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EXPONENTIABLE MORPHISMS: POSETS, SPACES, LOCALES, AND GROTHENDIECK TOPOSES

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ABSTRACT. In this paper, we consider those morphisms $p: P \longrightarrow B$ of posets for which the induced geometric morphism of presheaf toposes is exponentiable in the category of Grothendieck toposes. In particular, we show that a necessary condition is that the induced map $p^{\downarrow}: P^{\downarrow} \longrightarrow B^{\downarrow}$ is exponentiable in the category of topological spaces, where $P^{\downarrow}$ is the space whose points are elements of P and open sets are downward closed subsets of P. Along the way, we show that $p^{\downarrow}: P^{\downarrow} \longrightarrow B^{\downarrow}$ is exponentiable if and only if $p: P \longrightarrow B$ is exponentiable in the category of posets and satisfies an additional compactness condition. The criteria for exponentiability of morphisms of posets is related to (but weaker than) the factorization-lifting property for exponentiability of morphisms in the category of small categories (considered independently by Giraud [G] and Conduché [C]).

1. Introduction

Let $\mathbf{A}$ be a category with finite limits. Recall that an object X of $\mathbf{A}$ is called *exponentiable* if the functor $- \times X: \mathbf{A} \longrightarrow \mathbf{A}$ has a right adjoint (denoted by $(\ )^X$). The category $\mathbf{A}$ is called *cartesian closed* if every object is exponentiable, and it is called *locally cartesian closed* if the slice categories $\mathbf{A}/T$ are cartesian closed, for all objects T of $\mathbf{A}$, where $\mathbf{A}/T$ is the category whose objects are morphisms $p: X \longrightarrow T$ of $\mathbf{A}$ and morphisms are commutative triangles in $\mathbf{A}$. We shall say that a morphism $p: X \longrightarrow T$ of $\mathbf{A}$ is *exponentiable* if p is exponentiable in $\mathbf{A}/T$, or equivalently, if the pullback functor $p^*: \mathbf{A}/T \longrightarrow \mathbf{A}/T$ has a right adjoint. Note that we shall follow the customary abuse of notation and write $- \times_T X$ for $- \times p$, when the morphism p is unambiguous.

Exponentiability has been studied in many categories including

Top topological spaces and continuous maps

Cat small categories and functors

Pos partially-ordered sets and order-preserving maps

Loc locales and morphisms of locales

GTop Grothendieck toposes and geometric morphisms

Unif uniform spaces and uniformly continuous maps

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Aff affine schemes and morphism of affine schemes

This paper concerns the first five categories listed above. For the latter two, we refer the reader to [N3].

We began by asking to what extent the presheaf functor

$$\mathrm{PSh}\colon \mathbf{Cat} \longrightarrow \mathbf{GTop}$$

preserves and reflects exponentiability. This question arose when the author was working with Marta Bunge on [BN]. Since it is well-known that this functor preserves products and exponentiable objects, though not equalizers (see [J1]), we did not expect that exponentiable morphisms would be preserved, but we made little progress towards an answer. Subsequently, the author realized that this problem might be solved by first restricting to partially-ordered sets, solving the problem for the presheaf functor $\mathbf{Pos} \longrightarrow \mathbf{GTop}$, and later generalizing the results to **Cat**. The advantage of working with posets would be that one could exploit the equivalence between the category $\mathrm{PSh}(P)$ of presheaves on P and the category $\mathrm{Sh}(P^{\downarrow})$ of sheaves on the space $P^{\downarrow}$ whose points are elements of P and open sets are downward closed subsets, and thus make use of known results for topological spaces.

First, we consider some of the relevant history, partly to provide a context for the reader, but also to recall those results which will be used in this paper.

Interest in exponentiability of objects in non-cartesian closed categories is related to the study of suitable topologies on function spaces, for if X is exponentiable in **Top**, then taking $Y = 1$ in the natural bijection

$$\mathbf{Top}(Y \times X, Z) \cong \mathbf{Top}(Y, Z^X)$$

it is easy to see that Z^X can be identified with the set $\mathbf{Top}(X, Z)$ of continuous maps from X to Z. Perhaps the first definitive result in this area appeared in the 1945 paper "On topologies for function spaces" [F] by R. H. Fox, where he clearly stated the problem as follows:

> Given topological spaces X, T, and Y and a function h from $X \times T$ to Y which is continuous in x for each fixed t, there is associated a function h^* from T to $F = Y^X$, the space whose elements are continuous functions from X to Y. ... It would be desirable to so topologize F that the functions h^* which are continuous are precisely those which correspond to continuous h. It has been known for a long time that this is possible if X satisfies certain conditions, chief among which is the condition of local compactness.

and showed that a separable metrizable space is exponentiable if and only if it is locally compact.

The characterization of exponentiable spaces was finally achieved by Day and Kelly in their 1970 paper "On topological quotients preserved by pullback or products" [DK], where they proved that, for a space X, the functor

$$- \times X\colon \mathbf{Top} \longrightarrow \mathbf{Top}$$

preserves quotients if and only if the lattice $\mathcal{O}(X)$ of open sets of X is (what is now known as) a continuous lattice, in the sense of Scott [S], i.e., $\mathcal{O}(X)$ satisfies $V = \bigvee\{U | U << V\}$, where $U << V$ if every open cover of V has a finite subfamily that covers U. Since $- \times X$ preserves coproducts in any case, preservation of quotients (or equivalently, coequalizers) is necessary and sufficient for exponentiability in **Top**, by Freyd's Special Adjoint Functor Theorem [F]. Day and Kelly also showed that continuity of $\mathcal{O}(X)$ coincides with local compactness of X for Hausdorff spaces. For a thorough treatment of exponentiability in **Top** and related function space problems see Isbell's article [I] on the influence of the Day and Kelly paper.

For non-Hausdorff spaces, there were two nonequivalent definitions of local compactness under consideration in the 1970's. One hypothesized the existence of a compact neighborhood of each point (see [K]) while the other required arbitrarily small such neighborhoods (see [M]). In 1978, Hofmann and Lawson [HL] showed that the sober spaces satisfying the latter definition are those for which $\mathcal{O}(X)$ is continuous, and hence, precisely the sober spaces which are exponentiable in **Top**. Thus, the second definition seemed to be the appropriate one for non-Hausdorff spaces. Then in 1982, Eilenberg [E] proposed that continuity of $\mathcal{O}(X)$ be taken to be the *definition* of local compactness for a general space X. For this to make sense, he suggested that we need only rename the relation $<<$ by saying "U is compact in V" in place of "U is way-below in V." Thus, we could say X is called *locally compact* if every open set V can be covered by open sets U such that U is compact in V, and we have generalized the definition of local compactness in such a way that it coincides precisely with exponentiability. It is this definition of local compactness (for spaces and locales) that we shall follow in this paper.

In the decade following the Day/Kelly paper, the relationship between exponentiability and local compactness was generalized to locales and toposes. In [H], Hyland showed that a locale is exponentiable in **Loc** if and only if it is locally compact. Then Johnstone and Joyal generalized this result to Grothendieck toposes. They began with the definition of continuous posets based on the Hofmann and Stralka [HS] characterization of continuous lattices as those for which the map $\bigvee\colon \mathrm{Idl}(A) \longrightarrow A$ has a left adjoint. Using Grothendieck's [Gr] notion of the ind-completion $\mathrm{Ind}(\mathcal{E})$ of a locally small category $\mathcal{E}$, they defined $\mathcal{E}$ to be a *continuous category* if it has filtered colimits and the functor $\varinjlim\colon \mathrm{Ind}(\mathcal{E}) \longrightarrow \mathcal{E}$ has a left adjoint. They then proved that $\mathcal{E}$ is exponentiable in **GTop** if and only if it is a continuous category. In addition, they showed that for a locale X, the topos $\mathrm{Sh}(X)$ of sheaves on X is exponentiable in **GTop** if and only if X is metastably locally compact. This is a property implying local compactness but using a strengthening of $<<$ in its place.

The results of [DK] were also relativized by Niefield in [N3], where the characterization of exponentiable spaces was generalized to $\mathbf{Top}/T$. As corollaries, it was shown that if X is locally compact and T is Hausdorff, then any continuous map $X \longrightarrow T$ is exponentiable in **Top**, and if X is a subspace of T, then the inclusion $X \longrightarrow T$ is exponentiable in **Top** if and only if X is locally closed (i.e., the intersection of an open and a closed subspace of T). The latter was extended to **Loc** and **GTop** in [N2]. The results of [N3] were also used by the author in [N4] to show that for sober spaces X and T (in which points of T are locally closed), a continuous map $p: X \longrightarrow T$ is exponentiable in **Top** if and only if $\mathcal{O}(p): \mathcal{O}(X) \longrightarrow \mathcal{O}(T)$ is exponentiable in **Loc** if and only if $p_*(\Omega_X)$ is locally compact as an internal locale in $\mathrm{Sh}(T)$, where $p_*: \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(T)$ is the direct image of the geometric morphism induced by p.

There has also been considerable interest in exponentiability in **Cat**. It has long been known that **Cat** is cartesian closed but not locally cartesian closed. The exponentiable morphisms of **Cat** were characterized by Giraud [Gi] (and later rediscovered by Conduché [C]) as those satisfying a certain factorization lifting property. In [BN], Bunge and Niefield introduced the notion of a locally closed subcategory and showed that the inclusion of a full subcategory has this factorization lifting property if and only if it is locally closed if and only if the corresponding geometric morphism of presheaf toposes is locally closed.

Thus, we know that the presheaf functor $\mathbf{Pos} \longrightarrow \mathbf{GTop}$ preserves exponentiable objects (since every poset P is exponentiable as is every presheaf topos $\mathrm{PSh}(P)$) and exponentiable inclusions (since they are locally closed in both cases). The idea in the general case is to proceed as follows. Suppose $p: P \longrightarrow B$ is an order-preserving map for which the induced geometric morphism $p: \mathrm{PSh}(P) \longrightarrow \mathrm{PSh}(B)$ is exponentiable in **GTop**. Then using the equivalence $\mathrm{PSh}(P) \simeq \mathrm{Sh}(P^{\downarrow})$, we know that $p: \mathrm{Sh}(P^{\downarrow}) \longrightarrow \mathrm{Sh}(B^{\downarrow})$ is exponentiable as well, and since the Johnstone/Joyal work [JJ] is constructive and hence applies to the relative case, it follows that $p_*(\Omega_{P^{\downarrow}})$ is locally compact in the category $\mathbf{Loc}(\mathrm{Sh}(B^{\downarrow}))$ of internal locales in $\mathrm{Sh}(B^{\downarrow})$. But, $\mathbf{Loc}(\mathrm{Sh}(B^{\downarrow})) \simeq \mathbf{Loc}/\mathcal{O}(B^{\downarrow})$ (see [JT] or [J2]), and so using the results from [N4], it follows that $p^{\downarrow}: P^{\downarrow} \longrightarrow B^{\downarrow}$ is exponentiable in **Top**. Thus, we are led to considering the effect of the functor $(\ )^{\downarrow}: \mathbf{Pos} \longrightarrow \mathbf{Top}$ on exponentiability.

We begin, in §2, by showing that a weakened version of the Giraud-Conduché characterization of exponentiable morphisms in **Cat** can be used for the category **Pos**. In §3, we review the exponentiability criteria from [N3] for morphisms of **Top** and show that it applies to T_0-spaces, as well. We then use these results to show, in §4, that $p^{\downarrow}: P^{\downarrow} \longrightarrow B^{\downarrow}$ is exponentiable in **Top** if and only if $p: P \longrightarrow B$ is exponentiable in **Pos** and satisfies an additional compactness condition. In §5, we adapt these results to include locales, in the case where $B^{\downarrow}$ is a sober space. We conclude, in §6, by showing that these conditions on $p: P \longrightarrow B$ are necessary (but not sufficient) for the exponentiability of the corresponding geometric morphism $\mathrm{PSh}(P) \longrightarrow \mathrm{PSh}(B)$ in **GTop**. As a consequence, we give an example of a discrete opfibration $p: \mathbf{C} \longrightarrow \mathbf{B}$ for which the corresponding geometric morphism of presheaf toposes is not exponentiable in **GTop**, thus providing a counterexample to a "theorem" appearing in a preliminary version of [BN]. We also show that the presheaf functor $\mathbf{Cat} \longrightarrow \mathbf{GTop}$ neither preserves or nor reflects exponentiable morphisms.

2. Exponentiability in **Pos**

In this section, we characterize exponentiable morphism of the category of posets, and show that they include those which are exponentiable in **Cat**. Our proofs here are similar to those of Giraud [Gi] but we include them here for the sake of completeness, and since they are simpler than those for **Cat**.

We begin with some notation. For $x \leq y$ in a poset P, let $[x, y]$ denote the *interval* $\{z \in P | x \leq z \leq y\}$. Given $p\colon P \longrightarrow B$ in **Pos** and $b \in B$, the *fiber of P over b* is the subposet $P_b = p^{-1}(b)$ of P. For $A \subseteq P$, we shall write A_b for $A \cap P_b$.

2.1. Definition. *A poset morphism $p\colon P \longrightarrow B$ is called an interpolation-lifting map if $[x, y]_b \neq \emptyset$, for all $x \leq y$ in P and $b \in [px, py]$.*

Note that this says that given $x \leq y$ in P and $px \leq b \leq py$ in B, the following diagram can be completed:

$$\begin{array}{cccccccc} P & \qquad & x & \leq & z & \leq & y \\ \downarrow & & | & & | & & | \\ B & & px & \leq & b & \leq & py \end{array}$$

Thus, this is a consequence of the Giraud-Conduché factorization lifting property characterizing exponentiability in **Cat**, but there is no assumption of the connectedness of $[x, y]_b$ here.

2.2. Theorem. *A morphism $p\colon P \longrightarrow B$ is exponentiable in* **Pos** *if and only if it is an interpolation-lifting map.*

Proof. Suppose $p\colon P \longrightarrow B$ is an interpolation-lifting map. Given $q\colon Q \longrightarrow B$, let Q^P denote the set of pairs (σ, b), where $b \in B$ and $\sigma\colon P_b \longrightarrow Q_b$ is a morphism of posets. Define $(\sigma, b) \leq (\sigma', b')$ if $b \leq b'$ and $\sigma x \leq \sigma' x'$, for all $(x, x') \in P_b \times P_{b'}$ such that $x \leq x'$ in P. Then $\leq$ is clearly reflexive and antisymmetric. For transitivity, suppose $(\sigma, b) \leq (\sigma', b')$ and $(\sigma', b') \leq (\sigma'', b'')$. Then $b \leq b''$ since $b \leq b'$ and $b' \leq b''$. Given $(x, x'') \in P_b \times P_{b''}$ such that $x \leq x''$, since $px \leq b \leq px'$ and p is an interpolation-lifting map, there exists $x' \in P_{b'}$ such that $x \leq x' \leq x''$. Then $\sigma x \leq \sigma'' x''$, since $\sigma x \leq \sigma' x'$ and $\sigma x' \leq \sigma'' x''$, and it follows that $(\sigma, b) \leq (\sigma'', b'')$. Therefore, Q^P is a poset and the projection map $Q^P \longrightarrow B$ is clearly order-preserving. Moreover, one easily checks that $(\ )^P$ is a functor and the natural maps $\epsilon\colon Q^P \times_B P \longrightarrow Q$, $\epsilon((\sigma, b), x) = \sigma x$, and $\eta\colon Q \longrightarrow (Q \times_B P)^P$, $\eta(y) = ((y, -), qy)$, are order-preserving and satisfy the adjunction identities. Therefore, $p\colon P \longrightarrow B$ is exponentiable in **Pos**.

Conversely, suppose that $p\colon P \longrightarrow B$ is exponentiable in **Pos**. To show that p is an interpolation-lifting map, suppose $px \leq b \leq py$, where $x \leq y$ in P. Then the morphism $\mathbf{3} \longrightarrow B$ induced by $px \leq b \leq py$, gives rise to a pushout in $\mathbf{Pos}/B$ of the form

$$\begin{tikzcd} \mathbf{1} \arrow[r] \arrow[d] & \mathbf{2} \arrow[d] \\ \mathbf{2} \arrow[r] & \mathbf{3} \arrow[dr] \\ & & B \end{tikzcd}$$

where $\mathbf{1} = \{0\}$, $\mathbf{2} = \{0,1\}$, and $\mathbf{3} = \{0,1,2\}$ are the linearly ordered one, two, and three element posets, respectively. Since $- \times_B P$ preserves pushouts (being a left adjoint), it follows that the corresponding diagram

$$\begin{tikzcd} \mathbf{1} \times_B P \arrow[r] \arrow[d] & \mathbf{2} \times_B P \arrow[d] \\ \mathbf{2} \times_B P \arrow[r] & \mathbf{3} \times_B P \end{tikzcd}$$

is a pushout in **Pos**. It is not difficult to show that this pushout can be identified with the set

$$Q = P_{px\leq b} \bigcup_{P_b} P_{py\leq b}$$

where $P_{px\leq b}$ and $P_{py\leq b}$ correspond to the two copies of $\mathbf{2} \times_B P$ in the pushout above, and the order on Q is the smallest partial order containing that of $P_{px\leq b}$ and $P_{py\leq b}$. Thus, for $u \in P_{px}$ and $v \in P_{py}$, we have $u \leq v$ if there exists $w \in P_b$ such that $u \leq w$ and $w \leq v$. Since $Q \cong \mathbf{3} \times_B P$ and $(0,x) \leq (2,y)$ in $\mathbf{3} \times_B P$, there exists $(1,z)$ in $\mathbf{3} \times_B P$ such that $(0,x) \leq (1,z) \leq (2,y)$, and it follows that $p\colon P \longrightarrow B$ is an interpolation-lifting map, as desired. ∎

3. Exponentiability in Top

In this section, we recall the characterization of exponentiable morphisms of **Top** presented in [N1, N3] and show that it also applies to the category $\mathbf{Top}_0$ of T_0-spaces. The latter will be of interest in §4 since $P^{\downarrow}$ is T_0 for every poset P. The proof for $\mathbf{Top}_0$ is essentially the same as that of **Top** appearing in [N3]. We will include just enough here to show that the relevant spaces are T_0.

Recall that if X is a topological space, then $H \subseteq \mathcal{O}(X)$ is called *Scott-open* if it is upward closed, i.e., $U \in H$ and $U \subseteq V$ implies $V \in H$, and it satisfies the finite union property, $\{U_\alpha\}_{\alpha\in A} \subseteq \mathcal{O}(X)$ and $\bigcup_{\alpha\in A} U_\alpha \in H$ implies $\bigcup_{\alpha\in F} U_\alpha \in H$, for some finite $F \subseteq A$.

For a continuous map $p\colon X \longrightarrow T$ of spaces, families $H \subseteq \bigcup_{t\in T} \mathcal{O}(X_t)$ of open subsets of fibers are of interest. To simplify notation, we write $\bigcap H$ for the set whose fiber over $t \in T$ is given by

$$\left(\bigcap H\right)_t = \bigcap H_t = \bigcap\{V_t | V_t \in H_t\}$$

3.1. Theorem. *The following are equivalent for $p\colon X \longrightarrow T$ in $\mathbf{Top}_0$:*

(a) *p is exponentiable in* **Top**.

(b) *p is exponentiable in* $\mathbf{Top}_0$.

(c) *Given $x \in U_{px} \in \mathcal{O}(X_{px})$, there exists $H \subseteq \bigcup_{t\in T} \mathcal{O}(X_t)$ such that*

(i) $U_{px} \in H_{px}$

(ii) *H_t is Scott-open, for all $t \in T$*

(iii) *$\{t \in T | V_t \in H_t\}$ is open in T, for all $V \in \mathcal{O}(X)$*

(iv) *$\bigcap H$ is a neighborhood of x in X.*

PROOF. The proof of (a) $\Leftrightarrow$ (c) can be found in [N3] but is similar to that of (b) $\Leftrightarrow$ (c) presented below.

Suppose p satisfies (c). For a continuous map $q: Z \longrightarrow T$ of T_0-spaces, define Z^X as follows. The points of Z^X are pairs (σ, t), where $\sigma: X_t \longrightarrow Z_t$ is a continuous map and $t \in T$. The topology on Z^X is generated by sets of the form

$$\langle H, W\rangle = \{(\sigma, t) | \sigma^{-1}(W_t) \in H_t\}$$

where $H \subseteq \bigcup_{t \in T} \mathcal{O}(X_t)$ satisfies (ii) and (iii) above and W is open in Z. Now, the projection $\pi: Z^X \longrightarrow T$ is easily seen to be continuous. To see that Z^X is a T_0-space, suppose $(\sigma, t) \neq (\sigma', t')$ in Z^X. If $t \neq t'$, then there is an open set G of T containing one but not both of t and t', and so $\pi^{-1}(G)$ is an open set of Z^X containing one but not both of (σ, t) and (σ', t'). If $t = t'$, then $\sigma \neq \sigma'$, and so $\sigma x \neq \sigma' x$ for some $x \in X_t$. Since Z is T_0, there is an open set W of Z containing one but not both of σx and $\sigma' x$. Without loss of generality, assume $\sigma x \in W$ and $\sigma' x \notin W$. Taking $x \in U_t = \sigma^{-1}(W_t)$, we get H satisfying (i)-(iv) above. Then $(\sigma, t) \in \langle H, W\rangle$, since $U_t \in H_t$, and $(\sigma', t') \notin \langle H, W\rangle$, for otherwise $(\sigma')^{-1}(W_t) \in H_t$ and so $x \in \bigcap H_t \subseteq (\sigma')^{-1}(W_t)$ contradicting that $\sigma' x \notin W$. A straightforward calculation shows that $\epsilon: Z^X \times_T X \longrightarrow Z$, $\epsilon((\sigma, t), x) = \sigma x$ and $\eta: Z \longrightarrow (Z \times_T X)^X$, $\eta(z) = ((z, -), qz)$ are continuous, and that the adjunction identities hold. Note that the continuity of ϵ uses condition (c) but that of η holds in any case.

Conversely, suppose p is exponentiable in $\mathbf{Top}_0$. Proceeding as in [N3], we consider the projection $T \times \mathbf{2} \longrightarrow T$, where $\mathbf{2} = \{0, 1\}$ is the Sierpinski space with $\{0\}$ open but not $\{1\}$, and identify $(T \times \mathbf{2})^X$ with $\bigcup_{t \in T} \mathcal{O}(X_t)$. Then the counit $\epsilon: (T \times \mathbf{2})^X \times_T X \longrightarrow T \times \mathbf{2}$ is given by

$$\epsilon(U_t, x) = \begin{cases} (t, 0) & \text{if } x \in U_t \\ (t, 1) & \text{if } x \notin U_t \end{cases}$$

under this identification. To prove (c), suppose $x \in U_{px} \in \mathcal{O}(X_{px})$. Then by continuity of ϵ, there exists an open set $H \subseteq \bigcup_{t \in T} \mathcal{O}(X_t)$ and an open set $V \subseteq X$ such that $(U_{px}, x) \in H \times_T V$ and $\epsilon(H \times_T V) \subseteq T \times \{0\}$. One easily sees that H satisfies (i) and (iv). The proofs of (ii) and (iii) in [N3] make use of continuous maps $Z \longrightarrow (T \times \mathbf{2})^X$, for certain continuous maps $q: Z \longrightarrow T$. To see that this proof works for $\mathbf{Top}_0$, it suffices to show that the relevant spaces Z are T_0. But, they are T, $T \times \mathbf{2}$, and the spaces $\hat{A}$ defined as follows. Let A be any infinite set, and consider the space $\hat{A}$ whose points are finite subsets of A together with A itself, and open sets are $U \subseteq \hat{A}$ such that $A \notin U$, or $A \in U$ and $\uparrow F \subseteq U$, for some finite $F \subseteq A$. Now, T and $T \times \mathbf{2}$ are clearly T_0 since T is. To see that $\hat{A}$ is T_0, consider $F \neq G$ in $\hat{A}$. If both are finite subsets of A, then $U = \{F\}$ is an open subset of $\hat{A}$ containing F but not G. If only one is finite, say F, then $G = A$, and there exists $a \in A$ such that $a \notin F$. Then $\uparrow \{a\}$ is an open set of $\hat{A}$ containing G but not F, and so $\hat{A}$ is T_0, as desired. ■

Note that the proof of (b) $\Leftrightarrow$ (c) can be adapted to the category **Sob** of sober spaces since T, $T \times \mathbf{2}$, and the spaces $\hat{A}$ can easily seen to be sober when T is sober. For $\hat{A}$, one shows that the irreducible closed sets are of the form $\{A\}$ and $\{F\}$, where $F \subseteq A$ is finite. Thus, we see that if $p: X \longrightarrow T$ is exponentiable in **Sob**, then it is exponentiable in **Top**. With a further assumption on T, we can do even better [N4]. In particular, we need not assume that X is sober, only that $\tilde{p}: \widetilde{X} \longrightarrow T$ is exponentiable in **Sob** to get that $p: X \longrightarrow T$ is exponentiable in **Top**, where $\tilde{\ }: \mathbf{Top} \longrightarrow \mathbf{Sob}$ is the reflection of **Top** in **Sob**, i.e., the left adjoint to the inclusion $\mathbf{Sob} \longrightarrow \mathbf{Top}$.

Recall that T is called a *T_D-space* if points of T are locally closed. Note that for such a space, the inclusion $\{t\} \longrightarrow T$ is exponentiable in **Top**, for all $t \in T$, since locally closed inclusions are exponentiable [N3].

3.2. Corollary. *If T is a sober T_D-space and $\tilde{p}: \widetilde{X} \longrightarrow T$ is exponentiable in* **Sob**, *then $p: X \longrightarrow T$ is exponentiable in* **Top**.

Proof. By the above remarks, we know that $\tilde{p}: \widetilde{X} \longrightarrow T$ is exponentiable in **Top**, and hence satisfies (c) of Theorem 3.1. To see that p is exponentiable, we will show that $p: X \longrightarrow T$ also satisfies (c). Since the points of T are contained in those of $\widetilde{X}$ and $\mathcal{O}(X) \cong \mathcal{O}(\widetilde{X})$, it suffices to show that $\mathcal{O}(X_t) \cong \mathcal{O}(\widetilde{X}_t)$, for all $t \in T$. Note that $\tilde{\ }$ does not preserve pullbacks so the canonical map $\mathcal{O}(X_t) \longrightarrow \mathcal{O}(\widetilde{X}_t)$ need not be an isomorphism, in general. To see that it is, in this case, it suffices to show that the exponentials $(T \times \mathbf{2})^t$ in $\mathbf{Top}/T$ are sober spaces, for then we have natural isomorphisms

$$\begin{aligned} \mathbf{Top}/T(X_t, T \times \mathbf{2}) &\cong \mathbf{Top}/T(X \times_T t, T \times \mathbf{2}) \\ &\cong \mathbf{Top}/T(X, (T \times \mathbf{2})^t) \\ &\cong \mathbf{Top}/T(\widetilde{X}, (T \times \mathbf{2})^t) \\ &\cong \mathbf{Top}/T(\widetilde{X} \times_T t, T \times \mathbf{2}) \\ &\cong \mathbf{Top}/T(\widetilde{X}_t, T \times \mathbf{2}) \end{aligned}$$

and the desired result follows.

To see that $(T \times \mathbf{2})^t$ is sober, first note that it can be identified with $T \cup \{\star\}$, where $\star$ is an additional point over t. We will show that F is an irreducible closed subset of this space if (1) F is an irreducible closed subset of T and $\star \notin F$, or (2) $F = \mathrm{Cl}_T(t) \cup \{\star\}$, where Cl_T denotes the closure in T. In the first case, $F = \mathrm{Cl}_{T\cup\{\star\}}(u)$, where u is the unique point of T such that $F = \mathrm{Cl}_T(u)$. In the second case, $F = \mathrm{Cl}_{T\cup\{\star\}}(\star)$. Note that uniqueness is clear in both cases.

Using the description of the topology on $(T \times \mathbf{2})^t$ from the proof of Theorem 3.1, we see that closed sets F of $T \cup \{\star\}$ can be described as follows. If $\star \in F$, then $t \in F$, since the fibers of the complement of F must be Scott-open by (ii), and $F \setminus \{\star\}$ must be closed in T by (iii). If $\star \notin F$, then F must be closed in T, by (iii). Thus, closed sets F of $T \cup \{\star\}$ satisfy (1) F is closed in T and $\star \notin F$, or (2) $F = E \cup \{\star\}$, where E is closed in T. In the first case, F is clearly irreducible in $T \cup \{\star\}$ if and only if it is irreducible in T. In the second case, since $E \cup \{\star\} = E \cup (\mathrm{Cl}_T(t) \cup \{\star\})$, it will be irreducible if and only if $E = \mathrm{Cl}_T(t)$, and so F is of the desired form. ∎

In [N4], it was shown that the converse of this corollary also holds but the method of proof involved internal locales in the topos of sheaves on T, and so will be delayed until we consider locales in §5.

4. Posets and Spaces

In this section, we consider the extent to which the functor $(\ )^{\downarrow}\colon \mathbf{Pos} \longrightarrow \mathbf{Top}$ preserves and reflects exponentiability of morphisms. It is well-known that it preserves exponentiable objects. Indeed, every poset P is exponentiable, since $\mathbf{Pos}$ is cartesian closed, and $P^{\downarrow}$ is exponentiable in $\mathbf{Top}$ since it is easily seen to be locally compact.

4.1. Lemma. *If $p\colon P \longrightarrow B$ and $q\colon Q \longrightarrow B$ are in $\mathbf{Pos}$, then the induced map $(P \times_B Q)^{\downarrow} \longrightarrow P^{\downarrow} \times_{B^{\downarrow}} Q^{\downarrow}$ is an isomorphism of spaces.*

Proof. Since the map is clearly a bijection, it suffices to show that it is an open map. Suppose W is open in $(P \times_B Q)^{\downarrow}$. Then W is downward closed, and it easily following that $\downarrow x \ \times_{B^{\downarrow}} \downarrow y \subseteq W$, for all $(x, y) \in P^{\downarrow} \times_{B^{\downarrow}} Q^{\downarrow}$, and so W is open in $P^{\downarrow} \times_{B^{\downarrow}} Q^{\downarrow}$. ∎

Let X be a topological space and $x, y \in X$. Define $x \le y$, if $y \in U \Rightarrow x \in U$, for all U open in X. Then $\le$ is clearly reflexive and transitive, and it is antisymmetric if X is a T_0-space. Note that $\le$ is the opposite of the *specialization order* on X (in the sense of [J3]). Thus, X becomes a poset, which we denote by $X^{\bullet}$. Moreover, if $f\colon X \longrightarrow Z$, then $f^{\bullet}\colon X^{\bullet} \longrightarrow Z^{\bullet}$ is easily seen to be order-preserving.

4.2. Lemma. *This construction defines a functor $(\ )^{\bullet}\colon \mathbf{Top}_0 \longrightarrow \mathbf{Pos}$ which is a right inverse right adjoint to $(\ )^{\downarrow}\colon \mathbf{Pos} \longrightarrow \mathbf{Top}_0$.*

Proof. A straightforward calculation shows that $(P^{\downarrow})^{\bullet} = P$, for all posets P, and that the identity function $\epsilon\colon (X^{\bullet})^{\downarrow} \longrightarrow X$ is continuous, and so $(\ )^{\bullet}$ is right inverse right adjoint to $(\ )^{\downarrow}$ with counit ϵ and unit given by the identity function $\eta\colon P \longrightarrow (P^{\downarrow})^{\bullet}$. ∎

We shall see below that exponentiability of $p\colon P \longrightarrow B$ in $\mathbf{Pos}$ is necessary but not sufficient for the exponentiability of $p^{\downarrow}\colon P^{\downarrow} \longrightarrow B^{\downarrow}$ in $\mathbf{Top}$. An additional assumption of p will be needed.

4.3. Definition. *A poset morphism $p\colon P \longrightarrow B$ is called hereditarily compact if $\downarrow x \cap p^{-1}(\downarrow b)$ is compact in $P^{\downarrow}$, for all $x \in P$ and $b \le px$.*

4.4. Theorem. *The following are equivalent for $p\colon P \longrightarrow B$ in $\mathbf{Pos}$:*

(a) *$p\colon P \longrightarrow B$ is an interpolation-lifting map which is hereditarily compact.*

(b) *$p\colon P \longrightarrow B$ is exponentiable in $\mathbf{Pos}$ and hereditarily compact.*

(c) *$p^{\downarrow}\colon P^{\downarrow} \longrightarrow B^{\downarrow}$ is exponentiable in $\mathbf{Top}$.*

(d) *$p^{\downarrow}\colon P^{\downarrow} \longrightarrow B^{\downarrow}$ is exponentiable in $\mathbf{Top}_0$.*

PROOF. Since (a) $\Leftrightarrow$ (b) by Theorem 2.2 and (c) $\Leftrightarrow$ (d) by Theorem 3.1, it suffices to show that (d) $\Rightarrow$ (b) and (a) $\Rightarrow$ (c).

For (d) $\Rightarrow$ (b), suppose $p^{\downarrow}: P^{\downarrow} \longrightarrow B^{\downarrow}$ is exponentiable in $\mathbf{Top}_0$. Then

$$\begin{array}{rcl} \mathbf{Pos}/B\,(Q \times_B P, R) & \cong & \mathbf{Top}_0/B^{\downarrow}\left((Q \times_B P)^{\downarrow}, R^{\downarrow}\right) \\ & \cong & \mathbf{Top}_0/B^{\downarrow}\left(Q^{\downarrow} \times_{B^{\downarrow}} P^{\downarrow}, R^{\downarrow}\right) \\ & \cong & \mathbf{Top}_0/B^{\downarrow}\left(Q^{\downarrow}, (R^{\downarrow})^{P^{\downarrow}}\right) \\ & \cong & \mathbf{Pos}/(B^{\downarrow})^{\bullet}\left(Q, ((R^{\downarrow})^{P^{\downarrow}})^{\bullet}\right) \\ & \cong & \mathbf{Pos}/B\left(Q, ((R^{\downarrow})^{P^{\downarrow}})^{\bullet}\right) \end{array}$$

where all the bijections $\cong$ are natural, and the first one holds since $(\ )^{\downarrow}$ is full and faithful, the second by Lemma 4.1, the third by the exponentiability of $P^{\downarrow}$, the fourth since $(\ )^{\downarrow}$ is left adjoint to $(\ )^{\bullet}$, and the fifth since $(B^{\downarrow})^{\bullet} = B$. Therefore, $p: P \longrightarrow B$ is exponentiable in **Pos**.

To see that $p: P \longrightarrow B$ is hereditarily compact, suppose $b \leq px$ and consider

$$x \in ({\downarrow}x)_{px} \in \mathcal{O}(P^{\downarrow}_{px})$$

By Theorem 3.1, there exists $H \subseteq \bigcup_{t \in B^{\downarrow}} \mathcal{O}(P^{\downarrow}_t)$ such that

(i) $({\downarrow}x)_{px} \in H_{px}$

(ii) H_t is Scott-open, for all $t \in B^{\downarrow}$

(iii) $\{t \in B | V_t \in H_t\}$ is open in $B^{\downarrow}$, for all $V \in \mathcal{O}(X)$

(iv) $\bigcap H$ is a neighborhood of x in $P^{\downarrow}$

Since $({\downarrow}x)_{px} \in H_{px}$ and $b \leq px$, we know $({\downarrow}x)_b \in H_b$ by (iii). Since $({\downarrow}x)_b = \bigcup_{y \in ({\downarrow}x)_b} ({\downarrow}y)_b$, applying (ii) we see that

$$({\downarrow}y_1)_b \cup \ldots \cup ({\downarrow}y_n)_b \in H_b$$

for some $y_1, \ldots, y_n \in ({\downarrow}x)_b$. We claim that

$${\downarrow}x \cap p^{-1}({\downarrow}b) = ({\downarrow}y_1 \cup \ldots \cup {\downarrow}y_n) \cap p^{-1}({\downarrow}b)$$

which is compact in $P^{\downarrow}$, since $y_1, \ldots, y_n \in p^{-1}({\downarrow}b)$. Since $y_1, \ldots, y_n \leq x$, we know that

$$({\downarrow}y_1 \cup \ldots \cup {\downarrow}y_n) \cap p^{-1}({\downarrow}b) \subseteq {\downarrow}x \cap p^{-1}({\downarrow}b)$$

Since $({\downarrow}y_1 \cup \ldots \cup {\downarrow}y_n)_b \in H_b$, and open sets are downward closed in $B^{\downarrow}$, applying (iii) we see that $({\downarrow}y_1 \cup \ldots \cup {\downarrow}y_n)_t \in H_t$, for all $t \leq b$ in B. Since $\bigcap H$ is a neighborhood of x in $P^{\downarrow}$ by (iv), we know that ${\downarrow}x \subseteq \bigcap H$, and so $({\downarrow}x)_t \subseteq (\bigcap H)_t \subseteq ({\downarrow}y_1 \cup \ldots \cup {\downarrow}y_n)_t$, for all $t \leq b$. Therefore, ${\downarrow}x \cap p^{-1}({\downarrow}b) \subseteq ({\downarrow}y_1 \cup \ldots \cup {\downarrow}y_n) \cap p^{-1}({\downarrow}b)$, as desired.

For (a) $\Rightarrow$ (c), suppose $p: P \longrightarrow B$ is an interpolation-lifting map which is hereditarily compact. To show that the map $p^\downarrow: P^\downarrow \longrightarrow B^\downarrow$ is exponentiable in **Top**, we will show that it satisfies Theorem 3.1(c). Let $x \in U_{px} \in \mathcal{O}(P^\downarrow_{px})$, and consider

$$H = \begin{cases} \{W_t \in \mathcal{O}(P^\downarrow_t) | (\downarrow x)_t \subseteq W_t\} & \text{if } t \leq px \\ \emptyset & \text{otherwise} \end{cases}$$

Then clearly H satisfies (i) as well as the upward closure part of (ii), and $\bigcap H$ is a neighborhood of x in $P^\downarrow$, since $\downarrow x \cap p^{-1}(\downarrow px) \subseteq \bigcap H$ by definition of H.

To see that H satisfies the finite union part of (ii), suppose $\bigcup_{\alpha \in A} (U_\alpha)_t \in H_t$, for some $t \in B$. Then $t \leq px$ and $(\downarrow x)_t \subseteq \bigcup_{\alpha \in A} (U_\alpha)_t$. We claim that $\downarrow x \cap p^{-1}(\downarrow t) \subseteq \bigcup_{\alpha \in A} U_\alpha \cap p^{-1}(\downarrow t)$. Suppose $y \in \downarrow x \cap p^{-1}(\downarrow t)$. Then since p is an interpolation-lifting map, $y \leq x$, and $py \leq t \leq px$, there exists $z \in P_t$ such that $y \leq z \leq x$. Since $z \in (\downarrow x)_t$, we know $z \in \bigcup_{\alpha \in A} (U_\alpha)_t$, and since $\bigcup_{\alpha \in A} U_\alpha$ is downward closed, it follows that $y \in \bigcup_{\alpha \in A} U_\alpha \cap p^{-1}(\downarrow t)$, as desired. Now, since $\downarrow x \cap p^{-1}(\downarrow t)$ is compact, it follows that $\downarrow x \cap p^{-1}(\downarrow t) \subseteq \bigcup_{\alpha \in F} U_\alpha \cap p^{-1}(\downarrow t)$, for some finite $F \subseteq A$, and so $\bigcup_{\alpha \in F} (U_\alpha)_t \in H_t$, as desired.

It remains to show that (iii) holds. Consider $G = \{t \in B | V_t \in H_t\}$, where V is open in $P^\downarrow$. To show that G is open, suppose $t \in G$ and $u \leq t$. Then $V_t \in H_t$, and so $t \leq px$ and $(\downarrow x)_t \subseteq V_t$. We claim that $(\downarrow x)_u \subseteq V_u$. Let $y \in (\downarrow x)_u$. Then $y \leq x$ and $py = u \leq t \leq px$, and so, since p is an interpolation-lifting map, we know $t = pz$, were $y \leq z \leq x$. Then $z \in V$ (since $z \in (\downarrow x)_t \subseteq V_t$) and so $y \in V$ (since V is open in $P^\downarrow$). Therefore, $(\downarrow x)_u \subseteq V_u$, and it follows that $u \in G$, making G an open set, to complete the proof. ∎

We conclude this section with an example of an exponentiable morphism of **Pos** whose corresponding continuous map is not exponentiable in **Top**.

4.5. Example. *Let $B = \mathbf{2}$ and $P = \{x, y_1, y_2, \ldots\}$, where $y_i \leq x$, for all i, and there are no other comparable elements, and define $p: P \longrightarrow B$ by $px = 1$ and $py_i = 0$, for all i. Then p is exponentiable in* **Pos**, *but $p^\downarrow: P^\downarrow \longrightarrow B^\downarrow$ is not exponentiable in* **Top**.

Proof. Clearly, p is an interpolation-lifting map (it is a discrete opfibration) but $\downarrow x \cap p^{-1}(\downarrow 0) = \{y_1, y_2, \ldots\}$ is not compact. Therefore, $p^\downarrow: P^\downarrow \longrightarrow B^\downarrow$ is not exponentiable in **Top**, by Theorem 4.4. ∎

5. Posets and Locales

The main purpose of this section is to consider the extent to which

$$\mathbf{Pos} \xrightarrow{(\ )^\downarrow} \mathbf{Top} \xrightarrow{\mathcal{O}} \mathbf{Loc}$$

preserves and reflects exponentiable morphisms. As with $(\ )^\downarrow$, it preserves exponentiable objects, since $P^\downarrow$ is locally compact, for every poset P.

In [N4], it was shown that if T is a sober T_D-space and $p: X \longrightarrow T$ is in **Top**, then the conditions of Theorem 3.1 are equivalent to the exponentiability of $\tilde{p}: \widetilde{X} \longrightarrow T$ in **Sob**, as well as the corresponding morphism $\mathcal{O}(p): \mathcal{O}(X) \longrightarrow \mathcal{O}(T)$ in **Loc**. The latter is obtained using the equivalence (described in [JT]) of $\mathbf{Loc}/\mathcal{O}(T)$ with the category $\mathbf{Loc}(\mathrm{Sh}(T))$ of internal locales in $\mathrm{Sh}(T)$.

Now, $B^{\downarrow}$ is a T_D-space, for every poset B, and so if we assume that it is sober, then we can combine the results of Theorem 4.4 with this extended version of Theorem 3.1 to determine the relationship between exponentiability in **Pos** and **Loc**. Although this combination requires no further justification, we shall recall part of the approach in [N4] and present a simpler proof of Theorem 5.2 (a) $\Rightarrow$ (e) in the posetal case below.

The following two results appeared in [N4]. We omit the proof of the first as it would be identical to that of [N4] but we include the second as it is a slight variation.

5.1. Lemma. *If $p: X \longrightarrow T$ and $q: Y \longrightarrow T$ are morphisms of sober spaces such that $\mathcal{O}(p): \mathcal{O}(X) \longrightarrow \mathcal{O}(T)$ is exponentiable in* **Loc**, *then the induced morphism $\mathcal{O}(Y \times_T X) \longrightarrow \mathcal{O}(Y) \times_{\mathcal{O}(T)} \mathcal{O}(X)$ is an isomorphism in* **Loc**.

5.2. Proposition. *If $p: X \longrightarrow T$ is a morphism of sober spaces such that $\mathcal{O}(p): \mathcal{O}(X) \longrightarrow \mathcal{O}(T)$ is exponentiable in* **Loc**, *then $p: X \longrightarrow T$ is exponentiable in* **Sob**.

Proof. Suppose that $q: Y \longrightarrow T$ and $r: Z \longrightarrow T$ are morphisms of sober spaces. Then we have natural bijections

$$\begin{aligned}
\mathbf{Sob}/T\Big(Y \times_T X, Z\Big) &\cong \mathbf{Loc}/\mathcal{O}(T)\Big(\mathcal{O}(Y \times_T X), \mathcal{O}(Z)\Big)\\
&\cong \mathbf{Loc}/\mathcal{O}(T)\Big(\mathcal{O}(Y) \times_{\mathcal{O}(T)} \mathcal{O}(X), \mathcal{O}(Z)\Big)\\
&\cong \mathbf{Loc}/\mathcal{O}(T)\Big(\mathcal{O}(Y), \mathcal{O}(Z)^{\mathcal{O}(X)}\Big)\\
&\cong \mathbf{Sob}/T\Big(Y, pt(\mathcal{O}(Z)^{\mathcal{O}(X)})\Big)
\end{aligned}$$

where pt is the right adjoint to $\mathcal{O}: \mathbf{Sob} \longrightarrow \mathbf{Loc}$. ∎

Recall that if $p: \mathcal{E} \longrightarrow \mathcal{B}$ is a localic geometric morphism of toposes, then $p_*(\Omega)$ is an internal locale in $\mathcal{B}$ and $\mathcal{E}$ is equivalent to the topos of sheaves on the locale $p_*(\Omega)$. In particular, if $p: X \longrightarrow T$ is a continuous map of spaces and $p: \mathrm{Sh}(X) \longrightarrow \mathrm{Sh}(T)$ is the corresponding geometric morphism, then $p_*(\Omega_X)$ is the internal locale in $\mathrm{Sh}(T)$ whose value at an open set G of T is the locale $\mathcal{O}(p^{-1}(G))$, and for $G' \subseteq G$, the restriction map $\mathcal{O}(p^{-1}(G)) \longrightarrow \mathcal{O}(p^{-1}(G'))$ is given by $V|_{G'} = V \cap p^{-1}(G')$ (see [JT] or [J2]).

Following [J2] and [J4], local compactness in $\mathrm{Sh}(T)$ can by described as follows. Recall that for an ideal I of $p_*(\Omega_X)$ defined over an open set G of T, we have

$$\bigvee I = \bigcup \{U | U \in I(G'), \text{ for some } G' \subseteq G\}$$

Thus, to show that $p_*(\Omega_X)$ is locally compact it suffices to show that for all $x \in V \in \mathcal{O}(X)$, there exist open neighborhoods U and G of x and px, respectively, such that $G \subseteq [\![U << V]\!]$, where $[\![\phi]\!]$ denotes the truth-value of the formula ϕ in the topos $\mathrm{Sh}(T)$. To show that $G \subseteq [\![U << V]\!]$, it suffices to show that for all ideals I defined over an open set $G' \subseteq G$ of T,

$$G' \subseteq [\![V \subseteq \bigvee I]\!] \Rightarrow G' \subseteq [\![U \in I]\!]$$

or equivalently

$$V \cap p^{-1}(G') \subseteq \bigvee I \;\Rightarrow\; U \cap p^{-1}(G') \in I(G')$$

5.3. Theorem. *The following are equivalent for $p: P \longrightarrow B$ in* **Pos** *such that $B^{\downarrow}$ is a sober space:*

(a) *$p: P \longrightarrow B$ is an interpolation-lifting map which is hereditarily compact.*

(b) *$p: P \longrightarrow B$ is exponentiable in* **Pos** *and hereditarily compact.*

(c) *$p^{\downarrow}: P^{\downarrow} \longrightarrow B^{\downarrow}$ is exponentiable in* **Top**.

(d) *$p^{\downarrow}: P^{\downarrow} \longrightarrow B^{\downarrow}$ is exponentiable in* $\mathbf{Top}_0$.

(e) *$p_*(\Omega_{P^{\downarrow}})$ is locally compact in* $\mathbf{Loc}(\mathrm{Sh}(B^{\downarrow}))$.

(f) *$p_*(\Omega_{P^{\downarrow}})$ is exponentiable in* $\mathbf{Loc}(\mathrm{Sh}(B^{\downarrow}))$.

(g) *$\mathcal{O}(p^{\downarrow}): \mathcal{O}(P^{\downarrow}) \longrightarrow \mathcal{O}(B^{\downarrow})$ is exponentiable in* **Loc**.

(h) *$\widetilde{p^{\downarrow}}: \widetilde{P^{\downarrow}} \longrightarrow B^{\downarrow}$ is exponentiable in* **Sob**.

Proof. Note that (a) – (d) are equivalent by Theorem 4.4, (e) $\Rightarrow$ (f) by Hyland's theorem [H], (f) $\Rightarrow$ (g) follows from $\mathbf{Loc}(\mathrm{Sh}(B^{\downarrow})) \simeq \mathbf{Loc}/\mathcal{O}(B^{\downarrow})$, (g) $\Rightarrow$ (h) is a consequence of applying Proposition 5.2 to $\widetilde{p^{\downarrow}}: \widetilde{P^{\downarrow}} \longrightarrow B^{\downarrow}$, and (h) $\Rightarrow$ (c) holds by Corollary 3.2. Thus, it suffices to prove (a) $\Rightarrow$ (e).

Suppose $p: P \longrightarrow B$ satisfies (a). Then by the above description of local compactness, it suffices to show that if $x \in V \in \mathcal{O}(P^{\downarrow})$, then

$$\downarrow px \subseteq [\![\downarrow x << V]\!]$$

So, suppose that $V \cap p^{-1}(G) \subseteq \bigvee I$, for some ideal I defined over an open set $G \subseteq \downarrow px$. To see that $\downarrow x \cap p^{-1}(G) \in I(G)$, it suffices to show that $\downarrow x \cap p^{-1}(\downarrow b) \in I(\downarrow b)$, for all $b \in G$.

Since $\downarrow x \cap p^{-1}(\downarrow b)$ is compact in $P^{\downarrow}$, we know it is of the form $\downarrow y_1 \cup \ldots \cup \downarrow y_n$, for some $y_1, \ldots, y_n \in \downarrow x \cap p^{-1}(\downarrow b)$. We claim that we can assume $py_i = b$, for all i. Since $p: P \longrightarrow B$ is an interpolation-lifting map and $py_i \leq b \leq px$, we know there exists $z_i \in P$ such that $pz_i = b$ and $y_i \leq z_i \leq x$, for all i, and so replacing x_i by z_i gives the desired result.

Since $x \in V \subseteq \bigvee I = \bigcup\{U | U \in I(G'), \text{ for some } G' \subseteq \downarrow b\}$ and $y_i \leq x$, we know $y_i \in U_i$, for some $G'_i \subseteq \downarrow b$ and $U_i \in I(G'_i)$. But, then $b \in G'_i$ since $y_i \in U_i$ and $py_i = b$, and so $G'_i = \downarrow b$. Since I is an ideal, it follows that $\downarrow y_1 \cup \ldots \cup \downarrow y_n \in I(\downarrow b)$, and so $\downarrow x \cap p^{-1}(\downarrow b) \in I(\downarrow b)$, as desired. Therefore, $\downarrow px \subseteq [\![\downarrow x << V]\!]$, and so $p_*(\Omega_{P^{\downarrow}})$ is locally compact in $\mathbf{Loc}(\mathrm{Sh}(B^{\downarrow}))$. ∎

6. Posets and Toposes

In this section, we consider the extent to which the presheaf functor

$$\mathrm{PSh}\colon \mathbf{Pos} \longrightarrow \mathbf{GTop}$$

preserves and reflects exponentiable morphisms. Although we are able to give only a partial answer, we shall use these results to fully answer the analogous question for the presheaf functor defined on **Cat**.

If P is a poset, then it is not difficult to show that $\mathrm{PSh}(P)$ is a localic topos whose subobjects of 1 correspond to downward closed subsets of P, and so it follows that $\mathrm{PSh}(P)$ is equivalent to the topos $\mathrm{Sh}(P^{\downarrow})$ of sheaves on $P^{\downarrow}$. Moreover, if $p\colon P \longrightarrow B$ is a morphism of posets, then the geometric morphism $\mathrm{PSh}(P) \longrightarrow \mathrm{PSh}(B)$ is identified with the geometric morphism $\mathrm{Sh}(P^{\downarrow}) \longrightarrow \mathrm{Sh}(B^{\downarrow})$ via this equivalence. Since the latter is a localic geometric morphisms, we are led to consider exponentiability of localic toposes.

In [JJ], Johnstone and Joyal showed that a localic topos $\mathrm{Sh}(A)$ is exponentiable if and only if A is metastably locally compact, a condition that implies (but is stronger than) the local compactness of A. They also showed that every stably locally compact locale (in the sense of [J3]) satisfies this property. Note that although these results were proved for Grothendieck toposes, the authors remark that they hold for bounded toposes over any topos with a natural number object, and hence, they apply to $\mathbf{GTop}/\mathrm{Sh}(T)$, for any space T.

Using this description of exponentiable geometric morphisms, we will see, in the following example, that the presheaf functor $\mathbf{Cat} \longrightarrow \mathbf{GTop}$ does not reflect exponentiable morphisms.

6.1. Example. *Let $P = \mathbf{2} \times \mathbf{2}$, $B = \mathbf{3}$, and $p\colon P \longrightarrow B$ be given by $p(x, y) = x + y$. Then the induced geometric morphism $\mathrm{PSh}(P) \longrightarrow \mathrm{PSh}(B)$ is exponentiable in* **GTop** *but p is not exponentiable in* **Cat**.

Proof. First, p is not exponentiable in **Cat** since it does not satisfy the connectedness condition in the Giraud-Conduché lifting property. From the proof of (h) $\Rightarrow$ (a) in Theorem 5.3, wee see that $\downarrow x << \downarrow x$ in $p_*(\Omega_{P\downarrow}|_{\downarrow px})$, for all $x \in P$, and it follows that $p_*(\Omega_{P\downarrow})$ is stably locally compact in $\mathrm{Sh}(T)$, and so $\mathrm{PSh}(P) \longrightarrow \mathrm{PSh}(T)$ is exponentiable in **GTop**. ∎

6.2. Theorem. *Suppose $B^{\downarrow}$ is a sober space and $p\colon P \longrightarrow B$ is a morphism of posets such that the geometric morphism $p\colon \mathrm{PSh}(P) \longrightarrow \mathrm{PSh}(B)$ is exponentiable in* **GTop**. *Then $p\colon P \longrightarrow B$ satisfies the equivalent conditions of Theorem 5.3. In particular, $p\colon P \longrightarrow B$ is exponentiable in* **Pos** *and hereditarily compact.*

Proof. By the above remarks, we know that $p_*(\Omega_{P\downarrow})$ is locally compact in $\mathbf{Loc}(\mathrm{Sh}(B^{\downarrow}))$, and the desired result follows. ∎

6.3. Corollary. *The presheaf functor* $\mathbf{Pos} \longrightarrow \mathbf{GTop}$ *reflects exponentiable morphisms.*

6.4. Corollary. *There is a discrete opfibration $p\colon \mathbf{C} \longrightarrow \mathbf{B}$ such that the geometric morphism $p\colon \mathrm{PSh}(\mathbf{C}) \longrightarrow \mathrm{PSh}(\mathbf{B})$ is not exponentiable in* **GTop**.

PROOF. The poset morphism $p: P \longrightarrow B$ of Example 4.5 is clearly a discrete opfibration when considered as a morphism in **Cat**. The corresponding geometric morphism of presheaf toposes is not exponentiable in **GTop** since p is not hereditarily compact, as $\downarrow x \cap p^{-1}(\downarrow b) = \{y_1, y_2, \ldots\}$, when $b = 0$. ■

6.5. COROLLARY. *The presheaf functors* **Pos** $\longrightarrow$ **GTop** *and* **Cat** $\longrightarrow$ **GTop** *do not preserve exponentiable morphisms.*

PROOF. The poset morphism $p: P \longrightarrow B$ of Example 4.5 is exponentiable in **Pos** and **Cat** but, as seen in the proof of Corollary 6.4, the corresponding geometric morphism of presheaf toposes is not exponentiable in **GTop**. ■

It is well-known that the presheaf functor does not preserve pullbacks, since as noted in the introduction, it does not preserve equalizers [J1]. However, in the posetal case, we can use the equivalence of $\mathrm{PSh}(P)$ with $\mathrm{Sh}(P^{\downarrow})$ to get the following version of Lemma 5.1 for presheaf toposes.

6.6. COROLLARY. *Suppose $p: P \longrightarrow B$ is a morphism of posets such that $B^{\downarrow}$ is a sober space. If the geometric morphism $p: \mathrm{PSh}(P) \longrightarrow \mathrm{PSh}(B)$ is exponentiable in* **GTop**, *then the induced geometric morphism*

$$\mathrm{PSh}(Q \times_B P) \longrightarrow \mathrm{PSh}(Q) \times_{\mathrm{PSh}(B)} \mathrm{PSh}(P)$$

is an equivalence, for all $q: Q \longrightarrow B$ in **Pos**.

PROOF. It suffices to show that the geometric morphism

$$\mathrm{Sh}(Q^{\downarrow} \times_{B^{\downarrow}} P^{\downarrow}) \longrightarrow \mathrm{Sh}(Q^{\downarrow}) \times_{\mathrm{Sh}(B^{\downarrow})} \mathrm{Sh}(P^{\downarrow})$$

is an equivalence. Since the functor Sh: **Loc** $\longrightarrow$ **GTop** preserves pullbacks [JT], it suffices to show that the induced morphism

$$\mathcal{O}(Q^{\downarrow} \times_{B^{\downarrow}} P^{\downarrow}) \longrightarrow \mathcal{O}(Q^{\downarrow}) \times_{\mathcal{O}(B^{\downarrow})} \mathcal{O}(P^{\downarrow})$$

is an isomorphism in **Loc**. But, we know that $\mathcal{O}(p^{\downarrow}): \mathcal{O}(P^{\downarrow}) \longrightarrow \mathcal{O}(B^{\downarrow})$ is exponentiable in **Loc**, by Theorem 6.2, and so the desired result follows from Lemma 5.1. ■

We conclude with an example showing that the additional compactness condition on an exponentiable morphism $p: P \longrightarrow B$ of **Pos** is not sufficient to ensure the exponentiability in **GTop** of the corresponding geometric morphism of presheaf toposes. In particular, we give an example of a morphism $p: P \longrightarrow B$ such that $p_*(\Omega_{P^{\downarrow}})$ is locally compact but not metastably locally compact in **Loc**$(\mathrm{Sh}(B^{\downarrow}))$.

6.7. EXAMPLE. *Let $B = \mathbf{2}$ and $P = \{x, z_1, z_2, y_1, y_2, \ldots\}$, where $y_i \leq z_1 \leq x$ and $y_i \leq z_2 \leq x$, for all i, and there are no other comparable pairs, and define $p: P \longrightarrow B$ by $px = 1$, $pz_1 = pz_2 = 0$, and $py_i = 0$, for all i. Then $p_*(\Omega_{P^{\downarrow}})$ is locally compact but not metastably locally compact in* **Loc**$(\mathrm{Sh}(B^{\downarrow}))$.

PROOF. It is not difficult to show that $p_*(\Omega_{P^{\downarrow}})$ satisfies Theorem 5.3 (a), and so it is locally compact. To show that it is not metastably locally compact one shows that $[\![\downarrow x <<< \downarrow x]\!] \neq B$ using Proposition 5.11 of [JJ] and showing that the cover $\downarrow z_1 \cup \downarrow z_2$ defined over $\{0\}$ does not satisfy $\downarrow z_1 \cap \downarrow z_2 <<< \downarrow z_1 \cap \downarrow z_2$. ■

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