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COMBINATORICS OF BRANCHINGS IN HIGHER DIMENSIONAL AUTOMATA

PHILIPPE GAUCHER

ABSTRACT. We explore the combinatorial properties of the branching areas of execution paths in higher dimensional automata. Mathematically, this means that we investigate the combinatorics of the negative corner (or branching) homology of a globular ω-category and the combinatorics of a new homology theory called the reduced branching homology. The latter is the homology of the quotient of the branching complex by the sub-complex generated by its thin elements. Conjecturally it coincides with the non reduced theory for higher dimensional automata, that is ω-categories freely generated by precubical sets. As application, we calculate the branching homology of some ω-categories and we give some invariance results for the reduced branching homology. We only treat the branching side. The merging side, that is the case of merging areas of execution paths is similar and can be easily deduced from the branching side.

1. Introduction

After [22, 14], one knows that it is possible to model higher dimensional automata (HDA) using precubical sets (Definition 2.1). In such a model, a n-cube corresponds to a n-transition, that is the concurrent execution of n 1-transitions. This theoretical idea would be implemented later. Indeed a CaML program translating programs in Concurrent Pascal into a text file coding a precubical set is presented in [10]. At this step, one does not yet consider cubical sets with or without connections since the degenerate elements have no meaning at all from the point of view of computer-scientific modeling (even if in the beginning of [12], the notion of cubical sets is directly introduced by intellectual reflex).

In [14], the following fundamental observation is made : *given a precubical set $(K_n)_{n\geqslant 0}$ together with its two families of face maps (∂_i^α) for $\alpha \in \{-,+\}$, then both chain complexes $(\mathbb{Z}K_*, \partial^\alpha)$, where $\mathbb{Z}X$ means the free abelian group generated by X and where $\partial^\alpha = \sum_i (-1)^{i+1}\partial_i^\alpha$, give rise to two homology theories H_*^α for $\alpha \in \{-,+\}$ whose non-trivial elements model the branching areas of execution paths for $\alpha = -$ and the merging areas of execution paths for $\alpha = +$ in strictly positive dimension. Moreover the group H_0^- (resp. H_0^+) is the free abelian group generated by the final states (resp. the initial states) of the HDA.*

Consider for instance the 1-dimensional HDA of Figure 1. Then $u - w$ gives rise to a non-trivial homology class which corresponds to the branching which is depicted.

Then the first problem is that the category of precubical sets is not appropriate to

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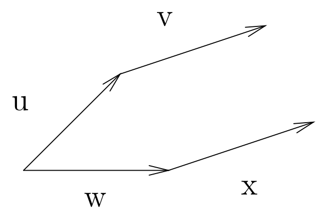


Figure 1: A 1-dimensional branching area

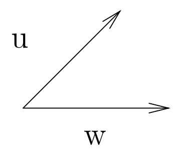


Figure 2: A 1-dimensional branching area

identify the HDA of Figure 1 with that of Figure 2 because there is no morphism between them preserving the initial state and both final states.

No matter : it suffices indeed to work with the category of precubical sets endowed with the $+_i$ cubical composition laws satisfying the axioms of Definition 2.4 and with the morphisms obviously defined. Now for any $n \geqslant 1$, there are n cubical composition laws $+_1, \dots, +_n$ representing the concatenation of n-cubes in the n possible directions. Let $X = u +_1 v$ and $Y := w +_1 x$. Then there is a unique morphism f in this new category of HDA from the HDA of Figure 2 to the HDA of Figure 1 such that $f : u \mapsto X$ and $f : w \mapsto Y$. However f is not invertible in the category of precubical sets equipped with cubical composition laws because there still does not exist any morphism from the HDA of Figure 1 to the HDA of Figure 2.

To make f invertible (recall that we would like to find a category where both HDA would be isomorphic), it still remains the possibility of formally adding inverses by the process of localization of a category with respect to a collection of morphisms. However a serious problem shows up : the non-trivial cycles $u - w$ and $X - Y$ of Figure 1 give rise to two distinct homology classes although these two distinct homology classes correspond to the same branching area. Indeed there is no chain in dimension 2 (i.e. $K_2 = \emptyset$), so no way to make the required identification !

This means that something must be added in dimension 2, but without creating additional homology classes. Now consider Figure 3. The element A must be understood as a *thin* 2-cube such that, with our convention of orientation, $\partial_1^- A = u$, $\partial_2^- A = u$, $\partial_1^+ A = \epsilon_1 \partial_1^- v$, $\partial_2^+ A = \partial_2^- B = \epsilon_1 \partial_1^- v$. And the element B must be understood as another *thin* 2-cube such that $\partial_2^- B = \epsilon_1 \partial_1^- v$, $\partial_2^+ B = \epsilon_1 \partial_1^+ v$ and $\partial_1^- B = \partial_1^+ B = v$. In such a situation, $\partial^-(A +_2 B) = u +_1 v - u$ therefore $u +_1 v$ and u become equal in the first homology group H_1^-. By adding this kind of thin 2-cubes to the chain complex $(\mathbb{Z}K_*, \partial^-)$, one can then identify the two cycles $u - w$ and $X - Y$. One sees that there are two kinds of thin cubes which are necessary to treat the branching case. The first kind is well-known in cubical set theory : this is for example $B = \epsilon_1 v$ or $\partial_1^+ A = \epsilon_1 \partial_1^- v$. The second kind is for example A which will be denoted by $\Gamma_1^- u$ and which corresponds to extra-degeneracy maps as defined in [6].

To take into account the symmetric problem of merging areas of execution paths, a third family Γ_i^+ of degeneracy maps will be necessary. In this paper, we will only treat the case of branchings. The case of mergings is similar and easy to deduce from the branching case. The solution presented in this paper to overcome the above problems is then as follows :

- One considers the free globular ω-category $F(K)$ generated by the precubical set K : it is obtained by associating to any n-cube x of K a copy of the free globular ω-category I^n generated by the faces of the n-cube (paragraph 3.1) ; the faces of this n-cube are denoted by $(x; k_1 \dots k_n)$; one takes the direct sum of all these cubes and one takes the quotient by the relations

$$(\partial_i^\alpha y; k_1 \dots k_n) \sim (y; k_1 \dots k_{i-1} \alpha k_i \dots k_n)$$

  for any $y \in K_{n+1}$, $\alpha \in \{-, +\}$ and $1 \leqslant i \leqslant n+1$.

- Then we take its cubical singular nerve $\mathcal{N}^\square(F(K))$ (which is equal also to the free cubical ω-category generated by K) ; the required thin elements above described (the three families ϵ_i, Γ_i^- and Γ_i^+) do appear in it as components of the algebraic structure of the cubical nerve (Definition 2.4 and Definition 3.3).

- The branching homology of $F(K)$ (Definition 3.5) is the solution for both following reasons :

  1. Let x and y be two n-cubes of the cubical nerve which are in the branching complex. If $x +_j y$ exists for some j with $1 \leqslant j \leqslant n$, then x and $x +_j y$ are equal modulo elements in the chain complex generated by the thin elements (Theorem 9.2) ;
  2. The chain complex generated by the thin elements is conjecturally acyclic in this situation, and so it does not create non-trivial homology classes (Conjecture 3.6).

We have explained above the situation in dimension 1. The 2-dimensional case is depicted in Figure 8. Additional explanations are available at the end of Section 9.

The branching homology (or negative corner homology) and the merging homology (or positive corner homology) were already introduced in [12]. This invariance with respect to the cubifications of the underlying HDA was already suspected for other reasons. The branching and merging homology theories are the solution to overcome the drawback of Goubault's constructions.

There are three key concepts in this paper which are not so common in the general literature and which we would like to draw to the reader's attention.

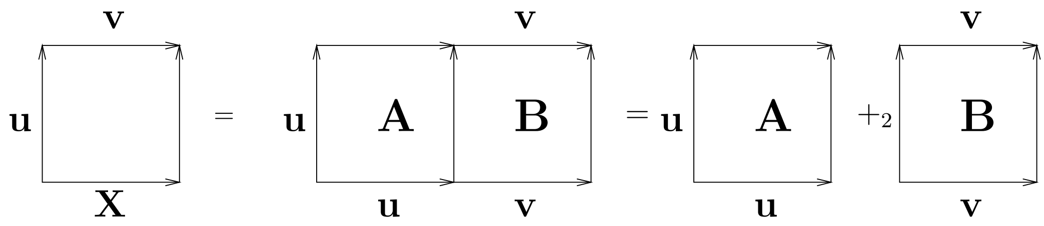


Figure 3: Identifying $u +_1 v$ and u

1. the extra structure of *connections* $\Gamma^{\pm}$ on cubical sets, which allow extra degenerate elements in which adjacent faces coincide. This structure was first introduced in [6].

2. the notion of *folding operator*. This was introduced in the groupoid context in [6], to fold down a cube to an element in a crossed complex, and in the category context in [1] to fold down a cube to an element in a globular category. Properties of this folding operator are further developed in [2]. This we call the 'usual folding operator'.

3. the notion of *thin cube*, namely a multiple composition of cubes of the form $\epsilon_i y$ or $\Gamma^{\pm} z$. A crucial result is that these are exactly the elements which fold down to 1 in the contained globular category.

So there are many ways of choosing a cycle in the branching complex for a given homology class, i.e. a given branching area, according to the choice of the cubification of the considered HDA. This possibility of choice reveals an intricate combinatorics. The most appropriate tool constructed in the mathematical theory of cubical sets to study this combinatorics is not relevant here. The machinery of folding operators [6, 1] does not work indeed for the study of the branching homology because the usual folding operators are not internal to the branching chain complex (see Section 6.2). The core of this paper is the proposal of a new folding operator adapted for the study of the branching complex (Section 6.5). This operator enables us to deduce several results on the reduced branching homology, the latter being obtained by taking the quotient of the former by the sub-complex generated by its thin elements. This sub-complex is conjecturally acyclic for a wide variety of ω-categories, including that freely generated by a precubical set or a globular set (Conjecture 3.6). Our main result is that the negative folding operator induces the identity map on the reduced branching complex (Corollary 8.4). Using some relations between the branching homology of some particular ω-categories and the usual simplicial homology of some associated ω-categories (Theorem 5.5), the behaviour of the composition maps (the globular and the cubical ones) modulo thin elements is completely studied (Section 9). All these results lead us to a question about the description of the reduced branching complex using globular operations by generators and relations (Proposition 9.4 and Question 9.6) and to two invariance results for the reduced branching homology (Proposition 11.1 and Theorem 11.2).

This paper is organized as follows. Section 2 recalls some important notations and conventions for the sequel. In Section 3, the branching homology and the reduced branching homology are introduced. In Section 4, the matrix notations for connections and degeneracies are described. Next in Section 5, the branching homology of some particular ω-categories (the ω-categories of length at most 1) is completely calculated in terms of the usual simplicial homology. In Section 6, the negative folding operators are introduced. In Section 7, the negative folding operators are decomposed in terms of elementary moves. In Section 8, we prove that each elementary move appearing in the decomposition of the folding operators induces the identity map on the reduced branching complex. Therefore the folding operators induce the identity map as well. In Section 9, the behaviour of the cubical and globular composition laws in the reduced branching complex is completely studied. In the following Section 10, some facts about the differential map in the reduced branching complex are exposed. In the last Section 11, some invariance results for the reduced branching homology are exposed and the reduced branching homology is calculated for some simple globular ω-categories.

2. Preliminaries : cubical set, globular and cubical category

Here is a recall of some basic definitions, in order to make precise some notations and some conventions for the sequel.

2.1. Definition. *[6] [16] A cubical set consists of a family of sets $(K_n)_{n\geqslant 0}$, of a family of face maps $K_n \xrightarrow{\partial_i^\alpha} K_{n-1}$ for $\alpha \in \{-,+\}$ and of a family of degeneracy maps $K_{n-1} \xrightarrow{\epsilon_i} K_n$ with $1 \leqslant i \leqslant n$ which satisfy the following relations*

1. $\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha$ *for all $i < j \leqslant n$ and $\alpha, \beta \in \{-,+\}$ (called sometimes the cube axiom)*
2. $\epsilon_i \epsilon_j = \epsilon_{j+1}\epsilon_i$ *for all $i \leqslant j \leqslant n$*
3. $\partial_i^\alpha \epsilon_j = \epsilon_{j-1}\partial_i^\alpha$ *for $i < j \leqslant n$ and $\alpha \in \{-,+\}$*
4. $\partial_i^\alpha \epsilon_j = \epsilon_j \partial_{i-1}^\alpha$ *for $i > j \leqslant n$ and $\alpha \in \{-,+\}$*
5. $\partial_i^\alpha \epsilon_i = Id$

A family $(K_n)_{n\geqslant 0}$ only equipped with a family of face maps ∂_i^α satisfying the same axiom as above is called a precubical set. An element of K_0 will be sometimes called a state, or a 0*-cube and an element of K_n a n-cube, or a n-dimensional cube.*

2.2. Definition. *Let $(K_n)_{n\geqslant 0}$ and $(L_n)_{n\geqslant 0}$ be two cubical (resp. precubical) sets. Then a morphism f from $(K_n)_{n\geqslant 0}$ to $(L_n)_{n\geqslant 0}$ is a family $f = (f_n)_{n\geqslant 0}$ of set maps $f_n : K_n \to L_n$ such that $f_n\partial_i^\alpha = \partial_i^\alpha f_n$ and $f_n\epsilon_i = \epsilon_i f_n$ (resp. $f_n\partial_i^\alpha = \partial_i^\alpha f_n$) for any i. The corresponding category of cubical sets is isomorphic to the category of pre-sheaves $Sets^{\square^{op}}$ over a small category $\square$. The corresponding category of precubical sets is isomorphic to the category of pre-sheaves $Sets^{\square^{pre\,op}}$ over a small category $\square^{pre}$.*

2.3. Definition. *[5] [26] [24] A (globular) ω-category is a set A endowed with two families of maps $(d_n^- = s_n)_{n\geqslant 0}$ and $(d_n^+ = t_n)_{n\geqslant 0}$ from A to A and with a family of partially defined 2-ary operations $(*_n)_{n\geqslant 0}$ where for any $n \geqslant 0$, $*_n$ is a map from $\{(a,b) \in A \times A, t_n(a) = s_n(b)\}$ to A ((a,b) being carried over $a *_n b$) which satisfy the following axioms for all α and β in $\{-,+\}$:*

1. $d_m^\beta d_n^\alpha x = \begin{cases} d_m^\beta x \text{ if } m < n \\ d_n^\alpha x \text{ if } m \geqslant n \end{cases}$
2. $s_n x *_n x = x *_n t_n x = x$
3. *if $x *_n y$ is well-defined, then $s_n(x *_n y) = s_n x$, $t_n(x *_n y) = t_n y$ and for $m \neq n$, $d_m^\alpha(x *_n y) = d_m^\alpha x *_n d_m^\alpha y$*
4. *as soon as the two members of the following equality exist, then $(x *_n y) *_n z = x *_n (y *_n z)$*
5. *if $m \neq n$ and if the two members of the equality make sense, then $(x *_n y) *_m (z *_n w) = (x *_m z) *_n (y *_m w)$*
6. *for any x in A, there exists a natural number n such that $s_n x = t_n x = x$ (the smallest of these numbers is called the dimension of x and is denoted by $dim(x)$).*

A globular set is a set A endowed with two families of maps $(s_n)_{n\geqslant 0}$ and $(t_n)_{n\geqslant 0}$ satisfying the same axioms as above [27, 21, 3]. We call $s_n(x)$ the n-source of x and $t_n(x)$ the n-target of x.

Notation. The category of all ω-categories (with the obvious morphisms) is denoted by ωCat. The corresponding morphisms are called ω-functors. The set of n-dimensional morphisms of $\mathcal{C}$ is denoted by $\mathcal{C}_n$. The set of morphisms of $\mathcal{C}$ of dimension lower or equal than n is denoted by $\mathrm{tr}_n\mathcal{C}$. The element of $\mathcal{C}_0$ will be sometimes called *states*. An *initial state* (resp. *final state*) of $\mathcal{C}$ is a 0-morphism α such that $\alpha = s_0 x$ (resp. $\alpha = t_0 x$) implies $x = \alpha$.

2.4. Definition. *[6, 1] A cubical ω-category consists of a cubical set*

$$((K_n)_{n\geqslant 0}, \partial_i^\alpha, \epsilon_i)$$

together with two additional families of degeneracy maps called connections

$$\Gamma_i^\alpha : K_n \longrightarrow K_{n+1}$$

with $\alpha \in \{-,+\}$, $n \geqslant 1$ and $1 \leqslant i \leqslant n$ and a family of associative operations $+_j$ defined on $\{(x,y) \in K_n \times K_n, \partial_j^+ x = \partial_j^- y\}$ for $1 \leqslant j \leqslant n$ such that

1. $\partial_i^\alpha \Gamma_j^\beta = \Gamma_{j-1}^\beta \partial_i^\alpha$ *for all $i < j$ and all $\alpha, \beta \in \{-,+\}$*

2. $\partial_i^\alpha \Gamma_j^\beta = \Gamma_j^\beta \partial_{i-1}^\alpha$ *for all* $i > j+1$ *and all* $\alpha, \beta \in \{-,+\}$
3. $\partial_j^\pm \Gamma_j^\pm = \partial_{j+1}^\pm \Gamma_j^\pm = Id$
4. $\partial_j^\pm \Gamma_j^\mp = \partial_{j+1}^\pm \Gamma_j^\mp = \epsilon_j \partial_j^\pm$
5. $\Gamma_i^\pm \Gamma_j^\pm = \Gamma_{j+1}^\pm \Gamma_i^\pm$ *if* $i \leqslant j$
6. $\Gamma_i^\pm \Gamma_j^\mp = \Gamma_{j+1}^\mp \Gamma_i^\pm$ *if* $i < j$
7. $\Gamma_i^\pm \Gamma_j^\mp = \Gamma_j^\mp \Gamma_{i-1}^\pm$ *if* $i > j+1$
8. $\Gamma_i^\pm \epsilon_j = \epsilon_{j+1} \Gamma_i^\pm$ *if* $i < j$
9. $\Gamma_i^\pm \epsilon_j = \epsilon_i \epsilon_i$ *if* $i = j$
10. $\Gamma_i^\pm \epsilon_j = \epsilon_j \Gamma_{i-1}^\pm$ *if* $i > j$
11. $(x +_j y) +_j z = x +_j (y +_j z)$
12. $\partial_j^-(x +_j y) = \partial_j^-(x)$
13. $\partial_j^+(x +_j y) = \partial_j^+(y)$
14. $\partial_i^\alpha(x +_j y) = \begin{cases} \partial_i^\alpha(x) +_{j-1} \partial_i^\alpha(y) \text{ if } i < j \\ \partial_i^\alpha(x) +_j \partial_i^\alpha(y) \text{ if } i > j \end{cases}$
15. $(x +_i y) +_j (z +_i t) = (x +_j z) +_i (y +_j t)$.
16. $\epsilon_i(x +_j y) = \begin{cases} \epsilon_i(x) +_{j+1} \epsilon_i(y) \text{ if } i \leqslant j \\ \epsilon_i(x) +_j \epsilon_i(y) \text{ if } i > j \end{cases}$
17. $\Gamma_i^\pm(x +_j y) = \begin{cases} \Gamma_i^\pm(x) +_{j+1} \Gamma_i^\pm(y) \text{ if } i < j \\ \Gamma_i^\pm(x) +_j \Gamma_i^\pm(y) \text{ if } i > j \end{cases}$
18. *If* $i = j$, $\Gamma_i^-(x +_j y) = \begin{bmatrix} \epsilon_{j+1}(y) & \Gamma_j^-(y) \\ \Gamma_j^-(x) & \epsilon_j(y) \end{bmatrix} \begin{array}{l} j \\ \uparrow\!\!\rightarrow j{+}1 \end{array}$
19. *If* $i = j$, $\Gamma_i^+(x +_j y) = \begin{bmatrix} \epsilon_j(x) & \Gamma_j^+(y) \\ \Gamma_j^+(x) & \epsilon_{j+1}(x) \end{bmatrix} \begin{array}{l} j \\ \uparrow\!\!\rightarrow j{+}1 \end{array}$
20. $\Gamma_j^+ x +_{j+1} \Gamma_j^- x = \epsilon_j x$ *and* $\Gamma_j^+ x +_j \Gamma_j^- x = \epsilon_{j+1} x$
21. $\epsilon_i \partial_i^- x +_i x = x +_i \epsilon_i \partial_i^+ x = x$

The corresponding category with the obvious morphisms is denoted by ∞Cat*.*

Without further precisions, the word ω-category is always supposed to be taken in the sense of globular ω-category. In [2], it is proved that the category of cubical ω-categories and the category of globular ω-categories are equivalent.

Notation. If S is a set, the free abelian group generated by S is denoted by $\mathbb{Z}S$. By definition, an element of $\mathbb{Z}S$ is a formal linear combination of elements of S.

2.5. Definition. *[12] Let $\mathcal{C}$ be an ω-category. Let $\mathcal{C}_n$ be the set of n-dimensional morphisms of $\mathcal{C}$. Two n-morphisms x and y are homotopic if there exists $z \in \mathbb{Z}\mathcal{C}_{n+1}$ such that $s_n z - t_n z = x - y$. This property is denoted by $x \sim y$.*

We have already observed in [12] that the corner homologies do not induce functors from ωCat to the category of abelian groups. A notion of non-contracting ω-functors was required.

2.6. Definition. *[12] Let f be an ω-functor from $\mathcal{C}$ to $\mathcal{D}$. The morphism f is non-contracting if for any 1-dimensional $x \in \mathcal{C}$, the morphism $f(x)$ is a 1-dimensional morphism of $\mathcal{D}$.*

The theoretical developments of this paper and future works in progress entail the following definitions too.

2.7. Definition. *Let $\mathcal{C}$ be an ω-category. Then $\mathcal{C}$ is non-contracting if and only if for any $x \in \mathcal{C}$ of strictly positive dimension, $s_1 x$ and $t_1 x$ are 1-dimensional (they could be a priori 0-dimensional as well).*

A justification of this definition among a lot of them is that if $\mathcal{C}$ is an ω-category which is not non-contracting, then there exists a morphism u of $\mathcal{C}$ such that $dim(u) > 1$ and such that for instance $s_1 u$ is 0-dimensional. For example consider the two-element set $\{A, \alpha\}$ with the rules $s_1 A = t_1 A = s_0 A = t_0 A = \alpha$ and $s_2 A = t_2 A = A$. This defines an ω-category which is not non-contracting. Then A is 2-dimensional though $s_1 A$ and $t_1 A$ are 0-dimensional. And in this situation $\square_2^-(A)$ defined in Section 6.5 is not an element of the branching nerve, and therefore for that $\mathcal{C}$, the morphism $CF_2^-(\mathcal{C})$ (see Proposition 9.4) to $CR_2^-(\mathcal{C})$ is not defined.

Notation. The category of non-contracting ω-categories with the non-contracting ω-functors is denoted by ωCat_1.

If f is a non-contracting ω-functor from $\mathcal{C}$ to $\mathcal{D}$, then for any morphism $x \in \mathcal{C}$ of dimension greater than 1, $f(x)$ is of dimension greater than one as well. This is due to the equality $f(s_1 x) = s_1 f(x)$.

All globular ω-categories that will appear in this work will be non-contracting.

3. Reduced branching homology

3.1. The globular ω-category I^n. We need first to describe precisely the ω-category associated to the n-cube. Set $\underline{n} = \{1, ..., n\}$ and let $\underline{cub}^n$ be the set of maps from $\underline{n}$ to $\{-, 0, +\}$ (or in other terms the set of words of length n in the alphabet $\{-, 0, +\}$). We say that an element x of $\underline{cub}^n$ is of dimension p if $x^{-1}(0)$ is a set of p elements. The set $\underline{cub}^n$ is supposed to be graded by the dimension of its elements. The set $\underline{cub}^0$ is the set of maps from the empty set to $\{-, 0, +\}$ and therefore it is a singleton. Let $y \in \underline{cub}^i$. Let r_y

be the map from $(\underline{cub}^n)_i$ to $(\underline{cub}^n)_{dim(y)}$ defined as follows (with $x \in (\underline{cub}^n)_i$) : for $k \in \underline{n}$, $x(k) \neq 0$ implies $r_y(x)(k) = x(k)$ and if $x(k)$ is the l-th zero of the sequence $x(1), ..., x(n)$, then $r_y(x)(k) = y(\ell)$. If for any ℓ between 1 and i, $y(\ell) \neq 0$ implies $y(\ell) = (-)^\ell$, then we set $b_y(x) := r_y(x)$. If for any ℓ between 1 and i, $y(\ell) \neq 0$ implies $y(\ell) = (-)^{\ell+1}$, then we set $e_y(x) := r_y(x)$. We have

If x is an element of $\underline{cub}^n$, let us denote by $R(x)$ the subset of $\underline{cub}^n$ consisting of $y \in \underline{cub}^n$ such that y can be obtained from x by replacing some occurrences of 0 in x by $-$ or $+$. For example, $-00++- \in R(-000+-)$ but $+000+- \notin R(-000+-)$. If X is a subset of $\underline{cub}^n$, then let $R(X) = \bigcup_{x \in X} R(x)$. Notice that $R(X \cup Y) = R(X) \cup R(Y)$.

3.2. Theorem. *There is one and only one ω-category I^n such that*

1. *the underlying set of I^n is included in the set of subsets of $\underline{cub}^n$*
2. *the underlying set of I^n contains all subsets like $R(x)$ where x runs over $\underline{cub}^n$*
3. *all elements of I^n are compositions of $R(x)$ where x runs over $\underline{cub}^n$*
4. *for x p-dimensional with $p \geqslant 1$, one has*
$$s_{p-1}(R(x)) = R(\{b_y(x), dim(y) = p-1\})$$
$$t_{p-1}(R(x)) = R(\{e_y(x), dim(y) = p-1\})$$
5. *if X and Y are two elements of I^n such that $t_p(X) = s_p(Y)$ for some p, then $X \cup Y \in I^n$ and $X \cup Y = X *_p Y$.*

Moreover, all elements X of I^n satisfy the equality $X = R(X)$.

The elements of I^n correspond to the loop-free well-formed sub pasting schemes of the pasting scheme $\underline{cub}^n$ [15] [9] or to the molecules of an ω-complex in the sense of [25]. The condition "$X *_n Y$ *exists if and only if* $X \cap Y = t_n X = s_n Y$" of [25] is not necessary here because the situation of [25] Figure 2 cannot appear in a composable pasting scheme.

The map which sends every ω-category $\mathcal{C}$ to $\mathcal{N}^\square(\mathcal{C})_* = \omega Cat(I^*, \mathcal{C})$ induces a functor from ωCat to the category of cubical sets. If x is an element of $\omega Cat(I^n, \mathcal{C})$, $\epsilon_i(x)$ is the ω-functor from I^{n+1} to $\mathcal{C}$ defined by $\epsilon_i(x)(k_1...k_{n+1}) = x(k_1...\widehat{k_i}...k_{n+1})$ for all i between 1 and $n+1$ and $\partial_i^\alpha(x)$ is the ω-functor from I^{n-1} to $\mathcal{C}$ defined by $\partial_i^\alpha(x)(k_1...k_{n-1}) = x(k_1...k_{i-1}\alpha k_i...k_{n-1})$ for all i between 1 and n.

The arrow ∂_i^α for a given i such that $1 \leqslant i \leqslant n$ induces a natural transformation from $\omega Cat(I^n, -)$ to $\omega Cat(I^{n-1}, -)$ and therefore, by Yoneda, corresponds to an ω-functor δ_i^α from I^{n-1} to I^n. This functor is defined on the faces of I^{n-1} by $\delta_i^\alpha(k_1...k_{n-1}) = R(k_1...[\alpha]_i...k_{n-1})$. The notation $[...]_i$ means that the term inside the brackets is at the i-th place.

3.3. Definition. *The cubical set $(\omega Cat(I^*, \mathcal{C}), \partial_i^\alpha, \epsilon_i)$ is called the cubical singular nerve of the ω-category $\mathcal{C}$.*

3.4. Remark. For $\alpha \in \{-,+\}$, and $x \in \omega Cat(I^n,\mathcal{C})$, let

$$\partial^\alpha x := \sum_{i=1}^{n} (-1)^{i+1} \partial_i^\alpha x$$

Because of the cube axiom, one has $\partial^\alpha \circ \partial^\alpha = 0$.

3.5. Definition. *[12] Let $\mathcal{C}$ be a non-contracting ω-category. The set of ω-functors $x \in \omega Cat(I^n,\mathcal{C})$ such that for any 1-morphism u with $s_0 u = -_{n+1}$, $x(u)$ is 1-dimensional (a priori $x(u)$ could be 0-dimensional as well) is denoted by $\omega Cat(I^n,\mathcal{C})^-$. Then*

$$\partial^-(\mathbb{Z}\omega Cat(I^{*+1},\mathcal{C})^-) \subset \mathbb{Z}\omega Cat(I^*,\mathcal{C})^-$$

by construction. We set

$$H_*^-(\mathcal{C}) = H_*(\mathbb{Z}\omega Cat(I^*,\mathcal{C})^-,\partial^-)$$

and we call this homology theory the branching homology of $\mathcal{C}$. The cycles are called the branchings of $\mathcal{C}$. The map H_*^- induces a functor from ωCat_1 to Ab.

The definition of $\omega Cat(I^n,\mathcal{C})^-$ is a little bit different from that of [12]. Both definitions coincide if $\mathcal{C}$ is the free ω-category generated by a precubical set or a globular set. This new definition ensures that the elementary moves introduced in Section 7 are well-defined on the branching nerve. Otherwise it is easy to find counterexample, even in the case of a non-contracting ω-category.

3.6. Conjecture. (About the thin elements of the branching complex) *Let $\mathcal{C}$ be a globular ω-category which is either the free globular ω-category generated by a precubical set or the free globular ω-category generated by a globular set. Let x_i be elements of $\omega Cat(I^n,\mathcal{C})^-$ and let λ_i be natural numbers, where i runs over some set I. Suppose that for any i, $x_i(0_n)$ is of dimension strictly lower than n (one calls it a thin element). Then $\sum_i \lambda_i x_i$ is a boundary if and only if it is a cycle.*

The thin elements conjecture is not true in general. Here is a counterexample. Consider an ω-category $\mathcal{C}$ constructed by considering I^2 and by dividing by the relations $R(-0) = R(0-)$ and $R(-0) *_0 R(0+) = R(0-) *_0 R(+0)$. Then the ω-functor $F \in \omega Cat(I^2,\mathcal{C})^-$ induced by the identity functor from I^2 to itself is a thin cycle in the branching homology. One can verify that this cycle would be a boundary if and only if $R(0+)$ was homotopic to $R(+0)$ in $\mathcal{C}$. This observation suggests the following questions.

3.7. Definition. *Let $\mathcal{C}$ be an ω-category. Then the n-th composition law is said to be left regular up to homotopy if and only if for any morphisms x, y and z such that $x *_n y = x *_n z$, then $y \sim z$.*

3.8. Question. *Does the thin elements conjecture hold for an ω-category $\mathcal{C}$ such that all composition laws $*_n$ for any $n \geqslant 0$ are left regular up to homotopy ?*

3.9. Question. *How may we characterize the ω-categories for which the thin elements conjecture holds ?*

3.10. Definition. *Let $M_n^-(\mathcal{C}) \subset \mathbb{Z}\omega Cat(I^n,\mathcal{C})^-$ be the sub-$\mathbb{Z}$-module generated by the thin elements (M for "mince" which means "thin" in French). Set*

$$CR_n^-(\mathcal{C}) = \mathbb{Z}\omega Cat(I^n,\mathcal{C})^-/(M_n^-(\mathcal{C}) + \partial^- M_{n+1}^-(\mathcal{C}))$$

where $M_n^-(\mathcal{C}) + \partial^- M_{n+1}^-(\mathcal{C})$ is the sub-$\mathbb{Z}$-module of $\mathbb{Z}\omega Cat(I^n,\mathcal{C})^-$ generated by $M_n^-(\mathcal{C})$ and the image of $M_{n+1}^-(\mathcal{C})$ by ∂^-. The differential map ∂^- induces a differential map

$$CR_{n+1}^-(\mathcal{C}) \longrightarrow CR_n^-(\mathcal{C})$$

This chain complex is called the reduced branching complex of $\mathcal{C}$. The homology associated to this chain complex is denoted by $HR_*^-(\mathcal{C})$ and is called the reduced branching homology of $\mathcal{C}$.

3.11. Proposition. *Conjecture 3.6 is equivalent to the following statement : if $\mathcal{C}$ is the free ω-category generated by a precubical set or by a globular set, then the canonical map from the branching chain complex to the reduced branching chain complex of $\mathcal{C}$ is a quasi-isomorphism.*

Proof. By the following short exact sequence of chain complexes

$$0 \longrightarrow M_*^-(\mathcal{C}) + \partial^- M_{*+1}^-(\mathcal{C}) \longrightarrow \mathbb{Z}\omega Cat(I^*,\mathcal{C})^- \longrightarrow CR_*^-(\mathcal{C}) \longrightarrow 0$$

the assumption $H_n^-(\mathcal{C}) \cong HR_n^-(\mathcal{C})$ for all n is equivalent to the acyclicity of the chain complex $(M_*^- + \partial^- M_{*+1}^-, \partial^-)$ (notice that $M_0^-(\mathcal{C}) = M_1^-(\mathcal{C}) = 0$).

Now if Conjecture 3.6 holds, then take an element $x \in M_n^-(\mathcal{C}) + \partial^- M_{n+1}^-(\mathcal{C})$ which is a cycle. Then $x = t_1 + \partial^- t_2$ where $t_1 \in M_n^-(\mathcal{C})$ and $t_2 \in M_{n+1}^-(\mathcal{C})$. Then t_1 is a cycle in $\mathbb{Z}\omega Cat(I^n,\mathcal{C})^-$ and a linear combination of thin elements. Therefore t_1 is a cycle in $\mathbb{Z}\omega Cat(I^n, \mathrm{tr}_{n-1}\mathcal{C})^-$. By Conjecture 3.6, $t_1 = \partial^- t_3$ where $t_3 \in \mathbb{Z}\omega Cat(I^{n+1}, \mathrm{tr}_{n-1}\mathcal{C})^-$. Therefore $t_1 \in \partial^- M_{n+1}^-(\mathcal{C})$. Conversely, suppose that the sub-complex generated by the thin elements is acyclic. Take a cycle t of $\mathbb{Z}\omega Cat(I^n,\mathcal{C})^-$ which is a linear combination of thin elements. Then t is a cycle of $M_n^-(\mathcal{C}) + \partial^- M_{n+1}^-(\mathcal{C})$, therefore there exists $t_1 \in M_{n+1}^-(\mathcal{C})$ and $t_2 \in M_{n+2}^-(\mathcal{C})$ such that $t = \partial^-(t_1 + \partial^- t_2) = \partial^- t_1$. ∎

3.12. Definition. *Let x and y be two elements of $\mathbb{Z}\omega Cat(I^n,\mathcal{C})^-$. Then x and y are T-equivalent (T for thin) if the corresponding elements in the reduced branching complex are equal, that means if $x - y \in M_n^-(\mathcal{C}) + \partial^- M_{n+1}^-(\mathcal{C})$. This defines an equivalence relation on $\mathbb{Z}\omega Cat(I^n,\mathcal{C})^-$ indeed.*

4. Matrix notation for higher dimensional composition in the cubical singular nerve

There exists on the cubical nerve $\omega Cat(I^*,\mathcal{C})$ of an ω-category $\mathcal{C}$ a structure of cubical ω-categories [12] by setting

$$\Gamma_i^-(x)(k_1 \dots k_n) = x(k_1 \dots \max(k_i, k_{i+1}) \dots k_n)$$
$$\Gamma_i^+(x)(k_1 \dots k_n) = x(k_1 \dots \min(k_i, k_{i+1}) \dots k_n)$$

with the order $- < 0 < +$ and with the proposition-definition :

4.1. Proposition. *[12] Let $\mathcal{C}$ be a globular ω-category. For any strictly positive natural number n and any j between 1 and n, there exists one and only one natural map $+_j$ from the set of pairs (x, y) of $\mathcal{N}^{\square}(\mathcal{C})_n \times \mathcal{N}^{\square}(\mathcal{C})_n$ such that $\partial_j^+(x) = \partial_j^-(x)$ to the set $\mathcal{N}^{\square}(\mathcal{C})_n$ which satisfies the following properties :*

$$\begin{aligned}
&\partial_j^-(x +_j y) = \partial_j^-(x) \\
&\partial_j^+(x +_j y) = \partial_j^+(x) \\
&\partial_i^\alpha(x +_j y) = \begin{cases} \partial_i^\alpha(x) +_{j-1} \partial_i^\alpha(y) \text{ if } i < j \\ \partial_i^\alpha(x) +_j \partial_i^\alpha(y) \text{ if } i > j \end{cases}
\end{aligned}$$

Moreover, these operations induce a structure of cubical ω-category on $\mathcal{N}^{\square}(\mathcal{C})$.

The sum $(x +_i y) +_j (z +_i t) = (x +_j z) +_i (y +_j t)$ if there exists will be denoted by

$$\begin{bmatrix} x & z \\ y & t \end{bmatrix} \begin{matrix} \mathrm{i} \\ \llcorner\!\!\rightarrow \mathrm{j} \end{matrix}$$

and using this notation, one can write

- If $i = j$, $\Gamma_i^-(x +_j y) = \begin{bmatrix} \epsilon_{j+1}(y) & \Gamma_j^-(y) \\ \Gamma_j^-(x) & \epsilon_j(y) \end{bmatrix} \begin{matrix} \mathrm{j} \\ \llcorner\!\!\rightarrow \mathrm{j+1} \end{matrix}$
- If $i = j$, $\Gamma_i^+(x +_j y) = \begin{bmatrix} \epsilon_j(x) & \Gamma_j^+(y) \\ \Gamma_j^+(x) & \epsilon_{j+1}(x) \end{bmatrix} \begin{matrix} \mathrm{j} \\ \llcorner\!\!\rightarrow \mathrm{j+1} \end{matrix}$

The matrix notation can be generalized to any composition like

$$(a_{11} +_i \ldots +_i a_{1n}) +_j \ldots +_j (a_{m1} +_i \ldots +_i a_{mn})$$

whenever the sources and targets of the a_{ij} match up in an obvious sense (this is not necessarily true). In that case, the above expression is equal by the interchange law to

$$(a_{11} +_j \ldots +_j a_{m1}) +_i \ldots +_i (a_{1n} +_j \ldots +_j a_{mn})$$

and we can denote the common value by

$$\begin{bmatrix} a_{m1} & \ldots & a_{mn} \\ \vdots & & \vdots \\ a_{11} & \ldots & a_{1n} \end{bmatrix} \begin{matrix} \mathrm{j} \\ \llcorner\!\!\rightarrow \mathrm{i} \end{matrix}$$

In such a matrix, an element like $\epsilon_i x$ is denoted by $\sqsupset$. An element like $\epsilon_j x$ is denoted by $||$. In a situation where $i = j + 1$, an element like $\Gamma_j^-(x)$ is denoted by $\urcorner$ and an element like $\Gamma_j^+(x)$ is denoted by $\llcorner$. An element like $\epsilon_j\epsilon_j x = \epsilon_{j+1}\epsilon_j x$ is denoted by $\square$. With $i = j + 1$, we can verify some of the above formulae :

$$\Gamma_j^-(x +_j y) = \begin{bmatrix} \sqsupset & \urcorner \\ \urcorner & || \end{bmatrix} = \begin{bmatrix} \epsilon_{j+1}(y) & \Gamma_j^-(y) \\ \Gamma_j^-(x) & \epsilon_j(y) \end{bmatrix}$$

$$\Gamma_j^+(x +_j y) = \begin{bmatrix} || & \llcorner \\ \llcorner & \sqsupset \end{bmatrix} = \begin{bmatrix} \epsilon_j(x) & \Gamma_j^+(y) \\ \Gamma_j^+(x) & \epsilon_{j+1}(x) \end{bmatrix}$$

4.2. Definition. *[6][1] A n-shell in the cubical singular nerve is a family of $2(n+1)$ elements $x_i^\pm$ of $\omega Cat(I^n,\mathcal{C})$ such that $\partial_i^\alpha x_j^\beta = \partial_{j-1}^\beta x_i^\alpha$ for $1 \leqslant i < j \leqslant n+1$ and $\alpha, \beta \in \{-,+\}$.*

4.3. Definition. *The n-shell $(x_i^\pm)$ is fillable if*

1. *the sets $\{x_i^{(-)^i}, 1 \leqslant i \leqslant n+1\}$ and $\{x_i^{(-)^{i+1}}, 1 \leqslant i \leqslant n+1\}$ have each one exactly one non-thin element and if the other ones are thin.*

2. *if $x_{i_0}^{(-)^{i_0}}$ and $x_{i_1}^{(-)^{i_1+1}}$ are these two non-thin elements then there exists $u \in \mathcal{C}$ such that $s_n(u) = x_{i_0}^{(-)^{i_0}}(0_n)$ and $t_n(u) = x_{i_1}^{(-)^{i_1+1}}(0_n)$.*

The following proposition is an analogue of [1] Proposition 2.7.3.

4.4. Proposition. *[12] Let $(x_i^\pm)$ be a fillable n-shell with u as above. Then there exists one and only one element x of $\omega Cat(I^{n+1},\mathcal{C})$ such that $x(0_{n+1}) = u$, and for $1 \leqslant i \leqslant n+1$, and $\alpha \in \{-,+\}$ such that $\partial_i^\alpha x = x_i^\alpha$.*

Proposition 4.4 has a very important consequence concerning the use of the above notations. In dimension 2, an expression A like (for example)

$$\left[\begin{matrix} \llcorner & \urcorner & \square & \shortparallel & \square \\ = & x & = & y & = \\ \square & \llcorner & \urcorner & \llcorner & = \end{matrix}\right] \begin{smallmatrix} 1 \\ \uparrow \\ \llcorner\!\to 2 \end{smallmatrix}$$

is necessarily equal to

$$\left[\begin{matrix} x & y \\ \shortparallel & \llcorner \end{matrix}\right] \begin{smallmatrix} 1 \\ \uparrow \\ \llcorner\!\to 2 \end{smallmatrix}$$

because the labels of the interior are the same ($A(00) = (x{+_2}y)(00)$) and because the shells of 1-faces are equal ($\partial_1^- A = \partial_1^- x$, $\partial_1^+ A = \partial_1^+ x +_1 \partial_1^+ y$, $\partial_2^- A = \partial_2^- x$, $\partial_2^+ A = \partial_1^- y +_1 \partial_2^+ y$) : the dark lines represent degenerate elements which are like mirrors reflecting rays of light. This is a fundamental phenomenon to understand some of the calculations of this work. Notice that $A \neq x +_2 y$ because $\partial_1^- A \neq \partial_1^-(x +_2 y)$.

All calculations involving these matrix notations are justified because the Dawson-Paré condition holds in 2-categories due to the existence of connections (see [11] and [7]). The Dawson-Paré condition stands as follows : *suppose that a square α has a decomposition of one edge a as $a = a_1 +_1 a_2$. Then α has a compatible composition $\alpha = \alpha_1 +_i \alpha_2$, i.e. such that α_j has edge a_j for $j = 1, 2$.* This condition can be understood as a coherence condition which ensures that all "compatible" tilings represent the same object.

Let us mention that these special 2-dimensional notations for connections and degeneracies first appeared in [8] and in [23].

5. Relation between the simplicial nerve and the branching nerve

5.1. Proposition. *[12] Let $\mathcal{C}$ be an ω-category and $\alpha \in \{-,+\}$. We set*

$$\mathcal{N}_n^-(\mathcal{C}) = \omega Cat(I^{n+1}, \mathcal{C})^-$$

and for all $n \geqslant 0$ and all $0 \leqslant i \leqslant n$,

$$\partial_i : \mathcal{N}_n^-(\mathcal{C}) \longrightarrow \mathcal{N}_{n-1}^-(\mathcal{C})$$

is the arrow ∂_{i+1}^-, and

$$\epsilon_i : \mathcal{N}_n^-(\mathcal{C}) \longrightarrow \mathcal{N}_{n+1}^-(\mathcal{C})$$

is the arrow Γ_{i+1}^-. We obtain in this way a simplicial set

$$(\mathcal{N}_*^-(\mathcal{C}), \partial_i, \epsilon_i)$$

called the branching simplicial nerve of $\mathcal{C}$. The non normalized complex associated to it gives exactly the branching homology of $\mathcal{C}$ (in degree greater than or equal to 1). The map $\mathcal{N}^-$ induces a functor from ωCat_1 to the category $Sets^{\Delta^{op}}$ of simplicial sets.

The globular ω-category Δ^n. Now let us recall the construction of the ω-category called by Street the n-th oriental [26]. We use actually the construction appearing in [17]. Let O^n be the set of strictly increasing sequences of elements of $\{0, 1, \ldots, n\}$. A sequence of length $p+1$ will be of dimension p. If $\sigma = \{\sigma_0 < \ldots < \sigma_p\}$ is a p-cell of O^n, then we set $\partial_j\sigma = \{\sigma_0 < \ldots < \widehat{\sigma_j} < \ldots < \sigma_k\}$. If σ is an element of O^n, let $R(\sigma)$ be the subset of O^n consisting of elements τ obtained from σ by removing some elements of the sequence σ and let $R(\Sigma) = \bigcup_{\sigma\in\Sigma} R(\sigma)$. Notice that $R(\Sigma \cup T) = R(\Sigma) \cup R(T)$.

5.2. Theorem. *There is one and only one ω-category Δ^n such that*

1. *the underlying set of Δ^n is included in the set of subsets of O^n*
2. *the underlying set of Δ^n contains all subsets like $R(\sigma)$ where σ runs over O^n*
3. *all elements of Δ^n are compositions of $R(\sigma)$ where σ runs over O^n*
4. *for σ p-dimensional with $p \geqslant 1$, one has*

$$s_{p-1}(R(\sigma)) = R\left(\{\partial_j\sigma, j \text{ is even}\}\right)$$
$$t_{p-1}(R(\sigma)) = R\left(\{\partial_j\sigma, j \text{ is odd}\}\right)$$

5. *if Σ and T are two elements of Δ^n such that $t_p(\Sigma) = s_p(T)$ for some p, then $\Sigma \cup T \in \Delta^n$ and $\Sigma \cup T = \Sigma *_p T$.*

Moreover, all elements Σ of Δ^n satisfy the equality $\Sigma = R(\Sigma)$.

If $\mathcal{C}$ is an ω-category and if $x \in \omega Cat(\Delta^n, \mathcal{C})$, then consider the labeling of the faces of respectively Δ^{n+1} and Δ^{n-1} defined by :

- $\epsilon_i(x)(\sigma_0 < \ldots < \sigma_r) = x(\sigma_0 < \ldots < \sigma_{k-1} < \sigma_k - 1 < \ldots < \sigma_r - 1)$ if $\sigma_{k-1} < i$ and $\sigma_k > i$.
- $x(\sigma_0 < \ldots < \sigma_{k-1} < i < \sigma_{k+1} - 1 < \ldots < \sigma_r - 1)$ if $\sigma_{k-1} < i$, $\sigma_k = i$ and $\sigma_{k+1} > i+1$.
- $x(\sigma_0 < \ldots < \sigma_{k-1} < i < \sigma_{k+2} - 1 < \ldots < \sigma_r - 1)$ if $\sigma_{k-1} < i$, $\sigma_k = i$ and $\sigma_{k+1} = i+1$.

and

$$\partial_i(x)(\sigma_0 < \ldots < \sigma_s) = x(\sigma_0 < \ldots < \sigma_{k-1} < \sigma_k + 1 < \ldots < \sigma_s + 1)$$

where $\sigma_k, \ldots, \sigma_s \geqslant i$ and $\sigma_{k-1} < i$.

It turns out that $\epsilon_i(x) \in \omega Cat(\Delta^{n+1}, \mathcal{C})$ and $\partial_i(x) \in \omega Cat(\Delta^{n-1}, \mathcal{C})$. See [19, 28] for further information about simplicial sets. One has :

5.3. Definition. *[26] The simplicial set $(\omega Cat(\Delta^n, \mathcal{C}), \partial_i, \epsilon_i)$ is called the simplicial nerve $\mathcal{N}(\mathcal{C})$ of the globular ω-category $\mathcal{C}$. The corresponding homology is denoted by $H_*(\mathcal{C})$.*

5.4. Definition. *Let $\mathcal{C}$ be a non-contracting ω-category. By definition, $\mathcal{C}$ is of length at most 1 if and only if for any morphisms x and y of $\mathcal{C}$ such that $x *_0 y$ exists, then either x or y is 0-dimensional.*

5.5. Theorem. *Let $\mathcal{C}$ be an ω-category of length at most 1. Denote by $\mathbb{P}\mathcal{C}$ the unique ω-category such that its set of n-morphisms is exactly the set of $(n+1)$-morphisms of $\mathcal{C}$ for any $n \geqslant 0$ with an obvious definition of the source and target maps and of the composition laws. Then one has the isomorphisms $H_n(\mathbb{P}\mathcal{C}) \cong H^-_{n+1}(\mathcal{C})$ for $n \geqslant 1$.*

Proof. We give only a sketch of proof. By definition, $H^-_{n+1}(\mathcal{C}) = H_n(\mathcal{N}^-(\mathcal{C}))$ for $n \geqslant 1$. Because of the hypothesis on $\mathcal{C}$, every element x of $\omega Cat(I^{n+1}, \mathcal{C})^-$ is determined by the values of the $x(k_1 \ldots k_{n+1})$ where $k_1 \ldots k_{n+1}$ runs over the set of words on the alphabet $\{0, -\}$. It turns out that there is a bijective correspondence between O^n and the word of length $n+1$ on the alphabet $\{0, -\}$: if $\sigma_0 < \ldots < \sigma_p$ is an element of O^n, the associated word of length $n+1$ is the word $m_0 \ldots m_n$ such that $m_{\sigma_i} = 0$ and if $j \notin \{\sigma_0, \ldots, \sigma_p\}$, then $m_j = -$. It is then straightforward to check that the simplicial structure of $\mathcal{N}^-(\mathcal{C})$ is exactly the same as the simplicial structure of $\omega Cat(\Delta^*, \mathbb{P}\mathcal{C})$ in strictly positive dimension [1]. ■

The above proof together with Proposition 5.1 gives a new proof of the fact that if $x \in \omega Cat(\Delta^n, \mathcal{C})$, the labelings $\partial_i(x)$ and $\epsilon_i(x)$ above defined yield ω-functors from Δ^{n-1} (resp. Δ^{n+1}) to $\mathcal{C}$.

Notice that the above proof also shows that $H_n(\mathbb{P}\mathcal{C}) \cong H^+_{n+1}(\mathcal{C})$ where H^+_* is the merging homology functor [2] This means that for an ω-category of length at most 1, $H^-_{n+1}(\mathcal{C}) \cong H^+_{n+1}(\mathcal{C})$ for any $n \geqslant 1$. In general, this isomorphism is false as shown by

[1]The latter point is actually detailed in [13].

[2]Like the branching nerve, the definition of the merging nerve needs to be slightly change, with respect to the definition given in [12]. The correct definition is : an ω-functor x from I^n to a non-contracting ω-category $\mathcal{C}$ belongs to $\omega Cat(I^n, \mathcal{C})^+$ if and only if for any 1-morphism γ of I^n such that $t_0(\gamma) = R(+_n)$, then $x(\gamma)$ is a 1-dimensional morphism of $\mathcal{C}$.

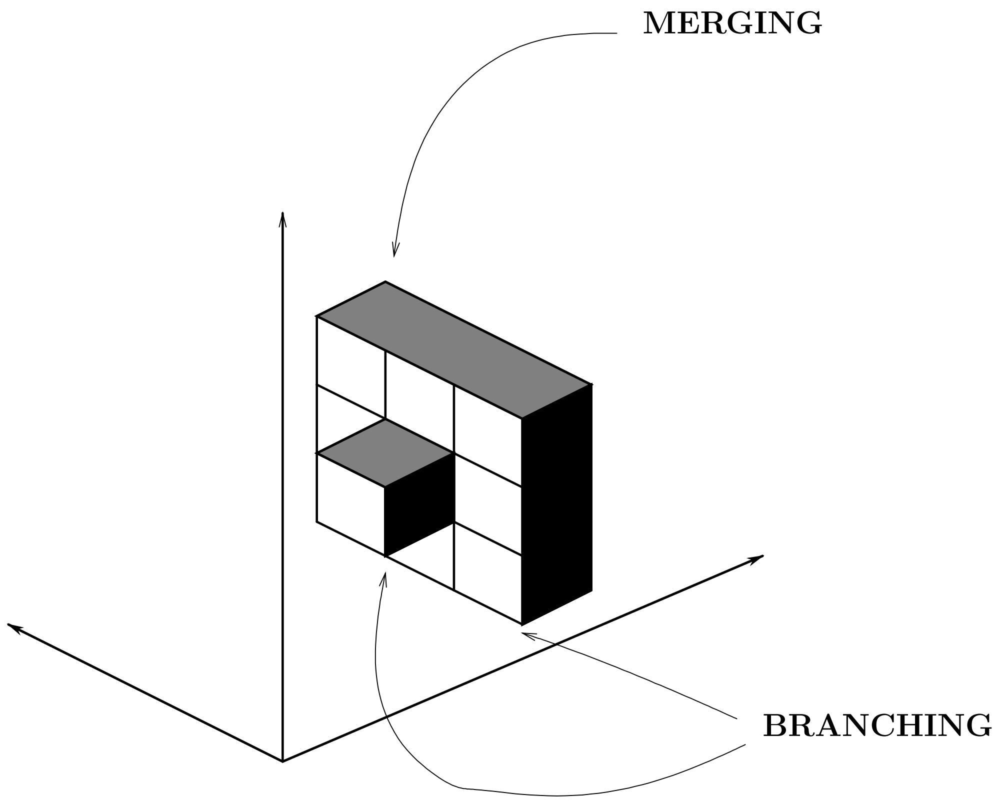


Figure 4: A case where branching and merging homologies are not equal in dimension 2

Figure 4. The precubical set we are considering in this figure is the complement of the depicted obstacle. Its branching homology is $\mathbb{Z} \oplus \mathbb{Z}$ in dimension two, and its merging homology is $\mathbb{Z}$ in the same dimension.

The result $H_n(\mathbb{P}\mathcal{C}) \cong H^-_{n+1}(\mathcal{C}) \cong H^+_{n+1}(\mathcal{C})$ for $\mathcal{C}$ of length at most one and for $n \geqslant 1$ also suggests that the program of constructing the analogue in the computer-scientific framework of usual homotopy invariants is complete for this kind of ω-categories. The simplicial set $\mathcal{N}(\mathbb{P}\mathcal{C})$ together with the graph obtained by considering the 1-category generated by the 1-morphisms of $\mathcal{C}$ up to homotopy contain indeed all the information about the topology of the underlying automaton. Intuitively the simplicial set $\mathcal{N}(\mathbb{P}\mathcal{C})$ is an *orthogonal section* of the automaton. Theorem 5.5 suggests that non-contracting ω-categories of length at most one play a particular role in this theory. This idea will be deepened in future works.

5.6. Corollary. *With the same notation, if $\mathbb{P}\mathcal{C}$ is the free globular ω-category generated by a composable pasting scheme in the sense of [15], then $H^-_{n+1}(\mathcal{C})$ vanishes for $n \geqslant 1$.*

Proof. By [25] Corollary 4.17 or by [17] Theorem 2.2, the simplicial nerve of the ω-category of any composable pasting scheme is contractible. ∎

5.7. Corollary. *Let 2_p be the free ω-category generated by a p-morphism. For any*

$p \geqslant 1$ and any $n \geqslant 1$, $H_n^-(2_p) = 0$.

PROOF. It is obvious for $n = 1$ and for $n \geqslant 2$, $H_n^-(2_p) \cong H_{n-1}(\mathbb{P}2_p)$. But $\mathbb{P}2_p = 2_{p-1}$, therefore it suffices to notice that the $(p-1)$-simplex is contractible. ■

5.8. COROLLARY. *For any $n \geqslant 1$, let $G_n\langle A, B\rangle$ be the ω-category generated by two n-morphisms A and B satisfying $s_{n-1}(A) = s_{n-1}(B)$ and $t_{n-1}(A) = t_{n-1}(B)$. Then*

$$H_p^-(G_n\langle A, B\rangle) = 0$$

for $0 < p < n$ or $p > n$ and

$$H_0^-(G_n\langle A, B\rangle) = H_n^-(G_n\langle A, B\rangle) = \mathbb{Z}.$$

PROOF. It suffices to calculate the simplicial homology of a simplicial set homotopic to a $(n-1)$-sphere. ■

Let S be a composable pasting scheme (see [15] for the definition and [17] for additional explanations). A reasonable conjecture is that the branching homology of the free ω-category $Cat(S)$ generated by any composable pasting scheme S vanishes in strictly positive dimension. By Conjecture 5.10, it would suffice for a given composable pasting scheme S to calculate the branching homology of the *bilocalization* $Cat(S)[I, F]$ of $Cat(S)$ with respect to its initial state I and its final state F, that is the sub-ω-category of $Cat(S)$ which consists of the p-morphisms x with $p \geqslant 1$ such that $s_0 x \in I$ and $t_0 x \in F$ and of the 0-morphism I and F. The question of the calculation of

$$H_{p+1}^-(Cat(S)[I, F]) \cong H_p(\mathbb{P}Cat(S)[I, F])$$

for $p \geqslant 1$ seems to be related to the existence of what Kapranov and Voevodsky call the derived pasting scheme of a composable pasting scheme [17]. It is in general not true that $\mathbb{P}Cat(S)[I, F]$ (denoted by $\Omega Cat(S)$ in their article) is the free ω-category generated by a composable pasting scheme. But we may wonder whether there is a "free cover" of $\Omega Cat(S)$ by some $Cat(T)$ for some composable pasting scheme T. This T would be the derived pasting scheme of S.

As for the n-cube I^n, its derived pasting scheme is the composable pasting scheme generated by the permutohedron [20, 4, 18]. Therefore one has

5.9. PROPOSITION. *Denote by $I^n[-_n, +_n]$ the bilocalization of I^n with respect to its initial state $-_n$ and its final state $+_n$, Then for all $p \geqslant 1$, $H_p^-(I^n[-_n, +_n]) = 0$.*

PROOF. It is clear that $H_1^-(I^n[-_n, +_n]) = 0$. For $p \geqslant 2$, $H_p^-(I^n[-_n, +_n]) \cong H_{p-1}(\Omega I^n)$ by Theorem 5.5. But ΩI^n is the free ω-category generated by the permutohedron, and with Corollary 5.6, one gets $H_p^-(I^n[-_n, +_n]) = 0$ for $p \geqslant 2$. ■

By filtrating the 1-morphisms of I^n by their length, it is possible to construct a spectral sequence abutting to the branching homology of I^n. More precisely a 1-morphism x is of length $\ell(x)$ if $x = R(x_1) *_0 \dots *_0 R(x_{\ell(x)})$ where $x_1, \dots, x_{\ell(x)} \in (\underline{cub}^n)_1$. Now let $F_p\omega Cat(I^*, I^n)^-$ be the subset of $x \in \omega Cat(I^*, I^n)^-$ such that for any $k_1 \dots k_* \in (\underline{cub}^*)_1$ such that $+ \in \{k_1, \dots, k_*\}$, $\ell(x(k_1 \dots k_*)) \leqslant p$. Then one gets a filtration on the branching complex of I^n such that

$$F_{-1}\mathbb{Z}\omega Cat(I^*, I^n)^- \subset F_0\mathbb{Z}\omega Cat(I^*, I^n)^- \subset \dots \subset F_n\mathbb{Z}\omega Cat(I^*, I^n)^-$$

with

$$\begin{aligned} F_{-1}\mathbb{Z}\omega Cat(I^*, I^n)^- &= 0 \\ F_0\mathbb{Z}\omega Cat(I^*, I^n)^- &= \mathbb{Z}\omega Cat(I^*, I^n(-_n, +_n))^- \\ F_n\mathbb{Z}\omega Cat(I^*, I^n)^- &= \mathbb{Z}\omega Cat(I^*, I^n)^-. \end{aligned}$$

One has $E^1_{pq} = H_{p+q}(F_p\mathbb{Z}\omega Cat(I^*, I^n)^-/F_{p-1}\mathbb{Z}\omega Cat(I^*, I^n)^-) \Longrightarrow H^-_{p+q}(I^n)$. By Proposition 5.9, $E_{0q} = 0$ if $q \neq 0$ and $E_{00} = \mathbb{Z}$.

The above spectral sequence probably plays a role in the following conjecture :

5.10. Conjecture. *Let $\mathcal{C}$ be a finite ω-category (that is such that the underlying set is finite). Let I be the set of initial states of $\mathcal{C}$ and let F be the set of final states of $\mathcal{C}$ (then $H^-_0(\mathcal{C}) = H^-_0(\mathcal{C}[I, F]) = \mathbb{Z}F$). If for any $n > 0$, $H^-_n(\mathcal{C}[I, F]) = 0$, then for any $n > 0$, $H^-_n(\mathcal{C}) = 0$.*

By [17], $\Omega\Delta^n = I^{n-1}$, therefore the vanishing of the branching homology of I^{n-1} in strictly positive dimension and Conjecture 5.10 would enable to establish that $H^-_p(\Delta^n) = 0$ for $p > 0$ and for any n.

6. About folding operators

The aim of this section is to introduce an analogue in our framework of the usual folding operators in cubical ω-categories. First we show how to recover the usual folding operators in our context.

The notations $\square_0$ or $\square^-_0$ (resp. $\square_1$ or $\square^-_1$) correspond to the canonical map from $\mathcal{C}_0$ to $\omega Cat(I^0, \mathcal{C})$ (resp. from $\mathrm{tr}_1\mathcal{C}$ to $\omega Cat(I^1, \mathcal{C})$). Now let us recall the construction of the operators $\square^-_n$ of [12].

6.1. Proposition. *[12] Let $\mathcal{C}$ be an ω-category and let $n \geqslant 1$. There exists one and only one natural map $\square^-_n$ from $tr_n\mathcal{C}$ to $\omega Cat(I^n, \mathcal{C})$ such that the following axioms hold :*

1. *one has $ev_{0_n}\square_n = Id_{tr_n\mathcal{C}}$ where $ev_{0_n}(x) = x(0_n)$ is the label of the interior of x.*
2. *if $n \geqslant 3$ and $1 \leqslant i \leqslant n-2$, then $\partial^\pm_i\square^-_n = \Gamma^-_{n-2}\partial^\pm_i\square^-_{n-1}s_{n-1}$.*
3. *if $n \geqslant 2$ and $n-1 \leqslant i \leqslant n$, then $\partial^-_i\square^-_n = \square^-_{n-1}d^{(-)^i}_{n-1}$ and $\partial^+_i\square^-_n = \epsilon_{n-1}\partial^+_{n-1}\square^-_{n-1}s_{n-1}$.*

Moreover for $1 \leqslant i \leqslant n$, we have $\partial^\pm_i\square^-_n s_n = \partial^\pm_i\square^-_n t_n$ and if x is of dimension greater or equal than 1, then $\square^-_n(x) \in \omega Cat(I^n, \mathcal{C})^-$.

6.2. THE USUAL FOLDING OPERATORS. One defines a natural map $\square_n$ from $\mathcal{C}_n$ to $\omega Cat(I^n, \mathcal{C})$ by induction on $n \geqslant 2$ as follows (compare with Proposition 6.1).

6.3. PROPOSITION. *For any natural number n greater or equal than 2, there exists a unique natural map $\square_n$ from $\mathcal{C}_n$ to $\omega Cat(I^n, \mathcal{C})$ such that*

1. *the equality $\square_n(x)(0_n) = x$ holds.*

2. *one has $\partial_1^\alpha \square_n = \square_{n-1} d_{n-1}^{(-)^\alpha}$ for $\alpha = \pm$.*

3. *for $1 < i \leqslant n$, one has $\partial_i^\alpha \square_n = \epsilon_1 \partial_{i-1}^\alpha \square_{n-1} s_{n-1}$.*

Moreover for $1 \leqslant i \leqslant n$, we have $\partial_i^\pm \square_n s_n u = \partial_i^\pm \square_n t_n u$ for any $(n+1)$-morphism u.

PROOF. The induction equations define a fillable $(n-1)$-shell (see Proposition 4.4). ∎

6.4. PROPOSITION. *For all $n \geqslant 0$, the evaluation map $ev_{0_n} : x \mapsto x(0_n)$ from $\omega Cat(I^n, \mathcal{C})$ to $\mathcal{C}$ induces a bijection from $\gamma\mathcal{N}^\square(\mathcal{C})_n$ to $tr_n\mathcal{C}$ where γ is the functor defined in [1].*

PROOF. Obvious for $n = 0$ and $n = 1$. Recall that γ is defined by

$$(\gamma G)_n = \{x \in G_n, \partial_j^\alpha x \in \epsilon_1^{j-1} G_{n-j} \text{ for } 1 \leqslant j \leqslant n, \alpha = 0, 1\}$$

Let us suppose that $n \geqslant 2$ and let us proceed by induction on n. Since $ev_{0_n} \square_n(u) = u$ by the previous proposition, then the evaluation map ev from $\gamma\mathcal{N}^\square(\mathcal{C})_n$ to $\mathrm{tr}_n\mathcal{C}$ is surjective. Now let us prove that $x \in \gamma\mathcal{N}^\square(\mathcal{C})_n$ and $y \in \gamma\mathcal{N}^\square(\mathcal{C})_n$ and $x(0_n) = y(0_n) = u$ imply $x = y$. Since x and y are in $\gamma\mathcal{N}^\square(\mathcal{C})_n$, then one sees immediately that the four elements $\partial_1^\pm x$ and $\partial_1^\pm y$ are in $\gamma\mathcal{N}^\square(\mathcal{C})_{n-1}$. Since all other $\partial_i^\alpha x$ and $\partial_i^\alpha y$ are thin, then $\partial_1^- x(0_{n-1}) = \partial_1^- y(0_{n-1}) = s_{n-1} u$ and $\partial_1^+ x(0_{n-1}) = \partial_1^+ y(0_{n-1}) = t_{n-1} u$. By induction hypothesis, $\partial_1^- x = \partial_1^- y = \square_{n-1}(s_{n-1} u)$ and $\partial_1^+ x = \partial_1^+ y = \square_{n-1}(t_{n-1} u)$. By hypothesis, one can set $\partial_j^\alpha x = \epsilon_1^{j-1} x_j^\alpha$ and $\partial_j^\alpha y = \epsilon_1^{j-1} y_j^\alpha$ for $2 \leqslant j \leqslant n$. And one gets $x_j^\alpha = (\partial_1^\alpha)^{j-1} \partial_j^\alpha x = (\partial_1^\alpha)^j x = (\partial_1^\alpha)^j y = y_j^\alpha$. Therefore $\partial_j^\alpha x = \partial_j^\alpha y$ for all $\alpha \in \{-, +\}$ and all $j \in [1, \ldots, n]$. By Proposition 4.4, one gets $x = y$. ∎

The above proof shows also that the map which associates to any cube x of the cubical singular nerve of $\mathcal{C}$ the cube $\square_{dim(x)}(x(0_{dim(x)}))$ is exactly the usual folding operator as exposed in [1].

Unfortunately, these important operators are not internal to the branching complex, due to the fact that an n-cube x of the cubical singular nerve is in the branching complex if and only for any 1-morphism γ of I^n starting from the initial state $-_n$ of the n-cube, $x(\gamma)$ is 1-dimensional (see Definition 3.5). But for example $(\square_n(x(0_n)))(- \ldots - 0)$ is 0-dimensional.

6.5. THE NEGATIVE FOLDING OPERATORS. The idea of the negative folding operator Φ_n^- is to "concentrate" a n-cube x of the cubical singular nerve of an ω-category $\mathcal{C}$ to the faces $\delta_{n-1}^-(0_{n-1})$ and $\delta_n^-(0_{n-1})$. Hence the following definition.

6.6. DEFINITION. *Set* $\Phi_n^-(x) = \square_n^-(x(0_n))$. *This operator is called the* n*-dimensional negative folding operator.*

It is clear that $x \in \omega Cat(I^n, \mathcal{C})^-$ implies $\Phi_n^-(x) \in \omega Cat(I^n, \mathcal{C})^-$. Therefore Φ_n^- yields a map from $\omega Cat(I^n, \mathcal{C})^-$ to itself.

Since $\partial_{n-1}^-\square_n^- = \square_{n-1}^- d_{n-1}^{(-)^{n-1}}$ and $\partial_n^-\square_n^- = \square_{n-1}^- d_{n-1}^{(-)^n}$, the effect of $\square_n^-(x(0_n))$ is indeed to concentrate the faces of the n-cube x on the faces $\delta_{n-1}^-(0_{n-1})$ and $\delta_n^-(0_{n-1})$. All the $(n-1)$-cubes $\partial_i^\alpha\square_n^-(x)$ for $(i,\alpha) \notin \{(n-1,-),(n,-)\}$ are thin. Of course there is not only one way of concentrating the faces of x on $\delta_{n-1}^-(0_{n-1})$ and $\delta_n^-(0_{n-1})$. But in some way, they are all equivalent in the branching complex (Corollary 8.4). We could decide also to concentrate the n-cubes for $n \geqslant 2$ on the faces $\delta_1^-(0_{n-1})$ and $\delta_2^-(0_{n-1})$, or more generally to concentrate the n-cubes on the faces $\delta_{p(n)}^-(0_{n-1})$ and $\delta_{q(n)}^-(0_{n-1})$ where $p(n)$ and $q(n)$ would be integers of opposite parity for all $n \geqslant 2$. Let us end this section by explaining precisely the structure of all these choices.

In an ω-category, recall that $d_n^- = s_n$, $d_n^+ = t_n$ and by convention, let $d_\omega^- = d_\omega^+ = Id$. All the usual axioms of globular ω-categories remain true with this convention and the partial order $n < \omega$ for any natural number n.

If x is an element of an ω-category $\mathcal{C}$, we denote by $\langle x\rangle$ the ω-category generated by x. The underlying set of $\langle x\rangle$ is $\{s_n x, t_n x, n \in \mathbb{N}\}$. We denote by 2_n any ω-category freely generated by one n-dimensional element.

Let $R(k_1 \dots k_n) \in I^n$ a face. Denote by $ev_{k_1\dots k_n}$ the natural transformation from $\omega Cat(I^n, -)$ to tr_ω which maps f to $f(R(k_1\dots k_n))$.

6.7. DEFINITION. *Let* $n \in \mathbb{N}$. *Recall that* tr_n *is the forgetful functor from* ω*-categories to sets which associates to any* ω*-category its set of morphisms of dimension lower or equal than* n *and let* i_n *be the inclusion functor from* tr_{n-1} *to* tr_n. *We call cubification of dimension* n*, or* n*-cubification a natural transformation* $\square$ *from* tr_n *to* $\omega Cat(I^n, -)$. *If moreover,* $ev_{0_n}\square = Id$*, we say that the cubification is thick.*

We see immediately that $\square_n^-$, $\square_n$ (and $\square_n^+$ of [12]) are examples of thick n-cubifications. By Yoneda the set of n-cubifications is in bijection with the set of ω-functors from I^n to 2_n. So for a given n, there is a finite number of n-cubifications.

6.8. PROPOSITION. *Let* f *be a natural transformation from* tr_m *to* tr_n *with* $m, n \in \mathbb{N} \cup \{\omega\}$. *Then there exists* $p \leqslant m$ *and* $\alpha \in \{-,+\}$ *such that* $f = d_p^\alpha$. *And necessarily,* $p \leqslant Inf(m,n)$.

PROOF. Denote by

$$<A> = 2_n \xrightarrow{g} <B> = 2_m$$

the ω-functor which corresponds to f by Yoneda. Then $g(A) = d_p^\alpha(B)$ for some p and some α. And necessarily, $p \leqslant \min(m,n)$ (where the notation *min* means the smallest element). ■

6.9. COROLLARY. *Let* $\square$ *be a* n*-cubification with* $n \geqslant 1$ *a natural number. Then for any* i *with* $1 \leqslant i \leqslant n$, $\partial_i^\pm\square s_n = \partial_i^\pm\square t_n$.

PROOF. We have

$$\partial_i^{\pm}\square s_n x(l_1 \ldots l_{n-1}) = ev_{l_1 \ldots [\pm]_i \ldots l_{n-1}}\square s_n(x)$$

But $ev_{l_1 \ldots [\pm]_i \ldots l_{n-1}}\square$ is a natural transformation from tr_n to tr_{n-1}. By Proposition 6.8, we get

$$\partial_i^{\pm}\square s_n x(l_1 \ldots l_{n-1}) = ev_{l_1 \ldots [\pm]_i \ldots l_{n-1}}\square t_n(x) = \partial_i^{\pm}\square t_n x(l_1 \ldots l_{n-1})$$

■

We arrive at a theorem which explains the structure of all cubifications :

6.10. THEOREM. *Let $\square$ be a thick n-cubification and let f be an ω-functor from I^{n+1} to I^n such that $f(R(0_{n+1})) = R(0_n)$. Denote by f^* the corresponding natural transformation from $\omega Cat(I^n, -)$ to $\omega Cat(I^{n+1}, -)$. Then there exists one and only one thick $(n+1)$-cubification denoted by $f^*.\square$ such that for $1 \leqslant i \leqslant n+1$,*

$$(f^*.\square)i_{n+1} = f^*\square$$

where i_{n+1} is the canonical natural transformation from tr_n to tr_{n+1}.

PROOF. One has

$$\begin{aligned} \partial_i^{\alpha}(f^*.\square) &= \partial_i^{\alpha}(f^*.\square)d_n^{(-)^i} \\ &= \partial_i^{\alpha}(f^*.\square)i_{n+1}d_n^{(-)^i} \\ &= \partial_i^{\alpha}f^*\square d_n^{(-)^i} \end{aligned}$$

Therefore if $x \in \mathcal{C}_{n+1}$ for some ω-category $\mathcal{C}$, then $\partial_i^{\alpha}(f^*.\square)x = \partial_i^{\alpha}f^*\square d_n^{(-)^i}x$ for $1 \leqslant i \leqslant n+1$ and we obtain a fillable n-shell in the sense of Proposition 4.4. ■

6.11. COROLLARY. *Let u be an ω-functor from I^n to 2_n which maps $R(0_n)$ to the unique n-morphism of 2_n (we will say that u is thick because the corresponding cubification is also thick). Let f be an ω-functor from I^{n+1} to I^n which maps $R(0_{n+1})$ to $R(0_n)$. Then there exists one and only one thick ω-functor v from I^{n+1} to 2_{n+1} such that the following diagram commutes :*

$$\begin{tikzcd} I^{n+1} \arrow[r, "v"] \arrow[d, "f"'] & 2_{n+1} \arrow[d] \\ I^n \arrow[r, "u"] & 2_n \end{tikzcd}$$

the arrow from 2_{n+1} to 2_n being the unique ω-functor which sends the $(n+1)$-cell of 2_{n+1} to the n-cell of 2_n.

If $\square$ is a n-cubification and f_i thick ω-functors from I^{n+i+1} to I^{n+i} for $0 \leqslant i \leqslant p$ then we can denote without ambiguity by $f_p.f_{p-1}.\ldots.f_0.\square$ the $(n+p)$-cubification $f_p.(f_{p-1}.(\ldots.f_0.\square))$. Let us denote by $\square_0$ the unique 0-cubification. We have the following formulas :

6.12. Proposition. *Let $x \in \mathcal{C}$ be a p-dimensional morphism with $p \geqslant 1$ and let $n \geqslant p$. Then*

$$\square_n^- x = \Gamma_{n-1}^- \dots \Gamma_p^- \square_p^- x$$

(by convention, the above formula is tautological for $n = p$)

Proof. We are going to show the formula by induction on n. The case $n = p$ is trivial. If $i \leqslant n-1$, then $\partial_i^\pm \square_{n+1}^- x = \Gamma_{n-1}^- \partial_i^\pm \square_n^- x = \partial_i^\pm \Gamma_n^- \Gamma_{n-1}^- \dots \Gamma_p^- \square_p^- x$. And if $i \geqslant n$, then

$$\partial_i^- \square_{n+1}^- x = \square_n^- x = \partial_i^- \Gamma_n^- \Gamma_{n-1}^- \dots \Gamma_p^- \square_p^- x$$

and

$$\partial_i^+ \square_{n+1}^- x = \epsilon_n \partial_i^+ n \square_n^- x = \partial_i^+ \Gamma_n^- \Gamma_{n-1}^- \dots \Gamma_p^- \square_p^- x.$$

So the labelings $\square_{n+1}^- x$ and $\Gamma_n^- \dots \Gamma_p^- \square_p^- x$ of I^{n+1} are the same ones. ∎

6.13. Proposition. *For $n \geqslant 1$, we have $\square_1^- = \epsilon_1 . \square_0$ and*

$$\square_n^- = \Gamma_{n-1}^- . \dots . \Gamma_1^- . \epsilon_1 . \square_0$$

Proof. It is an immediate consequence of Proposition 6.12 and of the uniqueness of Theorem 6.10. ∎

The converse of Theorem 6.10 is true. That is :

6.14. Proposition. *Let v be a thick ω-functor from I^{n+1} to 2_{n+1}. Then there exists an ω-functor f such that for any thick ω-functor u from I^n to 2_n, the following diagram commutes :*

$$\begin{tikzcd}
I^{n+1} \arrow[r, "v"] \arrow[d, "f"'] & 2_{n+1} \arrow[d] \\
I^n \arrow[r, "u"] & 2_n
\end{tikzcd}$$

Proof. Set $v(R(k_1 \dots k_{n+1})) = d_{n_{k_1 \dots k_{n+1}}}^{\alpha_{k_1 \dots k_{n+1}}}(A)$ where $\langle A \rangle = 2_{n+1}$ and set $\langle B \rangle = 2_n$. By hypothesis, the equality $f(0_{n+1}) = R(0_n)$ holds and let

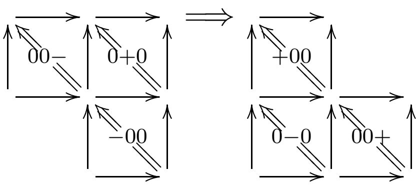
$$f(k_1 \dots k_{n+1}) = d_{n_{k_1 \dots k_{n+1}}}^{\alpha_{k_1 \dots k_{n+1}}}(R(0_n))$$

Take any thick ω-functor u from I^n to 2_n. Then

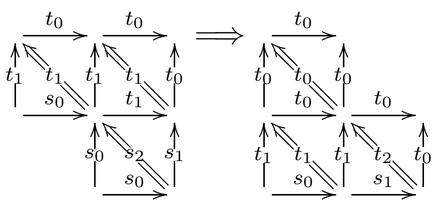
$$u \circ f(R(k_1 \dots k_{n+1})) = u(d_{n_{k_1 \dots k_{n+1}}}^{\alpha_{k_1 \dots k_{n+1}}}(R(0_n))) = d_{n_{k_1 \dots k_{n+1}}}^{\alpha_{k_1 \dots k_{n+1}}} u(R(0_n))$$
$$= d_{n_{k_1 \dots k_{n+1}}}^{\alpha_{k_1 \dots k_{n+1}}}(B)$$

By Proposition 4.4, it is clear that f induces an ω-functor. ∎

Here is an example of cubification : if the following picture depicts the 3-cube,



we can represent a 3-cubification $\square$ by indexing each face $k_1k_2k_3$ by the corresponding value of $ev_{k_1k_2k_3}\square i_3$ which is equal to s_d or t_d for some d between 0 and 2. So let us take $\square$ as follows :



We see that $\partial_1^-\square i_3 = \Gamma_1^+.\square_1$ and that $\partial_3^+\square i_3 = \Gamma_1^-.\square_1$.

Now let us come back to our choice. It is not completely arbitrary anyway because the operator $\square_n^-$ satisfies the following important property : if u is a n-morphism with $n \geqslant 2$, then $\square_n^-(u)$ is a simplicial homotopy within the branching nerve between $\square_{n-1}^-(s_{n-1}u)$ and $\square_{n-1}^-(t_{n-1}u)$. Moreover, the family of cubifications $(\square_n^-)_{n\geqslant 0}$ is the only family of cubifications which satisfies this property because it is equivalent to defining a n-shell for all n. However most of the results of the sequel can be probably adapted to any family of n-cubification, provided that they yield internal operations on the branching nerve (see Conjecture 7.7 and 7.8).

6.15. Characterization of the negative folding operators. Now here is a useful property of the folding operators :

6.16. Theorem. *Let $\mathcal{C}$ be an ω-category. Let x be an element of $\mathcal{N}_n^\square(\mathcal{C})$. Then the following two conditions are equivalent :*

1. *the equality $x = \Phi_n^-(x)$ holds*

2. *for $1 \leqslant i \leqslant n$, one has $\partial_i^+ x \in Im(\epsilon_1^{n-1})$, and for $1 \leqslant i \leqslant n-2$, one has $\partial_i^- x \in Im(\Gamma_{n-2}^- \dots \Gamma_i^-)$.*

Proof. If $x = \Phi_n^-(x)$, then $x = \square_n^-(x(0_n))$ and by construction of $\square_n^-$,

$$\partial_i^-\square_n^-(x(0_n)) = \Gamma_{n-2}^- \dots \Gamma_i^- \square_i^- d_i^{(-)^i} x(0_n)$$

for any $1 \leqslant i \leqslant n-2$ and $\partial_i^+ x$ is 0-dimensional for any $1 \leqslant i \leqslant n$. For n equal to 0, 1 or 2, the converse is obvious. Suppose the converse proved for $n-1 \geqslant 2$ and let us prove it by induction for $n \geqslant 3$. By hypothesis, as soon as $+ \in \{k_1, \dots, k_n\}$, then $x(k_1 \dots k_n)$ is 0-dimensional. For $1 \leqslant i \leqslant n-3$, one has

$$\begin{aligned}
\partial_i^- \partial_n^- x &= \partial_{n-1}^- \partial_i^- x \\
&= \partial_{n-1}^- \Gamma_{n-2}^- \dots \Gamma_i^- Y_i \text{ for some } Y_i \in \mathcal{N}_i^{\square}(\mathcal{C}) \\
&= \Gamma_{n-3}^- \dots \Gamma_i^- Y_i
\end{aligned}$$

and

$$\begin{aligned}
\partial_i^- \partial_{n-1}^- x &= \partial_{n-2}^- \partial_i^- x \\
&= \partial_{n-2}^- \Gamma_{n-2}^- \dots \Gamma_i^- Y_i \\
&= \Gamma_{n-3}^- \dots \Gamma_i^- Y_i
\end{aligned}$$

therefore $\partial_{n-1}^- x$ and $\partial_n^- x$ satisfy the induction hypothesis. So $\partial_{n-1}^- x = \Phi_{n-1}^-(\partial_{n-1}^- x)$ and $\partial_n^- x = \Phi_{n-1}^-(\partial_n^- x)$. Since the $\partial_i^- x$ are thin $(n-1)$-cubes for all i between 1 and $n-2$, then $d_{n-1}^{(-)^{n-1}}(x(0_n)) = \partial_{n-1}^- x(0_{n-1})$ and $d_{n-1}^{(-)^n}(x(0_n)) = \partial_n^- x(0_{n-1})$. Therefore $\partial_{n-1}^- x = \square_{n-1}^-(d_{n-1}^{(-)^{n-1}}(x(0_n)))$ and $\partial_n^- x = \square_{n-1}^-(d_{n-1}^{(-)^n}(x(0_n)))$. For $1 \leqslant i \leqslant n-2$, one has

$$\begin{aligned}
Y_i &= \partial_i^- \dots \partial_{n-2}^- \partial_i^- x \\
&= \partial_i^- \dots \partial_{n-1}^- x \\
&= \partial_i^- \dots \partial_{n-2}^- \square_{n-1}^-(d_{n-1}^{(-)^{n-1}}(x(0_n))) \\
&= \partial_i^- \dots \partial_{n-3}^- \square_{n-2}^-(d_{n-2}^{(-)^{n-2}}(x(0_n))) \\
&= (\dots) \\
&= \square_i^-(d_i^{(-)^i}(x(0_n)))
\end{aligned}$$

therefore an easy calculation shows that $x = \square_n^-(x(0_n))$. ∎

6.17. Corollary. *The folding operator Φ_n^- is idempotent.*

The end of this section is devoted to the description of Φ_2^- and Φ_3^-. Since $\partial_1^- \square_2^- = \square_1 s_1$ and $\partial_2^- \square_2^- = \square_1 t_1$, then one has for any ω-functor x from I^2 to $\mathcal{C}$

$$\Phi_2^-(x) = \begin{bmatrix} \urcorner & \square \\ x & \urcorner \end{bmatrix} \begin{array}{l} 1 \\ \uparrow\!\!\rightarrow 2 \end{array}$$

If x is an ω-functor from I^3 to $\mathcal{C}$, then

$$s_2(x(000)) = \begin{bmatrix} \partial_3^- x & \partial_2^+ x \\ \llcorner & \partial_1^- x \end{bmatrix} \begin{array}{l} 1 \\ \uparrow\!\!\rightarrow 2(00) \end{array}$$

because the 2-source of $R(000)$ in I^3 looks like

$$\begin{bmatrix} R(00-) & R(0+0) \\ \llcorner & R(-00) \end{bmatrix}$$

and

$$t_2(x(000)) = \left[\begin{array}{cc} \partial_1^+ x & \urcorner \\ \partial_2^- x & \partial_3^+ x \end{array}\right] \begin{array}{l} 1 \\ \llcorner\!\!\rightarrow 2 \end{array} (00)$$

because the 2-target of $R(000)$ in I^3 looks like

$$\left[\begin{array}{cc} R(+00) & \urcorner \\ R(0-0) & R(00+) \end{array}\right]$$

So by convention, an element x of $\omega Cat(I^3, \mathcal{C})$ will be represented as follows

$$x = \begin{array}{|c|c|}\hline A & B \\ \hline & C \\ \hline \end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline D & \\ \hline E & F \\ \hline \end{array}$$

where $A = \partial_3^- x$, $B = \partial_2^+ x$, $C = \partial_1^- x$, $D = \partial_1^+ x$, $E = \partial_2^- x$, $F = \partial_3^+ x$ and $x(000) = G$[3].

With this convention, $\Gamma_1^- y$ for $y \in \omega Cat(I^2, \mathcal{C})$ is equal to

$$\begin{array}{|c|c|}\hline \urcorner & \shortparallel \\ \hline & y \\ \hline \end{array} \overset{y(00)}{\Longrightarrow} \begin{array}{|c|c|}\hline \shortparallel & \\ \hline y & \urcorner \\ \hline \end{array}$$

One has $\partial_1^\pm \square_3^- = \Gamma_1^- \partial_1^\pm \square_2^- s_2$, $\partial_2^- \square_3^- = \square_2^- t_2$, $\partial_3^- \square_3^- = \square_2^- s_2$, $\partial_1^+ \square_3^- = \partial_2^+ \square_3^- = \partial_3^+ \square_3^- = \square_2^- t_0$ by definition of $\square_3^-$. Therefore

$$\begin{array}{rcl} \partial_i^+ \Phi_3^-(x) & = & \square_2^- t_0(G) \\ \partial_2^- \Phi_3^-(x) & = & \square_2^- t_2(G) \\ \partial_3^- \Phi_3^-(x) & = & \square_2^- s_2(G) \end{array}$$

and

$$\partial_1^\pm \Phi_3^-(x) = \Gamma_1^- \partial_1^\pm \square_2^- \left[\begin{array}{cc} A & B \\ \llcorner & C \end{array}\right] \begin{array}{l} 1 \\ \llcorner\!\!\rightarrow 2 \end{array}$$

$$= \Gamma_1^- \partial_1^\pm \left[\begin{array}{cccc} = & \urcorner & \square & \square \\ \urcorner & \shortparallel & \square & \square \\ A & B & = & \urcorner \\ \llcorner & C & \urcorner & \shortparallel \end{array}\right] \begin{array}{l} 1 \\ \llcorner\!\!\rightarrow 2 \end{array}$$

$$= \left\{ \begin{array}{c} \Gamma_1^-(\partial_1^- C +_1 \partial_2^+ C +_1 \partial_2^+ B) \text{ in the negative case} \\ \Gamma_1^- \partial_1^+ \partial_1^+ B \text{ in the positive case} \end{array} \right.$$

So if x is the above ω-functor from I^3 to $\mathcal{C}$, then

$$\Phi_3^-(x) = \begin{array}{|cccc|c|}\hline = & \urcorner & \square & \square & \\ \urcorner & \shortparallel & \square & \square & \\ A & B & = & \urcorner & \square \\ \llcorner & C & \urcorner & \shortparallel & \\ \hline & & & & \urcorner \\ \hline \end{array} \overset{G}{\Longrightarrow} \begin{array}{|ccc|c|}\hline & \square & & \\ \hline \urcorner & \square & \square & \\ D & \urcorner & \square & \square \\ E & F & \urcorner & \\ \hline \end{array}$$

[3]Beware of the fact that $A, \dots F$ are elements of the cubical singular nerve whereas G is an element of the ω-category we are considering.

7. Elementary moves in the cubical singular nerve

In this section, the folding operators Φ_n^- are decomposed in elementary moves. First of all, here is a definition.

7.1. Definition. *The elementary moves in the n-cube are one of the following operators (with $1 \leqslant i \leqslant n-1$ and $x \in \omega Cat(I^n, \mathcal{C})$) :*

1. $${}^v\psi_i^- x = \left[\begin{array}{c} \urcorner \\ x \end{array}\right] \begin{array}{l} i \\ \uparrow\!\!\rightarrow i{+}1 \end{array}$$
2. $${}^v\psi_i^+ x = \left[\begin{array}{c} x \\ \llcorner \end{array}\right] \begin{array}{l} i \\ \uparrow\!\!\rightarrow i{+}1 \end{array}$$
3. $${}^h\psi_i^- x = \left[\begin{array}{cc} x & \urcorner \end{array}\right] \begin{array}{l} i \\ \uparrow\!\!\rightarrow i{+}1 \end{array}$$
4. $${}^h\psi_i^+ x = \left[\begin{array}{cc} \llcorner & x \end{array}\right] \begin{array}{l} i \\ \uparrow\!\!\rightarrow i{+}1 \end{array}$$

Notation. One sets $\theta_i^- = {}^v\psi_{i+1}^- \, {}^v\psi_i^+$. This operator plays a central rôle in the sequel.

Proposition 7.2 expresses the elementary moves using the notation of the previous paragraph (only the operators used in the sequel are calculated).

7.2. Proposition. *Let*

$$x = \begin{array}{|c|c|}\hline A & B \\ \hline & C \\ \hline\end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline D & \\ \hline E & F \\ \hline\end{array}$$

be an element of $\omega Cat(I^3, \mathcal{C})$. Then one has

$${}^v\psi_1^+ x = \begin{array}{|c|c|}\hline \begin{array}{c} A \\ \llcorner \end{array} & \begin{array}{c} B \\ C \end{array} \\ \hline & \| \\ \hline\end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline \begin{array}{c} \| \\ D \end{array} & \\ \hline \begin{array}{c} E \\ \| \end{array} & \begin{array}{c} F \\ \llcorner \end{array} \\ \hline\end{array}$$

$${}^v\psi_2^+ x = \begin{array}{|c|c|}\hline = \; A & B \\ \hline & \begin{array}{c} C \\ \llcorner \end{array} \\ \hline\end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline \begin{array}{c} D \\ \llcorner \end{array} & \\ \hline = & E \;\; F \\ \hline\end{array}$$

$${}^v\psi_2^- x = \begin{array}{|c|c|}\hline A \;\; B & = \\ \hline & \begin{array}{c} \urcorner \\ C \end{array} \\ \hline\end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline \begin{array}{c} \urcorner \\ D \end{array} & \\ \hline E & F \; = \\ \hline\end{array}$$

$${}^v\psi_1^- x = \begin{array}{|c|c|}\hline \begin{array}{c} \urcorner \\ A \end{array} & \begin{array}{c} \| \\ B \end{array} \\ \hline & C \\ \hline\end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline \| & \\ \hline \begin{array}{c} D \\ E \end{array} & \begin{array}{c} \urcorner \\ F \end{array} \\ \hline\end{array}$$

$${}^h\psi_1^- x = \begin{array}{|c|c|}\hline A \; \urcorner & \| \\ \hline & \begin{array}{c} B \\ C \end{array} \\ \hline\end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline \begin{array}{c} \| \\ D \end{array} & \\ \hline E & F \; \urcorner \\ \hline\end{array}$$

$$ {}^h\psi_2^- x = \begin{array}{|c|c|}\hline A & B\ = \\ \hline & C\ \urcorner \\ \hline \end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline D\ \urcorner & \\ \hline E\ \ F & = \\ \hline \end{array} $$

$$ \theta_1^- x = \begin{array}{|c|c|}\hline \begin{array}{cc} A & B \\ \llcorner & C \end{array} & \begin{array}{c} = \\ = \\ = \end{array} \\ \hline & \begin{array}{c} \urcorner \\ \shortparallel \end{array} \\ \hline \end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline \begin{array}{c} \urcorner \\ D \end{array} & \\ \hline \begin{array}{c} E \\ \shortparallel \end{array} & \begin{array}{cc} F & = \\ & = \\ \llcorner & = \end{array} \\ \hline \end{array} $$

PROOF. One has ${}^v\psi_i^+ x = \Gamma_i^+ \partial_i^- x +_i x$. Therefore

$$
\begin{aligned}
\partial_1^- \ {}^v\psi_1^+ x &= \epsilon_1 \partial_1^- \partial_1^- x \\
\partial_1^+ \ {}^v\psi_1^+(x) &= \partial_1^+ x \\
\partial_2^- \ {}^v\psi_1^+ x &= \epsilon_1 \partial_1^- \partial_1^- x +_1 \partial_2^- x \\
\partial_2^+ \ {}^v\psi_1^+ x &= \partial_1^- x +_1 \partial_2^+ x \\
\partial_3^\pm \ {}^v\psi_1^+ x &= \partial_3^\pm \Gamma_1^+ \partial_1^- x +_1 \partial_3^\pm x
\end{aligned}
$$

So one has

$$ {}^v\psi_1^+ x = \begin{array}{|c|c|}\hline \begin{array}{c} A \\ \llcorner \end{array} & \begin{array}{c} B \\ C \end{array} \\ \hline & \shortparallel \\ \hline \end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline \begin{array}{c} \shortparallel \\ D \end{array} & \\ \hline \begin{array}{c} E \\ \shortparallel \end{array} & \begin{array}{c} F \\ \llcorner \end{array} \\ \hline \end{array} $$

And

$$
\begin{aligned}
\partial_1^- \ {}^v\psi_2^+ x &= \Gamma_1^+ \partial_1^- \partial_2^- x +_1 \partial_1^- x \\
\partial_2^- \ {}^v\psi_2^+ x &= \epsilon_2 \partial_2^- \partial_2^- x \\
\partial_3^- \ {}^v\psi_2^+ x &= \epsilon_2 \partial_2^- x +_2 \partial_3^- x \\
\partial_1^+ \ {}^v\psi_2^+ x &= \Gamma_1^+ \partial_1^+ \partial_2^- x +_1 \partial_1^+ x \\
\partial_2^+ \ {}^v\psi_2^+ x &= \partial_2^+ x \\
\partial_3^+ \ {}^v\psi_2^+ x &= \partial_2^- x +_2 \partial_3^+ x
\end{aligned}
$$

Consequently one has

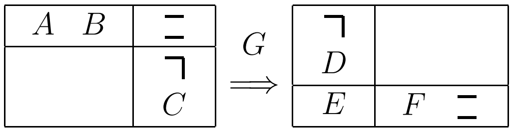
$$ {}^v\psi_2^+ x = \begin{array}{|c|c|}\hline =\ \ A & B \\ \hline & \begin{array}{c} C \\ \llcorner \end{array} \\ \hline \end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline \begin{array}{c} D \\ \llcorner \end{array} & \\ \hline = & E\ \ F \\ \hline \end{array} $$

One has ${}^v\psi_i^- x = x +_i \Gamma_i^- \partial_i^+ x$. Therefore

$$
\begin{aligned}
\partial_1^- \ {}^v\psi_2^- x &= \partial_1^- x +_1 \partial_1^- \Gamma_2^- \partial_2^+ x \\
\partial_1^+ \ {}^v\psi_2^- x &= \partial_1^+ x +_1 \partial_1^+ \Gamma_2^- \partial_2^+ x \\
\partial_2^- \ {}^v\psi_2^- x &= \partial_2^- x \\
\partial_2^+ \ {}^v\psi_2^- x &= \epsilon_2 \partial_2^+ \partial_2^+ x \\
\partial_3^- \ {}^v\psi_2^- x &= \partial_3^- x +_2 \partial_2^+ x \\
\partial_3^+ \ {}^v\psi_2^- x &= \partial_3^+ x +_2 \epsilon_2 \partial_2^+ \partial_2^+ x
\end{aligned}
$$

So

$$ {}^{v}\psi_2^- x = $$

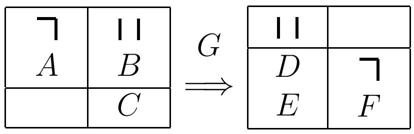


And

$$
\begin{aligned}
\partial_1^- \; {}^{v}\psi_1^- x &= \partial_1^- x \\
\partial_1^+ \; {}^{v}\psi_1^- x &= \epsilon_1 \partial_1^+ \partial_1^+ x \\
\partial_2^- \; {}^{v}\psi_1^- x &= \partial_2^- x +_1 \partial_1^+ x \\
\partial_2^+ \; {}^{v}\psi_1^- x &= \partial_2^+ x +_1 \epsilon_1 \partial_1^+ \partial_1^+ x \\
\partial_3^- \; {}^{v}\psi_1^- x &= \partial_3^- x +_1 \Gamma_1^- \partial_2^- \partial_1^+ x \\
\partial_3^+ \; {}^{v}\psi_1^- x &= \partial_3^+ x +_1 \Gamma_1^- \partial_2^+ \partial_1^+ x
\end{aligned}
$$

therefore

$$ {}^{v}\psi_1^- x = $$

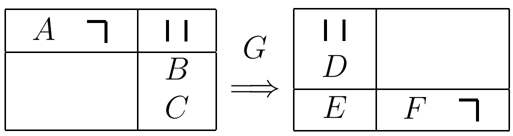


One has ${}^{h}\psi_1^- x = x +_2 \Gamma_1^- \partial_2^+ x$. Then

$$
\begin{aligned}
\partial_1^- \; {}^{h}\psi_1^- x &= \partial_1^- x +_1 \partial_2^+ x \\
\partial_1^+ \; {}^{h}\psi_1^- x &= \partial_1^+ x \\
\partial_2^- \; {}^{h}\psi_1^- x &= \partial_2^- x \\
\partial_2^+ \; {}^{h}\psi_1^- x &= \epsilon_1 \partial_1^+ \partial_2^+ x \\
\partial_3^- \; {}^{h}\psi_1^- x &= \partial_3^- x +_2 \Gamma_1^- \partial_2^- \partial_2^+ x \\
\partial_3^+ \; {}^{h}\psi_1^- x &= \partial_3^+ x +_2 \Gamma_1^- \partial_2^+ \partial_2^+ x
\end{aligned}
$$

So

$$ {}^{h}\psi_1^- x = $$



One has ${}^{h}\psi_2^- x = x +_3 \Gamma_2^- \partial_3^+ x$. Therefore

$$
\begin{aligned}
\partial_1^- \; {}^{h}\psi_2^- x &= \partial_1^- x +_2 \Gamma_1^- \partial_1^- \partial_3^+ x \\
\partial_1^+ \; {}^{h}\psi_2^- x &= \partial_1^+ x +_2 \Gamma_1^- \partial_1^+ \partial_3^+ x \\
\partial_2^- \; {}^{h}\psi_2^- x &= \partial_2^- x +_2 \partial_3^+ x \\
\partial_2^+ \; {}^{h}\psi_2^- x &= \partial_2^+ x +_2 \epsilon_2 \partial_2^+ \partial_3^+ x \\
\partial_3^- \; {}^{h}\psi_2^- x &= \partial_3^- x \\
\partial_3^+ \; {}^{h}\psi_2^- x &= \partial_3^+ \Gamma_2^- \partial_3^+ x = \epsilon_2 \partial_2^+ \partial_3^+ x
\end{aligned}
$$

so

$$ {}^h\psi_2^- = \begin{array}{|c|c|}\hline A & B \quad = \\ \hline & C \quad \urcorner \\ \hline\end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline D \quad \urcorner & \\ \hline E \quad F & = \\ \hline\end{array} $$

Now let us calculate $\theta_1^- x$. One has

$$
\begin{aligned}
\theta_1^- x &= {}^v\psi_2^- \; {}^v\psi_1^+ x \\
&= {}^v\psi_2^- \begin{array}{|c|c|}\hline \begin{array}{c} A \\ \llcorner \end{array} & \begin{array}{c} B \\ C \end{array} \\ \hline & \shortparallel \\ \hline\end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline \begin{array}{c} \shortparallel \\ D \end{array} & \\ \hline \begin{array}{c} E \\ \shortparallel \end{array} & \begin{array}{c} F \\ \llcorner \end{array} \\ \hline\end{array} \\
&= \begin{array}{|c|c|}\hline \begin{array}{cc} A & B \\ \llcorner & C \end{array} & \begin{array}{c} = \\ = \end{array} \\ \hline & \begin{array}{c} \urcorner \\ \shortparallel \end{array} \\ \hline\end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline \begin{array}{c} \urcorner \\ D \end{array} & \\ \hline \begin{array}{c} E \\ \shortparallel \end{array} & \begin{array}{cc} F & = \\ \llcorner & = \end{array} \\ \hline\end{array}
\end{aligned}
$$

■

The following proposition describes some of the commutation relations satisfied by the previous operators, the differential maps and the connection maps.

7.3. Proposition. *The following equalities hold (with $\alpha \in \{-,+\}$) :*

$$\partial_j^\alpha \; {}^v\psi_i^- = \begin{cases} {}^v\psi_{i-1}^- \partial_j^\alpha \text{ if } j < i \\ {}^v\psi_i^- \partial_j^\alpha \text{ if } j > i+1 \end{cases} \tag{1}$$

$$\partial_j^\alpha \; {}^h\psi_i^- = \begin{cases} {}^h\psi_{i-1}^- \partial_j^\alpha \text{ if } j < i \\ {}^h\psi_i^- \partial_j^\alpha \text{ if } j > i+1 \end{cases} \tag{2}$$

$$\partial_j^\alpha \theta_i^- = \begin{cases} \theta_{i-1}^- \partial_j^\alpha \text{ if } j < i \\ \theta_i^- \partial_j^\alpha \text{ if } j > i+2 \end{cases} \tag{3}$$

$$\theta_i^- \Gamma_j^- = \begin{cases} \Gamma_j^- \theta_{i-1}^- \text{ if } j < i \\ \Gamma_j^- \theta_i^- \text{ if } j > i+2 \end{cases} \tag{4}$$

$$\partial_i^- \; {}^v\psi_i^- = \partial_i^- \tag{5}$$

$$\partial_i^+ \; {}^v\psi_i^- = \epsilon_i \partial_i^+ \partial_i^+ \tag{6}$$

$$\partial_{i+1}^- \; {}^v\psi_i^- = \partial_{i+1}^- +_i \partial_i^+ \tag{7}$$

$$\partial_{i+1}^+ \; {}^v\psi_i^- = \partial_{i+1}^+ \tag{8}$$

$$\partial_i^- \; {}^h\psi_i^- = \partial_i^- +_i \partial_{i+1}^+ \tag{9}$$

$$\partial_i^+ \; {}^h\psi_i^- = \partial_i^+ \tag{10}$$

$$\partial_{i+1}^- \; {}^h\psi_i^- = \partial_{i+1}^- \tag{11}$$

$$\partial_{i+1}^+ \; {}^h\psi_i^- = \epsilon_i \partial_i^+ \partial_{i+1}^+ \tag{12}$$

$$\partial_i^- \theta_i^- = \Gamma_i^- \partial_i^- \partial_i^- \tag{13}$$

$$\partial_i^+ \theta_i^- = {}^v\psi_i^- \partial_i^+ \tag{14}$$

$$\partial^-_{i+1}\theta^-_i = \partial^-_{i+1} \tag{15}$$

$$\partial^+_{i+1}\theta^-_i = \epsilon_{i+1}\partial^+_{i+1}\partial^-_i +_i \epsilon_{i+1}\partial^+_{i+1}\partial^+_{i+1} \tag{16}$$

$$\partial^-_{i+2}\theta^-_i = \begin{bmatrix} \partial^-_{i+2} & \partial^+_{i+1} \\ \llcorner & \partial^-_i \end{bmatrix} \begin{array}{l} i \\ \uparrow\!\!\rightarrow i{+}1 \end{array} \tag{17}$$

$$\partial^+_{i+2}\theta^-_i = {}^v\psi^+_i\partial^+_{i+2} \tag{18}$$

$$\theta^-_i\Gamma^-_i = \Gamma^-_{i+1} \tag{19}$$

$$\theta^-_i\Gamma^-_{i+1} = \Gamma^-_{i+1} \tag{20}$$

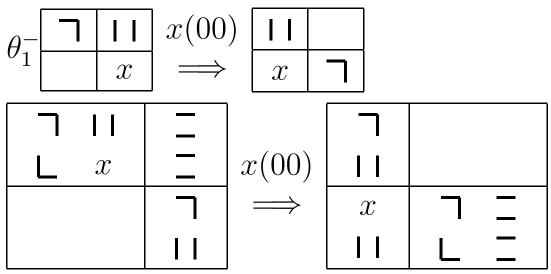
Proof. Equalities (1), (2), (3) and (4) are obvious. Equalities from (5) to (12) are immediate consequences of the definitions. With Proposition 7.2, one sees that

$$\begin{aligned}
\partial^-_1\theta^-_1 &= \Gamma^-_1\partial^-_1\partial^-_1 \\
\partial^+_1\theta^-_1 &= {}^v\psi^-_1\partial^+_1 \\
\partial^-_2\theta^-_1 &= \partial^-_2 \\
\partial^+_2\theta^-_1 &= \epsilon_2\partial^+_2\partial^-_1 +_1 \epsilon_2\partial^+_2\partial^+_2 \\
\partial^-_3\theta^-_1 &= \begin{bmatrix} \partial^-_3 & \partial^+_2 \\ \llcorner & \partial^-_1 \end{bmatrix} \begin{array}{l} 1 \\ \uparrow\!\!\rightarrow 2 \end{array} \\
\partial^+_3\theta^-_1 &= {}^v\psi^+_1\partial^+_3
\end{aligned}$$

For a given x, the above equalities are equalities in the free cubical ω-category generated by x. Therefore, they depend only on the relative position of the indices 1, 2 and 3 with respect to one another. Therefore, we can replace each index 1 by i, each index 2 by $i+1$ and each index 3 by $i+2$ to obtain the required formulae.

In the same way, it suffices to prove the last two formulae in lower dimension and for $i = 1$. One has

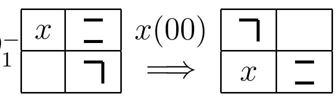
$\theta^-_1\Gamma^-_1 x = \theta^-_1$

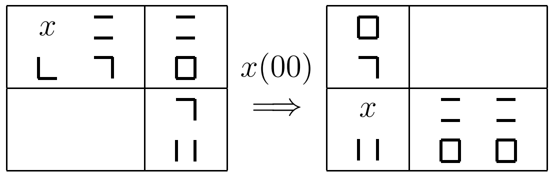


$= \Gamma^-_2 x$

and

$\theta^-_1\Gamma^-_2 x = \theta^-_1$



$=$ 

$= \Gamma_2^- x$

$\blacksquare$

7.4. Theorem. *Set* ${}^v\Psi_k^- = {}^v\psi_k^- \ldots {}^v\psi_1^-$ *and* ${}^h\Psi_k^- = {}^h\psi_k^- \ldots {}^h\psi_1^-$. *Then for* $n \geqslant 2$ *and* $1 \leqslant i \leqslant n$*, one has*

$$\partial_i^+({}^v\Psi_1^- \ {}^h\Psi_1^-) \ldots ({}^v\Psi_{n-1}^- \ {}^h\Psi_{n-1}^-) = \epsilon_1^{n-1}(\partial_1^+)^n.$$

Proof. It is obvious for $n = 2$. We are going to make an induction on n. Let $n \geqslant 2$ and $1 \leqslant i \leqslant n$. Then

$$\begin{aligned}
&\partial_i^+({}^v\Psi_1^- \ {}^h\Psi_1^-) \ldots ({}^v\Psi_n^- \ {}^h\Psi_n^-)\\
&= \epsilon_1^{n-1}(\partial_1^+)^n({}^v\Psi_n^- \ {}^h\Psi_n^-)\\
&= \epsilon_1^{n-1}(\partial_1^+)^{n-1} \ {}^v\Psi_{n-1}^-\epsilon_1\partial_1^+\partial_1^+ \ {}^h\Psi_n^-\\
&= \epsilon_1^{n-1}(\partial_1^+)^{n-2} \ {}^v\Psi_{n-2}^-(\epsilon_1\partial_1^+\partial_1^+)^2 \ {}^h\Psi_n^-\\
&= (\ldots)\\
&= \epsilon_1^{n-1}(\epsilon_1\partial_1^+\partial_1^+)^n \ {}^h\Psi_n^-\\
&= \epsilon_1^n(\partial_1^+)^{n+1} \ {}^h\Psi_n^-
\end{aligned}$$

The equality

$$\partial_i^+({}^v\Psi_1^- \ {}^h\Psi_1^-) \ldots ({}^v\Psi_n^- \ {}^h\Psi_n^-)x = \epsilon_1^n(\partial_1^+)^{n+1} \ {}^h\Psi_n^- x$$

makes sense if x is a $(n+1)$-cube. And in this case, $\epsilon_1^n(\partial_1^+)^{n+1} \ {}^h\Psi_n^- x$ is 0-dimensional and $\epsilon_1^n(\partial_1^+)^{n+1} \ {}^h\Psi_n^- x = \epsilon_1^n(\partial_1^+)^{n+1}x$. This equality holds in the free cubical ω-category generated by x, and therefore

$$\epsilon_1^n(\partial_1^+)^{n+1} \ {}^h\Psi_n^- = \epsilon_1^n(\partial_1^+)^{n+1}.$$

Now suppose that $i = n+1$. Then

$$\begin{aligned}
&\partial_{n+1}^+({}^v\Psi_1^- \ {}^h\Psi_1^-) \ldots ({}^v\Psi_n^- \ {}^h\Psi_n^-)\\
&= ({}^v\Psi_1^- \ {}^h\Psi_1^-) \ldots ({}^v\Psi_{n-1}^- \ {}^h\Psi_{n-1}^-)\partial_{n+1}^+({}^v\Psi_n^- \ {}^h\Psi_n^-)\\
&= ({}^v\Psi_1^- \ {}^h\Psi_1^-) \ldots ({}^v\Psi_{n-1}^- \ {}^h\Psi_{n-1}^-)({}^v\Psi_{n-1}^-\partial_{n+1}^+ \ {}^h\Psi_n^-)\\
&= ({}^v\Psi_1^- \ {}^h\Psi_1^-) \ldots ({}^v\Psi_{n-1}^- \ {}^h\Psi_{n-1}^-) \ {}^v\Psi_{n-1}^-\partial_{n+1}^+({}^h\psi_n^- \ldots \ {}^h\psi_1^-)\\
&= ({}^v\Psi_1^- \ {}^h\Psi_1^-) \ldots ({}^v\Psi_{n-1}^- \ {}^h\Psi_{n-1}^-) \ {}^v\Psi_{n-1}^-\epsilon_n\partial_n^+\partial_{n+1}^+({}^h\psi_{n-1}^- \ldots \ {}^h\psi_1^-)\\
&= ({}^v\Psi_1^- \ {}^h\Psi_1^-) \ldots ({}^v\Psi_{n-1}^- \ {}^h\Psi_{n-1}^-) \ {}^v\Psi_{n-1}^-\epsilon_n\partial_n^+({}^h\psi_{n-1}^- \ldots \ {}^h\psi_1^-)\partial_{n+1}^+\\
&= ({}^v\Psi_1^- \ {}^h\Psi_1^-) \ldots ({}^v\Psi_{n-1}^- \ {}^h\Psi_{n-1}^-) \ {}^v\Psi_{n-1}^-\epsilon_n\epsilon_{n-1}\partial_{n-1}^+\partial_n^+({}^h\psi_{n-2}^- \ldots \ {}^h\psi_1^-)\partial_{n+1}^+
\end{aligned}$$

$$\begin{aligned}
&= ({}^v\Psi_1^-\ {}^h\Psi_1^-)\dots({}^v\Psi_{n-1}^-\ {}^h\Psi_{n-1}^-)\ {}^v\Psi_{n-1}^-\epsilon_n\epsilon_{n-1}\partial_{n-1}^+({}^h\psi_{n-2}^-\dots\ {}^h\psi_1^-)\partial_n^+\partial_{n+1}^+\\
&= (\dots)\\
&= ({}^v\Psi_1^-\ {}^h\Psi_1^-)\dots({}^v\Psi_{n-1}^-\ {}^h\Psi_{n-1}^-)\ {}^v\Psi_{n-1}^-\epsilon_n\dots\epsilon_1\partial_1^+\dots\partial_{n+1}^+\\
&= (\epsilon_1)^n(\partial_1^+)^{n+1} \text{ for the same reason as above}
\end{aligned}$$

∎

Why does the proof of Theorem 7.4 work. The principle of the proof of Theorem 7.4 is the following observation (see in [2]): let $f_1,\dots,f_n$ be n operators such that (the product notation means the composition)

1. for any i, one has $f_if_i = f_i$ (the operators f_i are idempotent)
2. $|i-j|\geqslant 2$ implies $f_if_j = f_jf_i$
3. $f_if_{i+1}f_i = f_{i+1}f_if_{i+1}$ for any i

Then the operator $F = f_1(f_2f_1)\dots(f_nf_{n-1}\dots f_1)$ satisfies $f_iF = F$ for any i. This means that F enables to apply all f_i a maximal number of times. It turns out that the operators ${}^v\psi_i^\pm$ and ${}^h\psi_i^\pm$ satisfy the above relations :

7.5. Proposition. *The operators ${}^v\psi_i^\alpha$ and ${}^h\psi_j^\beta$ are idempotent. Moreover for any $i\geqslant 1$ and any $j\geqslant 1$, with $|i-j|\geqslant 2$, the following equalities hold :*

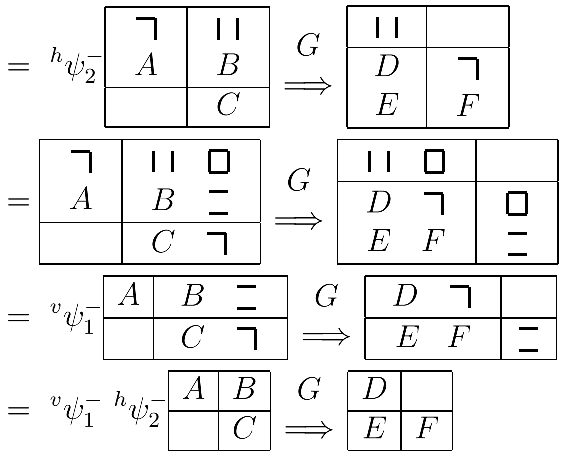
$$\begin{aligned}
{}^v\psi_i^\alpha\ {}^h\psi_j^\beta &= {}^h\psi_j^\beta\ {}^v\psi_i^\alpha \textit{ for } \alpha\in\{-,+\} &(21)\\
{}^v\psi_i^\alpha\ {}^h\psi_i^\alpha &= {}^h\psi_i^\alpha\ {}^v\psi_i^\alpha \textit{ for } \alpha\in\{-,+\} &(22)\\
{}^h\psi_{i+1}^\alpha\ {}^v\psi_i^\alpha &= {}^v\psi_i^\alpha\ {}^h\psi_{i+1}^\alpha \textit{ for } \alpha\in\{-,+\} &(23)\\
{}^a\psi_i^\alpha\ {}^a\psi_{i+1}^\alpha\ {}^a\psi_i^\alpha &= {}^a\psi_{i+1}^\alpha\ {}^a\psi_i^\alpha\ {}^a\psi_{i+1}^\alpha \textit{ for } a\in\{v,h\} \textit{ and } \alpha\in\{-,+\} &(24)
\end{aligned}$$

Proof. Equalities 21 and 22 are obvious.

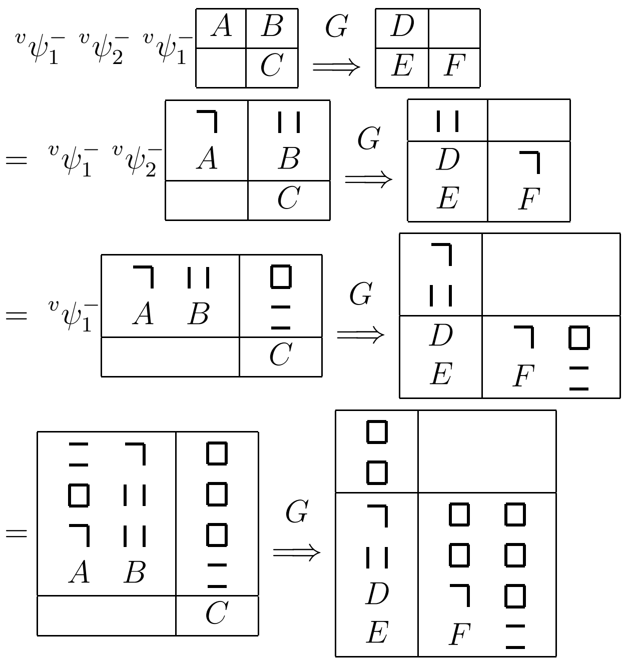
For the sequel, one can suppose $\alpha = -$. In the cubical singular nerve of an ω-category, two elements A and B of the same dimension n are equal if and only if $A(0_n) = B(0_n)$ and for $1\leqslant k\leqslant n$ and $\alpha\in\{-,+\}$, one has $\partial_k^\alpha A = \partial_k^\alpha B$.

Now we want to prove Equality 23. Since $({}^v\psi_i^\alpha x)(0_n) = ({}^h\psi_i^\alpha x)(0_n) = x(0_n)$, then ${}^h\psi_{i+1}^\alpha\ {}^v\psi_i^\alpha x = {}^v\psi_i^\alpha\ {}^h\psi_{i+1}^\alpha x$ for any x of dimension n (P_n) is equivalent to $\partial_k^\beta\ {}^h\psi_{i+1}^\alpha\ {}^v\psi_i^\alpha x = \partial_k^\beta\ {}^v\psi_i^\alpha\ {}^h\psi_{i+1}^\alpha x$ for $1\leqslant k\leqslant n$ and $\beta\in\{-,+\}(E_{k,n})$. Proposition 7.3 implies that $P_{n-1}\Longrightarrow E_{k,n}$ for $k<i$ or $k>i+2$. For $k\in\{i,i+1,i+2\}$, proving Equality $E_{k,n}$ is equivalent to proving it for the case $i=1$ and to replacing each index 1 by i, each index 2 in by $i+1$ and each index 3 by $i+2$. And in the case $i=1$, the equality is a calculation in the free cubical ω-category generated by x. So we can suppose that x is of dimension as low as possible. In our case, this equality makes sense if x is 3-dimensional. Therefore it suffices to verify Equality 23 in dimension 3 for $i=1$. And one has

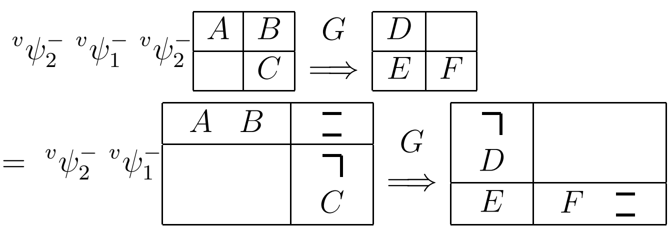
$${}^h\psi_2^-\ {}^v\psi_1^- \begin{array}{|c|c|}\hline A & B\\ \hline & C\\ \hline\end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|}\hline D & \\ \hline E & F\\ \hline\end{array}$$

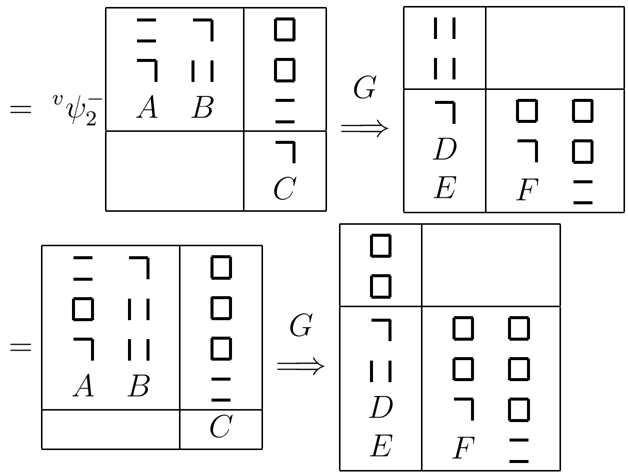


In the same way, to prove Equality 24, it suffices to prove it for $i = 1$ and in the 3-dimensional case. And one has

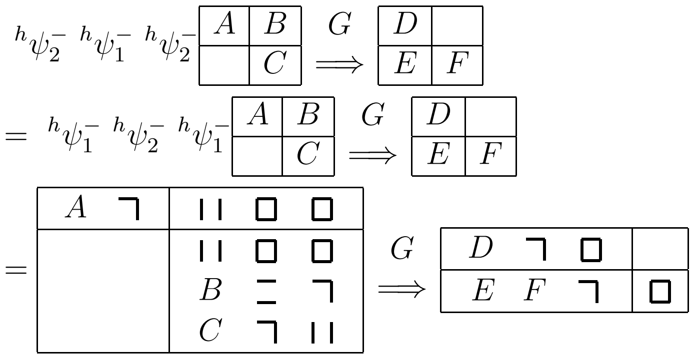


and





In the same way, one can verify that



$\blacksquare$

7.6. Theorem. *For any $n \geqslant 2$, Φ_n^- is a composition of ${}^v\psi_i^-$, ${}^h\psi_i^-$ and θ_i^-.*

Proof. It is easy to see that $\Phi_2^- = {}^v\psi_1^- \, {}^h\psi_1^- = {}^h\psi_1^- \, {}^v\psi_1^-$. Now we suppose that $n \geqslant 3$. Set $\Theta_k^{n-2} = \theta_k^- \dots \theta_{n-2}^-$.

We are going to prove that

$$\Phi_n^- = \Theta_{n-2}^{n-2}\Theta_{n-3}^{n-2} \dots \Theta_1^{n-2}({}^v\Psi_1^- \, {}^h\Psi_1^-) \dots ({}^v\Psi_{n-1}^- \, {}^h\Psi_{n-1}^-)$$

by verifying that the second member satisfies the characterization of Theorem 6.16. Let $x \in \omega Cat(I^n, \mathcal{C})^-$. Theorem 7.4 implies that for $1 \leqslant i \leqslant n$, the dimension of

$$\partial_i^+({}^v\Psi_1^- \, {}^h\Psi_1^-) \dots ({}^v\Psi_{n-1}^- \, {}^h\Psi_{n-1}^-)x$$

is zero (or equivalently that it belongs to the image of ϵ_1^{n-1}). With Proposition 7.3, one gets

$$\partial_i^+\Theta_{n-2}^{n-2}\Theta_{n-3}^{n-2} \dots \Theta_1^{n-2}({}^v\Psi_1^- \, {}^h\Psi_1^-) \dots ({}^v\Psi_{n-1}^- \, {}^h\Psi_{n-1}^-)x \in Im(\epsilon_1^{n-1})$$

for $1 \leqslant i \leqslant n$. It remains to prove that for $1 \leqslant k \leqslant n-2$,

$$\partial_k^-\Theta_{n-2}^{n-2}\Theta_{n-1}^{n-2} \dots \Theta_1^{n-2}y \in Im(\Gamma_{n-2}^- \dots \Gamma_k^-)$$

for any $y \in \omega Cat(I^n, \mathcal{C})^-$. One has

$$\begin{aligned}
&\partial_k^- \Theta_{n-2}^{n-2} \Theta_{n-3}^{n-2} \dots \Theta_1^{n-2} y \\
&= \left(\Theta_{n-3}^{n-3} \dots \Theta_k^{n-3}\right) \partial_k^- \Theta_k^{n-2} z \text{ with } z = \Theta_{k-1}^{n-2} \dots \Theta_1^{n-2} y \\
&= \left(\Theta_{n-3}^{n-3} \dots \Theta_k^{n-3}\right) \Gamma_k^- \partial_k^- \partial_k^- \Theta_{k+1}^{n-2} z \\
&= \Gamma_{n-2}^- \left(\Theta_{n-4}^{n-4} \dots \Theta_k^{n-4}\right) \partial_k^- \Theta_k^{n-3} \partial_k^- z \\
&= (\dots) \\
&= \left(\Gamma_{n-2}^- \dots \Gamma_{k+1}^-\right) \partial_k^- \theta_k^- (\partial_k^-)^{n-2-k} z \\
&= \left(\Gamma_{n-2}^- \dots \Gamma_k^-\right) (\partial_k^-)^{n-k} z
\end{aligned}$$

■

The operators ${}^v\psi_i^\pm$, ${}^h\psi_i^\pm$ and θ_i^- for $1 \leqslant i \leqslant n-1$ and Φ_n^- induce natural transformations of set-valued functors from $\omega Cat(I^n, -)^-$ to itself.

7.7. Conjecture. *Let f be an ω-functor from I^n to itself such that $f(0_n) = 0_n$ and such that the corresponding natural transformation from $\omega Cat(I^n, -)$ to itself induces a natural transformation Φ^- from $\omega Cat(I^n, -)^-$ to itself. Then Φ^- is a composition of ${}^v\psi_i^-$, ${}^h\psi_i^-$ and θ_i^- for $1 \leqslant i \leqslant n-1$.*

7.8. Conjecture. *Let Φ be a natural transformation from $\omega Cat(I^n, -)$ to itself such that the corresponding functor $(\Phi)^*$ from I^n to itself satisfies $(\Phi)^*(0_n) = 0_n$. Then Φ is a composition of ${}^v\psi_i^\pm$ and ${}^h\psi_i^\pm$ for $1 \leqslant i \leqslant n-1$.*

By Yoneda, the operators ${}^v\psi_i^\pm$ and ${}^h\psi_i^\pm$ for $1 \leqslant i \leqslant n-1$ induce ω-functors from I^n to itself denoted by $({}^v\psi_i^\pm)^*$ and $({}^h\psi_i^\pm)^*$. The dual conjecture is then

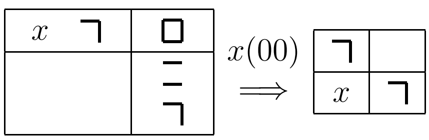
7.9. Conjecture. *Let f be an ω-functor from I^n to itself such that $f(0_n) = 0_n$. Then f is a composition of $({}^v\psi_i^\pm)^*$ and $({}^h\psi_i^\pm)^*$.*

8. Comparison of x and $\Phi_n^-(x)$ in the reduced branching complex

This section is devoted to proving that for any $x \in \omega Cat(I^n, \mathcal{C})^-$, x and $\Phi_n^-(x)$ are T-equivalent.

8.1. Proposition. *For any $i \geqslant 1$ and any $n \geqslant 2$, if $x \in \omega Cat(I^n, \mathcal{C})^-$, then ${}^h\psi_i^-(x)$ and x are T-equivalent.*

Proof. First let us make the proof for $i = 1$ and $n = 2$. Let us consider the following ω-functor from I^3 to $\mathcal{C}$:

$y_1 =$ 

Then $\partial^- y_1 = {}^h\psi_1^-(x) - x + t_1$ where t_1 is a thin element. Therefore x and ${}^h\psi_1^-(x)$ are T-equivalent.

We claim that the above construction is sufficient to prove that x and ${}^h\psi_1^-(x)$ are T-equivalent for any $x \in \omega Cat(I^n, \mathcal{C})^-$ and for any $n \geqslant 2$. The labeled 3-cube y_1 is actually a certain thin 3-dimensional element of the cubical ω-category $\mathcal{N}^\square(\mathcal{C})$ and it corresponds to the filling of a thin 2-shell. So

$$y_1 = f_1(\epsilon_1 x, \epsilon_2 x, \epsilon_3 x, \Gamma_1^- x, \Gamma_2^- x, \Gamma_1^+ x, \Gamma_2^+ x)$$

where f_1 is a function which only uses the operators $+_1$, $+_2$, and $+_3$. In this particular case, f_1 could be of course calculated. But it will not be always possible in the sequel to make such a calculation : this is the reason why no explicit formula is used here. And one has $\partial_2^- f_1(x) = x$, $\partial_3^- f_1(x) = {}^h\psi_1^-(x)$ and all other 2-faces $\partial_i^\alpha f_1(x)$ are (necessarily) thin 2-faces. The equalities $\partial_2^- f_1(x) = x$ and $\partial_3^- f_1(x) = {}^h\psi_1^-(x)$ do not depend on the dimension of x. Therefore for any $x \in \omega Cat(I^n, \mathcal{C})^-$ and for any $n \geqslant 2$, one gets $\partial^- y_1 = {}^h\psi_1^-(x) - x + t$ where t is a linear combination of thin elements.

Now we want to explain that the above construction is also sufficient to prove that x and ${}^h\psi_i^-(x)$ are T-equivalent for any $i \geqslant 1$ and any $x \in \omega Cat(I^n, \mathcal{C})^-$ and for any $n \geqslant 2$. The equalities $\partial_2^- f_1(x) = x$ and $\partial_3^- f_1(x) = {}^h\psi_1^-(x)$ do not depend on the absolute values 1, 2 3. But only on the relative values $1 = 3 - 2$, $2 = 3 - 1$ and $3 = 3 - 0$. So let us introduce a labeled $(n+1)$-cube $y_i = f_i(x)$ by replacing in f_1 any index 1 in by i, any index 2 by $i+1$ and any index 3 by $i+2$. Then one gets a thin $(n+1)$-cube $y_i = f_i(x)$ such that $\partial_{i+1}^- f_i(x) = x$ and $\partial_{i+2}^- f_i(x) = {}^h\psi_i^-(x)$.

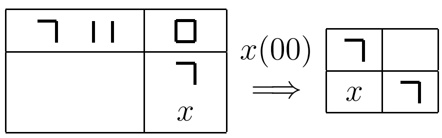
If the reader does not like this proof and prefers explicit calculations, it suffices to notice that $y_1 = {}^h\psi_1^- \Gamma_2^- x$ by Proposition 7.2. Set $y_i = {}^h\psi_i^- \Gamma_{i+1}^- x$. Then

$$\begin{aligned}\partial^-(y_i) = &\sum_{j<i} (-1)^{j+1}\, {}^h\psi_{i-1}^- \Gamma_i^- \partial_j^- x + (-1)^{i+1}(\Gamma_i^- \partial_i^- x +_i \epsilon_{i+1}\partial_{i+1}^+ x) + \\ &(-1)^{i+2}(x - {}^h\psi_i^- x) + \sum_{j>i+2} (-1)^{j+1}\, {}^h\psi_i^- \Gamma_{i+1}^- \partial_{j-1}^- x\end{aligned}$$

and one completes the proof by an easy induction on the dimension of $x(0_n)$. ∎

8.2. Proposition. *For any $i \geqslant 1$ and any $n \geqslant 2$, if $x \in \omega Cat(I^n, \mathcal{C})^-$, then ${}^v\psi_i^-(x)$ and x are T-equivalent.*

Proof. It suffices to make the proof for $i = 1$ and $n = 2$. And to consider the following thin 3-cube



Notice that the above 3-cube is exactly ${}^v\psi_2^- \Gamma_1^- x$ by Proposition 7.2. ∎

8.3. Proposition. *For any $i \geqslant 1$ and any $n \geqslant 3$, if $x \in \omega Cat(I^n, \mathcal{C})^-$, then $\theta_i^-(x)$ and x are T-equivalent.*

Proof. It suffices to make the proof for $i = 1$ and $n = 3$. Set

$$x = \begin{array}{|c|c|} \hline A & B \\ \hline & C \\ \hline \end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|} \hline D & \\ \hline E & F \\ \hline \end{array}$$

One has already seen that

$$\theta_1^- x = \begin{array}{|c|c|} \hline \begin{matrix} A & B \\ \llcorner & C \end{matrix} & \begin{matrix} = \\ = \end{matrix} \\ \hline & \begin{matrix} \urcorner \\ \mid\,\mid \end{matrix} \\ \hline \end{array} \overset{G}{\Longrightarrow} \begin{array}{|c|c|} \hline \begin{matrix} \urcorner \\ D \end{matrix} & \\ \hline \begin{matrix} E \\ \mid\,\mid \end{matrix} & \begin{matrix} F & = \\ \llcorner & = \end{matrix} \\ \hline \end{array}$$

It suffices to construct a thin 4-cube y such that $\partial_3^- y = x$ and $\partial_2^- y = \theta_1^- x$. If the 4-cube is conventionally represented by Figure 5, the thin labeled 4-cube of Figure 6 with $00{+}0 \mapsto (\partial_2^+ \partial_1^- x +_1 \partial_2^+ \partial_2^+ x)(0)$ meets the requirement. The latter labeled 4-cube can be defined as the unique thin 4-cube $\omega(x)$ which fills the 3-shell defined by

$$\begin{aligned}
\partial_1^- \omega(x) &= \Gamma_2^- \partial_1^- x \\
\partial_2^- \omega(x) &= \theta_1^-(x) \\
\partial_3^- \omega(x) &= x \\
\partial_4^- \omega(x) &= \begin{bmatrix} \Gamma_2^- \partial_3^- x & \epsilon_2 \partial_2^+ x \\ \Gamma_1^- \Gamma_1^+ \partial_1^- \partial_2^+ x +_2 \epsilon_1 \Gamma_1^- \partial_1^- \partial_2^+ x & \Gamma_1^- \partial_1^- x \end{bmatrix} \begin{smallmatrix} 1 \\ \uparrow \\ \llcorner\!\rightarrow 3 \end{smallmatrix} \\
\partial_1^+ \omega(x) &= {}^v\psi_2^- \Gamma_1^- \partial_1^+ x \\
\partial_2^+ \omega(x) &= \Gamma_2^- \partial_2^+ x \\
\partial_3^+ \omega(x) &= \epsilon_3 (\Gamma_1^- \partial_2^+ \partial_1^- x +_1 \epsilon_2 \partial_2^+ \partial_2^+ x) \\
\partial_4^+ \omega(x) &= {}^v\psi_2^+ \Gamma_2^- \partial_3^+ x
\end{aligned}$$

■

8.4. Corollary. *For any $n \geqslant 2$, for any $x \in \omega Cat(I^n, \mathcal{C})^-$, x and $\Phi_n^-(x)$ are T-equivalent and Φ_n^- is the identity map on the reduced branching complex.*

We have proved that for any $x \in \omega Cat(I^n, \mathcal{C})^-$, there exists $t_1 \in M_n$ and $t_2 \in M_{n+1}$ such that $\Phi_n^-(x) - x = t_1 + \partial^- t_2$. The proofs of this section use only calculations in the free cubical ω-category generated by x. This means that t_1 and t_2 can be formulated in terms of expressions in the same cubical ω-category. And so this means that t_1 and t_2 are linear combinations of expressions which use only x as variable and the operators $\partial_i^\pm$, $\Gamma_i^\pm$, ϵ_i and $+_i$. With Theorem 7.6 which allows to consider Φ_n^- like an operator defined in any cubical ω-category, one sees that Corollary 8.4 does make sense in an appropriate cubical setting. Moreover the terms t_1 and t_2 being elements of the free cubical ω-category generated by x, then t_1 and t_2 depend in a functorial way on x.

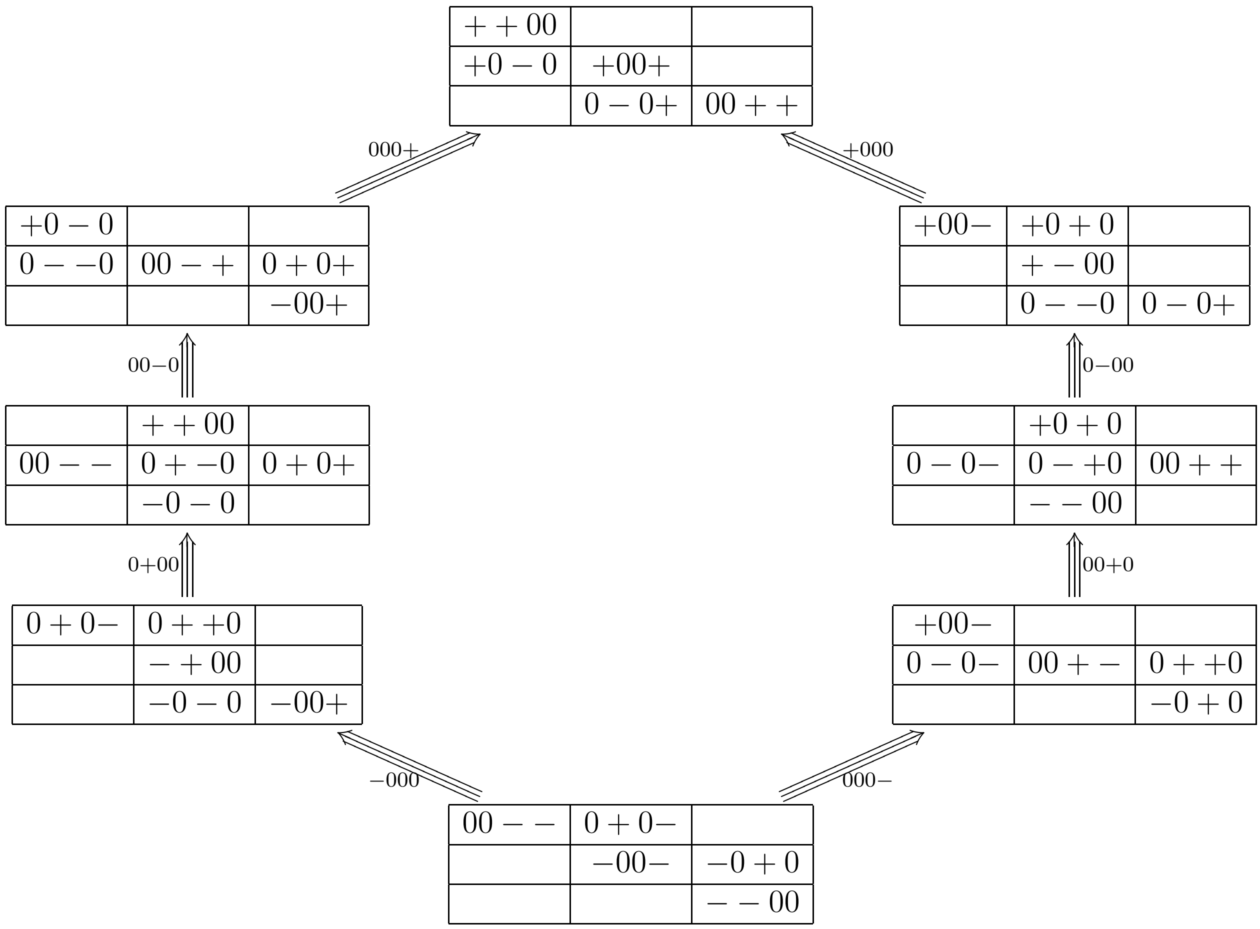


Figure 5: 2 -categorical representation of the 4 -cube

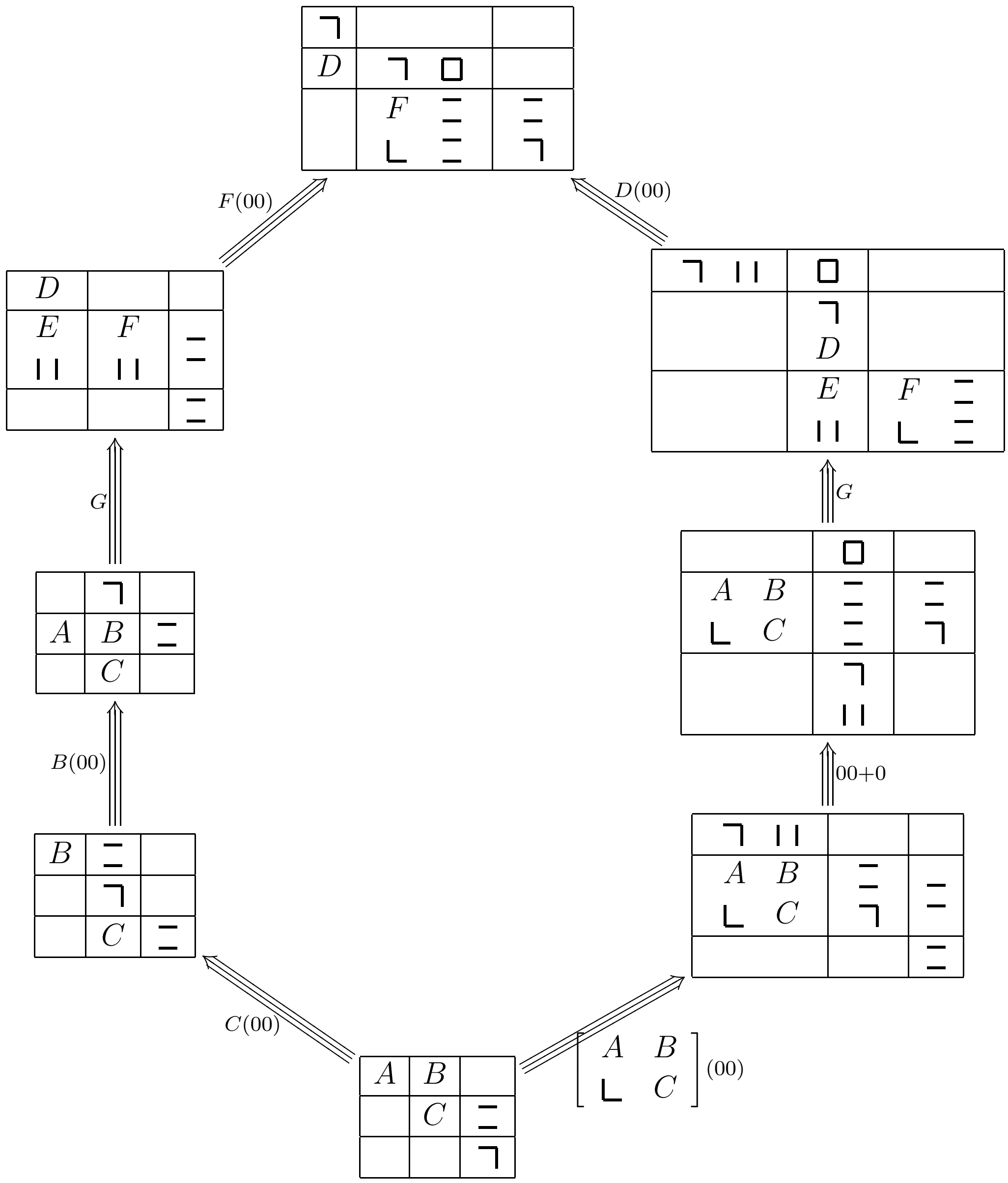


Figure 6: A labeled 4-cube

9. Folding operations and composition maps

9.1. Theorem. *Let x and y be two n-morphisms of $\mathcal{C}$ with $n \geqslant 2$.*

1. *if $x *_{n-1} y$ exists, then $\square_n^-(x *_{n-1} y) - \square_n^-(x) - \square_n^-(y)$ is a boundary in the normalized chain complex of the branching simplicial nerve of $\mathcal{C}$. Moreover, $\square_n^-(x *_{n-1} y)$ is T-equivalent to $\square_n^-(x) + \square_n^-(y)$.*

2. *if $1 \leqslant p \leqslant n-2$, then $\square_n^-(x *_p y)$ is T-equivalent to $\square_n^-(x) + \square_n^-(y)$.*

Proof. Let us denote by $P(h)$ the following property :

"for any $n \geqslant 2$ and with $p = n-h \geqslant 1$, for any n-morphisms x and y of any ω-category $\mathcal{C}$ such that $x *_p y$ exists, there exists a thin n-cube $A_p^n(x,y)$ and a thin $(n+1)$-cube $B_p^n(x,y)$ which lie in the cubical singular nerve of the free globular ω-category generated by x and y, and even in its branching nerve, such that

$$\square_n^-(x *_p y) = \square_n^-(x) + \square_n^-(y) + A_p^n(x,y) + \partial^- B_p^n(x,y)$$

in the normalized branching complex (i.e. the equality holds modulo degenerate elements of the branching simplicial nerve) and such that for any $(n+1)$-morphisms u and v, $A_p^n(s_n u, s_n v) = A_p^n(t_n u, t_n v)$."

Since

$$\begin{aligned}
&\partial^- \left(\square_n^-(x *_{n-1} y) - \square_n^-(x) - \square_n^-(y)\right) \\
&= \square_{n-1}^-(s_{n-1}x) - \square_{n-1}^-(t_{n-1}y) \\
&-\square_{n-1}^-(s_{n-1}x) + \square_{n-1}^-(t_{n-1}x) - \square_{n-1}^-(s_{n-1}y) + \square_{n-1}^-(t_{n-1}y) \\
&= \square_{n-1}^-(t_{n-1}x) - \square_{n-1}^-(s_{n-1}y) = 0
\end{aligned}$$

in the normalized chain complex of the branching simplicial nerve, then $\square_n^-(x *_{n-1} y) - \square_n^-(x) - \square_n^-(y)$ is a cycle in the branching homology of the free globular ω-category $\mathcal{D}$ generated by two n-morphisms such that $t_{n-1}x = s_{n-1}y$. The ω-category $\mathcal{D}$ is of length at most one and non-contracting. Therefore its branching nerve coincides with the simplicial nerve of $\mathbb{P}\mathcal{D}$, the latter being the globular ω-category freely generated by the composable pasting scheme whose total composition is $X *_{n-2} Y$ where X and Y are two $(n-1)$-dimensional cells. Therefore this simplicial nerve is contractible. Consequently there exists $B_{n-1}^n(x,y)$ lying in the cubical singular nerve of $\mathcal{D}$ (and also in its branching nerve) such that

$$\square_n^-(x *_{n-1} y) - \square_n^-(x) - \square_n^-(y) = \partial^- B_{n-1}^n(x,y).$$

The $(n+1)$-cube $B_{n-1}^n(x,y)$ is necessarily thin because there is no morphism of dimension $n+1$ in $\mathcal{D}$. By setting $A_{n-1}^n(x,y) = 0$, we obtain $P(1)$. We are going to prove $P(h)$ by induction on h. Suppose $P(h)$ proved for $h \geqslant 1$. Then

$$\begin{aligned}
&\partial^-(\square_n^-(x *_{n-h-1} y) - \square_n^-(x) - \square_n^-(y) - B_{n-h-1}^{n-1}(s_{n-1}x, s_{n-1}y) \\
&+B_{n-h-1}^{n-1}(t_{n-1}x, t_{n-1}y))
\end{aligned}$$

$$\begin{aligned}
&= \big(\square_n^-(s_{n-1}x *_{n-h-1} s_{n-1}y) - \square_n^-(s_{n-1}x) - \square_n^-(s_{n-1}y) - \partial^- B_{n-h-1}^{n-1}(s_{n-1}x, s_{n-1}y)\big) \\
&\quad - \big(\square_n^-(t_{n-1}x *_{n-h-1} t_{n-1}y) - \square_n^-(t_{n-1}x) - \square_n^-(t_{n-1}y) - \partial^- B_{n-h-1}^{n-1}(t_{n-1}x, t_{n-1}y)\big) \\
&= A_{n-h-1}^{n-1}(s_{n-1}x, s_{n-1}y) - A_{n-h-1}^{n-1}(t_{n-1}x, t_{n-1}y) \text{ by induction hypothesis} \\
&= 0 \text{ again by induction hypothesis}
\end{aligned}$$

Therefore we can set $A_{n-h-1}^n(x, y) = B_{n-h-1}^{n-1}(s_{n-1}x, s_{n-1}y) - B_{n-h-1}^{n-1}(t_{n-1}x, t_{n-1}y)$ and we have

$$\begin{aligned}
& A_{n-h-1}^n(s_n u, s_n v) - A_{n-h-1}^n(t_n u, t_n v) \\
= \; & B_{n-h-1}^{n-1}(s_{n-1}s_n u, s_{n-1}s_n v) - B_{n-h-1}^{n-1}(t_{n-1}s_n u, t_{n-1}s_n v) \\
& - B_{n-h-1}^{n-1}(s_{n-1}t_n u, s_{n-1}t_n v) + B_{n-h-1}^{n-1}(t_{n-1}t_n u, t_{n-1}t_n v) \\
= \; & 0
\end{aligned}$$

because of the globular equations. So we get a thin n-cube $A_{n-h-1}^n(x, y)$ such that

$$\square_n^-(x *_{n-h-1} y) - \square_n^-(x) - \square_n^-(y) - A_{n-h-1}^n(x, y)$$

is a cycle in the normalized chain complex associated to the branching simplicial nerve of $\mathcal{C}$. This cycle lies in the branching nerve of the free ω-category generated by two n-morphisms x and y such that $t_{n-h-1}x = s_{n-h-1}y$. This ω-category is of length at most one and non-contracting. Therefore its branching nerve is isomorphic to the simplicial nerve of the globular ω-category freely generated by the composable pasting scheme whose total composition is $X *_{n-h-2} Y$ where X and Y are two $(n-1)$-dimensional cells. Therefore it is contractible. Therefore there exists $B_{n-h-1}^n(x, y)$ such that

$$\square_n^-(x *_{n-h-1} y) - \square_n^-(x) - \square_n^-(y) - A_{n-h-1}^n(x, y) = \partial^- B_{n-h-1}^n(x, y).$$

The cube $B_{n-h-1}^n(x, y)$ is necessarily thin because there is no morphism of dimension $n+1$ in the cubical sub-ω-category generated by x and y. And $P(h+1)$ is proved. ∎

It turns out that the $(n+1)$-cube $B_{n-1}^n(x, y)$ can be explicitly calculated. One can easily verify that

$$B_{n-1}^n(x, y)_h^- = \Gamma_{n-1}^- \Gamma_{n-2}^- \dots \Gamma_h^- \square_h^- d_h^{(-)^h} x$$

for $1 \leqslant h \leqslant n-2$ (observe that in this case, $d_h^{(-)^h} x = d_h^{(-)^h} y$),

$$\begin{aligned}
B_{n-1}^n(x, y)_{n-1}^- &= \square_n^- y \\
B_{n-1}^n(x, y)_n^- &= \square_n^-(x *_{n-1} y) \\
B_{n-1}^n(x, y)_{n+1}^- &= \square_n^- x
\end{aligned}$$

and for all i between 1 and $n+1$,

$$B_{n-1}^n(x, y)_i^+ = \square_n^- t_0 x$$

is a solution. It suffices to prove that $(B_{n-1}^n(x, y)_i^\pm)_{1 \leqslant i \leqslant n+1}$ is a thin n-shell.

9.2. THEOREM. *Let $\mathcal{C}$ be a non-contracting ω-category. Let x and y be two elements of $\omega Cat(I^n, \mathcal{C})$ such that $x +_j y$ exists for some j between 1 and n and such that $dim(x(0_n)) \geqslant 1$, $dim(y(0_n)) \geqslant 1$ and $dim((x +_j y)(0_n)) \geqslant 1$. Then $\Phi_n^-(x +_j y)$ is T-equivalent to $\Phi_n^-(x)$ or $\Phi_n^-(y)$ or to $\Phi_n^-(x) + \Phi_n^-(y)$. If x is itself in the branching complex, then $\Phi_n^-(x +_j y)$ is T-equivalent to x.*

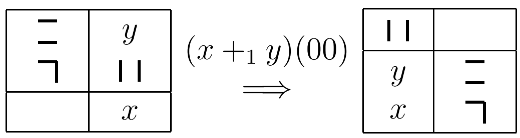
REMARK. The hypotheses about the dimension of $x(0_n)$, $y(0_n)$ and $(x +_j y)(0_n)$ are only to ensure that $\Phi_n^-(x)$, $\Phi_n^-(y)$ and $\Phi_n^-(x +_j y)$ are in the branching nerve. The hypothesis about the dimension of $(x +_j y)(0_n)$ is necessary because we do not assume that 1-morphisms in non-contracting ω-categories are not invertible. In dimension 1, the case $x(0) *_0 y(0) = (x +_1 y)(0) \in \mathcal{C}_0$ may happen.

PROOF. By definition, one has $\Phi_n^-(x +_j y) = \Box_n^-((x +_j y)(0_n))$. If $\mathcal{C}$ was equal to the globular sub-ω-category generated by

$$X = \{x(k_1 \dots k_n), k_1 \dots k_n \in \underline{cub}^n\} \cup \{y(k_1 \dots k_n), k_1 \dots k_n \in \underline{cub}^n\}$$

then $x +_j y$ still would exist in the cubical singular nerve. Therefore, $(x +_j y)(0_n)$ can be written as an expression using only the composition laws $*_n$ of $\mathcal{C}$ and the variables of X and moreover, the variables $x(0_n)$ and $y(0_n)$ can appear at most once. By Theorem 9.1, $\Box_n^-((x +_j y)(0_n))$ is therefore T-equivalent to $\Box_n^-(x(0_n))$, $\Box_n^-(y(0_n))$ or $\Box_n^-(x(0_n)) + \Box_n^-(y(0_n))$.

Now suppose that $x \in \omega Cat(I^n, \mathcal{C})^-$. Let $z = \Gamma_j^- x +_j \epsilon_{j+1} y \in \omega Cat(I^{n+1}, \mathcal{C})^-$. Then $\partial_j^- z = x$, $\partial_{j+1}^- z = x +_j y$ and $\partial_{j+1}^+ z = y$. Since z is a thin element, then all other faces $\partial_k^\pm z$ are thin (this can be verified directly by easy calculations). Therefore $\partial^- z$ is T-equivalent to $\pm(x +_j y - x)$. As illustration, let us notice that for $j = 1$ and $n = 2$, z is equal to



∎

9.3. THEOREM. *Let x and y be two morphisms of a non-contracting ω-category $\mathcal{C}$ such that $x *_0 y$ exists such that x and $x *_0 y$ are of dimension lower than n and of dimension strictly greater than 0. Then $\Box_n^-(x *_0 y)$ is T-equivalent to $\Box_n^-(x)$.*

PROOF. We need, only for this proof, the operator $\Box_n^+$ introduced in [12]. One has

$$\partial_1^+ \Box_n^-(x) = \epsilon_1^{n-1} \Box_0(t_0 x)$$

and

$$\partial_1^- \Box_n^+(y) = \epsilon_1^{n-1} \Box_0(s_0 y).$$

Therefore $\Box_n^-(x) +_1 \Box_n^+(y)$ exists and is T-equivalent to $\Box_n^-(x)$ by Theorem 9.2. If we work in the ω-category generated by x and y, then $\Box_n^-(x) +_1 \Box_n^+(y)$ is a well-defined element of the branching simplicial nerve of $\mathcal{D}$. And $\mathcal{D}$ is the free ω-category generated by a composable pasting scheme whose total composition is $x *_0 y$. Since union means composition in such a ω-category, then necessarily $(\square_n^-(x) +_1 \square_n^+(y))\,(0_n) = x *_0 y$. Since Φ_n^- is the identity map on the reduced branching complex, then $\square_n^-(x) +_1 \square_n^+(y)$ is T-equivalent to $\square_n^-(x *_0 y)$. ∎

The preceding formulae suggest another way of defining the reduced branching homology.

9.4. Proposition. *Set*

$$CF_n^-(\mathcal{C}) = \mathbb{Z}\mathcal{C}_n/\{x *_0 y = x, x *_1 y = x + y, \dots, x *_{n-1} y = x + y \ mod\ \mathbb{Z}tr_{n-1}\mathcal{C}\}.$$

Then $s_{n-1} - t_{n-1}$ from $CF_n^-(\mathcal{C})$ to $CF_{n-1}^-(\mathcal{C})$ for $n \geqslant 2$ and s_0 from $CF_1^-(\mathcal{C})$ to $CF_0^-(\mathcal{C})$ induce a differential map ∂_f^- on the $\mathbb{N}$-graded group $CF_*^-(\mathcal{C})$ and the chain complex one gets is called the formal branching complex. The associated homology is denoted by $HF_n^-(\mathcal{C})$ and is called the formal branching homology.

Proof. Obvious. ∎

A relation like $x *_0 y = x \bmod \mathbb{Z}\mathrm{tr}_{n-1}\mathcal{C}$ means that if x is for example a p-morphism for $p < n$ and y a n-morphism such that $x *_0 y$ exists, then in $CF_n^-(\mathcal{C})$, $x *_0 y = 0$.

9.5. Proposition. *Let $\mathcal{C}$ be a non-contracting ω-category. The linear map $\square_n^-$ from $\mathbb{Z}\mathcal{C}_n$ to $CR_n^-(\mathcal{C})$ induces a surjective morphism of chain complexes and therefore a morphism from $HF_*^-(\mathcal{C})$ to $HR_*^-(\mathcal{C})$.*

Proof. One has in the reduced branching complex $\square_n^-(x *_0 y) = \square_n^-(x)$ and $\square_n^-(x *_p y - x - y) = 0$ therefore $\square_n^-$ induces a linear map from $CF_n^-(\mathcal{C})$ to $CR_n^-(\mathcal{C})$. And $\square_{n-1}^-(\partial_f^-(x)) = \square_{n-1}^-(s_{n-1} - t_{n-1})(x) = \partial^-\square_n^-(x)$. Since Φ_n^- is the identity map on $CR_n^-(\mathcal{C})$, then $CR_n^-(\mathcal{C})$ is generated by the $\square_n^-(x)$ where x runs over $\mathcal{C}_n$. Therefore the induced morphism of chain complexes is surjective. ∎

9.6. Question. *When is the preceding map a quasi-isomorphism ?*

The meaning of the results of this section is that one homology class in branching homology corresponds really to one branching area. Here are some simple examples to understand this fact.

Figure 1 represents a 1-dimensional branching area. This branching area corresponds to one element in the reduced branching homology, that is

$$\square_1(u) - \square_1(w) = \square_1(u *_0 v) - \square_1(w) = \square_1(u) - \square_1(w *_0 x) = \square_1(u *_0 v) - \square_1(w *_0 x)$$

in homology. In fact, it even corresponds to one cycle in the reduced branching complex. The reason why it is more appropriate to work anyway with cycles modulo boundaries, and not only with cycles modulo boundaries of thin elements is illustrated in Figure 7. The two cycles $\square_1(u) - \square_1(v)$ and $\square_1(u) - \square_1(w)$ are in the same homology class as soon as u is homotopic to v.

These observations can be generalized in higher dimension but they are more difficult to draw. If u is a n-morphism, then, by definition of $\square_n^-$, $\square_n^-(u)$ is an homotopy between

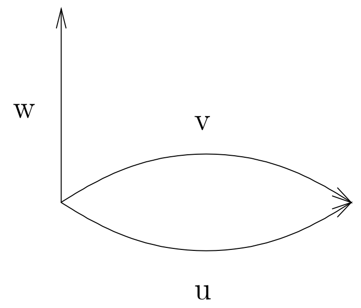


Figure 7: Another 1-dimensional branching area

$\square^-_{n-1} s_{n-1} u$ and $\square^-_{n-1} t_{n-1} u$ in the branching simplicial nerve. Figure 8 is an analogue of Figure 1 in dimension 2. Figure 8 represents a 2-dimensional branching area. In the branching complex, it corresponds to the cycles $(A) - (F) + (I)$, $(A,B,C,D) - (E,F,G,H) + (I,J,K,L)$, $(A) - (F,H) + (I,K)$, etc. In the reduced branching complex, there are even more possible cycles which correspond to this branching area. For example $(A,D) - (E,F) + (I,J,K,L)$, $(A) - (E,F) + (I,J)$, etc. In the branching homology, all these cycles are equivalent and therefore there is really one homology class which corresponds to one branching area. Or in other terms, the homology class does not depend on a cubification of the HDA.

10. Folding operations and differential map

Now we explore the relations between the folding operators and the differential map of the branching complex.

10.1. Proposition. *Let x be an element of $\omega Cat(I^n, \mathcal{C})^-$. Then*

$$\square^-_{n-1}(s_{n-1} - t_{n-1})(x(0_n)) = \square^-_{n-1}(\partial^- x)(0_{n-1}) = \sum_{p=1}^{n} (\partial^-_p x)(0_{n-1})$$

in $CR^-_{n-1}(\mathcal{C})$.

Proof. Since Φ^-_n induces the identity map on $CR^-_n(\mathcal{C})$, then $\Phi^-_{n-1}\partial^- = \partial^-\Phi^-_n = \partial^-$. Therefore

$$\square^-_{n-1}(\partial^- x)(0_{n-1}) = \Phi^-_{n-1}\partial^- x = \partial^-\Phi^-_n x = \partial^-\square^-_n(x(0_n)) = \square^-_{n-1}(s_{n-1} - t_{n-1})(x(0_n)).$$

∎

10.2. Proposition. *In the reduced branching homology of a given ω-category $\mathcal{C}$, one has*

1. *if $x \in \omega Cat(I^2, \mathcal{C})^-$, then $\square^-_1(s_1 x(00)) = \square^-_1 x(-0)$ and $\square^-_1(t_1 x(00)) = \square^-_1 x(0-)$*

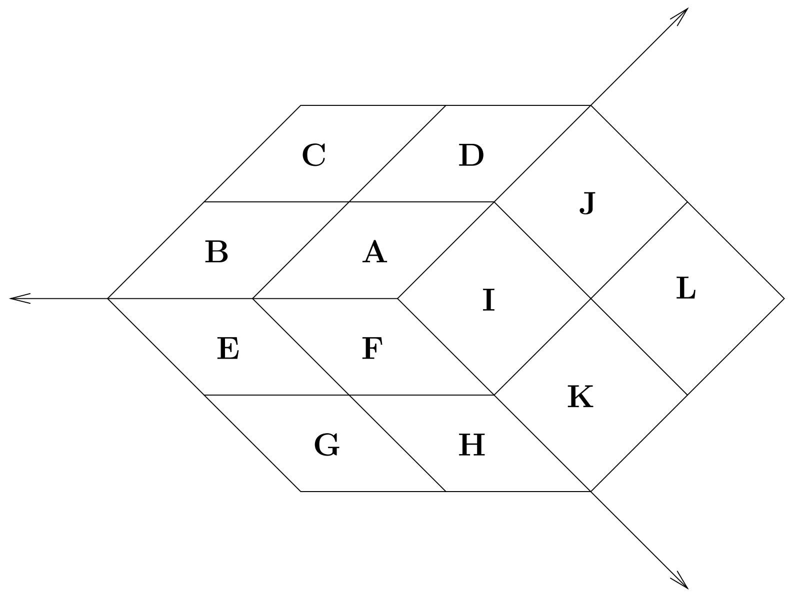


Figure 8: A 2-dimensional branching area

2. if $x \in \omega Cat(I^3, \mathcal{C})^-$*, then*

$$\square_2^-(s_2 x(000)) = \square_2^- x(-00) + \square_2^- x(-0-)$$
$$\square_2^-(t_2 x(000)) = \square_2^- x(0-0).$$

PROOF. One has

$$\square_1^-(s_1 x(00)) = \square_1^-(s_1(x(-0) *_0 x(0+))) = \square_1^- x(-0)$$

and

$$\square_1^-(t_1 x(00)) = \square_1^-(s_1(x(0-) *_0 x(+0))) = \square_1^- x(0-).$$

Now suppose that $x \in \omega Cat(I^3, \mathcal{C})^-$. Then

$$\begin{aligned}
&\square_2^-(s_2 x(000)) \\
&= \square_2^-\left((x(-00) *_0 x(0++)) *_1 (x(-0-) *_0 x(0+0)) *_1 (x(00-) *_0 x(++0))\right)) \\
&= \square_2^-(x(-00) *_0 x(0++)) + \square_2^-(x(-0-) *_0 x(0+0)) + \square_2^-(x(00-) *_0 x(++0))
\end{aligned}$$

So $\square_2^-(s_2 x(000)) = \square_2^-(x(-00)) + \square_2^-(x(00-))$.
In the same way, one has

$$\begin{aligned}
&\square_2^-(t_2 x(000)) \\
&= \square_2^-\left((x(--0) *_0 x(00+)) *_1 (x(0-0) *_0 x(+0+)) *_1 (x(0--) *_0 x(+00))\right)) \\
&= \square_2^-(x(--0) *_0 x(00+)) + \square_2^-(x(0-0) *_0 x(+0+)) + \square_2^-(x(0--) *_0 x(+00)) \\
&= \square_2^-(x(0-0))
\end{aligned}$$

∎

The preceding propositions can be in fact generalized as follows :

10.3. Theorem. *Let x be an element of $\omega Cat(I^n,\mathcal{C})^-$ with $n \geqslant 2$. Then in the reduced branching complex, one has*

$$\square^-_{n-1}(s_{n-1}x(0_n)) = \sum_{1\leqslant 2i+1\leqslant n} \square^-_{n-1}((\partial^-_{2i+1}x)(0_{n-1}))$$

$$\square^-_{n-1}(t_{n-1}x(0_n)) = \sum_{1\leqslant 2i\leqslant n} \square^-_{n-1}((\partial^-_{2i}x)(0_{n-1}))$$

Proof. For all n, we have seen that Φ^-_n induces the identity map on the reduced branching complex. Therefore for all $x \in \omega Cat(I^n,\mathcal{C})^-$, $\Phi^-_{n-1}\partial^- x = \partial^-\Phi^-_n x$. The latter equality can be translated into

$$\sum_{1\leqslant 2i+1\leqslant n} \Phi^-_{n-1}(\partial^-_{2i+1}x) - \sum_{1\leqslant 2i\leqslant n} \Phi^-_{n-1}(\partial^-_{2i}x) = \square^-_{n-1}s_{n-1}x(0_n) - \square^-_{n-1}t_{n-1}x(0_n).$$

If the above equality was in $\mathbb{Z}\omega Cat(I^{n-1},\mathcal{C})^-$, the proof would be complete. Unfortunately, we are working in the reduced branching chain complex, and so there exists $t_1 \in M^-_{n-1}$ and $t_2 \in M^-_n$ such that, in $\mathbb{Z}\omega Cat(I^{n-1},\mathcal{C})^-$

$$\sum_{1\leqslant 2i+1\leqslant n} \Phi^-_{n-1}(\partial^-_{2i+1}x) - \sum_{1\leqslant 2i\leqslant n} \Phi^-_{n-1}(\partial^-_{2i}x) = \square^-_{n-1}s_{n-1}x(0_n) - \square^-_{n-1}t_{n-1}x(0_n) + t_1 + \partial^- t_2.$$

Set $t_2 = \sum_{i\in I}\lambda_i T_i$ where T_i are thin elements of $\omega Cat(I^n,\mathcal{C})^-$. Each T_i corresponds to a thin $(n-1)$-cube in the free cubical ω-category generated by the n-cube x which will be denoted in the same way (see the last paragraph of Section 8). One can suppose that each $T_i(0_n)$ is $(n-1)$-dimensional. In the free cubical ω-category generated by x, either T_i is in the cubical ω-category generated by the $\partial_i^{(-)^i}x$ for $1\leqslant i\leqslant n$ (let us denote this fact by $T_i \leqslant s_{n-1}x(0_n)$), or T_i is in the cubical ω-category generated by the $\partial_i^{(-)^{i+1}}x$ for $1\leqslant i\leqslant n$ (let us denote this fact by $T_i \leqslant t_{n-1}x(0_n)$). Therefore one has

$$t_2 = \sum_{i\in I, T_i\leqslant s_{n-1}x(0_n)} \lambda_i T_i + \sum_{i\in I, T_i\leqslant t_{n-1}x(0_n)} \lambda_i T_i.$$

and

$$\begin{aligned}\partial^- t_2 = {} & \sum_{\substack{i\in I, T_i\leqslant s_{n-1}x(0_n)\\ 1\leqslant j\leqslant n, \partial^-_j T_i \text{ thin}}} (-1)^{j+1}\lambda_i\partial^-_j T_i + \sum_{\substack{i\in I, T_i\leqslant s_{n-1}x(0_n)\\ 1\leqslant j\leqslant n, \partial^-_j T_i \text{ non-thin}}} (-1)^{j+1}\lambda_i\partial^-_j T_i\\ & + \sum_{\substack{i\in I, T_i\leqslant t_{n-1}x(0_n)\\ 1\leqslant j\leqslant n, \partial^-_j T_i \text{ thin}}} (-1)^{j+1}\lambda_i\partial^-_j T_i + \sum_{\substack{i\in I, T_i\leqslant t_{n-1}x(0_n)\\ 1\leqslant j\leqslant n, \partial^-_j T_i \text{ non-thin}}} (-1)^{j+1}\lambda_i\partial^-_j T_i\end{aligned}$$

Because of the freeness of $\mathbb{Z}\omega Cat(I^{n-1},\mathcal{C})^{-}$, one gets

$$\sum_{1\leqslant 2i+1\leqslant n}\Phi^{-}_{n-1}(\partial^{-}_{2i+1}x)=\square^{-}_{n-1}s_{n-1}x(0_n)+\sum_{\substack{i\in I,T_i\leqslant s_{n-1}x(0_n)\\ 1\leqslant j\leqslant n,\partial^{-}_{j}T_i\text{ non-thin}}}(-1)^{j+1}\lambda_i\partial^{-}_{j}T_i$$

$$\sum_{1\leqslant 2i\leqslant n}\Phi^{-}_{n-1}(\partial^{-}_{2i}x)=\square^{-}_{n-1}t_{n-1}x(0_n)-\sum_{\substack{i\in I,T_i\leqslant t_{n-1}x(0_n)\\ 1\leqslant j\leqslant n,\partial^{-}_{j}T_i\text{ non-thin}}}(-1)^{j+1}\lambda_i\partial^{-}_{j}T_i$$

$$-t_1=\sum_{\substack{i\in I,T_i\leqslant s_{n-1}x(0_n)\\ 1\leqslant j\leqslant n,\partial^{-}_{j}T_i\text{ thin}}}(-1)^{j+1}\lambda_i\partial^{-}_{j}T_i+\sum_{\substack{i\in I,T_i\leqslant t_{n-1}x(0_n)\\ 1\leqslant j\leqslant n,\partial^{-}_{j}T_i\text{ thin}}}(-1)^{j+1}\lambda_i\partial^{-}_{j}T_i$$

■

11. Some consequences for the reduced branching homology

The following result generalizes the invariance result of [12] for the branching homology theory.

11.1. Proposition. *Let f and g be two non-contracting ω-functors from $\mathcal{C}$ to $\mathcal{D}$ satisfying the following conditions :*

- *for any 0-morphism x, $f(x)=g(x)$*
- *for any n-morphism x, $f(x)$ and $g(x)$ are two homotopic morphisms (and so of the same dimension).*

Then for any $n\geqslant 0$, $HR^{\pm}_{n}(f)=HR^{\pm}_{n}(g)$.

Proof. Consider the case of the reduced branching homology. Let $x\in CR^{-}_{n}(\mathcal{C})$. If

$$dim(f(x(0_n)))=dim(g(x(0_n))<n,$$

then $f(x)$ and $g(x)$ are two thin elements of $\omega Cat(I^n,\mathcal{C})^{-}$. Therefore $f(x)=g(x)$ in the reduced branching complex of $\mathcal{D}$. Now suppose that

$$dim(f(x(0_n)))=dim(g(x(0_n)))=n.$$

By hypothesis, there exists $z\in\mathcal{D}_{n+1}$ such that $f(x(0_n))-g(x(0_n))=(s_n-t_n)(z)$. Therefore in the reduced branching complex, one has $f(x)-g(x)=\square^{-}_{n}((s_n-t_n)(z))=\partial^{-}\square^{-}_{n+1}(z)$. So $f(x)-g(x)$ is a boundary. ■

We end up this section with another invariance result for the reduced branching homology and with some results related to Question 9.6.

11.2. THEOREM. *Let $\mathcal{C}$ and $\mathcal{D}$ be two ω-categories. Let f and g be two non 1-contracting ω-functors from $\mathcal{C}$ to $\mathcal{D}$ which coincide for the 0-morphisms and such that for any $n \geqslant 1$, there exists a linear map h_n from $CF_n^-(\mathcal{C})$ to $CF_{n+1}^-(\mathcal{D})$ such that for any $x \in CF_n^-(\mathcal{C})$, $h_{n-1}(s_{n-1} - t_{n-1}) + (s_n - t_n)h_n(x) = f(x) - g(x)$. Then $HR_n^-(f) = HR_n^-(g)$ for any $n \geqslant 0$.*

PROOF. Set $h_n^- x = \square_{n+1}^- h_n(x(0_n))$ for any $x \in \omega Cat(I^n, \mathcal{C})^-$. It is clear that $h_n^-(M_n^-(\mathcal{C})) = \{0\}$ in $CR_{n+1}^-(\mathcal{D})$. Now suppose that $x = \partial^- y$ for some $y \in M_{n+1}^-(\mathcal{C})$.

We already mentioned that $I^n[-_n, +_n]$ is the free ω-category generated by a composable pasting scheme in the proof of Corollary 5.9. It turns out that $s_n(R(0_{n+1}))$ and $t_n(R(0_{n+1}))$ belong to $I^n[-_n, +_n]$ and it is possible thereby to use the explicit combinatorial description of [18].

Set $I = \{1, 2, \ldots, n\}$ equipped with the total order $1 < 2 < \ldots < n$. Let $C(I, k)$ (or $C(n, k)$) be the set of all subsets of I of cardinality k. Let $\mathcal{P}$ an arbitrary subset of $C(I, k)$. There is a lexicographical order on $C(I, k)$ usually defined as follows : if $J = (j_1, \ldots, j_k)$ with $j_1 < \ldots < j_k$ and $J' = (j'_1, \ldots, j'_k)$ with $j'_1 < \ldots < j'_k$, then $J \leqslant J'$ means that either $j_1 < j'_1$, or $j_1 = j'_1$ and $j_2 < j'_2$, etc. If $K \in C(I, k+1)$, a K-packet is a set like $P(K) = \{J, J \in C(I, k), J \subset K\}$. If $K = (i_1, \ldots, i_{k+1})$ with $i_j < i_{j+1}$, then $P(K)$ consists of the sets $K_{\widehat{a}} = K - \{i_a\}$ for $a = 1, \ldots, k+1$. We have lexicographically

$$K_{\widehat{k+1}} < K_{\widehat{k}} < \ldots < K_{\widehat{1}}.$$

A total order σ on $C(I, k)$ will be denoted by $\sigma = J_1 J_2 \ldots J_N$ for $N = \begin{pmatrix} n \\ k \end{pmatrix}$, that is $J_i \sigma J_j$ for $i < j$. A total order is called *admissible* by Manin and Schechtman if on each packet it induces either a lexicographical order or the inverse lexicographical order. The set of admissible orders of $C(I, k)$ is denoted by $A(I, k)$ (or $A(n, k)$). Two total orders σ and σ' of $A(I, k)$ are called *elementary equivalent* if they differ by an interchange of two neighbours which do not belong to a common packet. The quotient of $A(I, k)$ by this equivalence relation is denoted by $B(I, k)$ (or $B(n, k)$). Suppose that for some $K \in C(I, k+1)$, the members of the packet $P(K)$ form a chain with respect to an admissible order σ of $A(I, k)$, i.e. any element of $C(I, k)$ lying between two elements of $P(K)$ belongs to $P(K)$. Define $p_K(\sigma)$ the admissible order in which this chain is reversed while all the rest elements conserve their positions. Then $p_K(\sigma)$ is still an admissible order and p_K passes to the quotient $B(I, k)$. The lemma on page 300 claims that $A(n, n-1) = B(n, n-1) = \{K_{\widehat{n}} \ldots K_{\widehat{1}}, K_{\widehat{1}} \ldots K_{\widehat{n}}\}$ where $K = (1, \ldots, n)$. And the poset $B(n, n-2)$ is described by the following picture :

$$
\begin{tikzcd}
& r_{min} \arrow[dl, "p_{K_{\widehat{n}}}"'] \arrow[dr, "p_{K_{\widehat{1}}}"] & \\
{} \arrow[d, "p_{K_{\widehat{n-1}}}"'] & & {} \arrow[d, "p_{K_{\widehat{2}}}"] \\
\vdots \arrow[d, "p_{K_{\widehat{2}}}"'] & & \vdots \arrow[d, "p_{K_{\widehat{n-1}}}"] \\
{} \arrow[dr, "p_{K_{\widehat{1}}}"'] & & {} \arrow[dl, "p_{K_{\widehat{n}}}"] \\
& r_{max} &
\end{tikzcd}
$$

It turns out that in the picture $B(n, n-2)$, the vertices are exactly the $(n-2)$-morphisms of $I^n[-_n, +_n]$ and the arrows are exactly the $(n-1)$-morphisms of $I^n[-_n, +_n]$. This explicit description shows therefore that $s_n(R(0_{n+1}))$ is equal to a composition $X_1 *_{n-1} \ldots *_{n-1} X_{n+1}$ where the only morphism of dimension n contained in X_j is $R(\delta_j^{(-)^j}(0_n))$. And the same description shows that $t_n(R(0_{n+1}))$ is equal to a composition $Y_{n+1} *_{n-1} \ldots *_{n-1} Y_1$ where the only morphism of dimension n contained in Y_j is $R(\delta_j^{(-)^{j+1}}(0_n))$. And one has

$$
\begin{aligned}
s_n(y(0_{n+1})) &= y(s_n(0_{n+1})) = y(X_1) *_{n-1} \ldots *_{n-1} y(X_{n+1}) \\
t_n(y(0_{n+1})) &= y(t_n(0_{n+1})) = y(Y_{n+1}) *_{n-1} \ldots *_{n-1} y(Y_1)
\end{aligned}
$$

Since y is thin, $s_n(y(0_{n+1})) = t_n(y(0_{n+1}))$. Since h_n is a map from $CF_n^-(\mathcal{C})$ to $CF_{n+1}^-(\mathcal{D})$, then

$$
\sum_{p=1}^{n+1} h_n(y(X_p)) = \sum_{p=1}^{n+1} h_n(y(Y_p))
$$

in $CF_{n+1}^-(\mathcal{D})$.

Since I^{n+1} is the free ω-category generated by the pasting scheme $\underline{cub}^{n+1}$, then for any p between 1 and $n+1$, X_p is a composition of $R(\delta_p^{(-)^p}(0_n))$ with other $R(k_1 \ldots k_{n+1})$ of dimension strictly lower than n. Suppose that p is odd. There exists $X_p^{(1)}$ and $X_p^{(1)\prime}$ such that $X_p = X_p^{(1)} *_{i_1} X_p^{(1)\prime}$ for some $0 \leqslant i_p \leqslant n-2$. If $i_p > 0$, then only one of the $X_p^{(1)}$ or $X_p^{(1)\prime}$ is of dimension n therefore $y(X_p) = y(X_p^{(1)})$ or $y(X_p) = y(X_p^{(1)\prime})$. If $i_p = 0$, then since $s_0 X_p = s_0 X_p^{(1)} = s_0 R(\delta_p^{(-)^p}(0_n))$ then in this case $X_p^{(1)}$ is n-dimensional and $X'^{(1)}_p$ is of dimension strictly lower than n. Therefore in this case $h_n(y(X_p)) = h_n(y(X_p^{(1)}))$. By repeating as many times as necessary the process, the number of cells $R(k_1 \ldots k_n)$ included in $y(X_p)$ decreases. And we obtain

$$
h_n(y(X_p)) = h_n(y(\delta_p^{(-)^p}(0_n))) = h_n((\partial_p^- y)(0_n)).
$$

Now suppose that p is even. Since $R(-_n) = s_0(X_p) \neq s_0 R(\delta_p^{(-)^p}(0_n))$, then necessarily at one step of the process, we have $i_h = 0$. Take the last h such that $i_h = 0$. Then

$h_n(y(X_p)) = h_n(y(X_p^{(h)}))$ and $X_p^{(h)} = X_p^{(h+1)} *_0 X_p^{(h+1)\prime}$. Since $s_0 X_p^{(h)} = s_0 X_p^{(h+1)} \neq s_0 R(\delta_p^{(-)^p}(0_n))$, then for this h, $X_p^{(h+1)}$ is of dimension strictly lower than n and $X_p^{(h+1)\prime}$ is of dimension n. Therefore $h_n(y(X_p)) = 0$.

In the same way, $h_n(y(Y_p)) = 0$ if p is odd and

$$h_n(y(Y_p)) = h_n(y(\delta_p^{(-)^{p+1}}(0_n))) = h_n((\partial_p^- y)(0_n))$$

if p is even. So

$$h_n^-(x) = \square_{n+1}^- h_n(x(0_n)) = \sum_{p=1}^{n+1} (-1)^{p+1} \square_{n+1}^- h_n((\partial_p^- y)(0_n)) = 0$$

in $CR_{n+1}^-(\mathcal{D})$ by Theorem 9.1. Therefore h_n^- induces a linear map from $CR_n^-(\mathcal{C})$ to $CR_{n+1}^-(\mathcal{D})$ still denoted by h_n^-. Take $x \in \omega Cat(I^n, \mathcal{C})^-$. Then in $CR_n^-(\mathcal{D})$, one has

$$\begin{aligned}
&\partial^- h_n^-(x) + h_{n-1}^- \partial^-(x) \\
&= \partial^- \square_{n+1}^- h_n(x(0_n)) + h_{n-1}^- \partial^- \Phi_n^-(x) \text{ since } \Phi_n^- \text{ is the identity map} \\
&= \square_n^-(s_n - t_n) h_n(x(0_n)) + h_{n-1}^- \partial^- \square_n^- x(0_n) \text{ by definition of } \Phi_n^- \\
&= \square_n^-(s_n - t_n) h_n(x(0_n)) + h_{n-1}^- \square_{n-1}^-(s_{n-1} x(0_n) - t_{n-1} x(0_n)) \\
&= \square_n^-(s_n - t_n) h_n(x(0_n)) + \square_n^- h_{n-1}(s_{n-1} x(0_n) - t_{n-1} x(0_n)) \text{ by definition of } h_n^- \\
&= \square_n^-(f(x)(0_n) - g(x)(0_n)) \text{ by hypothesis on } h_* \\
&= \Phi_n^-(f(x) - g(x)) \text{ by definition of } \Phi_n^- \\
&= f(x) - g(x) \text{ since } \Phi_n^- \text{ is the identity map}
\end{aligned}$$

∎

The proof of Theorem 11.2 provides another way of proving Theorem 10.3 and also establishes that Theorem 10.3 is still true for the formal branching homology.

11.3. Proposition. *Let $p \geqslant 1$ and let 2_p be the ω-category generated by a p-morphism A. Then $HF_n^-(2_p) = HR_n^-(2_p) = 0$ for $n > 0$ and $HF_0^-(2_p) = HR_0^-(2_p) = \mathbb{Z}$.*

Proof. The assertions concerning the formal branching homology are obvious. Since the negative folding operator induces the identity on the reduced branching complex, then $CR_n^-(2_p)$ is equal to 0 for $n > p$ and is generated by $\square_n^-(s_n A)$ and $\square_n^-(t_n A)$ for $0 \leqslant n \leqslant p$. The point is to prove that there is no relations between $\square_n^-(s_n A)$ and $\square_n^-(t_n A)$ for $1 \leqslant n < p$, that is $CR_n^-(2_p) = \mathbb{Z}\square_n^-(s_n A) \oplus \mathbb{Z}\square_n^-(t_n A) = CF_n^-(2_p)$. Suppose that there exists a linear combination of thin n-cubes t_1 and a linear combination of thin $(n+1)$-cubes t_2 such that for some integers λ and μ,

$$\lambda \square_n^-(s_n A) + \mu \square_n^-(t_n A) = t_1 + \partial^- t_2$$

in $C_n^-(2_p)$. Then $s_n t_2(0_{n+1}) = t_n t_2(0_{n+1})$ and so $\partial^- t_2$ is necessarily a linear combination of thin n-cube therefore $\lambda = \mu = 0$.

Another possible proof of this proposition is to use Theorem 11.2 and to use the homotopy equivalence of [12] Proposition 8.5 between 2_p and 2_1. ∎

11.4. PROPOSITION. *Let $p \geqslant 1$ and let $G_p\langle A,B\rangle$ be the ω-category generated by two non-homotopic p-morphisms A and B. Then $HF_n^-(G_p\langle A,B\rangle) = HR_n^-(G_p\langle A,B\rangle) = 0$ for $0 < n < p$ and $HF_0^-(G_p\langle A,B\rangle) = HR_0^-(G_p\langle A,B\rangle) = \mathbb{Z} = HF_p^-(G_p\langle A,B\rangle) = HR_p^-(G_p\langle A,B\rangle)$.*

PROOF. Analogous to the previous proof. ■

11.5. PROPOSITION. *Let $n \geqslant 0$. $HF_0^-(I^n) = \mathbb{Z}$ and for $p > 0$, $HF_p^-(I^n) = 0$.*

PROOF. We know that

$$CF_p^-(I^n) = \bigoplus_{R(k_1\dots k_n) \text{ of dimension } p} \mathbb{Z}\square_p^-(R(k_1\dots k_n)).$$

And the differential maps is also completely known. In the formal branching complex, one has

$$\partial^-\square_p^-(R(k_1\dots k_n)) = \sum_{1\leqslant j\leqslant p}(-1)^{j+1}\square_p^-(R(k_1\dots[-]_{n_j}\dots k_n))$$

where $k_{n_1},\dots,k_{n_p}$ are the 0's appearing in the word $k_1\dots k_n$ with $n_1 < \dots < n_p$. It follows that this chain complex can be split depending on the position and the number of the $+$'s, and that these positions and numbers are not modified by the differential maps. If the number of the $+$ signs is N, we are reduced to calculating the simplicial homology of the $(n-N)$-simplex which is known to vanish in dimension strictly greater than 0. ■

As for the calculation of $HR_*^-(I^n)$, the point is to prove as above for 2_p and $G_p\langle A,B\rangle$ that there is no additional relations between the $\square_p^-(R(k_1\dots k_n))$ in the reduced branching complex. Unfortunately, for a thin $(n+1)$-cube t_2 of the branching nerve of I^n, $\partial^- t_2$ is not necessarily a linear combination of thin n-cube. For example if a and b are two 1-morphisms of I^n such that $a*_0 b$ exists, then let t_2 the thin 2-cube such that $\partial_1^- t_2 = \square_1(a*_0 b)$, $\partial_2^+ t_2 = \square_1(t_0 b)$, $\partial_2^- t_2 = \square_1(a)$ and $\partial_1^+ t_2 = \square_1(b)$. Then $\partial^- t_2 = \square_1(a*_0 b) - \square_1(a)$.

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Institut de Recherche Mathématique Avancée
ULP et CNRS
7 rue René Descartes
67084 Strasbourg Cedex
France
Email: gaucher@irma.u-strasbg.fr

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