Theory and Applications of Categories, Vol. 7, No. 9, pp. 206–218.

NORMAL FUNCTORS AND STRONG PROTOMODULARITY

DOMINIQUE BOURN

Transmitted by Robert Rosebrugh

ABSTRACT. The notion of normal subobject having an intrinsic meaning in any protomodular category, we introduce the notion of normal functor, namely left exact conservative functor which reflects normal subobjects. The point is that for the category **Gp** of groups the change of base functors, with respect to the fibration of pointed objects, are not only conservative (this is the definition of a protomodular category), but also normal. This leads to the notion of strongly protomodular category. Some of their properties are given, the main one being that this notion is inherited by the slice categories.

Introduction

There are four general types of example of protomodular categories:

1) the varieties of classical algebraic structures, such as the category of groups, the category of rings, the category of associative or Lie algebras over a given ring A, the varieties of Ω-groups

2) the categories of internal algebraic structures of the previous kind in a left exact category $\mathbb{C}$

3) the non syntactical examples, such as the dual of any elementary topos $\mathbb{E}$

4) the constructible examples which inherit the property of being protomodular, such as the slice categories $\mathbb{C}/Z$ or the fibres $\mathrm{Pt}_Z\mathbb{C}$ of the fibration π of pointed objects.

We showed in [2] that the notion of protomodular category allows intrinsic definition of the concept of normal subobject, without any right exactness condition, in a way which allows recovery of the classical one in the setting of the category **Gp** of groups, and, more generally, of the classical concept of ideal in the setting of examples of type 1. The importance of this concept in examples of type 1 motivates its exploration in the other types of examples. The first aim of this paper was an attempt of a characterization of the normal subobjects in examples of types 2 and 4. For instance (type 4), when the basic protomodular category $\mathbb{C}$ is pointed or quasi-pointed (i.e. when the map $0 \to 1$ is monic), a subobject $j : (X, f) \to (Y, g)$ is normal in the slice category $\mathbb{C}/Z$ if and only if the map $j \cdot \mathrm{ker} f : K[f] \to X \to Y$ is normal in $\mathbb{C}$. The same result holds in the fibre $\mathrm{Pt}_Z\mathbb{C}$.

But this attempt led us mainly to focus on the notion of normal functor, namely a left exact conservative functor which reflects the normal subobjects. This kind of functor has still a meaning when $\mathbb{C}$ is only left exact and, even under this mild assumption, it has very

Received by the editors 1999 December 8 and, in revised form, 2000 June 19.
Published on 2000 July 7.
2000 Mathematics Subject Classification: 18D05,08B05,18G30,20L17.
Key words and phrases: abstract normal subobject, preservation and reflection of normal subobject, Mal'cev and protomodular categories.

nice properties. For instance, when a reflexive graph $\underline{X}_1$ has its image $F\underline{X}_1$ endowed with a groupoid structure, then $\underline{X}_1$ is itself endowed with a groupoid structure above the one on $F\underline{X}_1$. More specifically, when $\mathbb{C}$ is a left exact category and $\mathbf{Gp}\mathbb{C}$ the category of internal groups in $\mathbb{C}$, then the extension of the Yoneda embedding to the category of groups in $\mathbb{C}$, still denoted by $Y : \mathbf{Gp}\mathbb{C} \to \mathbf{Gp}^{\mathbb{C}^{op}}$, is normal, which gives a characterization (type 2) of normal subobjects in $\mathbf{Gp}\mathbb{C}$. The same result and characterization hold for internal rings.

The main point here will be the observation that the category $\mathbf{Gp}$ has not only its change of base functors, with respect to the fibration π of pointed objects, conservative (this is the definition of a protomodular category), but also normal. It is proved that a protomodular category does not necessarily satisfy this property, and I am indebted to G. Janelidze from whom I learned of the category of digroups which is the setting of the counterexample. This leads naturally to the definition of strongly protomodular categories as those which have their change of base functors, with respect to π, normal. Some properties of these categories are given, mainly the inheritance of the notion by slice categories and a characterization of internal groupoids.

The unexpected role of normal functors explains the organization of the paper along the following line:

1) Normal functors
2) Protomodularity and strong protomodularity
3) A counterexample
4) Example of strong protomodularity
5) Some properties of strongly protomodular categories

1. Normal functors

We shall suppose $\mathbb{C}$ a left exact category and denote Rel$\mathbb{C}$ the category of equivalence relations in $\mathbb{C}$. Let us recall that a map $f : R \to R'$ in Rel$\mathbb{C}$ is cartesian with respect to the forgetful functor U : Rel$\mathbb{C} \to \mathbb{C}$ associating with any equivalence relation R its underlying object X, when the following diagram is a joint pullback (in this case we shall often denote R by $f^{-1}(R')$ and call it the inverse image of R' by f). It is called fibrant when the square with the d_0 (or equivalently with the d_1) is a pullback:

$$\begin{tikzcd} R \arrow[r, "\phi"] \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] & R' \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\ X \arrow[r, "f"'] & X' \end{tikzcd}$$

Thus a map $f : X \to X'$ in $\mathbb{C}$ determines a map in Rel$\mathbb{C}$ if and only if it satisfies the property $R \subset f^{-1}(R')$. When it is the case, we shall use the same symbol f to denote the induced map $f : R \to R'$ in Rel$\mathbb{C}$.

A left exact functor $F : \mathbb{C} \to \mathbb{C}'$ between left exact categories extends to a functor F : Rel$\mathbb{C} \to$ Rel$\mathbb{C}'$ which preserves the cartesian and the fibrant maps. It reflects them as well, when it also reflects isomorphisms, since then F reflects the finite limits. In this situation, it also satisfies a nice property:

1.1. Proposition. *When a left exact functor $F : \mathbb{C} \to \mathbb{C}'$ between left exact categories reflects the isomorphisms, then each map in* $\mathrm{Rel}\mathbb{C}$ *is cocartesian (i.e. the dual of a cartesian map) with respect to the factorization functor $\Phi : \mathrm{Rel}\mathbb{C} \to \mathbb{C} \times_{\mathbb{C}} \mathrm{Rel}\mathbb{C}'$.*

Proof. Let us consider $f : R \to R'$ and $g : R \to R''$ two maps in $\mathrm{Rel}\mathbb{C}$ such that there is a map $k : X' \to X''$ in $\mathbb{C}$ satisfying $k \cdot f = g$ and determining a map $Fk : FR' \to FR''$ in $\mathrm{Rel}\mathbb{C}'$ such that $Fk \cdot Ff = Fg$.

Let us denote $S = k^{-1}(R'')$. Then $FS = (Fk)^{-1}(FR'')$. We have $Fk : FR' \to FR''$ and thus $FR' \subset FS$. In other words there is an isomorphism $FR' \cap FS \simeq FR'$. Consequently the inclusion $R' \cap S \hookrightarrow R'$ is mapped to an isomorphism. But F is conservative, whence an isomorphism $R' \cap S \simeq R'$ and the inclusion $R' \hookrightarrow S$ which determines a morphism $k : R' \to R''$. That $k \cdot f = g$ in $\mathrm{Rel}\mathbb{C}$ is a consequence of the fact that F, being conservative, is faithful. ■

Let us recall that a map $j : I \to X$ in $\mathbb{C}$ is *normal to an equivalence relation R* [2] when $j^{-1}(R)$ is the coarse relation grI on I (i.e. the kernel equivalence of the terminal map $I \to 1$) and the induced map $grI \to R$ in $\mathrm{Rel}\mathbb{C}$ is fibrant. This implies that j is necessarily a monomorphism. This definition gives an intrinsic way to express that I is an equivalence class of R. Clearly left exact functors preserve this kind of monomorphism.

1.2. Definition. *We shall call normal a left exact functor $F : \mathbb{C} \to \mathbb{C}'$ between left exact categories which reflects isomorphisms and normal monomorphisms: when j is such that Fj is normal to some equivalence relation S on FX, then there exists an equivalence relation R on X such that j is normal to R and $FR = S$.*

According to proposition 1.1, this R is unique up to isomorphism.

Examples.

1) Let $\mathbf{Gp}$ be the category of groups, Mag the category of magmas (sets endowed with a binary operation) and $U : \mathbf{Gp} \to Mag$ the forgetful functor. Then U is normal.

2) Let $\mathbb{C}$ be a left exact category and $\mathbf{Gp}\mathbb{C}$ the category of internal groups in $\mathbb{C}$. Then the extension of the Yoneda embedding to the category of groups in $\mathbb{C}$, still denoted by $Y : \mathbf{Gp}\mathbb{C} \to \mathbf{Gp}^{\mathbb{C}^{op}}$, is normal, [3] Prop. 5 and Theorem 6. This gives immediately a characterization of normal subobjects in the category $\mathbf{Gp}\mathbb{C}$, since the normal subobjects in a category $\mathbf{Gp}^{\mathbb{E}}$ are those subobjects which are componentwise normal. The same result and characterization hold for internal rings.

3) Any conservative left exact functor between additive categories with kernels is normal.

1.3. Proposition. *Suppose that $F : \mathbb{C} \to \mathbb{C}'$ is normal and $\mathbb{C}$ is quasi-pointed (i.e. it has an initial object 0 and the map $0 \to 1$ is a monomorphism), then the functor $\Phi : \mathrm{Rel}\mathbb{C} \to \mathbb{C} \times_{\mathbb{C}'} \mathrm{Rel}\mathbb{C}'$ is cofibrant on the fibrant morphisms of* $\mathrm{Rel}\mathbb{C}'$*: i.e. given an object R in* $\mathrm{Rel}\mathbb{C}$ *and a map (f, Ff) in $\mathbb{C} \times_{\mathbb{C}'} \mathrm{Rel}\mathbb{C}'$ with domain $\Phi(R)$ and such that the* *map Ff is fibrant in* Rel$\mathbb{C}'$*, there is a cocartesian map with respect to Φ above (f, Ff) in* Rel$\mathbb{C}$.

PROOF. According to proposition 1.1, it is sufficient to prove the existence of a map above (f, Ff).

Let R be an equivalence relation on X in Rel$\mathbb{C}$ and $f : X \to Y$ a map in $\mathbb{C}$ such that there is an equivalence relation S on FY which makes $Ff : FR \to S$ a fibrant map in Rel$\mathbb{C}'$. The category $\mathbb{C}$ being quasi-pointed, the map $0 \to 1$ is monic and consequently the kernel relation of any map $Z \to 0$ is the coarse relation grZ. So we can associate with R the normal monomorphism $j = d_1 \cdot k$ determined by the following pullback:

$$\begin{tikzcd} I \arrow[r, "k"] \arrow[d] & R \arrow[r, "d_1"] \arrow[d, "d_0"] & X \\ 0 \arrow[r] & X & \end{tikzcd}$$

This map j is normal to R and consequently Fj is normal to FR. But the morphism $Ff \cdot Fj : grFI \to S$ is fibrant in Rel$\mathbb{C}'$ as a composition of fibrant maps, and cartesian as any morphism with domain of the form grZ. Consequently the morphism $Ff \cdot Fj$ in $\mathbb{C}'$ is normal to S. The functor F being normal, the map $f \cdot j$ is normal to some S^* on Y in $\mathbb{C}$ such that $FS^* = S$. Proposition 1.1 asserts the existence of a factorization $f : R \to S^*$ which is necessarily fibrant since Ff is fibrant. ∎

We have also the following result:

1.4. PROPOSITION. *Suppose $\mathbb{C}$ and $\mathbb{C}'$ pointed, and $F : \mathbb{C} \to \mathbb{C}'$ normal. When a split epimorphism $(f, s) : X \to Y$ in $\mathbb{C}$ with kernel $K[f]$ is such that (Ff, Fs) is the canonical split epimorphism given by the projection $FK[f] \times FY \to FY$, then X is isomorphic to $K[f] \times Y$, and (f, s) is, up to isomorphism, the canonical split epimorphism $K[f] \times Y \to Y$.*

PROOF. For the sake of simplicity, let us denote by K the kernel $K[f]$ of the map f. Clearly the canonical inclusion $Fs = i_{FY} : FY \to FK \times FY$ is normal to the kernel relation $R[p_{FK}]$ of the projection $p_{FK} : FK \times FY \to FK$. The functor F being normal, let us denote by S the equivalence relation on X which makes s normal to S and such that $FS = R[p_{FK}]$. Now, if $R[f]$ denotes the kernel relation of the map f, let us consider $R[f]\square S$ the double relation on X, determined by the inverse image of $R[f] \times R[f]$ by the map $[d_0, d_1] : S \to X \times X$. It corresponds to the subobject of X^4 consisting of the quadruples (x, x', y, y') such that $xSx', ySy', xRy, x'Ry'$. This $R[f]\square S$ is sent by F to $R[p_{FY}]\square R[p_{FK}]$. Consequently the following diagram is a pullback since its image by F is clearly a pullback:

$$\begin{tikzcd} R[f]\square S \arrow[r, "d_0"] \arrow[d, "d_0"'] & R[f] \arrow[d, "d_0"] \\ S \arrow[r, "d_0"'] & X \end{tikzcd}$$

which, internally speaking, means that, for all x, x', y satisfying xRy, xSx',there is a unique y' such that $x'Ry'$ and ySy'. Then according to [2], theorem 11, there is a map $\phi : K \times Y \to X$ such that $\phi.i_K = \ker f$ and $\phi \cdot i_Y = s$. Its internally corresponds to the function which, in the set theoretical context, associates with (k, y) the unique x determined by the triple $e, s(y)$ and k, where e denotes the distinguished point of any object Z, since we do have eRk and $eSs(y)$ in X. But $F\phi$ is obviously an isomorphism and consequently ϕ is an isomorphism. ∎

We shall specifically need later on the following definition:

1.5. Definition. *A functor $F : \mathbb{C} \to \mathbb{C}'$ is called strongly normal when it is left exact, conservative and such that $\Phi : \mathrm{Rel}\mathbb{C} \to \mathbb{C} \times_{\mathbb{C}'} \mathrm{Rel}\mathbb{C}'$ is cofibrant on the fibrant morphisms of* $\mathrm{Rel}\mathbb{C}'$.

2. Protomodularity and strong protomodularity

We denote by $\mathrm{Pt}\mathbb{C}$ the category whose objects are the split epimorphisms in $\mathbb{C}$ with a given splitting and morphisms the commutative squares between these data. We denote by $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ the functor associating its codomain with any split epimorphism. As soon as $\mathbb{C}$ has pullbacks, the functor π is a fibration which is called the *fibration of pointed objects.*

Remark. A reflexive relation $(d_0, d_1) : R \hookrightarrow X \times X$ on an object X in a left exact category $\mathbb{C}$ determines, in the fibre $\mathrm{Pt}_X\mathbb{C}$ above X, a subobject (d_0, s_0) of the object $(p_0, s_0), p_0 : X \times X \to X$. The converse is true as well. Now, when R is an equivalence relation, $X \times R$ determines an equivalence relation on the object (p_0, s_0) in $\mathrm{Pt}_X\mathbb{C}$ to which the inclusion $(d_0, s_0) \hookrightarrow (p_o, s_0)$ is normal.

The category $\mathbb{C}$ is said to be *protomodular* [1] when π has its change of base functors conservative. When $\mathbb{C}$ is pointed, this condition is equivalent to the split short five lemma, which makes the category **Gp** of groups the leading example of this notion. As soon as a functor $F : \mathbb{C} \to \mathbb{C}'$ preserves pullbacks and is conservative, $\mathbb{C}'$ protomodular implies $\mathbb{C}$ protomodular. Accordingly any fibre $\mathrm{Pt}_X\mathbb{C}$ of π above an object X is protomodular, as well as any slice category $\mathbb{C}/X$.

Now in a protomodular category $\mathbb{C}$, when $j : I \to X$ is normal to some R, this R is unique up to isomorphism [2]. So the fact of being normal becomes an intrinsic property in $\mathbb{C}$. Of course, in the category **Gp** this notion coincides with the classical one, and in the category **Rng** of rings it coincides with the notion of two-sided ideals.

Remark. We know that a protomodular category is always Mal'cev (i.e. any reflexive relation is an equivalence relation), see [5] and [2]. The previous remark allows us to express the Mal'cev property in the following terms: *When $\mathbb{C}$ is protomodular, then any subobject of (p_0, s_0) in the fibre $Pt_X\mathbb{C}$ above X is normal in this fibre.*

We just saw that the fibres $\mathrm{Pt}_X\mathbb{C}$ and the slice categories $\mathbb{C}/X$ are still protomodular. It is then natural to ask whether it is possible to characterize the normal monomorphisms of these categories in terms of normality in $\mathbb{C}$. There is a simple answer when $\mathbb{C}$ is quasi-pointed.

2.1. Proposition. *When $\mathbb{C}$ is quasi-pointed and protomodular, a map $j : (X, f) \to (Y, g)$ is normal in $\mathbb{C}/Z$ if and only if the map $j \cdot \ker f : K[f] \to X \to Y$ is normal in $\mathbb{C}$. The same result holds in $\mathrm{Pt}_Z\mathbb{C}$.*

Proof. The map $0 \to 1$ is monic and consequently the kernel relation of any map $Z \to 0$ is the coarse relation grZ. Let us consider the following diagram in $\mathbb{C}$:

$$\begin{tikzcd}
K[f] \times K[f] \arrow[r] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & X \times_Z X \arrow[r] \arrow[d, shift right, "p_0"'] \arrow[d, shift left, "p_1"] & R \arrow[d, shift right, "d_0"'] \arrow[d, shift left, "d_1"] \\
K[f] \arrow[r, "\ker f"'] \arrow[d] & X \arrow[r, "j"'] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
0 \arrow[r] & Z \arrow[r] & Z
\end{tikzcd}$$

where R is the equivalence relation in $\mathbb{C}/Z$ to which j is normal and $K[f]$ the kernel of f, i.e. the pullback of the map f along the initial monomorphism $0 \to Z$. Thus the upper left hand side diagram determines a fibrant map in $\mathrm{Rel}\mathbb{C}$ and makes $j \cdot \ker f : grK[f] \to R$ a fibrant and cartesian map in $\mathrm{Rel}\mathbb{C}$. So $j \cdot \ker f$ is normal to R in $\mathbb{C}$.

Conversely, let us suppose the map $j \cdot \ker f$ normal in $\mathbb{C}$, and denote by R the equivalence relation to which it is normal. We must show that R lies in $\mathbb{C}/Z$ and that j is normal to R in this category. Let us denote by $R[f]$ and $R[g]$ the kernel equivalences of f and g. The map $j : R[f] \to R[g]$ is cartesian in $\mathrm{Rel}\mathbb{C}$, as well as $\ker f : grK[f] \to R[f]$. Thus $j \cdot \ker f : grK[f] \to R[g]$ is cartesian, while $j \cdot \ker f : grK[f] \to R$ is cartesian and fibrant. Consequently the following square is a pullback in $\mathrm{Rel}\mathbb{C}$:

$$\begin{tikzcd}
grK[f] \arrow[r] \arrow[d] & R \cap R[g] \arrow[d, "t"] \\
grK[f] \arrow[r] & R
\end{tikzcd}$$

Now the lower and thus the upper maps are fibrant. Consequently the change of base functor $(j \cdot \ker f)^*$ with respect to $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ maps t to an isomorphism. The category $\mathbb{C}$ being protomodular, the map t is itself an isomorphism, we have $R \subset R[g]$ and R belongs to $\mathbb{C}/Z$. Now let us consider the following pullbacks in $\mathrm{Rel}\mathbb{C}$:

$$\begin{tikzcd}
grK[f] \arrow[r] \arrow[d] & j^{-1}R \arrow[r] \arrow[d, "m"] & R \arrow[d] \\
grK[f] \arrow[r] & R[f] \arrow[r] & R[g]
\end{tikzcd}$$

The left hand side square is a pullback since the right hand side square and the total square are pullbacks. But $\ker f : grK[f] \to R[f]$ is fibrant, so the map $grK[f] \to j^{-1}R$

is fibrant too. Thus since $(\ker f)^*(m)$ is an isomorphism in $\mathrm{Pt}_{K[f]}\mathbb{C}$, the map m is itself an isomorphism. We have therefore $j : R[f] \to R$ which is certainly fibrant, since $\mathbb{C}$ is protomodular, and both $\ker f : grK[f] \to R[f]$ and $j \cdot \ker f : grK[f] \to R$ are fibrant. Thus j is normal to R in $\mathbb{C}/Z$. ■

REMARK. We do know the following result in the category **Gp** of groups: given a morphism between two exact sequences, when f'' is an isomorphism and f' a normal monomorphism, then $g' \cdot f'$ is a normal monomorphism (see [6] for instance, axiom 1.2 for a Moore category):

$$\begin{tikzcd} 1 \arrow[r] & H' \arrow[r, "h'"] \arrow[d, "f'"'] & H \arrow[r, "h"] \arrow[d, "f"] & H'' \arrow[r] \arrow[d, "f''"] & 1 \\ 1 \arrow[r] & G' \arrow[r, "g'"'] & G \arrow[r, "g"'] & G'' \arrow[r] & 1 \end{tikzcd}$$

Thus $f \cdot h' = f \cdot \ker h$ is normal in **Gp**. Suppose now f'' an identity. This result means that, the map h being a surjection, the map f is normal in $\mathbf{Gp}/H''$ if and only if the map f' is normal in **Gp**.

On the other hand, the homomorphisms h and g are certainly epimorphic when h is split. Whence the following theorem:

2.2. THEOREM. *In the category* **Gp** *of groups, the change of base functors with respect to* $\pi : \mathrm{Pt}\mathbf{Gp} \to \mathbf{Gp}$ *are normal.*

We shall study this specific property, so let us introduce:

2.3. DEFINITION. *A left exact category is said to be strongly protomodular when the change of base functors with respect to* $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ *are normal.*

Each fibre being pointed, this implies that each change of base functor is strongly normal.

3. A counterexample

A protomodular category is not necessarily strongly protomodular. Let us denote by $U : \mathbf{Gp} \to \mathbf{Set}^*$ the forgetful functor towards the category of pointed sets, associating with each group its underlying set pointed by the unit element. Let us call the category of *digroups* the category defined by the following pullback:

$$\begin{tikzcd} \mathbf{DiGp} \arrow[r, "p_1"] \arrow[d, "p_o"'] & \mathbf{Gp} \arrow[d, "U"] \\ \mathbf{Gp} \arrow[r, "U"'] & \mathbf{Set}^* \end{tikzcd}$$

The functor p_0 is clearly left exact and conservative. Thus the category **DiGp** is protomodular. On the other hand, it is clearly pointed. We are going to show that, however, it is not strongly protomodular.

In any pointed protomodular category, the diagonal $s_0 : X \to X \times X$ is normal if and only if X is an internal (abelian) group [2]. For that reason, given a digroup structure on a set G, its diagonal in **DiGp** is normal if and only if the two laws on G are abelian and coincide. Now let $(G, \cdot, \#)$ be a digroup. We shall denote respectively by x^{-1} and x° the inverse of an element x with respect to the two laws.

3.1. PROPOSITION. *A monomorphism $j : (G', \cdot, \#) \to (G, \cdot, \#)$ is normal in* **DiGp** *if and only if:*

1) $(G', \cdot)$ is normal in $(G, \cdot)$
2) $(G', \#)$ is normal in $(G, \#)$
3) $z \cdot x^{-1} \in G'$ if and only if $z\#x^\circ \in G'$.

PROOF. Let R be the equivalence relation in **DiGp** to which j is normal. Then $p_0(j)$ and $p_1(j)$ are normal to $p_0(R)$ and $p_1(R)$ in Gp and thus the two first statements are satisfied. The third is a consequence of the fact that: $z \cdot x^{-1} \in G' \Leftrightarrow zRx \Leftrightarrow z\#x^\circ \in G'$.

Conversely, let us suppose the three statements satisfied. Let us define zRx by $z \cdot x^{-1} \in G'$. The subgroup $(G', \cdot)$ being normal, R defines a subgroup of $(G \times G, \cdot)$. Now $z \cdot x^{-1}$ is equivalent to $z\#x^\circ \in G'$ and then R defines a subgroup of $(G \times G, \#)$. ∎

Now let us produce the counterexample. Let A be an abelian group, such that there is an element a with $a \neq -a$. Let us define $\theta : A \times A \to A \times A$ in the following way:
if $x \neq a$, then $\theta(z, x) = (z, x)$, if $x = a$ and $z \neq a$, $z \neq -a$, then $\theta(z, a) = (z, a)$
while $\theta(a, a) = (-a, a)$ and $\theta(-a, a) = (a, a)$.

We have then: $\theta \neq Id$, $\theta^2 = Id$, and $p_1 \cdot \theta = p_1$. Now let $\#$ be the transform along θ of the ordinary product law on $A \times A$. Whence $(z, x)\#(z', x') = \theta(\theta(z, x) + \theta(z', x'))$. Now $(A \times A, +, \#)$ is a digroup and $p_1 : (A \times A, +, \#) \to (A, +, +)$ a digroup homomorphism which is split in **DiGp** by the homomorphism s, with $s(z) = (0, z)$. Thus (p_1, s) is an object in Pt**DiGp** above $(A, +, +)$. Moreover the kernel of p_1 is again $(A, +, +)$.

Let $\omega_A : 1 \to (A, +, +)$ denote the initial map in **DiGp**. So, if s_0 denotes the diagonal of the object (p_1, s) in Pt_A**DiGp**, then $(\omega_A)^*(s_0)$ is the diagonal $s_0 : A \to A \times A$, which is normal in **DiGp** since the two laws on A are abelian and coincide. We are now going to prove that, however, this diagonal s_0 of (p_1, s) in Pt_A**DiGp** is not normal in this fibre and that, consequently, $(\omega_A)^*$ does not reflect the normal monomorphisms, which will mean that **DiGp** is not strongly protomodular. For that, according to proposition 2.1, we must check that $k = s_0 \cdot \ker p_1$ is not normal in **DiGp**. We have $k(z) = (z, z, 0)$. Note that $(-a, a)^\circ = (-a, -a)$. Then if $z \neq a$, $z \neq -a$, we have $(a, z, a)\#(-a, z + 2a, a)^\circ = (a, z, a)\#(-a, -z - 2a, -a) = (-2a, -2a, 0)$ which belongs consequently to $k(A)$, while $(a, z, a) + (a, -z - 2a, -a) = (2a, -2a, 0)$ does not. The third condition of proposition 2.1 is not fulfilled by $k(A)$, and k is not normal in **DiGp**.

4. Examples of strong protomodularity

Proposition 2.1 allows easily to check when a pointed or quasi-pointed protomodular category is strongly protomodular. So besides the category **Gp** of groups, the category

Rng of rings, for instance, is strongly protomodular. In the same way, any presheaf category of groups $\mathbf{Gp}^{\mathbb{E}^{op}}$ or of rings $\mathbf{Rng}^{\mathbf{E^{op}}}$ is strongly protomodular, since a morphism $j : F \to F'$ is normal in these categories if and only if, for every object X in $\mathbb{E}$, the homomorphism $j(X) : F(X) \to F'(X)$ is normal.

A category $\mathbb{C}$ is Naturally Mal'cev [7] when the fibres of $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ are additive. According to example 3 of normal functor, any protomodular Naturally Mal'cev category is strongly protomodular. As a particular case, any essentially affine category [1] (i.e. when $\pi : \mathrm{Pt}\mathbb{C} \to \mathbb{C}$ is trivial: any change of base functor with respect to π is an equivalence of categories), is strongly protomodular.

We have also a very helpful result:

4.1. Proposition. *Let $F : \mathbb{C} \to \mathbb{C}'$ be a strongly normal functor. Then:*

1) the functors $F_Z : \mathbb{C}/Z \to \mathbb{C}'/FZ$ are normal, and consequently the extension $\mathrm{Pt}F : \mathrm{Pt}\mathbb{C} \to \mathrm{Pt}\mathbb{C}'$ is fibrewise strongly normal

2) if $\mathbb{C}'$ is strongly protomodular, the category $\mathbb{C}$ is itself strongly protomodular.

Proof. The functor F, being left exact and conservative, produces left exact conservative functors $F_Z : \mathbb{C}/Z \to \mathbb{C}'/FZ$ and $F_Z : \mathrm{Pt}_Z\mathbb{C} \to \mathrm{Pt}_{FZ}\mathbb{C}'$.

1) Now let R be an equivalence relation on the object h in $\mathbb{C}/Z$. Let $f : h \to h'$ be a map in $\mathbb{C}/Z$ and $Ff : F(R) \to S$ a fibrant map in $\mathrm{Rel}(\mathbb{C}'/FZ)$. It is fibrant in $\mathrm{Rel}\mathbb{C}'$, and F being strongly normal, there is an equivalence relation S^* in $\mathrm{Rel}\mathbb{C}$ above S, making $f : R \to S^*$ a fibrant map in $\mathrm{Rel}\mathbb{C}$. We must show that S^* lies in $\mathbb{C}/Z$ on the object h'. So let us consider $S^* \cap R[h']$. It is mapped to $S \cap R[Fh']$ which is S since S is an equivalence relation on Fh' in $\mathbb{C}'/FZ$. Thus the inclusion $S^* \cap R[h'] \hookrightarrow S^*$ is mapped by F to an isomorphism and consequently is itself an isomorphism, which means that S^* lies in $\mathbb{C}/Z$ on the object h'. The same reasons hold for $F_Z : \mathrm{Pt}_Z\mathbb{C} \to \mathrm{Pt}_{FZ}\mathbb{C}'$.

2) Clearly, when $f : X \to X'$ is a map in $\mathbb{C}$, the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Pt}_{X'}C \arrow[r, "f^*"] \arrow[d, "F_{X'}"'] & \mathrm{Pt}_X C \arrow[d, "F_X"] \\
\mathrm{Pt}_{FX'}C' \arrow[r, "(Ff)^*"'] & \mathrm{Pt}_{FX}C'
\end{tikzcd}$$

Now when $\mathbb{C}'$ is strongly protomodular, the functor $(Ff)^*$ is strongly normal and consequently so is $F_X \cdot f^* = (Ff)^* \cdot F_{X'}$ since $F_{X'}$ is itself strongly normal. Now F_X is also strongly normal and thus f^* is strongly normal. ∎

As a corollary, we get:

4.2. Corollary. *When $\mathbb{C}$ is left exact, the category $\mathbf{Gp}\mathbb{C}$ of internal groups in $\mathbb{C}$ is strongly protomodular.*

Proof. The extension of the Yoneda embedding $Y : \mathbf{Gp}\mathbb{C} \to \mathbf{Gp}^{\mathbb{C}^{op}}$ is normal [3], but $\mathbf{Gp}\mathbb{C}$ is pointed and consequently Y is strongly normal. The presheaf category $\mathbf{Gp}^{\mathbb{C}^{op}}$ is strongly protomodular, and then so is $\mathbf{Gp}\mathbb{C}$. ∎

Suppose $\mathbb{C}$ left exact and denote $\mathbf{Grd}\mathbb{C}$ the category of internal groupoids in $\mathbb{C}$. Let $(\ )_0 : \mathbf{Grd}\mathbb{C} \to \mathbb{C}$ the functor associating with each groupoid its object of objects. It is a fibration. Clearly $\mathbf{Grd}_1\mathbb{C}$, the fibre above 1, is just $\mathbf{Gp}\mathbb{C}$.

But any fibre $\mathbf{Grd}_X\mathbb{C}$ is quasi-pointed and shares with $\mathbf{Grd}_1\mathbb{C}$ the property of being protomodular [1].

Actually, we have more:

4.3. Proposition. *Any fibre* $\mathbf{Grd}_X\mathbb{C}$ *is strongly protomodular.*

Proof. Again the extension of the Yoneda embedding $Y : \mathbf{Grd}\mathbb{C} \to \mathbf{Grd}^{\mathbb{C}^{op}}$, where $\mathbf{Grd}$ denotes the category of ordinary groupoids, is fibrewise normal: $\mathbf{Grd}_X\mathbb{C} \to (\mathbf{Grd}^{\mathbb{C}^{op}})_{Y_X}$ [3]. But each fibre is quasi-pointed, thus Y is fibrewise strongly normal. On the other hand $\mathbf{Grd}^{\mathbb{C}^{op}}$ has its fibres above $\mathbf{Set}^{\mathbb{C}^{op}}$ strongly protomodular, consequently, $\mathbf{Grd}\mathbb{C}$ is fibrewise (with respect to $(\ )_0 : \mathbf{Grd}\mathbb{C} \to \mathbb{C}$) strongly protomodular. ∎

5. Some properties of strongly protomodular categories

1) Abelian objects

We recalled that in a protomodular category $\mathbb{C}$ an object X has at most one internal group structure, which is necessarily abelian. In this situation we call X abelian. This is the case if and only if it has a point $1 \to X$ and its diagonal $s_0 : X \to X \times X$ is normal [2]. Then, given any map $f : X \to X'$ in a strongly protomodular category, the functor $f^* : \mathrm{Pt}_{X'}\mathbb{C} \to \mathrm{Pt}_X\mathbb{C}$ between the fibres reflects the abelian objects. In particular, when $\mathbb{C}$ is moreover pointed, we recover the well known result for the category $\mathbf{Gp}$ of groups following which a split epimorphism (h, t) with codomain Z determines a group structure in $\mathbf{Gp}/Z$ if and only if the kernel of h is an abelian group.

2) Internal groupoids

When the category $\mathbb{E}$ is left exact, an internal groupoid in $\mathbb{E}$ is a reflexive graph $\underline{X}_1$:

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1$$

endowed with a map $d_2 : R[d_0] \to X_1$, in such a way that all the simplicial identities hold as far as level 3 when the diagram is completed by the kernel pair of $d_0 : R[d_0] \to X_1$. Now the following identities: $d_0 \cdot d_2 = d_1 \cdot p_0$ and $d_2 \cdot s_0 \cdot s_0 = s_0 \cdot d_1 \cdot s_0 = s_0$ make $d_2 : (d_1 \cdot p_0, s_0 \cdot s_0) \to (d_0, s_0)$ a map in $\mathrm{Pt}_{X_0}\mathbb{E}$ which furthermore makes the factorization $[d_2, p_0] : (d_1 \cdot p_0, s_0 \cdot s_0) \to (d_0, s_0) \times (d_1, s_0)$ in $\mathrm{Pt}_{X_0}\mathbb{E}$ an isomorphism. This remark has interesting consequences:

5.1. Proposition. *Given a strongly normal functor* $F : \mathbb{E} \to \mathbb{E}'$ *between left exact categories and a reflexive graph* $\underline{X}_1$ *in* $\mathbb{E}$*. Then, when* $F(\underline{X}_1)$ *is endowed with a groupoid structure, there is a unique, up to isomorphism, groupoid structure on* $\underline{X}_1$ *above the one on* $F(\underline{X}_1)$*.*

PROOF. The extension $\mathrm{Pt}F : \mathrm{Pt}\mathbb{E} \to \mathrm{Pt}\mathbb{E}'$ is fibrewise strongly normal and consequently $F_{X_0} : \mathrm{Pt}_{X_0}\mathbb{E} \to \mathrm{Pt}_{FX_0}\mathbb{E}'$ is strongly normal. Now the object $(d_1 \cdot p_0, s_0 \cdot s_0)$ in $\mathrm{Pt}_{X_0}\mathbb{E}$ is mapped by F_{X_0} to the product $(Fd_0, Fs_0) \times (Fd_1, Fs_0)$ in $\mathrm{Pt}_{FX_0}\mathbb{E}'$, since $F(\underline{X}_1)$ has a groupoid structure, and the map $p_0 : (d_1 \cdot p_0, s_0 \cdot s_0) \to (d_1, s_0)$ is sent to the projection $p : (Fd_0, Fs_0)\times(Fd_1, Fs_0) \to (Fd_1, Fs_0)$. Then according to proposition 1.4, $(d_1{\cdot}p_0, s_0{\cdot}s_0)$ is isomorphic, in $\mathrm{Pt}_{X_0}\mathbb{E}$, to the product $\mathrm{Ker}p_0\times(d_1, s_0) = (d_0, s_0)\times(d_1, s_0)$, which produces a map $d_2 : (d_1 \cdot p_0, s_0 \cdot s_0) \to (d_0, s_0)$ in $\mathrm{Pt}_{X_0}\mathbb{E}$. This map d_2 in $\mathbb{E}$ satisfies the axiom of groupoids since $F(d_2)$ satisfies them in $\mathbb{E}'$ and F is faithful. ∎

On the other hand, the kernel of $d_2 : (d_1 \cdot p_0, s_0 \cdot s_0) \to (d_0, s_0)$ in the pointed category $\mathrm{Pt}_{X_0}\mathbb{E}$, is the map $s_0 : (d_1, s_0) \to (d_1 \cdot p_0, s_0 \cdot s_0)$, which is necessarily normal in $\mathrm{Pt}_{X_0}\mathbb{E}$ to the kernel equivalence $R[d_2]$ of d_2. Now, when $\mathbb{E}$ is protomodular, there is at most one groupoid structure on a given reflexive graph. In this situation the previous property is characteristic:

5.2. PROPOSITION. *Given a protomodular category $\mathbb{C}$, a reflexive graph $\underline{X}_1$ is a groupoid if and only if the map $s_0 : (d_1, s_0) \to (d_1 \cdot p_0, s_0 \cdot s_0)$ is normal in* $\mathrm{Pt}_{X_0}\mathbb{C}$.

PROOF. Let us suppose this map normal. So we have an epimorphism $p_0 : (d_1{\cdot}p_0, s_0{\cdot}s_0) \to (d_1, s_0)$, in the protomodular fibre $\mathrm{Pt}_{X_0}\mathbb{C}$, split by the normal monomorphism s_0. Then, according to [2], proposition 12, the object $(d_1 \cdot p_0, s_0 \cdot s_0)$ is isomorphic in $\mathrm{Pt}_{X_0}\mathbb{C}$ to the product $\mathrm{Ker}p_0 \times (d_1, s_0) = (d_0, s_0) \times (d_1, s_0)$. Whence a map $d_2 : (d_1 \cdot p_0, s_0 \cdot s_0) \to (d_0, s_0)$ in this category. The map d_2, in $\mathbb{C}$, is enough to make $\underline{X}_1$ a groupoid since, thanks to [5], theorem 2.2, and the fact that a protomodular category is always Mal'cev, this map satisfies necessarily the axioms of a groupoid. ∎

Two consequences follow, concerning the strongly protomodular categories:

A) If $\mathbb{C}$ is a pointed strongly protomodular category, a reflexive graph $\underline{X}_1$ is a groupoid if and only if the map $\omega^*(s_0) : K[d_1] \to K[d_1 \cdot p_0]$ is normal in $\mathbb{C}$, where ω denotes the initial map $1 \to X_0$.

REMARK. In the category **Gp** of groups, $K[d_1 \cdot p_0]$ is the group whose objects are the pairs (α, β) of arrows of the graph $\underline{X}_1$ with same domain and satisfying $d_1(\alpha) = 1$. To say that $\omega^*(s_0)$ is normal means that, for each arrow γ with codomain 1, we have: $\alpha\cdot\gamma\cdot\alpha^{-1} = \beta\cdot\gamma\cdot\beta^{-1}$. This is clearly equivalent to $(\beta^{-1}\cdot\alpha)\cdot\gamma = \gamma\cdot(\beta^{-1}\cdot\alpha)$ with $\beta^{-1}\cdot\alpha$ in $K[d_0]$. Thus to say that $\omega^*(s_0)$ is normal is equivalent to saying that $[K[d_0], K[d_1]] = 1$, which is precisely the condition given in [8] to define categorical groups.

B) Now suppose we are given any strongly protomodular category $\mathbb{C}$ and a morphism $\underline{f}_1 : \underline{X}_1 \to \underline{Y}_1$ of reflexive graphs such that the square with the d_0 and the square with the d_1 are pullbacks. Then if $\underline{X}_1$ is a groupoid, $\underline{Y}_1$ is itself a groupoid.

3) The slice categories $\mathbb{C}/X$

We are now going to prove that, when $\mathbb{C}$ is strongly protomodular, the slice categories $\mathbb{C}/X$ and the fibres $\mathrm{Pt}_X\mathbb{C}$ are still strongly protomodular. The forgetful functor $\mathbb{C}/X \to \mathbb{C}$ preserves the pullbacks, but no longer the products. However it satisfies the following definition:

5.3. Definition. *A functor $F : \mathbb{E} \to \mathbb{E}'$ is paraexact when it preserves the pullbacks and is such that the factorization $F(X \times X') \to F(X) \times F(X')$ is a monomorphism and the following square a pullback for any pair of maps (f, f') in $\mathbb{E} \times \mathbb{E}$:*

$$\begin{tikzcd} F(X \times X') \arrow[r, "F(f\times f')"] \arrow[d] & F(Y \times Y') \arrow[d] \\ FX \times FX' \arrow[r, "Ff\times Ff'"'] & FY \times FY' \end{tikzcd}$$

Remark. Then F extends to a functor $F : \mathrm{Rel}\mathbb{E} \to \mathrm{Rel}\mathbb{E}'$ which preserves the cartesian and fibrant maps. It preserves also the intersection of two equivalence relations defined on the same object X. But it does not preserve the coarse relation any more, and consequently does not preserves the fact that a map j would be normal to some equivalence relation R. However, when F is furthermore conservative, as this is the case for the forgetful functor $\mathbb{C}/X \to \mathbb{C}$, each map in $\mathrm{Rel}\mathbb{E}$ is cocartesian with respect to the factorization functor $\Phi : \mathrm{Rel}\mathbb{E} \to \mathbb{E} \times_{\mathbb{E}'} \mathrm{Rel}\mathbb{E}'$.

5.4. Definition. *A paraexact functor F is said to be paranormal when it is conservative and such that the factorization functor $\Phi : \mathrm{Rel}\mathbb{E} \to \mathbb{E} \times_{\mathbb{E}'} \mathrm{Rel}\mathbb{E}'$ is cofibrant on the fibrant morphisms of $\mathrm{Rel}\mathbb{E}'$.*

5.5. Proposition. *Let $F : \mathbb{C} \to \mathbb{C}'$ be a paranormal functor. Then:*

1) the functors $F_Z : \mathbb{C}/Z \to \mathbb{C}'/FZ$ are normal, and consequently the extension $\mathrm{Pt}F : \mathrm{Pt}\mathbb{C} \to \mathrm{Pt}\mathbb{C}'$ is fibrewise strongly normal

2) if $\mathbb{C}'$ is strongly protomodular, the category $\mathbb{C}$ is itself strongly protomodular.

Proof. The proof mimics exactly the proof of Proposition 4.1. ■

5.6. Corollary. *When $\mathbb{C}$ is strongly protomodular, this is also the case for the slice categories $\mathbb{C}/Z$ and the fibres $\mathrm{Pt}_Z\mathbb{C}$.*

Proof. When $\mathbb{C}$ is protomodular, the forgetful functor $\mathbb{C}/Z \to \mathbb{C}$ is not only paraexact, but also paranormal. Thus, when $\mathbb{C}$ is strongly protomodular, so is $\mathbb{C}/Z$. On the other hand, the forgetful functor $\mathrm{Pt}_Z\mathbb{C} \to \mathbb{C}/Z$ is strongly normal as soon as $\mathbb{C}$ is left exact. ■

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Université du Littoral,
220 av. de l'Université, BP 5526,
59379 Dunkerque Cedex, France
Email: bourn@lmpa.univ-littoral.fr

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