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ON THE OBJECT-WISE TENSOR PRODUCT OF FUNCTORS TO MODULES

MAREK GOLASIŃSKI

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ABSTRACT. We investigate preserving of projectivity and injectivity by the object-wise tensor product of $R\mathbb{C}$-modules, where $\mathbb{C}$ is a small category. In particular, let $\mathcal{O}(G,X)$ be the category of canonical orbits of a discrete group G, over a G-set X. We show that projectivity of $R\mathcal{O}(G,X)$-modules is preserved by this tensor product. Moreover, if G is a finite group, X a finite G-set and R is an integral domain then such a tensor product of two injective $R\mathcal{O}(G,X)$-modules is again injective.

1. Introduction

It is well-known that the tensor product of two projective R-modules is projective. One can easily check that the tensor product of two injective R-modules is injective for R being an integral domain or more generally, the product of a finite number of integral domains. But (even for a commutative ring R) this is not the case, in general. By [3], for a commutative noetherian ring R, the tensor product of any two injective R-modules is injective if and only if the local ring $R_{\mathfrak{p}}$ is quasi-Frobenius for any prime ideal $\mathfrak{p}$ in R.

Let G be a discrete group and $\mathcal{O}(G)$ the associated category of canonical orbits. The injectivity of the object-wise tensor product of injective functors from some categories associated with $\mathcal{O}(G)$ to vector spaces has been first extensively used but not proved in [4, 11] to study the equivariant rational homotopy theory and then applied in [5, 10] for further generalized investigations. The balance of the paper is devoted to preserving of the projectivity and injectivity by such a tensor product of functors from any small category $\mathbb{C}$ to R-modules, called $R\mathbb{C}$-modules.

Section 2 deals with contravariant functors from a small category $\mathbb{C}$ to R-modules. We start from the illuminating counterexample showing that the object-wise tensor product of two projective $R\mathbb{C}$-modules is not projective even for the category $\mathbb{C}$ associated with a finite partially ordered set. We observe that the tensor product of two such projective functors is projective if and only the functor $R(\mathbb{C}(C,-)\times\mathbb{C}(D,-))$ is projective for any objects C, D in the category $\mathbb{C}$. By [2], it is sufficient that the components of the comma category $Y\downarrow\mathbb{C}(C,-)\times\mathbb{C}(D,-)$ have right zeros, where $Y:\mathbb{C}\to\mathbb{S}et^{\mathbb{C}^{op}}$ is the Yoneda imbedding and $\mathbb{S}et$ the category of sets. Then we deduce that the object-wise tensor

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product of two projective $R\mathcal{O}(G,X)$-modules is projective, where $\mathcal{O}(G,X)$ denotes the category of canonical orbits of a discrete group G, over a G-set X.

In section 3 we move to covariant functors from $\mathbb{C}$ to R-modules and examine the dual problem, preserving of the injectivity by the object-wise tensor product of $R\mathbb{C}$-modules. We show that such a tensor product of two injective $R\mathcal{O}(G,X)$-modules is injective provided that the group G is finite, X a finite G-set and the tensor product of any two injective R-modules is injective. Then we deduce that the exterior, symmetric and tensor power constructions preserve injectivity of $R\mathcal{O}(G,X)$-modules, for R being a field of zero characteristic. At the end, for R being a field, we conclude that the projectivity (resp. injectivity) of functors from $\mathcal{O}(G,X)$ to linearly-compact vector R-spaces are preserved. That is the crucial fact in [5] to extend the equivariant homotopy theory on the disconnected case.

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2. Projective functors

Let ${}_R\mathrm{Mod}$ be the category of left R-modules over a commutative ring R with identity and $\mathbb{C}$ a small category with the set of objects $|\mathbb{C}|$. A covariant functor $\mathbb{C} \to {}_R\mathrm{Mod}$ is called a *left* $R\mathbb{C}$*-module* and the functor category ${}_{R\mathbb{C}}\mathrm{Mod}$ of all left such modules is called the *category of left* $\mathbb{C}$*-modules.*

The object of this section is the category $\mathrm{Mod}_{R\mathbb{C}}$ of contravariant functors $\mathbb{C} \to {}_R\mathrm{Mod}$, alias *right* $R\mathbb{C}$*-modules*, called in the sequel simply $R\mathbb{C}$-modules. Notions like coproducts, products, injective, projective etc. are defined as usual. In particular, an $R\mathbb{C}$-module P is projective if the following problem

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] & \\ M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

has a solution. For a set X, let $R(X)$ be the free R-module generated by X. Thus, a contravariant functor $F : \mathbb{C} \to \mathbb{S}et$ to the category $\mathbb{S}et$ of sets gives a rise to the right $R\mathbb{C}$-functor RF such that $(RF)(C) = R(F(C))$, for any object $C \in |\mathbb{C}|$. In particular, for any object $C \in |\mathbb{C}|$, we get the *representable* $R\mathbb{C}$-modules $R\mathbb{C}(-,C))$. Here $\mathbb{C}(D,C)$ is the set of morphisms from D to C in the category $\mathbb{C}$. An $R\mathbb{C}$-module is called to be *free* if it is isomorphic to a coproduct of representable functors. Then the notion of *finitely generated* has its usual meaning. For $C \in |\mathbb{C}|$, let T_C be the evaluation functor, i.e., $T_C(M) = M(C)$ for any $R\mathbb{C}$-module M. Then the left adjoint of T_C is the $R\mathbb{C}$-module given by

$$S_C(M)(D) = \bigoplus_{D \to C} M = R(\mathbb{C}(D,C)) \otimes_R M$$

for any R-module M. Observe that $S_C(M)$ is a projective $R\mathbb{C}$-module provided that M is a projective R-module. For an $R\mathbb{C}$-module M and $x \in M(C)$ denote C by $|x|$. Them there is an epimorphism

$$\bigoplus_{x \in M} R(\mathbb{C}(-, |x|)) \longrightarrow M \to 0.$$

In particular, it follows that a projective $R\mathbb{C}$-module is a direct summand of a free $R\mathbb{C}$-module.

If $\mathbb{C}$ is an EI-category (i.e., any endomorphism in $\mathbb{C}$ is an isomorphism) then by [8, Theorem 9.39] any projective $R\mathbb{C}$-module can be split into a direct sum of projective $R\mathbb{C}$-modules living over various group rings $R[\mathrm{Aut}(C)]$ for $C \in |\mathbb{C}|$, where $\mathrm{Aut}(C)$ is the automorphism group of the object C. In particular, it follows the result stated in [9]: if $\mathbb{C}$ *is a one way category and finite from below then any projective* $\mathbb{C}$*-module is of the form* $\bigoplus_{C \in |\mathbb{C}|} S_C(P_C)$ *for some projective* R*-modules* P_C.

If M_1 and M_2 are $R\mathbb{C}$-modules then their *object-wise tensor product* $M_1 \otimes_R M_2$ is defined to be the composition

$$\mathbb{C} \xrightarrow{\Delta} \mathbb{C} \times \mathbb{C} \xrightarrow{M_1 \times M_2} {}_R\mathrm{Mod} \times {}_R\mathrm{Mod} \xrightarrow{\otimes_R} {}_R\mathrm{Mod},$$

where $\Delta : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ is the diagonal functor. As it was observed by Bousfield [1], the object-wise tensor product $M_1 \otimes_R M_2$ is not projective in general, for any projective $R\mathbb{C}$-modules M_1 and M_2.

2.1. Example. Let $\mathbb{C}$ be the category associated with the partially order set $(\{a, b, c, d, e\}, <)$ with the relations: $a < b$, $a < c$, $b < d$, $c < d$, $b < e$ and $c < e$. This set can be also given by the following oriented graph

$$\begin{tikzcd}
 & e & \\
 & d & \\
b \arrow[uur] \arrow[ur] & & c \arrow[ul] \arrow[uul] \\
 & a. \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Consider the projective $R\mathbb{C}$-modules $S_d(R)$ and $S_e(R)$ for a nonzero ring R which can be also presented by the commutative diagrams

$$
\begin{tikzcd}
& 0 \arrow[ddl] \arrow[ddr] & \\
& R \arrow[dl, equal] \arrow[dr, equal] & \\
R \arrow[dr, equal] & & R \arrow[dl, equal] \\
& R &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& R \arrow[ddl, equal] \arrow[ddr, equal] & \\
& 0 \arrow[dl] \arrow[dr] & \\
R \arrow[dr, equal] & & R \arrow[dl, equal] \\
& R. &
\end{tikzcd}
$$

Then their object-wise tensor product $S_d(R) \otimes_R S_e(R)$ is given by the commutative diagram

$$
\begin{tikzcd}
& 0 \arrow[ddl] \arrow[ddr] & \\
& 0 \arrow[dl] \arrow[dr] & \\
R \arrow[dr, equal] & & R \arrow[dl, equal] \\
& R. &
\end{tikzcd}
$$

Take two other projective $R\mathbb{C}$-modules $S_b(R)$ and $S_c(R)$ given by the commutative diagrams

$$
\begin{tikzcd}
& 0 \arrow[ddl] \arrow[ddr, equal] & \\
& 0 \arrow[dl] \arrow[dr, equal] & \\
R \arrow[dr, equal] & & 0 \arrow[dl] \\
& R &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
& 0 \arrow[ddl, equal] \arrow[ddr] & \\
& 0 \arrow[dl, equal] \arrow[dr] & \\
0 \arrow[dr] & & R \arrow[dl, equal] \\
& R. &
\end{tikzcd}
$$

Let $\beta : S_b(R) \to S_d(R) \otimes_R S_e(R)$ and $\gamma : S_c(R) \to S_d(R) \otimes_R S_e(R)$ be the maps of$R\mathbb{C}$-modules determined by the identity on 0 and R, respectively. Then the map $p = (\beta, \gamma) : S_b(R) \oplus S_c(R) \to S_d(R) \otimes_R S_e(R)$ is surjective for which does not exist any splitting $q : S_d(R) \otimes_R S_e(R) \to S_b(R) \oplus S_c(R)$. To show this observe that any map $\varphi : S_d(R) \otimes_R S_e(R) \to S_b(R)$ is determined by its values $\varphi(a)$, $\varphi(b)$ and $\varphi(c)$. From the commutativity of the diagram

$$\begin{tikzcd}
& & 0 \arrow[d, equal] \arrow[dddll] \arrow[dddrr] & & \\
& & 0 \arrow[ddl] \arrow[ddr] & & \\
& & 0 \arrow[dl] \arrow[dr, equal] & & \\
R \arrow[r, "\varphi(b)"] \arrow[ddrr, equal] & R \arrow[dr, equal] & & 0 \arrow[dl] & R \arrow[l, "\varphi(c)"'] \arrow[ddll, equal] \\
& & R & & \\
& & R \arrow[u, "\varphi(a)"'] & &
\end{tikzcd}$$

it follows that $\varphi(c) = \varphi(a) = 0$. Thus $\varphi(b) = 0$ and finally we get that $\varphi = 0$. In the same way one can show that any map $\psi : S_d(R) \otimes_R S_e(R) \to S_c(R)$ is also trivial and the result follows.

Since a projective $R\mathbb{C}$-module is a direct summand of a free $R\mathbb{C}$-module we can state

2.2. Lemma. *The object-wise tensor product of any two projective $R\mathbb{C}$-modules is projective if and only if the $R\mathbb{C}$-module $S_C(R) \otimes_R S_D(R) = R(\mathbb{C}(-, C) \times \mathbb{C}(-, D))$ is projective, for all $C, D \in |\mathbb{C}|$.*

By [8, p. 186] a functor $F : \mathbb{A} \to \mathbb{B}$ between EI-categories is called to be *admissible* if the induced restriction functor $Res_F : R\mathbb{B}$-modules $\to R\mathbb{A}$-modules sends finitely generated free resp. projective $R\mathbb{B}$-modules to finitely generated free resp. projective $R\mathbb{A}$-modules for any commutative ring R with unit. Therefore, Lemma 2.2 yields

2.3. Remark. The object-wise tensor product of any two projective $R\mathbb{C}$-modules is projective if and only if the diagonal functor $\Delta : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ is admissible.

Let $Y : \mathbb{C} \to \mathbb{S}et^{\mathbb{C}^{op}}$ be the Yoneda imbedding. For a functor $F : \mathbb{C}^{op} \to \mathbb{S}et$, let $Y \downarrow F$ denote the associated comma category. In the light of [2] the sufficient condition for the $R\mathbb{C}$-module $S_C(R) \otimes_R S_D(R) = R(\mathbb{C}(-, C) \times \mathbb{C}(-, D))$ to be projective is that the components of the comma category $Y \downarrow \mathbb{C}(-, C) \times \mathbb{C}(-, D)$ have right zeros.

Let now G be a discrete group and $\mathcal{O}(G)$ the category of its canonical orbits. For any subgroups $H_1, H_2 \subseteq G$, the G-set $G/H_1 \times G/H_2$ is in one-one correspondence with a disjoint union $\bigcup_{\alpha \in A} G/L_\alpha$, for some subgroups $L_\alpha \subseteq G$. Thus, the set $\mathcal{O}(G)(G/K, G/H_1) \times \mathcal{O}(G)(G/K, G/H_2)$ is in one-one correspondence with the disjoint union $\bigcup_{\alpha \in A} \mathcal{O}(G)(G/K, G/L_\alpha)$ and there is an isomorphism $R(\mathcal{O}(G)(-, G/H_1)) \otimes_R (\mathcal{O}(G)R(-, G/H_2)) \approx \bigoplus_{\alpha \in A} R(\mathcal{O}(G)(-, G/L_\alpha))$ of $R\mathcal{O}(G)$-modules.

More generally, for a G-set X, consider the category $\mathcal{O}(G, X)$ of canonical orbits over X. Objects in $\mathcal{O}(G, X)$ are G-maps $x : G/H \to X$ for an arbitrary subgroup $H \subseteq G$. A morphism from $x : G/H \to X$ to $y : G/K \to X$ is a G-map $\sigma : G/H \to G/K$ such that $y\sigma = x$. Thus objects in $\mathcal{O}(G,X)$ correspond to points in fixed point subsets of the G-set X.

2.4. Proposition. *If G is a discrete group and X is a G-set then the set $\mathcal{O}(G,X)((G/K,y),(G/H_1,x_1)) \times \mathcal{O}(G)((G/K,y),(G/H_2,x_2))$ is in one-one correspondence with the disjoint union $\bigcup_{\alpha\in A} \mathcal{O}(G,X)((G/K,y),(G/L_\alpha, gx_1 = g'x_2))$, where the union runs over all distinct isotropy subgroups L_α of the points $(gH_1, g'H_2) \in G/H_1 \times G/H_2$ with $gx_1 = g'x_2$.*

Proof. Let $G/H_1 \times G/H_2 = \bigcup_{\alpha\in A} G(a_\alpha H_1, b_\alpha H_2)$ be the disjoint union and L_α the isotropy subgroup of the point $(a_\alpha H_1, b_\alpha H_2) \in G/H_1 \times G/H_2$, for $\alpha \in A$. Consider two maps $\varphi_1 : (G/K,y) \to (G/H_1,x_1)$, $\varphi_2 : (G/K,y) \to (G/H_2,x_2)$ in the category $\mathcal{O}(G,X)$ with $\varphi(K) = g_1H_1$, $\varphi_2(K) = g_2H_2$. Then $g_1H_1 = ga_\alpha H_1$ and $g_2H_2 = gb_\alpha H_2$, for some $\alpha \in A$. Hence $y = g_1x_1 = ga_\alpha x_1$, $y = g_2x_2 = gb_\alpha x_2$ and thus $a_\alpha x_1 = b_\alpha x_2 = g^{-1}y$. If L_α is the isotropy group of the point $(a_\alpha H_1, b_\alpha H_2)$ in the G-set $G/H_1 \times G/H_2$ then $a_\alpha^{-1}L_\alpha a_\alpha \subseteq H_1$, $b_\alpha^{-1}L_\alpha b_\alpha \subseteq H_2$ and $g^{-1}Kg \subseteq L_\alpha$. Consequently the maps φ_1 and φ_2 determine a map $\psi : (G/K,y) \to (G/L_\alpha, a_\alpha x_1 = b_\alpha x_2)$ with $\psi(K) = gL_\alpha$.

Conversely, let L_α be the isotropy group of the point $(a_\alpha H_1, b_\alpha H_2) \in G/H_1 \times G/H_2$ and $\psi : (G/K,y) \to (G/L_\alpha, a_\alpha x_1 = b_\alpha x_2)$ a map in the category $\mathcal{O}(G,X)$. Then we can consider the maps $\eta_1 : (G/L_\alpha, a_\alpha x_1 = b_\alpha x_2) \to (G/H_1,x_1)$ and $\eta_2 : (G/L_\alpha, a_\alpha x_1 = b_\alpha x_2) \to (G/H_2,x_2)$ with $\eta_1(L_\alpha) = a_\alpha H_1$ and $\eta_2(L_\alpha) = b_\alpha H_2$. Hence, we get the maps $\varphi_1 = \eta_1\psi : (G/K,y) \to (G/H_1,x_1)$ and $\varphi_2 = \eta_2\psi : (G/K,y) \to (G/H_2,x_2)$ and the result follows.

■

Consequently, in the light of Lemma 2.2, we may state

2.5. Theorem. *If G is a discrete group, X a G-set and M_1, M_2 are projective $R\mathcal{O}(G,X)$-modules then their object-wise tensor product $M_1 \otimes_R M_2$ is projective, for any commutative ring R.*

In particular, $\mathcal{O}(G,*) = \mathcal{O}(G)$, the orbit category of the discrete group G for a single point set $*$. Thus, the object-wise tensor product $M_1 \otimes_R M_2$ of projective $R\mathcal{O}(G)$-modules M_1 and M_2 is projective.

3. Injective functors

The object of this section is the category ${}_{R\mathbb{C}}\mathrm{Mod}$ alias left $R\mathbb{C}$-modules. For $C \in |\mathbb{C}|$, the right adjoint of the evaluation functor T'_C is the $R\mathbb{C}$-module given by

$$S'_C(M)(D) = \prod_{D\to C} M = \mathrm{Hom}_R(R(\mathbb{C}(D,C)), M),$$

for any R-module M and $D \in |\mathbb{C}|$. Observe that $S'_i(M)$ is an injective $R\mathbb{C}$-module provided that M is an injective R-module. Given an $R\mathbb{C}$-module M, fix an R-monomorphism

$0 \to M(C) \to Q_C$ for any object $C \in |\mathbb{C}|$, where Q_C is an injective R-module. Then there is a monomorphism of $R\mathbb{C}$-modules

$$0 \to M \longrightarrow \prod_{x \in M} \mathrm{Hom}_R(R(\mathbb{C}(-, |x|)), Q_C).$$

In particular, it follows that an injective $R\mathbb{C}$-module is a direct summand of an injective $R\mathbb{C}$-module $\prod_{x \in M} \mathrm{Hom}_R(R(\mathbb{C}(C, -)), Q_C)$, where Q_C are injective R-modules. A full characterization of $R\mathbb{C}$-modules for an EI-category $\mathbb{C}$, has been presented in [6].

If $\mathbb{C}$ is a finite category (i.e., with a finite set of morphisms) then $\mathrm{Hom}_R(R(\mathbb{C}(C, -)), M) \otimes_R \mathrm{Hom}_R(R(\mathbb{C}(D, -)), N) \approx \mathrm{Hom}_R(R(\mathbb{C}(C, -) \times \mathbb{C}(D, -)), M \otimes_R N)$ for any R-modules M and N. One can easily check that the tensor product of two injective R-modules is injective for R being an integral domain or more generally, the product of a finite number of integral domains. But (even for a commutative ring R) this is not the case, in general. This problem, for noetherian commutative rings R, has been intensively studied in [3]: *The tensor product of two injective R-modules is injective if and only if the localization $R_{\mathfrak{p}}$ is a quasi-Frobenius ring, for any prime ideal $\mathfrak{p}$ in the ring R.* Consequently, in the light of the above facts and Section 2 we can state

3.1. Proposition. *Let R be a commutative ring with unit and such that the tensor product of any two injective R-modules in injective. If G is a finite group, X a finite G-set and M_1, M_2 are injective $R\mathcal{O}(G, X)$-modules then the object-wise tensor product $M_1 \otimes_R M_2$ is also injective. In particular, if R is an integral domain, then the tensor product of injective $R\mathcal{O}(G, X)$-modules is injective.*

Let G be a finite group, X a finite G-set, k a field and $M = \{M_i\}_{i \geq 0}$ a graded left $k\mathcal{O}(G, X)$-module. Then for any $(G/H, x) \in |\mathcal{O}(G, X)|$ we get a graded left k-module $M(G/H, x) = \{M_i(G/H, x)\}_{i \geq 0}$. Write $|m| = i$, for $m \in M_i(G/H, x)$. We define graded left $k\mathcal{O}(G, X)$-modules $T^n M$ and $S^n M$ (called the n'th *tensor* and *symmetric power*, respectively) as follows:

$$(T^n M)_i(G/H, x) = \bigoplus_{i_1 + \cdots i_n = i} M_{i_1}(G/H, x) \otimes_k \cdots \otimes_k M_{i_n}(G/H, x)$$

and

$$(S^n M)_i(G/H, x) = (T^n M)_i(G/H, x)/(R^n M)_i(G/H, x)$$

for $i, n \geq 0$, where $(R^n M)_i(G/H, x)$ is the homogeneous k-submodule of $(T^n M)_i(G/H, x)$ generated by elements $m_1 \otimes \cdots \otimes m_n - (-1)^{|m_l||m_{l+1}|} m_1 \otimes \cdots \otimes m_{l+1} \otimes m_l \otimes \cdots \otimes m_n$ for $m_l \in M_{|m_l|}(G/H, x)$ and $l = 1, \ldots, n$.

Let $\pi_{(G/H,x)} : T^n M(G/H, x) \to T^n M(G/H, x)/R^n M(G/H, x)$ be the natural canonical map. If the characteristic of k is zero then there is a natural map

$$\sigma_{(G/H,x)} : S^n M(G/H, x) \longrightarrow T^n M(G/H, x)$$

such that

$$\sigma_{(G/H,x)}([m_1 \otimes \cdots \otimes m_n]) = \frac{1}{n!} \sum_{\tau \in S_n} \epsilon(\tau) m_{\tau(1)} \otimes \cdots \otimes m_{\tau(n)}$$

for $m_l \in M_{|m_l|}(G/H, x)$ with $(G/H, x) \in |\mathcal{O}(G, X)|$ and $l = 1, \ldots, n$, where S_n is the n'th symmetric group and $\epsilon : S_n \to \{-1, +1\}$ the sign map. Then $\pi_{(G/H,x)} \circ \sigma_{(G/H,x)} = \mathrm{id}_{S^n M(G/H,x)}$ for $(G/H, x) \in |\mathcal{O}(G, X)|$ and consequently $S^n M$ is a direct summand of $T^n M$. Moreover, we define TM and SM, the graded *tensor* and *symmetric left* $k\mathcal{O}(G, X)$-*algebra*, where $(TM)_i = \bigoplus_{n\geq 0}(T^n M)_i$ and $(SM)_i = \bigoplus_{n\geq 0}(S^n M)_i$ for $i \geq 0$. Observe that $SM = TM/RM$, where RM is the homogeneous ideal of TM generated by elements $x \otimes y - (-1)^{|x||y|} y \otimes x$ for $x, y \in TM$. Then from Proposition 3.1 we can deduce

3.2. Corollary. *Let G be a finite group and X a finite G-set. If $M = \{M_i\}_{i\geq 0}$ is a graded component-wise injective left $k\mathcal{O}(G, X)$-module, then the graded left $k\mathcal{O}(G, X)$-modules $T^n M$, $S^n M$ and $TM = \{(T^n M)\}_{n\geq 0}$, $SM = \{(S^n M)\}_{n\geq 0}$ are component-wise injective $k\mathcal{O}(G, X)$-modules, for $n \geq 0$.*

For $n \geq 0$, let $\Lambda^n M$ denote the n'th *exterior power* and ΛM the *exterior algebra*, respectively of a graded left $k\mathcal{O}(G, X)$-module $M = \{M_i\}_{i\geq 0}$. We can proceed in the same way as above to get

3.3. Remark. If $M = \{M_i\}_{i\geq 0}$ is a graded component-wise injective left $k\mathcal{O}(G, X)$-module, then the graded $k\mathcal{O}(G, X)$-modules $\Lambda^n M$ and $\Lambda M = \{(\Lambda^n M)\}_{n\geq 0}$ are component-wise injective left $k\mathcal{O}(G, X)$-modules, for $n \geq 0$.

At the end, we move to the dual category $({}_k\mathrm{Mod})^{op}$ which, in view of [7], is isomorphic to the category ${}_k\mathrm{Mod}^c$ of linearly compact k-modules. Recall some properties of these modules (see [5, 7] for the details).

(1) A linearly topological k-module M is linearly compact if and only if its topological dual M^* is discrete.

(2) For a linearly compact (resp. discrete) k-modules M and N their complete tensor product $M\widehat{\otimes}_k N$ is linearly compact (resp. discrete) and there is a topological isomorphism $M^*\widehat{\otimes}_k N^* \xrightarrow{\approx} (M\widehat{\otimes}_k N)^*$.

(3) If $\{M_i\}_{i\in I}$ and $\{N_j\}_{j\in J}$ are collections of linearly compact k-modules then there exists a topological isomorphism

$$\prod_{i\in I} M_i \widehat{\otimes}_k \prod_{j\in J} N_j \xrightarrow{\approx} \prod_{i\in I,\, j\in J} M_i \widehat{\otimes}_k N_j.$$

Then we may dualize Theorem 2.5 and Proposition 3.1 to summarize our considerations by the following conclusion being the crucial fact in [5] to extend the equivariant homotopy theory on the disconnected case.

3.4. Corollary. (1) *Let G be a discrete group and X a G-set. If $M_1, M_2 : \mathcal{O}(G, X) \to {}_k\mathrm{Mod}^c$ are injective $k\mathcal{O}(G, X)$-modules then the object-wise tensor product $M_1\widehat{\otimes}_k M_2$ is also an injective $k\mathcal{O}(G, X)$-module.*

(2) *Let G be a finite group and X a finite G-set. If $M_1, M_2 : \mathcal{O}(G, X) \to {}_k\mathrm{Mod}^c$ are projective $k\mathcal{O}(G, X)$-modules then the object-wise tensor product $M_1\widehat{\otimes}_k M_2$ is also a projective $k\mathcal{O}(G, X)$-module.*

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Faculty of Mathematics and Informatics
Nicholas Copernicus University
87-100 Toruń, Chopina 12/18
Poland

Email: `marek@mat.uni.torun.pl`

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