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A SHEAF-THEORETIC VIEW OF LOOP SPACES

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Abstract. The context of enriched sheaf theory introduced in the author's thesis provides a convenient viewpoint for models of the stable homotopy category as well as categories of finite loop spaces. Also, the languages of algebraic geometry and algebraic topology have been interacting quite heavily in recent years, primarily due to the work of Voevodsky and that of Hopkins. Thus, the language of Grothendieck topologies is becoming a necessary tool for the algebraic topologist. The current article is intended to give a somewhat relaxed introduction to this language of sheaves in a topological context, using familiar examples such as n-fold loop spaces and pointed G-spaces. This language also includes the diagram categories of spectra from [19] as well as spectra in the sense of [17], which will be discussed in some detail.

1. Introduction

An excellent description of the formalism of enriched sheaf theory under certain technical assumptions on the base category is given in [5]. Unfortunately, the category of topological spaces (which will refer to compactly generated, weak Hausdorff spaces with basepoints) does not satisfy these technical assumptions. Specifically, limits and colimits do not interact particularly well in topological spaces. Thus, the author was forced to develop a notion of enriched sheaves of topological spaces in [15]. The main purpose was to pin down Peter May's suggestion that infinite loop spaces might be viewed as something akin to sheaves of spaces. In fact, the technical details of this statement will be one of the main results of the current article.

Both the notion of a stack and a simplicial sheaf involve the notion of a site or a Grothendieck topology. The importance of both stacks and simplicial sheaves alone should justify some attempt to give a gradual introduction to the language of sheaves, intended for topologists. Thus, using examples familiar to topologists, such as categories of n-fold loop spaces and pointed G-spaces, the basic definitions will be discussed.

The work of [19] deals with diagram categories of spectra. However, diagram categories will be shown to be a particularly nice type of sheaf category. Thus, their work as well as the viewpoint of [17] is unified with that of n-fold loop spaces and pointed G-spaces in the language of sheaf categories. Unfortunately, there seems to be significant uncertainty about the "fancy new models" for the stable homotopy category such as symmetric or orthogonal spectra, (see [13] or [19]). The last section of this article will address the technical differences between these models.

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The article is organized as follows: section 2 recalls some basic information about topological diagram categories, section 3 introduces the notions of Grothendieck topology and Grothendieck basis with a number of motivating examples. Section 4 defines the notion of sheaf on either a topology or a basis and culminates with the proof in our cases that sheaves on a basis agree with sheaves on the topology generated by the basis. The focus shifts to stable homotopy theory in section 5, where the example of spectra in the sense of [17] is shown to be a category of sheaves in the usual category of prespectra. Finally, in section 6 the different models for the stable homotopy category are discussed, with a comparison of their technical properties.

2. Topological Diagrams

The language of enriched categories, while intimidating, becomes fairly familiar in topological contexts. Essentially, the question becomes one of considering topologies on morphism sets so that standard maps become continuous.

Once again, (pointed) topological spaces will mean τ_*, the category of compactly generated, weak Hausdorff spaces with basepoints. This category is closed symmetric monoidal under the compactly generated smash product.

2.1. Definition. *The category $\mathcal{C}$ is a (pointed)* **topological category** *provided the morphism sets are equipped with topologies which make them topological spaces such that the composition maps $\mathcal{C}(B, C) \wedge \mathcal{C}(A, B) \to \mathcal{C}(A, C)$ are continuous in these topologies.*

2.2. Example.

1. The most basic example of a topological category will be the category of pointed spaces τ_*, with the Kelley functor applied to the compact-open topologies on mapping spaces.

2. Any standard category may be made topological by giving the morphism spaces discrete topologies and adding disjoint basepoints.

3. A full subcategory of a topological category remains topological, although the question is a bit more delicate for subcategories where only a subset of maps are considered. This follows from the fact that subspaces may not be compactly generated (see Warning 3.1).

4. The category of pointed G-spaces for a compact Lie group G, as discussed in Example 2.8.2.

5. The category of n-fold loops spaces and n-fold loop maps, as indicated in Example 4.5.1.

All of the examples discussed throughout this article will be (pointed) topological categories. The general viewpoint is that τ_* has become the "ground ring" in the algebraic sense. Further discussion of this type of abstract viewpoint can be found in [16].

2.3. Definition. *A functor between topological categories is a* **topological functor** *provided the induced maps of morphism spaces $\mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$ are continuous.*

Similarly, natural transformations will be defined by assuming the usual commutative squares on morphism spaces consist of continuous maps.

2.4. Warning. In general this disagrees with the usual meaning in an enriched category, however these two notions will agree for all of the examples considered in this article.

A pair of topological functors F and G are topological adjoints provided the usual bijection of morphism sets extends to a homeomorphism of spaces. That is, $\mathcal{C}(X, G(Y))$ must be naturally homeomorphic to $\mathcal{D}(F(X), Y)$ rather than assuming a natural bijection as would be the case for a standard adjunction. In particular, topological adjoints are standard adjoints.

This leads one to wonder how much more structure a standard adjoint pair must preserve in order to qualify as a topological adjoint pair. The most convenient answer is given in section II.6 of [3] and involves so-called tensors and cotensors. The statement is that a standard adjoint pair is a topological adjoint pair if the left adjoint preserves tensors, or equivalently, if the right adjoint preserves cotensors.

A topological category $\mathcal{C}$ is tensored over τ_* provided there exist natural objects $X \otimes M \in \mathcal{C}$ for each $X \in \mathcal{C}$ and $M \in \tau_*$ with the property that the space $\mathcal{C}(X \otimes M, Y)$ is naturally homeomorphic to $\tau_*(M, \mathcal{C}(X, Y))$ for arbitrary $Y \in \mathcal{C}$. Another way to describe the situation is that $(X \otimes ?, \mathcal{C}(X, ?))$ becomes a topological adjoint pair between τ_* and $\mathcal{C}$, which is also natural in the argument X.

There is a dual notion of cotensor, or hom object, which requires natural homeomorphisms $\tau_*(M, \mathcal{C}(X, Y)) \approx \mathcal{C}(X, \hom(M, Y))$ for any $M \in \tau_*$ and $X, Y \in \mathcal{C}$. Notice in particular that, for $M \in \tau_*$, the functors $? \otimes M$ and $\hom(M, ?)$ form a topological adjoint pair between $\mathcal{C}$ and itself whenever $\mathcal{C}$ is both tensored and cotensored over τ_*.

2.5. Example.

1. The category τ_* is tensored and cotensored over itself, where the tensor is defined as the compactly-generated smash product and the cotensor is the compactly-generated mapping space.

2. The category of loop spaces is cotensored over spaces, with $\hom(M, Y)$ again defined as the compactly-generated mapping space. This is a loop space because of the exponential law for τ_*. In fact, this category is also tensored by later results, in a much more complicated way.

3. The usual notion of loops on a spectrum is simply $\hom(S^1, Y)$.

Next is the main definition of this section.

2.6. Definition. *Suppose $\mathcal{C}$ is a small topological category. Then the category of* **diagrams indexed by** $\mathcal{C}$ *will denote the category of topological functors from $\mathcal{C}$ to τ_*, with topological natural transformations as morphisms.*

2.7. Remark. Small in this context refers to the object class of $\mathcal{C}$ being a set.

2.8. Example.

1. Choose $\mathcal{C}_1$ to be the category with one object, whose morphism space is S^0. Then the category of diagrams indexed by $\mathcal{C}_1$ is τ_* itself. To see this, notice a topological functor in this case consists of a space together with a continuous map $S^0 \to \tau_*(X, X)$. However, since this must be a pointed map and functors must preserve identities the map must be the adjoint of the identity $X \wedge S^0 \approx X \to X$.

2. More generally, suppose G is a topological group and let $\mathcal{G}$ denote the category with one object whose morphism space is G_+. (Here G_+ denotes G with a disjoint basepoint added.) The composition law should be given by the group multiplication via $G_+ \wedge G_+ \approx (G \times G)_+ \to G_+$. Then the category of diagrams indexed by $\mathcal{G}$ is the category of pointed left G-spaces. The category of diagrams indexed on the opposite of $\mathcal{G}$ is the category of pointed right G-spaces.

3. Let $\mathcal{E}$ be the topological category whose objects are the nonnegative integers so that $\mathcal{E}(n, n) = (\Sigma_n)_+$ and all other morphism spaces are the basepoint. The composition law $\mathcal{E}(n, n) \wedge \mathcal{E}(n, n) \to \mathcal{E}(n, n)$ is given by the multiplication in Σ_n. Then the category of diagrams indexed by $\mathcal{E}$ consists of sequences of spaces $\{X_n\}$ such that X_n is a pointed Σ_n-space and morphisms are sequences of maps that are Σ_n-equivariant at the nth entry.

4. Let $\mathcal{A}$ be the topological category whose objects are the nonnegative integers so that
$$\mathcal{A}(m, n) = \overbrace{S^1 \wedge \cdots \wedge S^1}^{m-n}$$
with composition given by associativity, unit or zero maps (if $m < n$, S^{m-n} is a basepoint, while S^0 is intended for m=n). Then the category of diagrams indexed on the opposite of $\mathcal{A}$ is called prespectra (on the canonical indexing sequence) in [17] or often naive spectra in other sources. To see this, consider the pointed continuous map $\overbrace{S^1 \wedge \cdots \wedge S^1}^{m-n} \to \tau_*(X_n, X_m)$ which is adjoint to the usual structure map $\Sigma^{m-n}X_n \to X_m$.

5. Let Σ be a topological category with objects the nonnegative integers. Clearly, there is an action of the symmetric group Σ_{m-n} on $\mathcal{A}(m, n)$ by permuting smash factors. This extends naturally to an action of the larger group Σ_m on $(\Sigma_m)_+ \wedge_{\Sigma_{m-n}} S^{m-n}$. One defines the spaces $\Sigma(m, n)$ to be $(\Sigma_m)_+ \wedge_{\Sigma_{m-n}} S^{m-n}$. The associativity and unit maps which induce the composition law in $\mathcal{A}$ extend by naturality of these extended actions to make Σ a small topological category. The category of diagrams indexed on the opposite of Σ is then the "intersection" of the previous two examples. That

is, objects of this new diagram category are prespectra such that the n-th entry has a Σ_n action, while the structure maps become equivariant in an appropriate sense. This category is called symmetric spectra and is discussed in the later portions of [13] (the early portions are devoted to the simplicial version, which is quite similar) or in [19]. The main technical point is that symmetric spectra form a closed symmetric monoidal category, i.e. carry a "good" smash product on the point-set level.

6. This is the "coordinate-free" version of the previous example. Let $\mathcal{U}$ denote a countable-dimensional real inner-product space. Let $\mathcal{O}$ denote a topological category whose objects are the finite linear subspaces of $\mathcal{U}$. Given objects V and W, define S^{W-V} to be the one-point compactification of $W \perp V$, the orthogonal complement of V in W. Note $W \perp V$ is defined to be empty (hence S^{W-V} is the basepoint) if $V \nsubseteq W$ and contains only zero (hence S^{W-V} is S^0) if $W = V$. Clearly, the orthogonal group $\mathbb{O}(W-V)$ acts naturally on S^{W-V}. However, as in the previous example one defines $\mathcal{O}(W,V)$ to be $\mathbb{O}(W)_+ \wedge_{\mathbb{O}(W-V)} S^{W-V}$ with composition law given by extension of the natural associativity, unit or zero maps $S^{W-W'} \wedge S^{W'-V} \to S^{W-V}$. Thus, $\mathcal{O}$ becomes a small topological category and the category of diagrams indexed on the opposite of $\mathcal{O}$ is referred to as orthogonal spectra in [19]. As with symmetric spectra, the category of orthogonal spectra carries a closed symmetric monoidal structure (or "good" smash product). The increased complexity of the (non-finite) orthogonal groups is justified by the coordinate-free nature of this construction as well as the fact that stable weak equivalences will be unambiguously defined (unlike in the case of symmetric spectra).

Given a topological functor $F : \mathcal{C} \to \mathcal{D}$, precomposition defines a topological functor from the category of diagrams indexed on $\mathcal{D}$ to that indexed on $\mathcal{C}$. In fact, this precomposition functor will have both a left and a right topological adjoint by the usual Kan extension formulae.

For each nonnegative integer n, there is an obvious topological functor from $\mathcal{C}_1$ (of Example 1) to $\mathcal{A}$ (of Example 4) sending the unique object of $\mathcal{C}_1$ to n and acting as the identity on morphism spaces. The comments above imply that precomposition by this functor will have both a left and a right topological adjoint. However, consideration of Examples 1 and 4 implies this precomposition functor is isomorphic to the evaluation at the nth entry of a prespectrum. Hence, one has both left and right topological adjoints to evaluation, which implies both limits and colimits are defined entrywise in prespectra. This case is general in the sense that left and right adjoints to evaluation can be produced in this manner for any diagram category.

In fact, one can describe these adjoints for the current situation, while they can become quite complicated for larger indexing categories. The left adjoint to evaluation at n is usually labeled the nth desuspension of a space. The kth entry of $\mathbb{L}_n(X)$ is $\Sigma^{k-n}X$, where negative suspensions indicate a basepoint. The right adjoint is less well-known, primarily because it yields stably trivial objects. Given $M \in \tau_*$, define a diagram $\mathbb{R}_n(M)$ indexed on the opposite of $\mathcal{A}$ with the kth entry of $\mathbb{R}_n(M)$ defined as $\Omega^{n-k}M$ (which should be taken to mean a basepoint when $k > n$). This is clearly natural and yields the expected right adjoint to evaluation at n without much difficulty.

In the case of Example 5, the entries of $\mathbb{L}_n(M)$ look like $(\Sigma_k)_+ \wedge_{\Sigma_{k-n}} S^{k-n} \wedge M$. An entry of $\mathbb{R}_n(M)$ is given by $(\mathrm{map}(S^{k-n}, \mathrm{map}((\Sigma_n)_+, X)))^{\Sigma_{k-n}}$, or the space of Σ_{k-n}-equivariant maps from S^{k-n} to the cofree pointed Σ_n-space on M. A similar modification involving orthogonal rather than symmetric groups yields formulae for the adjoints in the case of Example 6. One should notice that even the nth space of $\mathbb{R}_n(M)$ is not M, but rather the cofree space associated to M, $\mathrm{map}((\Sigma_n)_+, M)$. Only the zeroth entry yields a standard loop space on M, namely $\Omega^n M$.

By now the reader will have noticed that Examples 2.8.4,5 and 6 were all described as categories of diagrams indexed on the opposite of a certain category. This is in line with the following definition.

2.9. Definition. *Suppose $\mathcal{C}$ is a small topological category. Then the category of (enriched)* **presheaves on** $\mathcal{C}$ *is the category of diagrams indexed on the opposite of* $\mathcal{C}$.

Thus, from the examples above, presheaves on $\mathcal{A}$ yields the category of prespectra, presheaves on Σ yields the category of symmetric spectra, presheaves on $\mathcal{O}$ yields the category of orthogonal spectra and presheaves on $\mathcal{G}$ yields the category of pointed right G-spaces.

Clearly, for $Z \in \mathcal{C}$ the formula $H_Z(Y) = \mathcal{C}(Y, Z)$ defines a diagram H_Z or $\mathcal{C}(?, Z)$ indexed on the opposite of $\mathcal{C}$, hence a presheaf. Presheaves of the form $\mathcal{C}(?, Z)$ are commonly referred to as representable presheaves. In fact, the Yoneda embedding which sends Z to $\mathcal{C}(?, Z)$ is an equivalence of $\mathcal{C}$ with the full subcategory of representable presheaves. This equivalence follows from the fact that naturality implies any morphism is completely determined by where it sends the identity map on the target object. (The reason for defining presheaves in terms of diagrams indexed on an opposite category is to make this a covariant equivalence.)

3. Grothendieck Topologies and Bases

Unfortunately, the definition of a Grothendieck topology is quite daunting to many, primarily because of the generality inherent in the definition. A series of examples will be discussed to motivate the definitions. A Grothendieck topology may be viewed as a possible answer to the question, "which subobjects of a representable presheaf might be viewed as 'filling up' the representable in some natural sense ?" The reader should keep in mind that the motivation is closely tied to the notion of sheaves to be introduced in the next section.

The approach to Grothendieck topologies taken here, following [4], will depend upon the notion of a subobject of a presheaf. Ordinarily, category theorists define subobjects as (isomorphism classes of) monomorphisms into a fixed object.

However, it is more natural in topological situations to consider subsets equipped with the subspace topology.

3.1. Warning. Recall that the naive subspace topology may not be compactly generated, hence "the subspace topology" refers to applying the Kelley functor k to this naive construction. If the naive subspace is closed, for example, the functor k is not required.

Thus, a subobject of a presheaf X will consist of another presheaf Y together with a morphism of presheaves $Y \to X$ so that each evaluation functor sends this morphism to the inclusion of a subspace (in this compactly generated sense). Thus, for example, taking Y to be X with finer topologies on the entries in such a way that Y remains a presheaf would **NOT** yield a subobject of X in this sense. However, it should be clear that the constant diagram on the basepoint is a subobject of every presheaf.

Because the axioms are often confusing at first glance, the reader should keep in mind that they bear some resemblance to familiar properties of open covers in basic topology. The first axiom is essentially the statement that the identity map is an open cover, the second that restriction to a subspace preserves open covers, and the third reflects transitivity of open covers. (See [2] for a different version of a Grothendieck topology which reflects this analogy more closely.)

A sieve refers to a subobject of a representable functor in a presheaf category. Suppose $r : R \to \mathcal{C}(?, C)$ is a sieve and $D \in \mathcal{C}$. Then a point $x \in R_D$ gives a point $r(x) \in \mathcal{C}(D, C)$, hence a corresponding morphism $x_r : D \to C$.

3.2. Definition. *Suppose $\mathcal{C}$ is a small topological category. Then a* **Grothendieck topology** *on $\mathcal{C}$ is the choice, for every object $C \in \mathcal{C}$ of a family $\top(C)$ of sieves in $\mathcal{C}(?, C)$ (often called the covering sieves) which satisfy the following axioms:*

- *each $\mathcal{C}(?, C)$ is in $\top(C)$ for $C \in \mathcal{C}$;*
- *given a covering sieve $r : R \to \mathcal{C}(?, C)$ and a morphism $f \in \mathcal{C}(D, C)$, the sieve $f^{-1}(R)$ must cover $\mathcal{C}(?, D)$, where $f^{-1}(R)$ is defined by pulling back R over the morphism f;*

$$\begin{tikzcd} f^{-1}(R) \arrow[r] \arrow[d] & R \arrow[d] \\ \mathcal{C}(?, D) \arrow[r, "f_*"'] & \mathcal{C}(?, C) \end{tikzcd}$$

- *suppose $s : S \to \mathcal{C}(?, C)$ is a covering sieve, and $r : R \to \mathcal{C}(?, C)$ is a sieve such that $f^{-1}(R)$ covers $\mathcal{C}(?, D)$ for each morphism f of the form x_s, corresponding to $x \in S_D$. Then the sieve R must be a covering sieve in $\mathcal{C}(?, C)$ as well.*

As usual, there is also a notion of basis for a Grothendieck topology. In fact, the most natural description of the example of spectra in this framework is as a category of sheaves on a basis (see Corollary 5.2). The examples will be discussed after the definition of a basis.

In order to define a basis, one needs appropriate notions of image of a morphism of presheaves and union of such images. Since morphisms are actually natural transformations, the image of a natural transformation may be defined entrywise and will remain a presheaf.

For a union of images, simply take the subspace topology on the underlying set images, equipped with the "structure maps" of the target object.

3.3. Definition. *An (enriched)* **Grothendieck basis** *on $\mathcal{C}$ will consist of a family of sieves satisfying the following axioms:*

- *each identity $1 : \mathcal{C}(?, C) \to \mathcal{C}(?, C)$ is a basis cover;*
- *given a basis cover $s : S \to \mathcal{C}(?, C)$ and a morphism $g : D \to C$, there exists some basis cover $r : R \to \mathcal{C}(?, D)$ which is a subobject of $g^{-1}(S)$, i.e. which factors through $g^{-1}(S) \to \mathcal{C}(?, D)$;*

$$\begin{tikzcd} R \arrow[d] & \\ g^{-1}(S) \arrow[r] \arrow[d] & S \arrow[d] \\ \mathcal{C}(?, D) \arrow[r, "g_*"'] & \mathcal{C}(?, C) \end{tikzcd}$$

- *suppose $s : S \to \mathcal{C}(?, C)$ is a basis cover and $\{R_x \to \mathcal{C}(?, D_x)\}$ is a family of basis covers indexed over elements $x \in S_{D_x}$. Then the union of their images $\bigcup_{x \in S} x_s(R_x)$ must be a basis cover of $\mathcal{C}(?, C)$.*

The topology associated to a basis will consist of the sieves $r : R \to \mathcal{C}(?, C)$ which factor some inclusion of a basis cover. The following shows this process actually yields a Grothendieck topology.

3.4. Proposition. *Any basis induces a Grothendieck topology, with $R \to \mathcal{C}(?, C)$ a cover in the topology precisely when there is an element of the basis R' such that R' is a subobject of R.*

Proof. Since pullbacks preserve subobjects by definition, the first two axioms for a Grothendieck topology are obvious from those for a basis. It is not immediately obvious how to apply the third axiom for a basis in order to verify that for a topology. Thus, given a covering sieve S from the topology and an arbitrary sieve $R \to \mathcal{C}(?, C)$, choose a subobject S' of S which is a basis cover. Given any morphism $f : D \to C$ of the form x_s, one assumes that $f^{-1}(R)$ is a cover of D. Hence, there exists a covering sieve in the basis, S_x, which is a subobject of $f^{-1}(R)$. Now, the third axiom for the basis implies $\bigcup_{x \in S'} f(S_x)$ is a cover of C. However, since the topology on R is inherited from $\mathcal{C}(?, C)$, the union $\bigcup_{x \in S'} f(S_x)$ is a subobject of R by construction. This implies R is a cover in the topology by definition. ∎

3.5. EXAMPLE.

1. The category $\mathcal{C}_1$ from Example 2.8.1 supports only two topologies. The first consists of both the basepoint and the identity as covers. The second consists of only the identity as a cover. These examples are completely general and play the role of the discrete and indiscrete topologies. By declaring every sieve to be a cover, the axioms are clearly satisfied for an arbitrary indexing category. Later, it will become clear that the only sheaf on this topology is the basepoint. Similarly, declaring only isomorphisms to be covering sieves one exhibits a topology on an arbitrary indexing category. Here the condition of being a sheaf will become vacuous, so that presheaf categories will be examples of sheaf categories using this indiscrete topology.

2. Let $\mathcal{A}_n$ denote the category with two objects 0 and n, and morphism spaces given by $\mathcal{A}_n(n,0) = S^n$, $\mathcal{A}_n(0,n)$ the basepoint and S^0 as endomorphism spaces, with topological unit maps for composition. This is a small topological category, hence yields a topological presheaf category $\mathcal{P}_n$. By analogy with Example 2.8.4, this category $\mathcal{P}_n$ consists of pairs of spaces X_0 and X_n together with a continuous map $\Sigma^n X_0 \to X_n$. One may form a basis for a topology here in a manner quite similar to the previous example. The symbol S^{-n} will denote the representable on the object n, i.e. the object written $\Sigma^n * \to S^0$. Similarly, S^{-0} will denote the representable on the object 0, i.e. the object written $\Sigma^n S^0 \to S^n$ (where the map is the expected isomorphism). In line with later notation, $S^{-n} \otimes S^n$ corresponds to the object $\Sigma^n * \to S^n$. There is an obvious inclusion of $S^{-n} \otimes S^n$ as a subobject of $\Sigma^n S^0 \to S^n$ which will be indicated as $i_n : S^{-n} \otimes S^n \to S^{-0}$ (the subspace is clearly closed). The claim is that i_n together with the identities form a basis for a topology. As before, the first axiom is trivial. For the second axiom, one must consider several cases of pulling back over different morphisms. In any case, pulling back a monomorphism over the zero map will yield an isomorphism, which is a cover by assumption. Pulling back over identities is clearly going to yield nothing new, hence one need only consider pulling back over some non-zero morphism $S^{-n} \to S^{-0}$. Notice the 0 entry of the preimage $f^{-1}(S^{-n} \otimes S^n)$ is the basepoint in any case, as a subobject of $(S^{-n})_0 = *$. Since i_n is an isomorphism at entry n, the n entry of the preimage will be isomorphic to the n entry of S^{-n} which is simply S^0. In other words, $f^{-1}(S^{-n} \otimes S^n) \approx S^{-n}$ which is a cover by assumption.

   Finally, for the covering sieve S in the third axiom notice that the only choices are $S^{-n} \otimes S^n$ or S^{-0} over S^{-0} and S^{-n} over S^{-n}. From the existence of identity maps, it should be clear that the axiom is always satisfied for all S but $S^{-n} \otimes S^n \to S^{-0}$. In this case, choose $R \to S^{-0}$ a sieve such that $f^{-1}(R)$ covers S^{-n} for $f = x_s$ and $x \in S_n$. Since i_n is an isomorphism at entry n, this implies $R_n \approx (S^{-0})_n$ (the only cover of S^{-n} is an isomorphism). However, this implies $R \approx S^{-n} \otimes S^n$ or $R \approx S^{-0}$ since these are the only such sieves. In particular, R is a covering sieve by definition, which verifies the third axiom.

4. Sheaves on a Site or a Basis

Let $\mathcal{C}$ denote a small topological category, together with a Grothendieck topology $\top$ on $\mathcal{C}$, where $\mathcal{P}$ is the presheaf category indexed on $\mathcal{C}$. The pair $(\mathcal{C}, \top)$ is referred to as a Grothendieck site.

4.1. Definition. *Suppose $(\mathcal{C}, \top)$ is a Grothendieck site.*

- *A presheaf $Y \in \mathcal{P}$ will be called a* **sheaf** *on this site provided each precomposition $\mathcal{P}(r, Y) : \mathcal{P}(\mathcal{C}(?, C), Y) \to \mathcal{P}(R, Y)$ by a covering sieve $r : R \to \mathcal{C}(?, C)$ is a homeomorphism.*

- *A* **separated presheaf** *on a site indicates $\mathcal{P}(r, Y)$ is a monomorphism for each covering sieve R.*

- *The topology $\top$ will be called a* **subcanonical topology** *provided each representable presheaf $\mathcal{C}(?, C)$ is a sheaf on this site.*

One should keep in mind that the nature of monomorphisms implies that the class of separated presheaves on a site will be closed under subobjects.

Now suppose $\mathcal{B}$ is a basis for a topology on $\mathcal{P}$. The obvious variations of the previous definitions, considering only basis covers yield notions of sheaves on a basis, etc.

4.2. Definition. *A subcategory is called:*

- **reflective** *provided there exists a left adjoint to the inclusion of the subcategory, and the left adjoint is called the reflector.*

- **strongly reflective** *if the reflector preserves monomorphisms between sieves.*

The following result is highly suggestive of the relation of sheaf categories to localizations.

4.3. Lemma. *Suppose the subcategory of sheaves on a basis is reflective. Then the reflector sends each inclusion of a covering sieve in the basis $r' : R' \to \mathcal{C}(?, C)$ to an isomorphism.*

Proof. Naturality of the reflector yields the following commutative diagram of isomorphisms.

$$\begin{tikzcd} \mathcal{P}(\mathcal{C}(?, C), Y) \arrow[r, "(r')^*"] \arrow[d, "\approx"'] & \mathcal{P}(R', Y) \arrow[d, "\approx"] \\ \mathcal{S}(\mathbb{S}(\mathcal{C}(?, C)), Y) \arrow[r, "\mathbb{S}(r')^*"'] & \mathcal{S}(\mathbb{S}(R'), Y) \end{tikzcd}$$

Hence, the morphism $\mathbb{S}(r')$ induces an isomorphism under precomposition against every object in the subcategory. It is a brief exercise in diagram chasing to see $\mathbb{S}(r')$ must then be an isomorphism of sheaves. ∎

In order to avoid the technically difficult construction of the associated sheaf functor (see [15]), the following is phrased conditionally. The usual role of the associated sheaf functor is to be the strong reflector for the category of sheaves.

4.4. Proposition. *Suppose the subcategory of sheaves on a basis $\mathcal{B}$ is strongly reflective. Then it agrees with the category of sheaves on the topology generated by the basis.*

Proof. It should be clear that any sheaf on the topology generated by a basis must be a sheaf on the basis by definition. Since both subcategories are full, it suffices to show any sheaf on the basis is a sheaf in the topology. Thus, suppose Y is a sheaf on the basis while $r : R \to \mathcal{C}(?, C)$ is the inclusion of a cover in the topology generated by the basis. If the reflector $\mathbb{S}$ applied to r is an isomorphism, then the following diagram implies $\mathcal{P}(r, Y)$ is an isomorphism.

$$\begin{tikzcd}
\mathcal{P}(\mathcal{C}(?,C),Y) \arrow[r, "r^*"] \arrow[d, "\approx"'] & \mathcal{P}(R,Y) \arrow[d, "\approx"] \\
\mathcal{S}(\mathbb{S}(\mathcal{C}(?,C)),Y) \arrow[r, "\mathbb{S}(r)^*"'] & \mathcal{S}(\mathbb{S}(R),Y)
\end{tikzcd}$$

However, by assumption there exists a commutative diagram

$$\begin{tikzcd}
R' \arrow[d] \arrow[dr, "r'"] & \\
R \arrow[r] & \mathcal{C}(?,C)
\end{tikzcd}$$

of subobjects where $r' : R' \to \mathcal{C}(?, C)$ is a cover in the basis. Applying $\mathbb{S}$ to this diagram then yields a factorization of an isomorphism as two monomorphisms (by the assumption of a strong reflector), which must then be isomorphisms for formal reasons. ■

In fact, the associated sheaf functor often allows one to recover the topology as those $r : R \to \mathcal{C}(?, C)$ with $\mathbb{S}(r)$ an isomorphism, see [5], [18] (or [15]) for details.

4.5. Example.

1. Consider the category $\mathcal{P}_n$ together with the basis for a topology introduced in Example 3.5.2. By construction the diagram

$$\begin{tikzcd}
\mathcal{P}_n(S^{-0},Y) \arrow[r, "i_n^*"] \arrow[d, "\approx"'] & \mathcal{P}_n(S^{-n}\otimes S^n,Y) \arrow[d, "\approx"] \\
Y_0 \arrow[r] & \Omega^n Y_n
\end{tikzcd}$$

   commutes, where the bottom map is adjoint to $\Sigma^n Y_0 \to Y_n$. Since both vertical maps are homeomorphisms, requiring Y to be a sheaf in the topology discussed in Example 3.5.2 is equivalent to demanding that Y_0 is homeomorphic to $\Omega^n Y_n$ via the adjoint structure map. Hence, the category of sheaves on this basis is equivalent to the category of n-fold loop spaces. Notice Proposition 4.4 then implies the category of sheaves on the topology generated by this basis is another description of n-fold loop spaces. (The strong reflector is given by the functor sending $\Sigma^n X_0 \to X_n$ to $\Sigma^n(\Omega^n X_n) \to X_n$.)

2. There is a somewhat more natural variation on the previous example which is closer to Example 2.8.4. Thus, let $\mathcal{A}_0^n$ consist of the full subcategory of $\mathcal{A}$ whose objects are the nonnegative integers between 0 and n (inclusive). This may be equipped with a basis for a topology consisting of all inclusions $S^{-k} \otimes S^{k-m} \to S^{-m}$ with $n \geq k \geq m \geq 0$. (The case $n = \infty$ will be proven below for the interested reader.) As in the previous example, the category of sheaves on this basis is equivalent to the category of n-fold loop spaces as well as being strongly reflective. Consider the functor which takes an arbitrary presheaf Y to the diagram whose kth entry is $\Omega^{n-k} Y_n$. This construction clearly yields a sheaf and a simple exercise verifies the universal property of a reflector, hence Proposition 4.4 applies.

Combining the previous two examples, one can see that more than one site may be used to model the same category of sheaves up to topological equivalence. This suggests that one attempt to find presheaf categories which are closer to a given category of sheaves. Since colimits in the category of sheaves are formed by applying an associated sheaf functor to the colimit in a presheaf category, one would like to minimize the damage done by this associated sheaf functor in order to produce tractable models for such colimits.

The technique is based on a relatively simple idea. Given a sheaf category $\mathbb{S}$ described in terms of a site $(\mathcal{C}, \top)$, form a new indexing category $\mathcal{E}$ which consists of the full subcategory of "representable sheaves", i.e. the associated sheaves of any representable functors in the presheaf category. Thus, the objects of $\mathcal{E}$ are in 1-1 correspondence with the objects of $\mathcal{C}$, but the morphism spaces of $\mathcal{E}$ are, in general, richer than those of $\mathcal{C}$. In particular, there is a natural inclusion functor $J : \mathcal{E} \to \mathbb{S}$ which may be used to construct an (topological) adjoint pair between $\mathcal{P}_{\mathcal{E}}$ and $\mathbb{S}$. The left adjoint $\mathbb{L} : \mathcal{P}_{\mathcal{E}} \to \mathbb{S}$ will be defined as the left (topological) Kan extension of J over the Yoneda embedding $\mathcal{E} \to \mathcal{P}_{\mathcal{E}}$. The right adjoint is then given by precomposition with J, i.e. $\mathbb{R}(X)_E = \mathbb{S}(J(E), X)$ for each $X \in \mathbb{S}$ and $E \in \mathcal{E}$. Notice, this is completely analogous to the construction of the geometric realization/singular set pair between simplicial sets and spaces, hence $\mathbb{L}$ is generally called the realization functor and $\mathbb{R}$ its associated singular functor.

In fact, the functor $\mathbb{R}$ is an equivalence onto its image, because every element of $\mathbb{S}$ is an (indexed) colimit of elements in the image of the functor J. The question then becomes one of constructing a Grothendieck topology for $\mathcal{P}_{\mathcal{E}}$ where the image of $\mathbb{R}$ is the category of sheaves on this topology. This construction is possible by exploiting the fact that $\mathbb{RL}$ is left adjoint to the inclusion of the image of $\mathbb{R}$. In line with the comment following Proposition 4.4, this adjunction usually allows one to build a topology consisting of those inclusions $r : R \to \mathcal{E}(?, E)$ which are sent to an isomorphism by $\mathbb{RL}$.

4.6. Example.

1. Consider the category of n-fold loop spaces as described in Example 4.5.1 above. This approach suggests defining a new indexing category $\mathcal{E}_n$ with 0 and n as objects, and morphism spaces given as in Example 3.5.2 other than the fact that $\mathcal{E}_n(0,0)$ should be the space of topological endomorphisms of S^n rather than simply S^0. Hence the objects of $\mathcal{P}_{\mathcal{E}_n}$ will be those elements X of $\mathcal{P}_n$ which, in addition, have a natural action of this endomorphism space (considered as a pointed topological monoid) on X_0. Notice the inclusion will not be full, since all morphisms in $\mathcal{P}_{\mathcal{E}_n}$ must satisfy an additional equivariance condition. As expected from the discussion above, this diagram category more nearly approximates n-fold loop spaces than the category $\mathcal{P}_n$.

2. Similarly, one can improve upon the indexing category considered in Example 4.5.2 in this manner. The resulting diagram category consists of elements of the former diagram category which support an action of the topological endomorphisms of S^{n-k} on the k-th entry, as well as various additional structure maps. For example, there must be a structure map $\Omega^n\Sigma^{n-k}X_k \to X_0$ which would correspond to Ω^n on the usual structure map $\Sigma^{n-k}X_k \to X_n$ when $X_0 \approx \Omega^n X_n$. Once again, the inclusion will not be a full functor as more equivariance conditions are imposed in the new category.

These examples are suggestive of the usual practice of imposing an action of the endomorphisms operad in the study of infinite loop spaces.

5. Spectra as Sheaves

The motivating example for the author's work in the first chapter of [15] is summarized in the following few results.

5.1. Lemma. *On the category $\mathcal{P}_{\mathcal{A}}$, the collection of morphisms $S^{-k} \otimes S^{k-m} \to S^{-m}$ with $k \geq m \geq 0$ form a basis for a topology.*

Proof. First, the case $k = m$ is included to ensure that each identity is a basis cover. For the second axiom, choose any morphism $f : S^{-l} \to S^{-m}$ and consider the preimage $f^{-1}(S^{-k} \otimes S^{k-m})$. If f is the zero map, the fact that $S^{-k} \otimes S^{k-m} \to S^{-m}$ is a monomorphism implies this preimage is isomorphic to S^{-l} itself. However, each non-zero f is itself a monomorphism. This implies that for entries below k, the preimage is the basepoint. Since the inclusion $S^{-k} \otimes S^{k-m} \to S^{-m}$ is an isomorphism for entries greater than or equal to k, the preimage will also be isomorphic to S^{-l} for entries greater than or equal to k. Because the structure maps (and topologies since all inclusions are closed embeddings) are inherited as a subobject of S^{-l}, this uniquely determines the preimage as $S^{-k} \otimes S^{k-l}$ for $k > l$ or S^{-l} itself if $k \leq l$. In either case, the preimage is also a basis element.

The third axiom is accessible primarily because the composition law in the category $\mathcal{A}$ consists of isomorphisms. Hence, every morphism $S^{-l} \to S^{-m}$ factors through each intermediate S^{-j} for $l \geq j \geq m$. Also, the entries of any $S^{-k} \otimes S^{k-m}$ consist of either the zero map or all possible maps. Thus, taking the union of all possible images of a family of $S^{-j} \otimes S^{j-l}$ will yield some $S^{-k} \otimes S^{k-m}$ sitting inside S^{-m}. ∎

5.2. Corollary. *The category of spectra (on the canonical indexing category, in the sense of [17]) is the category of sheaves on the basis given by Lemma 5.1, as well as being strongly reflective.*

Proof. As in Example 4.5.1, considering diagrams of the form

$$\begin{tikzcd}
\mathcal{P}_n(S^{-n}, Y) \arrow[r, "i_n^*"] \arrow[d, "\approx"'] & \mathcal{P}_n(S^{-k} \otimes S^{k-n}, Y) \arrow[d, "\approx"] \\
Y_n \arrow[r] & \Omega^{k-n} Y_k
\end{tikzcd}$$

implies the sheaf condition is equivalent to saying each adjoint structure map is a homeomorphism. The fact that the category of spectra is strongly reflective is the focus of the first section of the appendix to [17]. ∎

5.3. Definition. *A* **cofinal sieve** *will refer to a sieve $r : R \to S^{-n}$ such that each evaluation r_j is an isomorphism for some choice of k and all $j \geq k$.*

(Compare this with the definition of cofinal given in [1].) The term cofinal topology was suggested by R. Bruner.

5.4. Theorem. *The collection of cofinal sieves form a Grothendieck topology on $\mathcal{P}_{\mathcal{A}}$, the cofinal topology. Furthermore, the category of spectra (on the canonical indexing sequence) is equivalent to the category of sheaves on the cofinal topology.*

Proof. The previous corollary reduces the proof of the theorem to the identification of the cofinal sieves as those sieves containing an element of the basis from Lemma 5.1.

Since the inclusion $i_n^k : S^{-k} \otimes S^{k-n} \to S^{-n}$ is an isomorphism above entry k and the inclusion of a basepoint below, any cofinal sieve contains some $S^{-k} \otimes S^{k-n}$ by definition.

Conversely, suppose a sieve $r : R \to S^{-n}$ contains some $S^{-k} \otimes S^{k-n}$. Then evaluation at each entry $j \geq k$ leads to a factorization of an isomorphism as a pair of monomorphisms $(S^{-k} \otimes S^{k-n})_j \to R_j \to S_j^{-n}$. However, this implies both maps are actually homeomorphisms, hence r_j is a homeomorphism for all $j \geq k$ as desired. ∎

6. Building the Stable Homotopy Category

The reader will notice that all of the current topological models for the stable homotopy category, with the exception of the S-modules of [7], have been placed in the framework of enriched sheaf categories. As presheaf (or diagram) categories, prespectra, symmetric and orthogonal spectra are all categories of sheaves on “indiscrete” topologies. However, some form of localization is required in order to produce a model for the stable category from any of these as a starting point. The category of spectra has just been exhibited as a proper sheaf category, or categorical localization of prespectra. A similar homotopy-theoretic localization of prespectra also yields the stable category (see Theorem 6.8 below), in a manner more familiar to students of Adams's model from [1]. Analogous homotopy-theoretic localizations of symmetric and orthogonal spectra produce models for the stable category. However, this structure on orthogonal spectra may be produced more easily by direct comparison with the stable structure on prespectra, as in Theorem 6.10 below. The intent of this section is to contrast the technical properties of these models.

The language of homotopy theory used here will be that of Quillen's homotopical algebra [20]. Those unfamiliar with this language may want to glance at the recent books [9] or [11]. For full details of the existence of the model structures described below see a more general case in [12] or [14] or the author's original viewpoint in [15].

Recall that a model structure consists of choosing three classes of maps: weak equivalences, cofibrations and fibrations, which must satisfy a series of axioms generalizing standard notions such as the homotopy extension property. The following definition describes one standard method of building new model categories from old ones, originally due to Quillen.

6.1. Definition. *Suppose $\mathcal{C}$ is a model category while $\mathbb{L} : \mathcal{C} \to \mathcal{D}$ and $\mathbb{R} : \mathcal{D} \to \mathcal{C}$ form an adjoint pair. Then the model structure is said to* **lift over the adjoint pair** *provided $\mathcal{D}$ is a model category with a morphism $h \in \mathcal{D}$ a weak equivalence or fibration precisely when $\mathbb{R}(h) \in \mathcal{C}$ is a weak equivalence or fibration, respectively.*

6.2. Remark.

1. In practice, this technique is usually applied to cofibrantly generated model categories. Then an additional condition is that the generating cells in $\mathcal{D}$ are the set of morphisms $\mathbb{L}(g)$ with g a generating cell in $\mathcal{C}$.

2. The generalization of this technique to include a set of adjoint pairs is described in [14] and implies the existence of the strict structures described below.

6.3. Example.

1. The usual model structure on pointed spaces is lifted from unpointed spaces in this sense, where the adjoint pair is adding a disjoint basepoint or forgetting the basepoint. This explains why basepoints other than the fixed basepoint must be considered in defining a weak homotopy equivalence of pointed spaces.

2. The usual structure on spaces may be lifted to G-spaces via the free functor/forgetful functor pair. However, this is generally not an important structure there, as it essentially ignores the orbit data. The usual structure on G-spaces comes from a similar trick involving a set of adjoint pairs, or lifting a set of times and intersecting the various structures. See [14] for details of this approach.

One useful point about lifting model structures is that it simplifies the question of whether the adjoint pair is a Quillen equivalence. This is exploited in the proof of Theorem 6.10 below.

6.4. Definition. *Suppose $\mathcal{C}$ and $\mathcal{D}$ are both model categories with $(\mathbb{L}, \mathbb{R})$ an adjoint pair and $\mathbb{L} : \mathcal{C} \to \mathcal{D}$.*

1. *The pair is called a (strong)* **Quillen pair** *if $\mathbb{L}$ preserves cofibrations and $\mathbb{R}$ preserves fibrations.*

2. *A Quillen pair is called a* **Quillen equivalence** *if for each cofibrant $X \in \mathcal{C}$ and fibrant $Y \in \mathcal{D}$ the morphism $\mathbb{L}(X) \to Y$ is a weak equivalence in $\mathcal{D}$ precisely when its adjoint map $X \to \mathbb{R}(Y)$ is a weak equivalence in $\mathcal{C}$.*

6.5. Remark. The condition of being a Quillen pair is sufficient to imply the adjunction descends to an adjunction on the associated homotopy categories. A Quillen pair descends to an adjoint equivalence of homotopy categories if and only if it is a Quillen equivalence.

The following is generally referred to as the "strict structure", in line with the notation of [6]. The adjectives in the statement are technically important but the casual reader may want to disregard them.

6.6. Proposition. *The categories of prespectra, symmetric spectra and orthogonal spectra each carry a cofibrantly generated, proper, topological model structure with fibrations and weak equivalences defined entrywise.*

The following is essentially a corollary, because the structure on prespectra is transported to the subcategory of spectra by lifting over the associated spectrum functor (and its right adjoint, the inclusion of spectra in prespectra).

6.7. Corollary. *The category of spectra carries a cofibrantly generated, topological model structure with fibrations and weak equivalences defined entrywise.*

There are two key technical elements, both due to Lewis in [17]. First, cofibrations in spectra will be entrywise closed embeddings. Second, sequential colimits over entrywise closed embeddings between spectra do not require an application of the associated spectrum functor. Thus, the small object argument may be applied, despite the fact that evaluations do not preserve colimits, in general.

This is the first example so far of a model for the stable homotopy category. Thus, a categorical localization of the category of prespectra yields such a model and one is lead to wonder if a homotopical localization might yield a model as well.

Consider the map f in $\mathcal{P}_{\mathcal{A}}$ which is defined as the coproduct of all morphisms $S^{-k} \wedge S^{k-n} \to S^{-n}$ with $k \geq n$. There is a general approach to inverting a map in the homotopy category of a model category, commonly called a localization of the model category. (See [10].)

6.8. Theorem. *The f-localization of the strict structure on prespectra is Quillen equivalent to the structure on spectra given by Corollary 6.7. Furthermore, the f-localization of the strict structure yields the same homotopy theory as Adams's model for the stable homotopy category.*

The key point in the proof of this theorem is that the associated spectrum functor acts as an f-local replacement functor, at least for cofibrant prespectra. (See Lemma 4.3.5 in [15].) The comparison with Adams's model then follows almost immediately from Theorem III.3.4 in [1].

There are natural forgetful functors from orthogonal to symmetric spectra and from symmetric spectra to prespectra. Each of these may be written as an enriched precomposition, hence has both left and right adjoints via Kan extensions, which are topological functors. In particular, there are $\mathbb{L} : \mathcal{P}_{\mathcal{A}} \to \mathcal{P}_{\Sigma}$ and $\mathbb{L}' : \mathcal{P}_{\mathcal{A}} \to \mathcal{P}_{\mathcal{O}}$ each left adjoint to a forgetful functor, which preserve the entrywise smash product with a space (or tensor) and preserve representable functors.

The following is phrased differently than the approach in [13], but is essentially equivalent.

6.9. Theorem. *The functor $\mathbb{L}$ together with the forgetful functor forms a (topological) Quillen equivalence between the f-localization of the strict structure on prespectra and the $\mathbb{L}(f)$-localization of the strict structure on symmetric spectra.*

Unfortunately, the $\mathbb{L}(f)$-local equivalences are not determined by the usual stable homotopy groups, unless the objects in question are $\mathbb{L}(f)$-local. There was originally much confusion about the definition of stable weak equivalences in symmetric spectra, which seems relatively straightforward from this description. A functorial way of dealing with the difference between stable weak equivalences and maps between symmetric spectra inducing isomorphisms on stable homotopy groups has been introduced in [22].

By contrast, the situation in orthogonal spectra is simpler, primarily because a local replacement operation is not necessary.

6.10. Theorem. *The f-localization of the strict structure on prespectra lifts over the adjoint pair consisting of $\mathbb{L}'$ and the forgetful functor to yield a Quillen equivalent (topological) structure on orthogonal spectra.*

The key difference between $\mathbb{L}$ and $\mathbb{L}'$ stems from the fact that the connectivity of orthogonal groups remains constant, while that of symmetric groups increases in n. (Both of these left adjoints are built from free functors, whose connectivity depends on that of the group involved.) Hence, one may verify directly that $\mathbb{L}'(f)$ is a stable homotopy equivalence as a map of prespectra (see [19]) and the rest follows formally. (See [15].)

In particular, it is true that a morphism g of orthogonal spectra is an $\mathbb{L}'(f)$-local equivalence precisely when g is a stable homotopy equivalence (considered as a morphism of prespectra). This is one of the technical advantages of orthogonal spectra, which justifies the added complexity of dealing with the compact topological groups $\mathbb{O}_V$ rather than the finite groups Σ_n.

The category of prespectra plays a pivotal role as the intermediate model that all the other models may be compared with. In fact, this is not a surprise from the viewpoint of [16], where prespectra is shown to be initial among stable topological model categories, in a certain sense. The Bousfield-Friedlander category has been shown to satisfy a similar property by [21] as a key step along the way to their classification of simplicial models for the stable homotopy category.

The category of spectra is quite useful because stable homotopy equivalences and entrywise weak equivalences coincide. It is often quite useful to have all objects fibrant, which is also the case for spectra. For example, much of the recent work on multiplicative stable homotopy by Goerss and Hopkins [8] has been done in this framework.

Symmetric spectra is the simplest model which carries a symmetric monoidal structure reflecting the smash product in the stable homotopy category, as described by Boardman. This yields point-set models for constructions such as THH and function spectra. The hard question in this category is to determine whether a morphism is a stable weak equivalence.

Finally, orthogonal spectra also carries a symmetric monoidal structure reflecting the smash product in the stable homotopy category, as described by Boardman. The difficulty of determining stable weak equivalences is avoided, since they are precisely the stable homotopy equivalences in the usual sense. However, the indexing category is somewhat more intimidating. Fortunately, most formulae seem to be determined in symmetric spectra and then translated into the appropriate orthogonal analog.

Thus, the symmetric spectra continue to play a vital role in the discussion of orthogonal spectra, just as prespectra are technically vital to understand point-set constructions in spectra.

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