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# SOME EPIMORPHIC REGULAR CONTEXTS

*Dedicated to Joachim Lambek on the occasion of his 75th birthday*

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ABSTRACT. A von Neumann regular extension of a semiprime ring naturally defines a epimorphic extension in the category of rings. These are studied, and four natural examples are considered, two in commutative ring theory, and two in rings of continuous functions.

## 1. Introduction

In this note we study certain extensions of commutative semiprime rings which we call epimorphic regular contexts, or sometimes simply contexts. Work by Olivier shows that there exists a universal such context for each semiprime ring. It turns out that these extensions are "tangible" (cf. Theorem 2.2 and Remark 3.4) and this facilitates their study in concrete situations. We present four instances of epimorphic regular contexts. The first is the epimorphic hull due to Storrer, a ring that is defined by a universal property whose formulation is categorical. The second is the minimal regular rings as studied by Pierce and Burgess. The last two are topological in nature. As a detailed instance of the first example one considers the epimorphic hull of a ring of continuous functions as studied in [19]. Lastly one considers a general context (currently being studied in [7]) which can be defined naturally on any topological space.

The theme of this note is that categorical methods can stimulate fruitful inquiry in other mathematical disciplines—in our case ring theory and topology. In the studies we cite, new notions were defined and new examples were discovered. Sometimes the notions are fully or partially "internal" to their discipline. Yet it seems most unlikely that they would have been uncovered without the "external" stimulus from category theory.

There were several motivations for writing this note. There has been considerable recent work on epimorphisms and rings of quotients in topological settings ([7], [19], [20], [21], [22]) and non-algebraists might profit from an introduction to some of the ring-theoretic and categorical aspects. As well, some of the "folklore" material is in danger of being lost if it is not recorded. There is also the hope that others will be encouraged to solve the open problems and to communicate additional instances of epimorphic regular

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contexts among the rings that arise naturally in mathematics. Most of the open questions have been considered either with Storrer or in the collaborations with Henriksen, Macoosh, and Woods. I am indebted to all of them for many stimulating conversations in algebra and in topology.

## 2. Epimorphisms

Recall that a map $f\colon R \to S$ is called an epimorphism if $g$ and $h$ must coincide if they are maps from $S \to T$ whose compositions with $f$ agree. A good introduction to general epimorphisms is found in [1]. In concrete categories, such as categories of rings, an onto map is an epimorphism, but so is the inclusion of $Z$ into $Q$. There is a substantial literature on ring epimorphisms with many results of interest. To mention just two— an epimorphic extension of a commutative ring must be commutative [24, 1.3], and an epimorphic extension of a finite ring need not be finite [10, p. 268].

We are interested in the case of epimorphisms between rings which are commutative with identity, and semiprime, meaning that 0 is the only nilpotent element. Regular rings are pertinent to the study of epimorphisms as we shall presently see. A ring R is called regular in the sense of von Neumann if for all $a \in R$ there is a $b \in R$ such that $a = aba$. The element $b$ is called a quasi-inverse for $a$. As pointed out in [11, p. 36 Ex. 3] each element a has a unique quasi-inverse $a^*$ that also satisfies $a^* = a^*aa^*$.

2.1. Von Neumann Regular extensions and the universal regular ring of Olivier. Suppose that $R$ is a semiprime subring of a general commutative regular ring $S$. Let $G(S)$ be the intersection of the commutative regular subrings of $S$ that contain $R$. From the existence of the unique quasi-inverses in the commutative case, it is easy to see that $G(S)$ is itself a regular ring, and it is clearly the smallest regular subring of $S$ that contains $R$. What is much less evident, and crucial to our studies, is the following:

2.2. Theorem. [Olivier, [15]]

($i$) *The ring $G(S)$ is generated by $R$ and the unique quasi-inverses of the elements of $R$ in $S$; i.e. each element of $G(S)$ is a finite sum of products of the form $cd^*$ where $c, d \in R$, and * denotes the unique quasi-inverse.*

*(ii) The embedding of $R$ into $G(S)$ is an epimorphism of rings.*

One attractive way of seeing ($ii$) once one has ($i$) is to note that each $d^*$ satisfies a simple form of the zig-zag condition due to Isbell [23, 3.3]. Part ($i$) of the theorem follows from the following key result.

2.3. Theorem. [Olivier, [15]] *Given a commutative ring $R$, there is a universal object $T(R)$ among epimorphic regular extensions of $R$. The ring $T(R)$ maps (over $R$) canonically onto any epimorphic regular extension of $R$.*

2.4. Remark. The functor $T$ is the left adjoint to the inclusion of the category of von Neumann regular rings into the category of commutative rings. Olivier's construction of $T(R)$ is built by adjoining to $R$ one indeterminate for each element of $R$, and then dividing out the minimal relations needed to ensure regularity for the elements of $R$. A localization argument shows that the ring $T(R)$ is itself regular [15]. Each element of $T(R)$ is finitely expressible in terms of the elements of $R$ and their quasi-inverses. Since $T(R)$ maps onto $G(S)$ over $R$, one has part $(i)$ of Theorem 2.2. Notice that it also shows:

2.5. COROLLARY. *If $R$ is an infinite commutative semiprime ring and $S$ is an epimorphic regular extension of $R$, then $R$ and $S$ have the same cardinality.*

2.6. DEFINITION. *By an epimorphic regular context we will mean a map that is both a monomorphism and an epimorphism from a commutative semiprime ring into a regular ring. Note that there is no suggestion that the monomorphism is regular in the (categorical) sense of being a coequalizer.*

It is clear from the above discussion that any extension of a semiprime ring by a regular ring gives rise to an epimorphic regular context. We shall presently examine several natural cases of this phenomenon.

2.7. THE QUESTION OF REGULARITY DEGREE. Let us return to the fact that each element in $G(S)$ has a representation of the form indicated in Theorem 2.2. Adopting the terminology of [7] let us define the regularity degree of an element $x$ of $G(S)$ with respect to $R$ to be the least number of terms of the form $cd^*, c, d \in R$, needed to represent $x$. The regularity degree of $G(S)$ is the supremum of the regularity degrees of its elements. Having regularity degree 0 means that $R$ itself is regular. Having regularity degree infinite means that there is no supremum for the regularity degrees of the individual elements of $R$.

There are some natural questions which arise:

2.8. PROBLEM. Can one have an epimorphic regular extension $R \to S$ of infinite regularity degree, or must every such extension be of finite regularity degree?

2.9. PROBLEM. Suppose that every epimorphic regular extension is of finite degree. Are there such extensions of arbitrary large regularity degree, or is there a global bound for these degrees as one ranges over all epimorphic regular extensions of semiprime rings? In view of Theorem 2.3 it suffices to determine the regularity degrees for Olivier's extensions $R \to T(R)$.

Problem 2.9 is closely related to the following:

2.10. PROBLEM. Are direct products of epis emanating from commutative rings, still epi? Stated precisely if $R_i$, $S_i$ are families of rings so that each $S_i$ is an epimorphic extension of $R_i$, is $\prod S_i$ an epimorphic extension of $\prod R_i$? The answer is known to be negative in the non-commutative case [24, 2].

It is easy to see there are implications between Problems 2.8, 2.9, and 2.10. In view of Theorems 2.2(i) and 2.3 the existence of a global bound on regularity degrees implies that direct products of epimorphic regular extensions are epis. And if there are epimorphic regular extensions of unbounded degree then Problem 2.10 has a negative answer by an easy argument using direct products and Theorem 2.2(i) above. For the same reason a positive reply to Problem 2.8 means a negative answer to Problem 2.10. From recent joint work with Henriksen and Woods it appears that there are examples using rings of functions for which the degrees are finite, but unbounded. This predicts a negative answer to Problem 2.10. There remains the question of getting "algebraic" examples as answers to these problems.

2.11. Overview. In studying the context $R \to G(S)$ determined by a particular regular extension $R \to S$ one has considerable machinery at one's disposal. First of all Olivier's universal ring is close-by, as is the the epimorphic hull, another universal object, due to Storrer, which will be described presently. Secondly, regular rings have many attractive properties in their own right. Thirdly, epimorphisms impose added structure, for instance, a result due to Lazard [14, 1.6] says that the contraction map $Spec\, G(S) \to Spec(R)$ must be one-to-one. Lastly, one has a "handle" on the elements of $G(S)$ in that they have the finite representation mentioned in Theorem 2.2(i). Although this representation is not unique, its very existence is useful.

## 3. Examples of epimorphic regular contexts

There are two possible ways of finding a context. It may be that we are given both a semiprime ring $R$ and a regular overring $S$, from which we build $G(S)$. Or it maybe that $S$ itself is built by a natural construction that begins with $R$. We shall encounter examples of both sorts. Our first examples are algebraic.

3.1. The epimorphic hull of a commutative semiprime ring [Storrer]. To define this ring we need another categorical notion. A monomorphism $f\ A \to B$ is called essential if each map $g\colon B \to C$ must be a monomorphism whenever $g \circ f$ is a monomorphism. Usually one discusses essentiality for extensions of modules but the definition makes sense in any category (cf [23]).

An epimorphic hull $E(A)$ for an object $A$ is an essential monoepi $f\colon A \to E(A)$ with the property that if $h\colon A \to B$ is an essential monoepi, there is a map $g\colon B \to E(A)$ so that $g \circ h = f$. Storrer [23] showed that epimorphic hulls are unique up to isomorphism and studied the epimorphic hulls of commutative semiprime rings in detail. Let us recall that if $R$ is a semiprime ring and $S$ is an extension of $R$, then $S$ is called a ring of quotients of $R$ if given $s \neq 0 \in R$, there exists a $t \in S$ so that $st \in R, st \neq 0$. The simplest example of a ring of quotients is the passage from $Z$ to $Q$, or, indeed, from any domain to its field of fractions. Rings of quotients have been studied in many ways and from many points of view. A ring of quotients is an essential extension in the category of rings. Every semiprime ring $R$ has a maximal (or "complete") ring of quotients denoted $Q(R)$ which is unique up to isomorphism over $R$ [11, 2.3]. It is a regular ring [11, 2.4] so one has the following:

3.2. Theorem. [[23]] *Let $R$ be a commutative semiprime ring. Then $R$ has an epimorphic hull in the category of rings, denoted $E(R)$. It is a ring of quotients of $R$ and is the* *epimorphic regular context determined by R and its complete ring of quotients $Q(R)$. It is (up to isomorphism) the only epimorphic regular extension which is essential.*

Thus we have two important rings associated with a general commutative semiprime ring $R$, — $T(R)$, due to Olivier, and $E(R)$, due to Storrer. Each has a universal property. The two rings are related as follows:

3.3. Theorem. [Folklore] *Let $R$ be a commutative semiprime ring. Let $T(R)$ be its universal epimorphic regular extension, and $E(R)$ be its epimorphic hull. There is a homomorphism from $T(R)$ onto $E(R)$ that fixes $R$. (Indeed any epimorphic regular extension of $R$ maps onto $E(R)$ over $R$.)*

Proof. For one thing, such a map must exist by the universal property of $T(R)$. But in fact, the nicer way to see this is to use Zorn's lemma to choose in $T(R)$ an ideal $I$ maximal with respect to having (0) intersection with $R$. Although the ideal $I$ is far from unique, the ring $T(R)/I$ is a regular essential mono-epi, and hence a copy of $E(R)$. Note that there are certain topological situations when the choice of the ideal $I$ is "canonical". ∎

3.4. Remark. I am indebted to Hans Storrer for pointing out to me in private correspondence the following fascinating point concerning $E(S)$. Since $r^{**} = r$, and $(rs)^* = r^* s^*$ one can establish part $(i)$ of Theorem 2.2 for $E(S)$ if one knows an expression for a quasi-inverse of $\sum_{i=1}^n r_i s_i^*$ and this is given by

$$\left(\sum_{i=1}^{n} r_i s_i^*\right)^* = \sum_M \prod_{i\in M}(1 - s_i s_i^*) \prod_{j\notin M} s_j \left(\sum_{k\notin M}\left(\prod_{l\notin(M\cup\{k\})} s_l\right) r_k\right)^*$$

where $M$ runs through all subsets of $\{1, 2, \ldots, n\}$.

3.5. Minimal regular rings (Pierce, Burgess). In order to present this example, we must recall some notions from studies by Pierce [17], Burgess [1], and Burgess and Stewart [3].

Suppose that $S$ is a ring. The characteristic subring of $S$, denoted $\kappa(S)$, is the maximal epimorphic extension of the canonical image of $Z$ in $S$. By $B(S)$ one denotes the algebra of central idempotents of $S$. The minimal subring of $S$ (denoted $P(S)$ in [1]) is the subring of $S$ that is generated by $\kappa(S)$ and $B(S)$. When $P(S) = S$, $S$ is called a minimal ring. The minimal subring of a regular ring is itself regular. A regular ring is minimal, exactly when its field images are primitive fields. Minimal regular rings were studied by Pierce [17] who showed that they form a category that is contravariantly equivalent to the category of boolean spaces over the one-point compactification of the set of primes called the category of "labelled Boolean spaces" (LBS). The category of minimal regular rings has some interesting properties, for example, it is complete and co-complete.

To relate these notions to epimorphic regular contexts we suppose that the ring $S$ is regular, and that $R$ is the subring generated by $B(S)$ and the image of $Z$ in $S$. Then one has the following:

3.6. Proposition. *(i) $P(S) = G(S)$,*

*(ii) $S$ is the classical ring of quotients of $R$ and hence the regularity degree of $G(S)$ with respect to $R$ is one.*

Proof. (i) Since $G(S)$ is the smallest regular ring between $R$ and $S$, and $P(S)$ is regular, one knows that $G(S)$ is a subring of $P(S)$. On the other hand $G(S)$ and $P(S)$ have the same idempotents, so when these regular rings are represented as sheaves over Boolean spaces using the Pierce sheaf [16] they have a common base space $X = \mathrm{S}pecB(S)$. For each $x \in X$ the stalks $G(S)_x$ and $P(S)_x$ are fields, and $G(S)_x$ is a subfield of $P(S)_x$. Since the stalks of $P(S)$ are primitive fields, the stalks of the two rings coincide, as must the rings themselves.

(ii) Suppose that $r \in R$. By the regularity of $S$, the annihilator of $r$ in $S$ has the form $eS$ for some idempotent $e$ of $S$. Since $B(S) = B(R)$, the annihilator of $r$ in $R$ is $eR$ and it is easy to check that this makes $Q_{\mathrm{cl}}(R)$ regular. Furthermore each non zero-divisor in $R$ is a non zero-divisor, and hence invertible in $S$. The universal property of $Q_{\mathrm{cl}}(R)$ implies that it maps into $S$ over $R$. Since its image is a regular ring between $R$ and $G(S)$, it is $G(S)$. The kernel of the map is clearly trivial, establishing the isomorphism. Since the elements in $Q_{\mathrm{cl}}(R)$ have the form $ab^{-1}, a, b \in R$, the regularity degree of $G(S)$ with respect to $R$ is one. ∎

3.7. Remark. It follows that one cannot build a counterexample to Problem 2.10 using the rings of Proposition 3.6.

3.8. Rings of continuous functions. One is led naturally to consider rings of the form $C(X)$, the algebra of all continuous functions on a (completely regular) Hausdorff space. There are two reasons for this: first, if the algebraic notions are valid, they may have interpretations of interest in rings of functions; and second, rings of functions are sufficiently complicated that they are likely to provide examples and counterexamples for algebraic questions. This was already clear when Storrer studied the epimorphic hull [23, 11.6]

Let us first recall some basic notions, bearing in mind that [5] is the standard reference for the topic. If $X$ is a topological set and $f: X \to R$ is a continuous function, then the zero-set of $f$ is $\{x \in X \mid f(x) = 0\}$. A subset of $X$ is a $G_\delta$ if it is a countable intersection of open sets. A point $p \in X$ is called a $P$-point of $X$ if the zero-set of $f$ is a neighbourhood of $p$ whenever $f \in C(X)$ and $f(p) = 0$.

A topological space $X$ is called completely regular [5] if it is a Hausdorff space in which any point and any closed set disjoint from it can be completely separated by a continuous real-valued function on $X$. When considering the ring of all continuous functions on a topological space one can assume, without loss of generality that $X$ is completely regular [5, Chapter 3].

3.9. $\phi$-algebras. There are several simultaneous structures on $C(X)$—that of a commutative algebra over $\mathbb{R}$, that of a partially ordered set, and that of a lattice. In [8] these essential features were abstracted to the notion of a $\phi$-algebra whose definition requires the following notions. Let $A$ denote a lattice ordered ring, and let $A^+ = \{a \in A: a \geq 0\}$.

A lattice-ordered ring $A$ is called Archimedian if, for each element $a$ which is different from 0 the set $\{na : n \in Z\}$ has no upper bound in $A$. An element $a \in A^+$ is called a weak order unit of $A$ of if $b \in A$ and $a \vee b = 0$ imply $b = 0$. Lastly, an ideal of $A$ is called an $l$-ideal, if $a \in I, b \in A$, and $|b| \leq |a|$ imply $b \in I$. A $\phi$-algebra is an Archimedean lattice-ordered algebra over $\mathbb{R}$ in which 1 is a weak order unit. Each $C(X)$ is a $\phi$-algebra but the converse is far from true. If $A$ is a $\phi$-algebra then $M(A)$ denotes the set of maximal $l$-ideals in $A$ endowed with the hull-kernel topology. For each $M \in M(A)$, $A/M$ is a totally ordered field containing $\mathbb{R}$. The ideal $M$ is called real or hyper-real accordingly as $A/M$ coincides with $\mathbb{R}$ or properly contains it. The (possibly empty) subset of real maximal $l$-ideals is denoted $R(A)$, and if $\bigcap R(A) = (0)$, then $A$ is a subdirect product of copies of $\mathbb{R}$ and one calls $A$ a $\phi$-algebra of real-valued functions.

Henriksen and Johnson [8, 2.3] showed that any $\phi$-algebra $A$ has a representation as an algebra of continuous functions to the extended reals defined on the space $M(A)$. Each element of $A$ is real on a dense open subset of $M(A)$ and functions are identified when they agree on the intersection of the sets where they are real-valued. There is a natural (supremum) metric on $A$, so that it makes sense to speak of $\phi$-algebras being uniformly closed, (meaning that Cauchy sequences converge).

A natural and recurring question is whether a given $\phi$-algebra is isomorphic to a ring of the form $C(X)$. Henriksen and Johnson [8, 5.2] found conditions which are necessary and sufficient—the ring must be a $\phi$-algebra of real-valued functions, it must be uniformly closed, be closed under inversion, and $R(A)$ must be $z$-embedded in $M(A)$.

3.10. $P$-SPACES. A topological space is called a $P$-space if every $G_\delta$ is open, equivalently, if every zero-set is open. Compact $P$-spaces are necessarily finite and countable $P$-spaces are discrete, but there exist $P$-spaces with no isolated points. These spaces are pertinent because $X$ is a $P$-space precisely when $C(X)$ is its own epimorphic hull.

3.11. ALMOST $P$-SPACES. A space $X$ is almost $P$ if every non-empty zero-set has a non-empty interior, equivalently every nonempty $G_\delta$ set has non-empty interior. Unlike $P$ spaces, almost $P$ spaces can be compact and infinite, though they also are discrete if they are countable. Watson [25] has constructed in ZFC a compact almost $P$ space with no $P$-points. There is also an interesting space due to Levy [12] who invented these spaces [13]. Once again there is an algebraic characterization: it is precisely when $C(X) = Q_{\mathrm{cl}}(X)$ that $X$ is almost $P$.

3.12. THE DELTA TOPOLOGY. [18, 1W] There is a canonical way of making a space $X$ into a $P$-space by (if necessary) strengthening its topology. One takes as a base for the new topology all $G_\delta$ sets in the original space. It is equivalent to take the set of zero-sets as a base for a new topology. The new space, denoted $X_\delta$, is called the $P$-space coreflection of $X$. If $j$ is the continuous identity map $X_\delta \to X$ then one has the following universal property: if $Y$ is a Tychonoff $P$-space and $f: Y \to X$ is a continuous map, then there is a continuous map $k: Y \to X_\delta$ so that $j \circ k = f$. In categorical terms, $P$-spaces form a coreflective subcategory of the category of Tychonoff spaces.

3.13. The epimorphic hull of C(X). The classical and complete rings of quotients of a $C(X)$ were considered as instances of the general algebraic study by Fine, Gillman, and Lambek in their now classic [4]. We will use their notation for these rings, $Q_{\mathrm{cl}}(X)$, and $Q(X)$ respectively. The main result reads:

3.14. Theorem. [Fine, Gillman, Lambek, [4, 2.6]] *The ring $Q(X)$ is the set of all continuous functions defined on dense open sets of $X$ modulo the equivalence relation that identifies two functions that agree on the intersection of their domains. The ring $Q_{cl}(X)$ is the algebra of all continuous functions on dense cozero sets of $X$ modulo the same equivalence relation.*

Note that $Q_{\mathrm{cl}}(X)$ and $Q(X)$ are easily seen to be $\phi$-algebras. Hager showed that $Q(X)$ is rarely a ring of functions as follows:

3.15. Theorem. [Hager [6]]
*Suppose that the spectrum of $Q(X)$ is of non-measurable cardinality. Then $Q(X)$ is isomorphic to a $C(Y)$ iff $X$ has a dense set of isolated points.*

The epimorphic hull of $C(X)$ was studied in [19] but it was denoted $H(X)$ because the symbol $E(X)$ had an existing and conflicting meaning in topology. The epimorphic hull was shown to be a $\phi$-algebra for all spaces $X$. The most natural (and still open) question was:

3.16. Problem. Let $X$ be a Tychonoff space. Find necessary and sufficient conditions on $X$ for $H(X)$ to be isomorphic to a $C(Y)$.

3.17. Subproblem. Find necessary and sufficient conditions on $X$ for the $\phi$-algebra $H(X)$ to be uniformly closed.

Problem 3.16 has never been solved. Indeed, one does not even know whether $X$ must contain a $P$-point if $H(X)$ is a $C(Y)$, although this was recently established positively for basically disconnected spaces [7]. The presence of a dense set of $P$-points, indeed of isolated points, does not suffice to give that $H(X)$ is a $C(Y)$ [7].

Here is a summary of the main things one does know [19]:

1. If $H(X)$ is a $C(Y)$, then compact subspaces of $X$ are scattered.
2. If $X$ has a countable dense set of isolated points then $H(X) = C(N)$, where $N$ denotes a countable discrete space.
3. One has a characterization of when $Q_{\mathrm{cl}}(X)$ is a $C(Y)$.
4. One has many examples of spaces, almost $P$ and not, for which $H(X)$ is a $C(Y)$, as well as examples where some but not all of the three rings of quotients $Q_{\mathrm{cl}}(X)$, $H(X)$ and $Q(X)$ are isomorphic to rings of continuous functions.

An important tool for studying $H(X)$ is the definition of the following "test" subspace:

3.18. Definition. *By $gX$ denote the intersection of the dense cozero sets of a realcompact Tychonoff space $X$.*

One shows the following: let $X$ be realcompact and Tychonoff:

(i) The space $gX$ is a realcompact almost $P$ space.

(ii) If $X$ has a dense almost $P$ space it has a largest one and it is $gX$.
(iii) The following are equivalent:
(a) $H(X)$ is a $\phi$-algebra of real-valued functions,
(b) $gX$ is dense in $X$,
(c) $C(X)$ has a regular ring of quotients of the form $C(Y)$.
This shows that (cf Problem 3.16):
(iv) If $H(X)$ is a $C(Y)$ then $Y = (gX)_\delta$, ($gX$ with the delta topology).

3.19. A CONTEXT OF REAL-VALUED FUNCTIONS. More recently there has been parallel work undertaken by Henriksen, Raphael, and Woods which drops the rings of quotients aspect but keeps the discussion very concrete. The idea is the following: Let $X$ be a Tychonoff space and observe that $C(X)$ lies in the natural von Neumann regular ring $F(X)$ of all real-valued functions defined on $X$. Let $G(X)$ be the context determined by $C(X)$ and $F(X)$. Like $H(X)$, the ring $G(X)$ is always a $\phi$-algebra but unlike $H(X)$ it is always a $\phi$-algebra of real-valued functions.

When $X$ is an almost $P$ space, $H(X)$ and $G(X)$ coincide. In general these rings differ but $H(X)$ is always a homomorphic image of $G(X)$.

One is no longer dealing with rings of quotients, but still has very natural problems such as:

3.20. PROBLEM. Find necessary and sufficient conditions on $X$ so that $G(X)$ be a ring of continuous functions.

This is known to occur for some non $P$ spaces, but it occurs so rarely that one raises a more general question. Let $G^u(X)$ denote the set of limits of Cauchy sequences from $G(X)$.

3.21. PROBLEM. Find necessary and sufficient conditions on $X$ for $G^u(X)$ to be a ring.

[Of course there is the parallel problem for $H(X)$]. It is easy to see that $G^u(X)$ is closed under sums, finite sups and infs, and scalar multiples. Thus in the cases where it is a ring, it is a $\phi$-algebra. It is also easy to see that being a ring is equivalent to being closed under forming squares. Note that Isbell gave an example [9, 1.8] of an algebra of real valued functions whose uniform closure is not closed under taking squares.

Lastly returning to Olivier's work, there is the question

3.22. PROBLEM. Describe $T(C(X))$ where $X$ is an arbitrary Tychonoff space. Certainly it has both $G(X)$ and $H(X)$ as a homomorphic image. But when is it $G(X)$? When is it a $\phi$-algebra? When is it a $\phi$-algebra of real-valued functions or a uniformly closed $\phi$-algebra?

AFTERWORD Recent joint work with W.D. Burgess using algebraic methods seems to give progress on some of the problems presented above.

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