr{V}$; that is to say, $\bar{G}$ is the 2-by-2 matrix $\begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix}$. To give a morphism $(2, \bar{G}) \to (X, A)$ of $\mathscr{V}$-graphs is just to give a pair $x, y \in X$ and a morphism $u : G \to A(x, y)$ in $\mathscr{V}$.

The forgetful functor $P : \mathscr{V}\text{-}\mathbf{Gph} \to \mathbf{Set}$ sending the $\mathscr{V}$-graph (X, A) to X clearly has a left adjoint D sending the set X to the $\mathscr{V}$-graph $(X, 0_X)$, where 0_X is the initial object of $\mathscr{V}\text{-}\mathbf{Mat}(X, X)$ given by $0_X(x, x') = 0$.

4.1. Lemma. *A morphism $(f, \alpha) : (X, A) \to (Y, B)$ in $\mathscr{V}$-**Gph** is monomorphic if and only if $f : X \to Y$ is an injective function and each $\alpha_{x,x'} : A(x, x') \to B(fx, fx')$ is a monomorphism in $\mathscr{V}$ (that is, in $\mathscr{V}_0$).*

Proof. The "if" part being clear from the definition of composition in $\mathscr{V}$-**Gph**, it suffices to prove the "only if" part; so suppose that (f, α) is monomorphic in $\mathscr{V}$-**Gph**. Then f is injective because $P : \mathscr{V}\text{-}\mathbf{Gph} \to \mathbf{Set}$, having a left adjoint, preserves monomorphisms. Suppose that, for some $x, x' \in X$, maps $\beta, \gamma : G \to A(x, x')$ in $\mathscr{V}$ satisfy $\alpha_{x,x'}\beta = \alpha_{x,x'}\gamma$, and define $g : 2 \to X$ by setting $g0 = x$ and $g1 = x'$; now the morphisms $(g, \beta), (g, \gamma) : (2, \bar{G}) \to (X, A)$ have the same composite with $(f, \alpha) : (X, A) \to (Y, B)$, whence $\beta = \gamma$. Thus $\alpha_{x,x'}$ is indeed monomorphic. ∎

4.2. Lemma. *If a set $\mathscr{G}$ of objects constitutes a strong generator of $\mathscr{V}_0$, then the set $\{(2, \bar{G}) \mid G \in \mathscr{G} \text{ or } G = 0\}$ constitutes a strong generator of $\mathscr{V}$-**Gph**.*

Proof. We prove the assertion in the equivalent form — see Section 2 above — that the totality of maps in $\mathscr{V}$-**Gph** into the object (Y, B) having domain one of the $(2, \bar{G})$ with $G \in \mathscr{G} \cup \{0\}$ factorizes through no proper subobject of (Y, B) and is therefore jointly a strong epimorphism. Suppose then that $(f, \alpha) : (X, A) \to (Y, B)$ is a monomorphism in

$\mathscr{V}$-**Gph** through which every $(g,\beta):(2,\bar{G})\to(Y,B)$ with $G\in\mathscr{G}\cup\{0\}$ factorizes. To give a map from $(2,\bar{0})=(2,0_2)$ into (Y,B) is just to give two elements of Y; and since every such map factorizes through (f,α), the injection f is in fact a bijection. Since every $(g,\beta):(2,\bar{G})\to(Y,B)$ with $G\in\mathscr{G}$ factorizes through (f,α), every map $G\to B(fx,fx')$ in $\mathscr{V}$ factorizes through the monomorphism $\alpha_{x,x'}:A(x,x')\to B(fx,fx')$, which is therefore invertible, because $\mathscr{G}$ is a strong generator for $\mathscr{V}_0$. Thus the monomorphism (f,α) is indeed invertible. ∎

We now examine the "presentability" of such a strong generator for $\mathscr{V}$-**Gph**. Note that $\mathscr{V}_0(0,-):\mathscr{V}_0\to\mathbf{Set}$ is the functor constant at 1, which preserves all connected colimits; so that the object 0 of $\mathscr{V}_0$ is λ-presentable for any regular cardinal λ.

4.3. Lemma. *If, for some regular cardinal λ, the object G is λ-presentable in $\mathscr{V}_0$, then $(2,\bar{G})$ is λ-presentable in $\mathscr{V}$-**Gph**.*

Proof. Consider as in Section 3 above the colimit cone $(q_j,\alpha_j):(X_j,A_j)\to(\bar{X},\bar{A})$ of a functor $(X,A):\mathscr{J}\to\mathscr{V}$-**Gph**, where the category $\mathscr{J}$ is λ-filtered; we are to show that the functor $\mathscr{V}$-**Gph**$((2,\bar{G}),-):\mathscr{V}$-**Gph** $\to\mathbf{Set}$ preserves every such colimit; equivalently, we are to prove bijective the canonical comparison

$$\kappa:\mathrm{colim}_{k\in\mathscr{J}}\mathscr{V}\text{-}\mathbf{Gph}((2,\bar{G}),(X_k,A_k))\to\mathscr{V}\text{-}\mathbf{Gph}((2,\bar{G}),(\bar{X},\bar{A})$$

of sets. We begin by proving κ surjective; that is to say, that every map $(g,\tau):(2,\bar{G})\to(\bar{X},\bar{A})$ factorizes through some $(q_k,\alpha_k):(X_k,A_k)\to(\bar{X},\bar{A})$. To give (g,τ) is to give a function $g:2\to\bar{X}$ picking out elements $x,y\in\bar{X}$ and to give a map $\tau:G\to\bar{A}(x,y)$ in $\mathscr{V}$. We recall from Section 3, however, that the $(\widehat{\alpha}_j)_{x,y}:\widetilde{A}_j(x,y)\to\bar{A}(x,y)$ constitute a colimit cone for the functor $\widetilde{A}(x,y):\mathscr{J}\to\mathscr{V}$; so, G being λ-presentable in $\mathscr{V}_0$, the map $\tau:G\to\bar{A}(x,y)$ factorizes as

$$G\xrightarrow{\sigma}\widetilde{A}_j(x,y)\xrightarrow{(\widehat{\alpha}_j)_{x,y}}\bar{A}(x,y)$$

for some $j\in\mathscr{J}$. Here $\widetilde{A}_j$ is the object $(q_j)_*A_j(q_j)^*$ of $\mathscr{V}$-$\mathbf{Mat}(\bar{X},\bar{X})$, so that

$$\widetilde{A}_j(x,y)=\sum_{\substack{q_jt=x\\ q_js=y}}A_j(t,s).$$

This coproduct, however, is the λ-filtered colimit of its sub-coproducts indexed by subsets of $q_j^{-1}(x)\times q_j^{-1}(y)$ of cardinality less than λ; so, G being λ-presentable, σ factorizes through such a sub-coproduct, say $\sum_{\nu\in N}A_j(t_\nu,s_\nu)$ where $q_jt_\nu=x$ and $q_js_\nu=y$ for all $\nu\in N$ and where $\mathrm{card}N<\lambda$. Using yet again the λ-filteredness of $\mathscr{J}$, there is some arrow $\theta:j\to k$ in $\mathscr{J}$ for which all the $X_\theta t_\nu$ are equal and all the $X_\theta s_\nu$ are equal: say

$$X_\theta t_\nu=\bar{t}\in X_k \text{ and } X_\theta s_\nu=\bar{s}\in X_k \text{ for all } \nu\in N.$$

Using the factorization above of τ, we have

$$\tau=(\widehat{\alpha}_j)_{x,y}\sigma=(\widehat{\alpha}_k)_{x,y}(\widetilde{A}_\theta)_{x,y}\sigma=(\widehat{\alpha}_k)_{x,y}\rho,$$

where $\rho = (\widetilde{A}_\theta)_{x,y}\sigma$, which by Section 3 is in fact $\Big((q_k)_*\widehat{A}_\theta(q_k)^*\Big)_{x,y}\sigma$. The point is that this map

$$\rho : G \to \widetilde{A}_k(x,y) = \sum_{\substack{q_k t'=x \\ q_k s'=y}} A_k(t',s')$$

factorizes through the coprojection of a single summand $A_k(\bar{t},\bar{s})$, say as the composite of this coprojection with the map $\varphi : G \to A_k(\bar{t},\bar{s})$ of $\mathscr{V}$. Now the pair $(\bar{t},\bar{s})$ determines a function $h : 2 \to X_k$, so that we have in $\mathscr{V}$**-Gph** the map $(h,\varphi) : (2,\bar{G}) \to (X_k, A_k)$; but the composite of this with (q_k,α_k) is (g,τ). So (g,τ) does indeed factorize through some (q_k,α_k), which completes the proof that the canonical comparison κ is surjective.

It remains to show that κ is injective. Suppose then that $(h,\varphi) : (2,\bar{G}) \to (X_k, A_k)$ and $(h',\varphi') : (2,\bar{G}) \to (X_{k'}, A_{k'})$ are maps in $\mathscr{V}$**-Gph** with $(q_k,\alpha_k)(h,\varphi) = (q_{k'},\alpha_{k'})(h',\varphi')$. We are to show that there exist $\theta : k \to j$ and $\theta' : k' \to j$ in $\mathscr{J}$, with $(X_\theta, \widetilde{A}_\theta)(h,\varphi) = (X_{\theta'}, \widetilde{A}_{\theta'})(h',\varphi')$. Since $\mathscr{J}$ is λ-filtered, there certainly do exist maps $\theta : k \to j$ and $\theta'_0 : k' \to j$, and so without loss of generality we may suppose that $k = k'$.

Write (t,s) for $(h0,h1)$ and (t',s') for $(h'0,h'1)$, so that $\varphi : G \to A_k(t,s)$ and $\varphi' : G \to A_k(t',s')$. Since $q_k h = q_k h'$, there is some $\theta : k \to j$ with $X_\theta t = X_\theta t'$ and $X_\theta s = X_\theta s'$; in other words, we may suppose without loss of generality that $t = t'$ and $s = s'$. Now, therefore, $h = h' : 2 \to X_k$ corresponds to $(t,s) \in X_k$, and $\varphi,\varphi' : G \to A_k(t,s)$ have $\alpha_k\varphi = \alpha_k\varphi'$. Write $\bar{t}$ for $q_k t \in \bar{X}$, and $\bar{s}$ for $q_k s$, and recall from Section 3 that we have in $\mathscr{V}$ the colimit cone $((\widehat{\alpha}_j)_{\bar{t},\bar{s}} : \widetilde{A}_j(\bar{t},\bar{s}) \to \bar{A}(\bar{t},\bar{s}))_{j\in\mathscr{J}}$, where

$$\widetilde{A}_j(\bar{t},\bar{s}) = \sum_{\substack{q_j\widetilde{t}=\bar{t} \\ q_j\widetilde{s}=\bar{s}}} A_j(\widetilde{t},\widetilde{s}).$$

Composing φ and φ' with the (t,s)-coprojection for $\widetilde{A}_k$ gives us two maps $\psi,\psi' : G \to \widetilde{A}_k(\bar{t},\bar{s})$ in $\mathscr{V}$ with $(\widehat{\alpha}_k)_{\bar{t},\bar{s}}\psi = (\widehat{\alpha}_k)_{\bar{t},\bar{s}}\psi'$. Because $\mathscr{J}$ is λ-filtered and G is λ-presentable in $\mathscr{V}_0$, there is some $\theta : k \to j$ in $\mathscr{J}$ for which $\widetilde{A}_\theta(\bar{t},\bar{s})\psi = \widetilde{A}_\theta(\bar{t},\bar{s})\psi'$. When we recall the definition of the functor $\widetilde{A}$, we see that this gives exactly the equality $(X_\theta, A_\theta)(h,\varphi) = (X_\theta, A_\theta)(h,\varphi')$ that we need for the injectivity of κ. ■

Since, as we have remarked, the object 0 of $\mathscr{V}_0$ is λ-presentable for any regular cardinal λ, it follows from Lemmas 4.2 and 4.3 that, under the standing hypotheses of this section:

4.4. Proposition. $\mathscr{V}$**-Gph** *is locally λ-presentable when $\mathscr{V}_0$ is so.* ■

Combining this with Theorem 3.3 and using [5, Satz 10.3], we conclude that :

4.5. Theorem. *If $\mathscr{V}$ is a monoidal closed category whose underlying ordinary category $\mathscr{V}_0$ is locally λ-presentable, then* $\mathscr{V}$**-Cat** *is also locally λ-presentable.*

We can be more specific, in the sense of actually exhibiting a strong generator for $\mathscr{V}$**-Cat** consisting of λ-presentable objects. We make use of the following simple and well-known general observations:

4.6. Lemma. *Let $F \dashv U : \mathscr{A} \to \mathscr{B}$ where $\mathscr{A}$ and $\mathscr{B}$ are cocomplete categories. Then (i) for an object G of $\mathscr{B}$, the object FG of $\mathscr{A}$ is λ-presentable if G is λ-presentable and U preserves λ-filtered colimits; and (ii) if a small set $\mathscr{G}$ of objects of $\mathscr{B}$ constitutes a strong generator of $\mathscr{B}$, and if U reflects isomorphisms (as it surely does whenever it is monadic), then the set $\{FG \mid G \in \mathscr{G}\}$ constitutes a strong generator of $\mathscr{A}$.*

Let us use F now for the left adjoint of the forgetful $U : \mathscr{V}\textbf{-Cat} \to \mathscr{V}\textbf{-Gph}$. When $\mathscr{V}_0$ is locally λ-presentable, we can take for $\mathscr{G}$ the full subcategory $\mathscr{V}_\lambda$ of $\mathscr{V}_0$ given by the λ-presentable objects, noting that it contains the initial object 0. By Lemmas 4.2 and 4.3, the $(2, \bar{G})$ for $G \in \mathscr{V}_\lambda$ constitute a strong generator of $\mathscr{V}\textbf{-Gph}$ consisting of λ-presentable objects. By Lemma 4.6 and Corollary 3.4, therefore, the $F(2, \bar{G})$ for $G \in \mathscr{V}_\lambda$ constitute a strong generator of $\mathscr{V}\textbf{-Cat}$ consisting of λ-presentable objects. In future we shall write $\mathfrak{2}_G$ for the $\mathscr{V}$-category $F(2, \bar{G})$; it is characterized by the observation that to give a $\mathscr{V}$-functor from $\mathfrak{2}_G$ to a $\mathscr{V}$-category B is to give objects x and y of B along with a map $G \to B(x, y)$ in $\mathscr{V}$. The reader will easy verify that $\mathfrak{2}_G$ has two objects 0 and 1, with $\mathfrak{2}_G(0,0) = \mathfrak{2}_G(1,1) = I$, $\mathfrak{2}_G(0,1) = G$, $\mathfrak{2}_G(1,0) = 0$, and with the evident composition.

A standard result from the theory of locally presentable categories now gives:

4.7. Proposition. *When $\mathscr{V}_0$ is locally λ-presentable, the class of λ-presentable objects in $\mathscr{V}\textbf{-Cat}$ is the closure in $\mathscr{V}\textbf{-Cat}$ under λ-small colimits of the $\mathscr{V}$-categories $\mathfrak{2}_G$, where G is a λ-presentable object of $\mathscr{V}$.* ∎

Recall from Section 2 above that a monoidal closed category $\mathscr{V}$ is locally λ-presentable as a closed category when its underlying ordinary category $\mathscr{V}_0$ is locally λ-presentable, and the λ-presentable objects of $\mathscr{V}_0$ are closed under the monoidal structure. Although our interest in local presentability for closed categories is rather secondary, we nonetheless record:

4.8. Proposition. *If the symmetric monoidal closed category $\mathscr{V}$ is locally λ-presentable as a closed category, then so is $\mathscr{V}\textbf{-Cat}$.*

Proof. We must show that the λ-presentable $\mathscr{V}$-categories are closed under tensor product. By [9, (5.2)] it suffices to show that $\mathfrak{2}_G \otimes \mathfrak{2}_H$ is λ-presentable for all $G, H \in \mathscr{V}_\lambda$.

Write $\mathscr{I}$ for the $\mathscr{V}$-category with a single object $*$ and $\mathscr{I}(*, *) = I$; to give a $\mathscr{V}$-functor $\mathscr{I} \to \mathscr{A}$ is just to give an object of $\mathscr{A}$. Thus $\mathscr{V}\textbf{-Cat}(\mathscr{I}, -) : \mathscr{V}\textbf{-Cat} \to \textbf{Set}$ is the functor sending a $\mathscr{V}$-category to its set of objects. This has a right adjoint, and so preserves all colimits, whence $\mathscr{I}$ is certainly λ-presentable.

The $\mathscr{V}$-category $\mathscr{C} = \mathfrak{2}_G \otimes \mathfrak{2}_H$ has four objects: $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$, and hom-objects

$$\mathscr{C}((i,j),(i',j')) = \begin{cases} G \ \text{ if } i = 0,\ i' = 1,\ j = j' \\ H \ \text{ if } i = i',\ j = 0,\ j' = 1 \\ G \otimes H \ \text{ if } i = i' = 0,\ j = j' = 1 \\ I \ \text{ if } i = i',\ j = j' \\ 0 \ \text{ otherwise} \end{cases}$$

with the obvious composition maps. To give a $\mathscr{V}$-functor $T : \mathbb{2}_G \otimes \mathbb{2}_H \to \mathscr{A}$, therefore, is to give four objects $A = S(0,0)$, $B = S(0,1)$, $C = S(1,0)$, $D = S(1,1)$ of $\mathscr{A}$, along with maps $\alpha : G \to \mathscr{A}(A,C)$, $\beta : G \to \mathscr{A}(B,D)$, $\gamma : H \to \mathscr{A}(A,B)$, and $\delta : H \to \mathscr{A}(C,D)$ in $\mathscr{V}$ rendering commutative the diagram

$$\begin{tikzcd} G \otimes H \arrow[rr, "\beta\otimes\gamma"] \arrow[d, "\tau"'] & & \mathscr{A}(B,D) \otimes \mathscr{A}(A,B) \arrow[d, "M"] \\ H \otimes G \arrow[r, "\delta\otimes\alpha"'] & \mathscr{A}(C,D) \otimes \mathscr{A}(A,C) \arrow[r, "M"'] & \mathscr{A}(A,D) \end{tikzcd} \qquad (*)$$

wherein τ denotes the symmetry isomorphism and M the composition maps in $\mathscr{A}$.

To give A, C, D along with α and δ is to give a $\mathscr{V}$-functor $R : \mathbb{3}_{G,H} \to \mathscr{A}$, where $\mathbb{3}_{G,H}$ is the pushout

$$\begin{tikzcd} \mathscr{I} \arrow[r, "0"] \arrow[d, "1"'] & \mathbb{2}_H \arrow[d] \\ \mathbb{2}_G \arrow[r] & \mathbb{3}_{G,H} \end{tikzcd}$$

in $\mathscr{V}$**-Cat**. Similarly to give A, B, D along with β and γ is to give a $\mathscr{V}$-functor $S : \mathbb{3}_{H,G} \to \mathscr{A}$. To give $T : \mathbb{2}_G \otimes \mathbb{2}_H \to \mathscr{A}$, therefore, is to give R and S with the same A and D and satisfying $(*)$; which is to say that $\mathbb{2}_G \otimes \mathbb{2}_H$ is the pushout

$$\begin{tikzcd} \mathbb{2}_{G\otimes H} \arrow[r, "N"] \arrow[d, "M"'] & \mathbb{3}_{H,G} \arrow[d] \\ \mathbb{3}_{G,H} \arrow[r] & \mathbb{2}_G \otimes \mathbb{2}_H \end{tikzcd}$$

in $\mathscr{V}$**-Cat**, where M and N are the evident $\mathscr{V}$-functors. Since $\mathscr{I}$, $\mathbb{2}_G$, and $\mathbb{2}_H$ are λ-presentable, and since the λ-presentables are closed under finite colimits, it follows that $\mathbb{2}_G \otimes \mathbb{2}_H$ is λ-presentable, as desired. ∎

5. $\mathscr{V}$**-Cat** is locally bounded if $\mathscr{V}$ is so

For the first part of this section we suppose only that $\mathscr{V}$ is cocomplete and monoidal closed. The functor ob : $\mathscr{V}$**-Cat** $\to$ **Set** sending a $\mathscr{V}$-category to its set of objects then has both adjoints: the left adjoint D sends a set X to the "discrete" $\mathscr{V}$-category with object-set X and $DX(x,y)$ equal to 0 unless $x = y$ in which case it is the unit I, and the right adjoint C sends X to the "chaotic" $\mathscr{V}$-category with object-set X and $CX(x,y) = 1$.

For a $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$, we consider the set K of pairs (A,B) of objects of $\mathscr{A}$ with $FA = FB$, and form the coequalizer $Q : \mathscr{A} \to \mathscr{C}$ in $\mathscr{V}$**-Cat** of the two "projections" $P_1, P_2 : DK \to \mathscr{A}$. Since $FP_1 = FP_2$, there is a unique $\mathscr{V}$-functor $I : \mathscr{C} \to \mathscr{B}$ satisfying $IQ = F$. Clearly Q is invertible if and only if F is injective on objects; if I is invertible then we say that F is a *quotient on objects.* Since the "congruence" K arising from F

is the same as that arising from Q, we see that Q is a quotient on objects, while I is injective on objects by construction. The factorization is clearly functorial, and so we obtain a factorization system $(\mathscr{Q}, \mathscr{I})$ on $\mathscr{V}$**-Cat** in which $\mathscr{Q}$ consists of the quotients on objects, and $\mathscr{I}$ consists of those $\mathscr{V}$-functors that are injective on objects. (Although each $Q \in \mathscr{Q}$ is an epimorphism in $\mathscr{V}$**-Cat**—in fact a regular one — there can be $\mathscr{V}$-functors which are injective on objects but not monomorphic, and so $(\mathscr{Q}, \mathscr{I})$ is not proper.)

Before turning to the main results of the section, recall that if $\mathscr{H}$ is a class of arrows in $\mathscr{V}$, a $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$ is said to be *locally in* $\mathscr{H}$ if each $F : \mathscr{A}(A, B) \to \mathscr{B}(FA, FB)$ is in $\mathscr{H}$.

We now suppose that $\mathscr{V}$ is locally λ-bounded as a closed category, with respect to the proper factorization system $(\mathscr{E}, \mathscr{M})$.

Write $\mathscr{M}'$ for the class of $\mathscr{V}$-functors which are injective on objects and locally in $\mathscr{M}$; clearly every such $\mathscr{V}$-functor is a monomorphism. Because of the $(\mathscr{E}, \mathscr{M})$-generator $\mathscr{G}$, an object of $\mathscr{V}$ has only a small set of $\mathscr{M}$-subobjects, from which it follows that an object of $\mathscr{V}$**-Cat** has only a small set of $\mathscr{M}'$-subobjects, and therefore admits arbitrary intersections of $\mathscr{M}'$-subobjects. Since $\mathscr{M}'$ is clearly closed under composition and intersections, and stable under pullback, it forms, by [3, Lemma 3.1], part of a factorization system $(\mathscr{E}', \mathscr{M}')$ on $\mathscr{V}$**-Cat**. Since $(\mathscr{E}, \mathscr{M})$ is proper, every coretraction in $\mathscr{V}$ lies in $\mathscr{M}$, and one now easily shows that every coretraction in $\mathscr{V}$**-Cat** lies in $\mathscr{M}'$, and so that $(\mathscr{E}', \mathscr{M}')$ is proper.

It takes a little work to compute $\mathscr{E}'$, although it is easy to see that a $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$ in $\mathscr{E}'$ must be surjective on objects, since otherwise it would factorize through some non-invertible $J : \mathscr{C} \to \mathscr{B}$ which is injective on objects and fully faithful, and therefore lies in $\mathscr{M}'$. Now consider, for an $F : \mathscr{A} \to \mathscr{B}$ that *is* surjective on objects, its $(\mathscr{Q}, \mathscr{I})$-factorization $F = IQ$. Since here I, like F, is surjective on objects, it is in fact bijective on objects. The quotient-on-objects $\mathscr{V}$-functor Q, being a regular epimorphism in $\mathscr{V}$**-Cat**, certainly lies in the $\mathscr{E}'$ of the proper factorization system $(\mathscr{E}', \mathscr{M}')$; whence it follows by [6, Proposition 2.1.1] that F lies in $\mathscr{E}'$ if and only if I lies in $\mathscr{E}'$. The case of a bijective-on-objects $\mathscr{V}$-functor, however, is dealt with in the following:

5.1. Lemma. *A $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$ which is bijective on objects lies in $\mathscr{E}'$ if and only if it is locally in $\mathscr{E}$.*

Proof. Suppose that $F : \mathscr{A} \to \mathscr{B}$ is bijective on objects and in $\mathscr{E}'$; without loss of generality we may suppose F to be the *identity* on objects. Let the $(\mathscr{E}, \mathscr{M})$-factorization of $F : \mathscr{A}(A, B) \to \mathscr{B}(A, B)$ be

$$\mathscr{A}(A, B) \xrightarrow{E_{A,B}} \mathscr{D}(A, B) \xrightarrow{M_{A,B}} \mathscr{B}(A, B).$$

For objects A, B, C of $\mathscr{A}$, we have $E_{B,C} \otimes E_{A,B}$ in $\mathscr{E}$, since the class $\mathscr{E}$ is by assumption closed under tensor products, and we also have $M_{A,C}$ in $\mathscr{M}$; thus there is a unique map

M' making commutative the diagram

$$\begin{tikzcd}
\mathscr{A}(B,C) \otimes \mathscr{A}(A,B) \arrow[r, "E_{B,C} \otimes E_{A,B}"] \arrow[d, "M"'] & \mathscr{D}(B,C) \otimes \mathscr{D}(A,B) \arrow[r, "M_{B,C} \otimes B_{A,B}"] \arrow[d, "M'"'] & \mathscr{B}(B,C) \otimes \mathscr{B}(A,B) \arrow[d, "M''"] \\
\mathscr{A}(A,C) \arrow[r, "E_{A,C}"'] & \mathscr{D}(A,C) \arrow[r, "M_{A,C}"'] & \mathscr{B}(A,C),
\end{tikzcd}$$

in which M and M'' are the composition maps for $\mathscr{A}$ and $\mathscr{B}$. The M' give to the $\mathscr{D}(A,B)$ the structure of a $\mathscr{V}$-category $\mathscr{D}$, for which the $E_{A,B}$ constitute a $\mathscr{V}$-functor $M : \mathscr{D} \to \mathscr{B}$ which is the identity on objects: the point is that the $\mathscr{V}$-category axioms for $\mathscr{D}$ follow from those for $\mathscr{B}$, since the $M_{A,B}$ are monomorphic. Now the $E_{A,B}$ constitute a $\mathscr{V}$-functor $E : \mathscr{A} \to \mathscr{D}$ which is the identity on objects, and $F = ME$ provides a factorization of F with $M \in \mathscr{M}'$. Since $F \in \mathscr{E}'$, this implies that M is invertible, and in particular that each $M_{A,B}$ is so, so that $F_{A,B} = M_{A,B}E_{A,B}$ lies in $\mathscr{E}$ as required.

Conversely, a $\mathscr{V}$-functor which is bijective on objects and locally in $\mathscr{E}$ factorizes through no proper $\mathscr{M}'$-subobject, and so must be in $\mathscr{E}'$. ∎

This now gives:

5.2. Proposition. *A $\mathscr{V}$-functor $F : \mathscr{A} \to \mathscr{B}$ is in $\mathscr{E}'$ if and only if it can be written as $F = IQ$ where Q is a quotient on objects and I is bijective on objects and locally in $\mathscr{E}$.*

As a first step to proving that $\mathscr{V}$**-Cat** is locally λ-bounded with respect to $(\mathscr{E}', \mathscr{M}')$, we prove:

5.3. Lemma. $\mathscr{V}$**-Cat** *admits arbitrary cointersections of maps in $\mathscr{E}'$.*

Proof. Let $(E_i : \mathscr{A} \to \mathscr{B}_i)_{i \in I}$ be a family of $\mathscr{V}$-functors, each lying in $\mathscr{E}'$. By well-ordering the indexing set I, we can write these instead in the form $(E_\alpha : \mathscr{A} \to \mathscr{B}_\alpha)_{\alpha < \delta}$ for some initial ordinal δ. We set out to define by transfinite induction a “descending” family $F_\alpha : \mathscr{A} \to \mathscr{C}_\alpha$ of $\mathscr{V}$-functors in $\mathscr{E}'$. We set $F_0 : \mathscr{A} \to \mathscr{C}_0$ to be equal to $1 : \mathscr{A} \to \mathscr{A}$. We take for $F_{\alpha+1} : \mathscr{A} \to \mathscr{C}_{\alpha+1}$ the cointersection of $F_\alpha : \mathscr{A} \to \mathscr{C}_\alpha$ and $E_\alpha : \mathscr{A} \to \mathscr{B}_\alpha$; it lies in $\mathscr{E}'$, because $\mathscr{E}'$ is closed under any cointersections that exist. Finally, for a limit ordinal α, we take for $F_\alpha : \mathscr{A} \to \mathscr{C}_\alpha$ the cointersection of all the F_β with $\beta < \alpha$, provided that this exists; then $F_\delta : \mathscr{A} \to \mathscr{C}_\delta$ is clearly the required cointersection of the $E_\alpha : \mathscr{A} \to \mathscr{B}_\alpha$, if it exists.

Suppose it does not. Let γ be the first ordinal for which F_γ fails to exist. Then γ cannot be of the form $\alpha + 1$, since binary cointersections certainly exist; thus γ is a limit ordinal. It cannot be small, since small cointersections exist. Since $\mathrm{ob}\mathscr{A}$ has only a small set of epimorphic images in **Set**, the surjections $\mathrm{ob}F_\alpha : \mathrm{ob}\mathscr{A} \to \mathrm{ob}\mathscr{C}_\alpha$ have become constant at some ordinal $\beta < \gamma$; so that the comparison functor $F_\rho^\sigma : \mathscr{C}_\rho \to \mathscr{C}_\sigma$ is bijective on objects whenever $\beta \leq \rho < \sigma < \gamma$. Since F_ρ^σ is in $\mathscr{E}'$ by [6, Proposition 2.1.1], it is locally in $\mathscr{E}$ by Lemma 5.1. But now the non-existence of the cointersection $F_\gamma : \mathscr{A} \to \mathscr{C}_\gamma$ contradicts the hypothesis that $\mathscr{V}_0$ admits arbitrary cointersections of maps in $\mathscr{E}$. ∎

We have seen that $\mathscr{V}$**-Cat** is a cocomplete category with a proper factorization system $(\mathscr{E}', \mathscr{M}')$ for which $\mathscr{V}$**-Cat** admits arbitrary $\mathscr{E}'$-cointersections. It will therefore be locally λ-bounded if it has an $(\mathscr{E}', \mathscr{M}')$-generator consisting of λ-bounded objects. Let $\mathscr{G}$ be an $(\mathscr{E}, \mathscr{M})$-generator for $\mathscr{V}_0$ consisting of λ-bounded objects; without loss of generality we may suppose that $\mathscr{G}$ contains the initial object 0. Write $\mathscr{G}'$ for the set of those $\mathscr{V}$-categories of the form $\mathfrak{2}_G$ for some $G \in \mathscr{G}$. The reader will easily verify that $\mathscr{G}'$ is an $(\mathscr{E}', \mathscr{M}')$-generator for $\mathscr{V}$**-Cat**: the argument is essentially that used to prove Lemma 4.2. A little more work is required in showing that $\mathfrak{2}_G$ is λ-bounded in $\mathscr{V}$**-Cat** when G is so in $\mathscr{V}_0$, since we first need the following lemma:

5.4. Lemma. *Consider a small filtered family $(F_j : \mathscr{A}_j \to \mathscr{B})_{j \in J}$ in $\mathscr{M}'$; without loss of generality we take the functions $\mathrm{ob}F_j : \mathrm{ob}\mathscr{A}_j \to \mathrm{ob}\mathscr{B}$ to be set-inclusions. Then $(F_j : \mathscr{A}_j \to \mathscr{B})_{j \in J}$ is an $\mathscr{M}'$-union in $\mathscr{V}$**-Cat** precisely when $\mathrm{ob}\mathscr{B}$ is the union in **Set** of the $\mathrm{ob}\mathscr{A}_j$ and, for each pair X, Y of objects of $\mathscr{B}$, the family $(F_j : \mathscr{A}_j(X,Y) \to \mathscr{B}(X,Y))_{j \in J_{X,Y}}$ is an $\mathscr{M}$-union in $\mathscr{V}_0$, where $J_{X,Y}$ is $\{j \in J | X \text{ and } Y \text{ lie in } \mathrm{ob}\mathscr{A}_j\}$.*

Proof. The F_j are an $\mathscr{M}'$-union if and only if the induced $\mathscr{V}$-functor $F : \mathrm{colim}_j \mathscr{A}_j \to \mathscr{B}$ lies in $\mathscr{E}'$. This means in particular that it is surjective on objects; but F is in any case injective on objects, since the F_j are so, and $\mathscr{J}$ is filtered. Thus F would need to be bijective on objects, and we saw in Lemma 5.1 that such a $\mathscr{V}$-functor lies in $\mathscr{E}'$ if and only if it is locally in $\mathscr{E}$. Thus the F_j are an $\mathscr{M}'$-union if and only if $\mathrm{ob}\mathscr{B}$ is the union of the $\mathrm{ob}\mathscr{A}_j$ and F is locally in $\mathscr{E}$.

Since $\mathscr{J}$ is filtered, the colimit of the $\mathscr{A}_j$ is preserved by $U : \mathscr{V}\textbf{-Cat} \to \mathscr{V}\textbf{-Gph}$. Thus

$$(\mathrm{colim}_j \mathscr{A}_j)(X,Y) = \mathrm{colim}_j \sum_{\substack{F_j A_j = X \\ F_j B_j = Y}} \mathscr{A}_j(A_j, B_j),$$

but then to say that $F : \mathrm{colim}_j \mathscr{A}_j(X,Y) \to \mathscr{B}(X,Y)$ is in $\mathscr{E}$ for all X and Y is just to say that $(F_j : \mathscr{A}_j(X,Y) \to \mathscr{B}(X,Y))_{j \in J_{X,Y}}$ is an $\mathscr{M}$-union in $\mathscr{V}_0$. ∎

It now follows easily that $\mathscr{V}\textbf{-Cat}(\mathfrak{2}_G, -) : \mathscr{V}\textbf{-Cat} \to \textbf{Set}$ preserves λ-filtered $\mathscr{M}'$-unions if $\mathscr{V}_0(G, -) : \mathscr{V}_0 \to \textbf{Set}$ preserves λ-filtered $\mathscr{M}$-unions, and so we have:

5.5. Proposition. *$\mathscr{V}$**-Cat** is locally λ-bounded with respect to $(\mathscr{E}', \mathscr{M}')$.*

Finally, we look at the closed structure of $\mathscr{V}$**-Cat** in this context:

5.6. Theorem. *If $\mathscr{V}$ is a symmetric monoidal closed category which is locally λ-bounded as a closed category with respect to the proper factorization system $(\mathscr{E}, \mathscr{M})$, then $\mathscr{V}$**-Cat** is locally λ-bounded as a closed category with respect to the proper factorization system $(\mathscr{E}', \mathscr{M}')$.*

Proof. We must prove for each $\mathscr{V}$-category $\mathscr{X}$ that $\mathscr{X} \otimes E$ is in $\mathscr{E}'$ if E is so; or, equivalently, that $[\mathscr{X}, M]$ is in $\mathscr{M}'$ if M is so. Suppose then that $\mathscr{X}$ is a $\mathscr{V}$-category and that $M : \mathscr{A} \to \mathscr{B}$ lies in $\mathscr{M}'$. An object of $[\mathscr{X}, \mathscr{A}]$ is a $\mathscr{V}$-functor from $\mathscr{X}$ to $\mathscr{A}$; since M is a monomorphism, $[\mathscr{X}, M] : [\mathscr{X}, \mathscr{A}] \to [\mathscr{X}, \mathscr{B}]$ is injective on objects. To see that $[\mathscr{X}, M]$ is locally in $\mathscr{M}$, let $F, G : \mathscr{X} \to \mathscr{A}$ be $\mathscr{V}$-functors. Then the hom-object

$[\mathscr{X},\mathscr{A}](F,G)$ is given by the end

$$\int_{X\in\mathscr{X}} \mathscr{A}(FA,GA).$$

Each $M : \mathscr{A}(FX,GX) \to \mathscr{B}(MFX,MGX)$ lies in $\mathscr{M}$, and $\mathscr{M}$ is closed under limits; it follows that

$$\int_{X\in\mathscr{X}} M : \int_{X\in\mathscr{X}} \mathscr{A}(FX,GX) \to \int_{X\in\mathscr{X}} \mathscr{B}(MFX,MGX)$$

lies in $\mathscr{M}$; that is, that $M : \mathscr{A}(F,G) \to \mathscr{B}(MF,MG)$ does so. ∎

References

[1] J. Adámek and J. Rosický, *Locally presentable categories and accessible categories*, LMS Lecture Notes Series 189, Cambridge University Press, Cambridge, 1994.

[2] R. Betti, A. Carboni, R. Street, and R. Walters, Variation through enrichment, *J. Pure Appl. Alg.* **29**(1983), 109–127.

[3] C. Cassidy, M. Hébert, and G.M. Kelly, Reflective subcategories, localizations and factorization systems, *J. Austral. Math. Soc.*(A) **38**(1985), 287–329.

[4] S. Eilenberg and G.M. Kelly, Closed categories, *Proceedings of the Conference on Categorical Algebra* (La Jolla, 1965), 421–562, Springer-Verlag, 1966.

[5] P. Gabriel and F. Ulmer, *Lokal-Präsentierbare Kategorien*, Lecture Notes in Mathematics 221, Springer-Verlag, 1971.

[6] P. Freyd and G.M. Kelly, Categories of continuous functors I, *J. Pure Appl. Alg.*, **2**(1972), 169–191.

[7] André Joyal and Ross Street, Braided tensor categories, *Adv. Math.* **102**(1993), 20-78.

[8] G.M. Kelly, *Basic Concepts of Enriched Category Theory*, LMS Lecture Notes Series 64, Cambridge University Press, Cambridge, 1982.

[9] G.M. Kelly, Structures defined by finite limits in the enriched context I, *Cah. de Top. et Géom. Diff.*, **XXIII**(1982), 3–42.

[10] Michael Makkai and Robert Paré, *Accessible categories: The foundations of categorical model theory*, Contemporary Mathematics 104, Amer. Math. Soc., Providence, 1989.

[11] Harvey Wolff, V-cat and V-graph, *J. Pure Appl. Alg.* **4**(1974), 123–135.

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