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# A USEFUL CATEGORY FOR MIXED ABELIAN GROUPS.

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ABSTRACT. All the useful categories in the study of the mixed abelian groups (e.g. **Warf** and **Walk)** ignore the torsion. We introduce a new category denoted $\mathcal{A}$ which ignores the torsion-freeness and could characterize some classes of nonsplitting mixed groups with the aid of **Walk.**

## 1. Introduction

The categories **Warf** , first introduced as $\mathcal{H}$ in [7] and **Walk,** first introduced as $\mathcal{C}$ in [2] have useful applications in the theory of the mixed abelian groups. In what follows we introduce the category $\mathcal{A}$ whose objects are all the abelian groups (i.e. $Ob(\mathcal{A})$ $=Ob(\mathbf{Ab})$) and whose morphisms, are $\mathcal{A}(G,H) = \mathbf{Ab}(G,H)/J(G,H)$ where

$$J(G,H) = \{f : G \to H | T(G) \leq ker(f)\},$$

for $G, H \in Ob(\mathcal{A})$, study its categorical properties and establish connections with the above mentioned category **Walk**. Finally, some results that justify the utility of this category are given.

Needless to say, all the groups considered will be abelian.

## 2. The categorical structure

For two groups $G$ and $H$, we consider on the abelian group $\mathbf{Ab}(G,H)$ the binary relation $\rho_{G,H}$ defined by $(f,g) \in \rho_{G,H} \Leftrightarrow T(G) \subseteq ker(f-g)$ where $G \underset{g}{\overset{f}{\rightrightarrows}} H$.

2.1. LEMMA. *For $\alpha, \beta \in \mathbf{Ab}(G,H)$ the inclusion $ker\alpha \cap ker\beta \subseteq ker(\alpha+\beta)$ holds.* ■

2.2. PROPOSITION. *The relation $\rho_{G,H}$ is a congruence relation.*

PROOF. Indeed, using 2.1 two times, the relation $\rho_{G,H}$ is :

- *reflexive* $T(G) \subseteq G = ker(0) = ker(f-f) \Rightarrow (f,f) \in \rho_{G,H}, \forall f \in \mathbf{Ab}(G,H)$
- *symmetric* $(f,g) \in \rho_{G,H} \Rightarrow T(G) \subseteq ker(f-g) = ker(g-f) \Rightarrow (g,f) \in \rho_{G,H}$

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- *transitive* $(f,g),(g,h)\in\rho_{G,H}\Rightarrow T(G)\subseteq ker(f-g),ker(g-h)\Rightarrow T(G)\subseteq ker(f-g)\cap ker(g-h)\subseteq ker((f-g)+(g-h))=ker(f-h)\Rightarrow(f,h)\in\rho_{G,H}$.

Moreover, if $(f,g),(f_1,g_1)\in\rho_{G,H}$ then $T(G)\subseteq ker(f-g)\cap ker(f_1-g_1)\Rightarrow T(G)\subseteq ker((f-g)+(f_1-g_1))=ker((f+f_1)-(g+g_1))\Rightarrow((f+f_1),(g+g_1))\in\rho_{G,H}$. ■

There is a well-known order isomorphism between congruences and subgroups: $J(G,H)=\rho_{G,H}\langle 0\rangle=\{f\in\mathbf{Ab}(G,H)|T(G)\subseteq ker(f)\}$ is the corresponding subgroup.

Elementary: $T(G)\subseteq ker(f)\cap ker(g)\subseteq ker(f\pm g)$, so that

2.3. Remark. For every $f,g\in J(G,H)$ also $f\pm g\in J(G,H)$ holds.

Clearly, if $T$ is a torsion group, $J(T,H)=\{0\}$ for every group $H$ and so $\mathcal{A}(T,H)=\mathbf{Ab}(T,H)$.

2.4. Lemma. *(a)* $ker\alpha\subseteq ker(\beta\circ\alpha)$; *(b) For* $G\xrightarrow{\alpha}H\xrightarrow{\beta}K$, $T(H)\subseteq ker\beta\Rightarrow T(G)\subseteq ker(\beta\circ\alpha)$.

Proof. (b) Indeed, $x\in T(G)\Rightarrow\alpha(x)\in T(H)\subseteq ker\beta\Rightarrow x\in ker(\beta\circ\alpha)$. ■

2.5. Proposition. *The relations* $\{\rho_{G,H}|G,H\in Ob(\mathbf{Ab})\}$ *are compatible with composition.*

Proof. Indeed, using Lemma 2.4 (a) and (b), one has: $(f,g)\in\rho_{G,H},(f,'g')\in\rho_{H,K}\Rightarrow ker(f\circ f'-g\circ g')=ker((f'\circ(f-g)+(f'-g')\circ g)\supseteq$

$ker(f'\circ(f-g))\cap ker((f'-g')\circ g)\overset{2.4}{\supseteq}ker(f-g)\cap T(G)\supseteq T(G)\Rightarrow$

$(f'\circ f,g'\circ g)\in\rho_{G,K}$. ■

Hence, we define the category $\mathcal{A}$, as a quotient category of $\mathbf{Ab}$ whose objects are all the abelian groups (i.e. $Ob(\mathcal{A})=Ob(\mathbf{Ab})$) and whose morphisms, for each two groups $G,H$ are given by $\mathcal{A}(G,H)=\mathbf{Ab}(G,H)/J(G,H)$ (or $\mathbf{Ab}(G,H)/\rho_{G,H}$). We shall denote the classes $\overline{f}=f+J(G,H)$ in $\mathcal{A}(G,H)$. The composition in $\mathcal{A}$ is well-defined according to the above Proposition and $1_G+J(G,G)$ is the identity morphism. Associativity and bilinearity are easily verified (using 2.2) so that

2.6. Theorem. *$\mathcal{A}$ is an additive category.* ■

For the following elementary results we use the notation: if $f:G\to H$ then $f|_{T(G)}:T(G)\to H$ and $\widetilde{f|_{T(G)}}:T(G)\to T(H)$ (because $im(f|_{T(G)})\subseteq T(H)$).

2.7. Proposition. *(a)* $f+J(G,G)$ *is the identity in* $\mathcal{A}(G,G)$ *iff* $f|_{T(G)}:T(G)\to G$ *is the inclusion (i.e.* $f$ *fixes the finite order elements);*

*(b) if* $f|_{T(G)}$ *or* $\widetilde{f|_{T(G)}}$ *is a monomorphism in* $\mathbf{Ab}$ *then* $f+J(G,H)$ *is a monomorphism in* $\mathcal{A}(G,H)$*;*

*(c) if* $\widetilde{f|_{T(G)}}$ *is an epimorphism in* $\mathbf{Ab}$ *then* $f+J(G,H)$ *is an epimorphism in* $\mathcal{A}(G,H)$*. If* $H$ *splits over* $T(H)$*, the converse also holds.*

PROOF. Clearly, **the equality in $\mathcal{A}$** is characterized as follows: $\overline{f} = f + J(G,H) = \overline{g} = g + J(G,H) \Leftrightarrow f - g \in J(G,H) \Leftrightarrow T(G) \leq ker(f-g) \Leftrightarrow$
$(f-g)(T(G)) = 0 \Leftrightarrow f|_{T(G)} = g|_{T(G)}$.

Hence, for (a) it suffices to observe that $1_G|_{T(G)} : T(G) \to G$ is the inclusion.

(b) For $L \underset{\beta}{\overset{\alpha}{\rightrightarrows}} G \xrightarrow{f} H$ and $f|_{T(G)}$ monic in **Ab** suppose $\overline{f} \circ \overline{\alpha} = \overline{f} \circ \overline{\beta}$. Then $\overline{f \circ \alpha} = \overline{f \circ \beta}$ and $f \circ \alpha|_{T(L)} = f \circ \beta|_{T(L)}$. Using $\alpha\widetilde{|_{T(L)}} : T(L) \to T(G)$ (indeed, $im(\alpha|_{T(L)}) \subseteq T(G)$) and $f \circ \alpha|_{T(L)} = f|_{T(G)} \circ \alpha\widetilde{|_{T(L)}}$ we derive $\alpha\widetilde{|_{T(L)}} = \beta\widetilde{|_{T(L)}}$ or $\alpha|_{T(L)} = \beta|_{T(L)}$. Hence $\overline{\alpha} = \overline{\beta}$.

(c) For $G \xrightarrow{f} H \underset{\beta}{\overset{\alpha}{\rightrightarrows}} L$ and $f\widetilde{|_{T(G)}}$ epic in **Ab** suppose $\overline{\alpha} \circ \overline{f} = \overline{\beta} \circ \overline{f}$. Then $\overline{\alpha \circ f} = \overline{\beta \circ f}$ and $\alpha \circ f|_{T(G)} = \beta \circ f|_{T(G)}$. As above $\alpha \circ f|_{T(G)} = \alpha|_{T(H)} \circ f\widetilde{|_{T(G)}}$ so that $\alpha|_{T(H)} = \beta|_{T(H)}$ and $\overline{\alpha} = \overline{\beta}$.

If $T(H)$ is a direct summand of $H$, all homomorphisms $\sigma, \tau : T(H) \to L$ extend to morphisms $\sigma_1, \tau_1 : H \to L$. Now, set $T(G) \overset{f\widetilde{|_{T(G)}}}{\longrightarrow} T(H) \underset{\tau}{\overset{\sigma}{\rightrightarrows}} L$ such that $\sigma \circ f\widetilde{|_{T(G)}} = \tau \circ f\widetilde{|_{T(G)}}$. As before, using any extensions $\sigma_1, \tau_1$ we derive $\sigma_1 \circ f|_{T(G)} = \sigma_1|_{T(H)} \circ f\widetilde{|_{T(G)}} = \sigma \circ f\widetilde{|_{T(G)}} = \tau_1 \circ f|_{T(G)}$ or $\overline{\sigma_1 \circ f} = \overline{\tau_1 \circ f}$. Hence $\overline{\sigma_1} = \overline{\tau_1}$ or $\sigma_1|_{T(H)} = \tau_1|_{T(H)}$ and $\sigma = \tau$. ∎

2.8. REMARK. The groups $G$ such that for every group $H$, each homomorphism $\sigma : T(G) \to H$ extends to a homomorphism $\sigma_1 : G \to H$ are exactly the splitting ones.

Indeed, for $H = T(G)$ and $\sigma = 1_{T(G)}$ there is an extension $u : G \to T(G)$ such that $u \circ i = 1_{T(G)}$, where $i : T(G) \to G$ is the inclusion. ∎

2.9. REMARK. $\mathcal{A}$ is not balanced and so, not normal nor conormal.

PROOF. Consider the inclusion $i : T(G) \to G$ of the torsion part of a nonsplitting mixed group $G$ such that $T(G)$ is no epimorphic image of $G$ (e.g. $\prod_{p \in \mathbf{P}} \mathbf{Z}(p) \notin \mathcal{M}_1$ (see [9])). According to the proposition above $\overline{i} \in \mathcal{A}(T(G), G)$ is a monomorphism and an epimorphism but not an isomorphism in $\mathcal{A}$. Indeed, if $\overline{i}$ should be an isomorphism in $\mathcal{A}$ there would exist a morphism $\pi : G \to T(G)$ in **Ab** such that $\overline{\pi} \circ \overline{i} = \overline{1}_{T(G)}, \overline{i} \circ \overline{\pi} = \overline{1}_G$ in $\mathcal{A}$. Hence $\pi|_{T(G)} = 1_{T(G)}$ and so $\pi$ would be an epimorphism. ∎

2.10. THEOREM. *In $\mathcal{A}$ the torsionfree groups are exactly the zero objects. In particular, all the torsionfree groups are $\mathcal{A}$-isomorphic.*

PROOF. A group $G$ is an initial object in $\mathcal{A}$ iff $\mathbf{Ab}(G,H) = J(G,H)$ holds for each group $H$. Hence $G$ is initial iff $T(G) \leq ker(f)$ holds for each group $H$ and each homomorphism $f : G \to H$. Taking $f$ any injective homomorphism we obtain $T(G) = 0$. Conversely, if $T(G) = 0$ surely $T(G) \leq ker(f)$ holds for every $H$ and every $f$. Hence $J(G,H) = \mathbf{Ab}(G,H)$ and $\mathcal{A}(G,H) = \mathbf{Ab}(G,H)/\mathbf{Ab}(G,H) = \left\{\overline{0}\right\}$.

Further, $G$ is a terminal object in $\mathcal{A}$ iff $\mathbf{Ab}(H,G) = J(H,G)$ holds for each group $H$. Hence $G$ is terminal iff $T(H) \leq ker(f)$ holds for each group $H$ and each homomorphism

$f : H \to G$. Taking $H = G, f = 1_G$ we obtain $T(G) = 0$. Conversely, $T(G) = 0$ implies $T(H) \leq ker(f)$ for each group $H$ and each homomorphism $f : H \to G$. Indeed, $f(T(H)) \subseteq T(G)$ implies $f(T(H)) = 0$ and so $T(H) \leq ker(f)$.

Hence the zero objects in $\mathcal{A}$ are the torsionfree groups. ∎

2.11. Theorem. *$\mathcal{A}$ has cokernels.*

Proof. Finally, for $f + J(G,H) \in \mathcal{A}(G,H)$, if $p : H \to \overline{H} = H/(f(T(G))$ denotes the canonical projection, we verify that $p + J(H,\overline{H}) = coker(f)$.

First, $p \circ f \in J(G,\overline{H})$. Indeed, $T(G) \leq ker(p \circ f) \Leftrightarrow (p \circ f)(T(G)) = 0 \Leftrightarrow p(f(T(G)) = 0$, which clearly holds. Next, if the following diagram commutes

$$\begin{tikzcd}
 & L \\
G \arrow[ur, "0"] \arrow[r, "f"] \arrow[dr, "0"'] & H \arrow[u, "g"'] \arrow[d, "p"] \\
 & \overline{H}
\end{tikzcd}$$

there is a unique homomorphism $h : \overline{H} \to L$ such that the following triangle commutes

$$\begin{tikzcd}
L & \\
H \arrow[u, "g"] \arrow[r, "p"'] & \overline{H} \arrow[ul, "h"']
\end{tikzcd}$$

Indeed, $g \circ f = 0$ in $\mathcal{A}$ iff $g \circ f \in J(G,L)$. This is consequently equivalent to $T(G) \leq ker(g \circ f) \Leftrightarrow f(T(G)) \leq ker(g)$ and so, to $ker(p) \leq ker(g)$. Hence a unique homomorphism $h : \overline{H} \to L$ exists such that the above triangle commutes. ∎

2.12. Remark. For each $G, H$ the group $\mathcal{A}(G,H)$ can be identified with a subgroup of $\mathbf{Ab}(T(G), T(H))$.

Indeed, first observe that $J(G,H)$ can be identified with $\mathbf{Ab}(G/T(G), H)$. Indeed, $T(G) \leq ker(f)$ implies that there is a unique homomorphism $\underline{f} : G/T(G) \to H$ with $f = p_{T(G)} \circ \underline{f}$. Next, use the left exactness of the contravariant functor $\mathbf{Ab}(-,H)$ for the short exact sequence $0 \to T(G) \to G \to G/T(G) \to 0$. We obtain the exact

sequence $0 \to \mathbf{Ab}(G/T(G), H) \to \mathbf{Ab}(G, H) \xrightarrow{t} \mathbf{Ab}(T(G), H)$ and then $\mathcal{A}(G, H) = \mathbf{Ab}(G, H)/J(G, H) \cong \mathbf{Ab}(G, H)/\mathbf{Ab}(G/T(G), H) \cong$
$\mathbf{Ab}(G, H)/ker(t) \cong im(t)$, which can be identified with a subgroup of
$\mathbf{Ab}(T(G), T(H))$. ■

2.13. THEOREM. *The category $\mathcal{A}$ has products.*

PROOF. Let $\{f_i + J(G, G_i) : G \to G_i\}$ be a family of morphisms in $\mathcal{A}$ and $\{p_j : \prod_{i \in I} G_i \to G_j, \forall j \in I\}$ the canonical projections for the direct product (from $\mathbf{Ab}$). Clearly there is a unique $f : G \to \prod_{i \in I} G_i$ such that $f_i = p_i \circ f$. One easily checks that $\forall i \in I : g_i \in f_i + J(G, G_i)$, $g_i = p_i \circ g$ implies $g \in f + J(G, \prod_{i \in I} G_i)$.

Indeed, $T(G) \leq ker(g_i - f_i)$, $\forall i \in I \Rightarrow T(G) \leq ker(g - f)$ because $ker(g - f) = \bigcap_{i \in I} ker(g_i - f_i)$.

Clearly, there is a unique factorization $f_i + J(G, G_i) = (p_i + J(\prod_{i \in I} G_i, G_j)) \circ (f + J(G, \prod_{i \in I} G_i))$. ■

Notice that in $\mathcal{A}$ there are finite direct sums (products) defined as usually in $\mathbf{Ab}$ (as objects). Moreover

2.14. THEOREM. *$\mathcal{A}$ has infinite coproducts (direct sums).*

PROOF. Let $\{f_i + J(G_i, G) : G_i \to G\}$ be a family of morphisms in $\mathcal{A}$. The proof is similar to the previous one: it reduces to the inclusion $\bigoplus_{i \in I} ker(f_i) \leq ker(f)$ and the equality $T\left(\bigoplus_{i \in I} G_i\right) = \bigoplus_{i \in I} T(G_i)$, where $f_i = f \circ q_i$ gives the unique decomposition with $\{q_j : G_j \to \bigoplus_{i \in I} G_i, \forall j \in I\}$ the canonical injections into the coproduct (direct sum). ■

2.15. REMARK. $\mathcal{A}$ does not have kernels.

Indeed, for a morphism $\overline{f} \in \mathcal{A}(G, H)$, $(T(G) \cap ker(f), \overline{incl})$ must be the kernel in $\mathcal{A}$. But this is not the case in general.

2.16. THEOREM. *Two groups $G$ and $H$ are isomorphic in $\mathcal{A}$ iff there are two torsion-free groups $U$ and $V$ such that $G \oplus U \cong H \oplus V$.*

PROOF. The condition is sufficient: first, notice that if $U$ is torsion-free, the canonical projection $p_G : G \oplus U \to G$, respectively injection $e_G : G \to G \oplus U$ have mutually inverse classes in $\mathcal{A}$. Indeed, $p_G \circ e_G = 1_G$ implies $\overline{p_G} \circ \overline{e_G} = \overline{1_G}$ in $\mathcal{A}$ and conversely, $(e_G \circ p_G, 1_{G \oplus U}) \in \rho_{G \oplus U, G \oplus U}$, this being justified as follows: $ker(1_{G \oplus U} - e_G \circ p_G) = ker\,(e_U \circ p_U) = G \geq T(G) = T(G \oplus U)$. Then $G \overset{\mathcal{A}}{\cong} G \oplus U$ and one uses also $G \oplus U \cong H \oplus V$ and similarly $H \oplus V \overset{\mathcal{A}}{\cong} H$.

The condition is also necessary: suppose $G \overset{\mathcal{A}}{\cong} H$, that is, there are homomorphisms $f : G \to H$ and $g : H \to G$ such that $f \circ g - 1_H = s \in J(H, H)$ and $g \circ f - 1_G = t \in J(G, G)$.

First observe that the restrictions $f|_{T(G)} : T(G) \to T(H), g|_{T(H)} : T(H) \to T(G)$ are mutually inverses in **Ab** (indeed, e.g. $f|_{T(G)} \circ g|_{T(H)}(h) = 1_H(h) + s(h) = h = 1_{T(H)}(h), \forall h \in T(H)$ using $T(H) \leq ker(s)$). Define $P$ as the pushout in the following commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & T(G) \arrow[r, phantom, "\overset{f|_{T(G)}}{\cong}"] \arrow[d, "in"'] & T(H) \arrow[r, "in"] & H \arrow[r, "pr"] \arrow[d] \arrow[lld, "g"'] & H/T(H) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[rr] & & P \arrow[r] & H/T(H) \arrow[r] & 0
\end{tikzcd}
$$

A well-known exercise from abelian category theory shows that the bottom line is also exact . As $g : H \to G$ renders the upper triangle commutative, the bottom line splits and so $P \cong G \oplus H/T(H)$. Using the $3 \times 3-$lemma the same pushout may be used

$$
\begin{tikzcd}
& 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T(G) \arrow[r, phantom, "\overset{f|_{T(G)}}{\cong}"] \arrow[d, "in"'] & T(H) \arrow[r, "in"] & H \arrow[r, "pr"] \arrow[d] & H/T(H) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G \arrow[rr] \arrow[d, "pr"'] & & P \arrow[r] \arrow[d] & H/T(H) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G/T(G) \arrow[rr, phantom, "="] \arrow[d] & & G/T(G) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
& 0 & & 0 & 0 &
\end{tikzcd}
$$

in order to prove that $P \cong H \oplus G/T(G)$. Hence $G \oplus H/T(H) \cong H \oplus G/T(G)$ with torsion-free groups $H/T(H)$ and $G/T(G)$. ∎

Similarly to [7], one can write down an isomorphism in terms of the given functions $f$ and $g$ : indeed

$$G \oplus H/T(H) \longrightarrow H \oplus G/T(G)$$

$$(x, y + T(H)) \longmapsto (f(x - g(y)) + y, x - g(y) + T(G))$$

and

$$H \oplus G/T(G) \longrightarrow G \oplus H/T(H)$$

$$(y, x + T(G)) \longmapsto (g(y - f(x)) + x, y - f(x) + T(H))$$

define group morphisms that are inverses of one another.

As in [3], one can also prove the above result by $G \oplus H/ker(s) \cong H \oplus G/ker(t)$.

Then

2.17. COROLLARY. *$G \overset{\mathcal{A}}{\cong} H$ and $G \overset{\mathbf{Walk}}{\cong} H$ iff there are torsion groups $S, T$ and torsion-free groups $U, V$ such that $G \oplus S \cong H \oplus T$ and $G \oplus U \cong H \oplus V$.*

## 3. A full embedding

As in [4] there is a natural embedding of **To**, the full subcategory of **Ab** which consists of all the torsion abelian groups, into $\mathcal{A}$.

3.1. THEOREM. *The functor $I : \mathbf{To} \to \mathcal{A}$, defined by $I(T) = T$ on objects and $I_{TS} : \mathbf{To}(T, S) \to \mathcal{A}(T, S), I_{TS}(f) = f + J(T, S) = \{f\}$ on morphisms, is a full embedding.*

PROOF. Indeed, as we already have noticed for any $T \in Ob(\mathbf{To})$, $J(T, G) = \{0\}$ and hence $\mathcal{A}(T, S) = \{\{f\} \,|f \in \mathbf{Ab}(T, S)\}$. ■

3.2. THEOREM. *$I$ has an adjoint (to the right): $K : \mathcal{A} \to \mathbf{To}$, defined $K(G) = T(G)$ on objects and $K_{GH}(\overline{f}) = \widetilde{f|_{T(G)}}$ on morphisms.*

PROOF. First of all, notice that $K$ is well-defined (see the characterization of the equality of the morphisms in $\mathcal{A}$ : for each $G, H \in \mathcal{A}$ we can consider $K_{GH}(\overline{f}) = \widetilde{f|_{T(G)}}$ because $\overline{f} = \overline{g} \Leftrightarrow f|_{T(G)} = g|_{T(G)} \Leftrightarrow \widetilde{f|_{T(G)}} = \widetilde{g|_{T(G)}}$). Next, for the adjoint situation the *unit* $\eta : 1_{\mathbf{To}} \to K \circ I$ is trivially given by the identity $1_T : T \to K(I(T)) = T$ for each $T \in Ob(\mathbf{To})$, and the *counit* $\varepsilon : I \circ K \to 1_{\mathcal{A}}$ is given by the inclusion $\varepsilon_G : I(K(G)) = T(G) \to G$ for each $G \in Ob(\mathcal{A})$, all these being natural transformations. Moreover, one easily verifies $K \overset{\eta \cdot K}{\to} K \circ I \circ K \overset{K \cdot \varepsilon}{\to} K = 1_K$ and $I \overset{I \cdot \eta}{\to} I \circ K \circ I \overset{\varepsilon \cdot I}{\to} I = 1_I$. [Another proof: one verifies the natural equivalence of abelian group-valued bifunctors $\alpha_{T,G} : \mathcal{A}(I(T), G) \to \mathbf{To}(T, K(G)), \forall T \in Ob(\mathbf{To}), \forall G \in Ob(\mathcal{A})$]. ■

3.3. COROLLARY. *$K$ is a limit preserving monofunctor and $I$ is a colimit preserving epifunctor.*

Indeed, this is a known property of functors which admit an adjoint to the right.

3.4. COROLLARY. *$I$ also reflects colimits.*

Use the dual of Ex. 27H(c), p.204,[1].

3.5. REMARK. $I$ is not an equivalence of categories.

Indeed, $I$ is an equivalence $\Leftrightarrow$ $I$ is dense (representative) $\Leftrightarrow$ $\forall G \in Ob(\mathbf{Ab}) = Ob(\mathcal{A}), \exists T \in Ob(\mathbf{To}) : T = I(T) \overset{\mathcal{A}}{\cong} G$.

This is also equivalent with the existence of two torsion-free groups $U, V$ such that $G \oplus U \cong T \oplus V$. Taking the torsion parts (we apply the functor $\overline{T} : \mathbf{Ab} \to \mathbf{Ab}$, $\overline{T}(G) = T(G), \forall G \in \mathbf{Ab}$) of these groups we observe that $T(G) \cong T$.

Hence, $I$ is an equivalence $\Leftrightarrow$ $\forall G \in Ob(\mathbf{Ab}), \exists U, V$ torsion-free groups : $G \oplus U \cong T(G) \oplus V$.

For splitting mixed groups this last condition holds when

**a)** $G$ is torsion: obviously $U = V = 0$;

**b)** $G$ is torsion-free: obviously $U = T(G) = 0, V = G$;

**c)** $G$ is splitting mixed, say $G = T \oplus F$ : obviously $U = 0, V = F$.

So in order to prove that $I$ is not an equivalence an example of non-splitting mixed group $M$ such that $\forall U, V$ torsion-free groups, $M \oplus U \ncong T(M) \oplus V$ suffices.

## 4. Relations with **Walk**

The category **Walk** was also defined as a quotient category of **Ab** by $Ob(\mathbf{Walk}) = Ob(\mathbf{Ab})$ and $\mathbf{Walk}(G, H) = \mathbf{Ab}(G, H)/I(G, H)$ where

$I(G, H) = \{f \in \mathbf{Ab}(G, H) | im(f) \subseteq T(H)\}$ or, similarly with $\mathcal{A}$, with the aid of a congruence relation $\omega_{G,H}$ defined by $(f, g) \in \omega_{G,H} \Leftrightarrow im(f - g) \subseteq T(H)$.

Notice that $f \in I(G, H) \Leftrightarrow G = \overset{-1}{f}(T(H))$.

4.1. REMARK. $I(G, H) \cap J(G, H)$ can be identified with $\mathbf{Ab}(G/T(G), T(H))$.

Indeed, $f \in I(G, H) \cap J(G, H)$ iff there is a unique $f_1 \in \mathbf{Ab}(G/T(G), T(H))$ such that $f = p_{T(G)} \circ f_1 \circ in$, the inclusion $in : T(H) \to H$. The situation is described in the following canonical commutative diagram

$$\begin{tikzcd}
& G \arrow[r, "f"] \arrow[dl, "p_{T(G)}"'] \arrow[d] & H & \\
G/T(G) \arrow[r, "pr"'] & G/ker f \arrow[r, "f_0"'] & im f \arrow[u] \arrow[r, "incl"'] & T(H) \arrow[ul, "in"']
\end{tikzcd}$$

with $f_1 = incl \circ f_0 \circ pr$(as for the converse, $im(p_{T(G)} \circ f_1 \circ in) \leq im(in) = T(H)$ resp. $T(G) = ker(p_{T(G)}) \leq ker(p_{T(G)} \circ f_1 \circ in)$).

4.2. Remark. $\{f \in \mathbf{Ab}(G,H) | f^{-1}(T(H))$ is a direct summand of $G\} \leq I(G,H) + J(G,H)$.

Proof. Indeed, for any $f \in \mathbf{Ab}(G,H)$ set $S = f^{-1}(T(H)) = \{x \in G | f(x) \in T(H)\}$, the preimage. Surely, $T(G) \leq S$ and $ker(f) = f^{-1}(0) \leq S$. If $S$ is a direct summand and $G = S \oplus K$, consider $g \in I(G,H), g(s+k) = f(s), \forall s \in S, k \in K$ (i.e. $img \leq f(S) \leq T(H)$) and $h \in J(G,H), h(s+k) = f(k), \forall s \in S, k \in K$ (i.e. $T(G) \leq S \leq kerh$). Clearly $f = g + h$. ■

A more categorical proof was pointed out by the referee: if $G = S \oplus K$ and $i_S, i_K$ respectivelly $p_S, p_K$ denote the canonical injections respectivelly projections then $i_S \circ p_S + i_K \circ p_K = 1_G$ so that $f = f \circ i_S \circ p_S + f \circ i_K \circ p_K$. Clearly, $f \circ i_S \circ p_S \in I(G,H)$ and $f \circ i_K \circ p_K \in J(G,H)$.

## 5. Endomorphism rings in $\mathcal{A}$

In **Warf** and **Walk** the endomorphism rings for torsion-free rank one groups are characterized (see [8] and [5]).

If we denote $End_{\mathcal{A}}(G) = \mathcal{A}(G,G)$ for any group $G$ then

5.1. Theorem. *The map* $\alpha : End_{\mathcal{A}}(G) \to End_{\mathbf{Ab}}(T(G)), \alpha(f + J(G,G)) = \widetilde{f|_{T(G)}}$ *is a ring embedding. If $G$ splits, this is a ring isomorphism.*

Proof. Indeed, $g \in f + J(G,G)) \Leftrightarrow f|_{T(G)} = g|_{T(G)}$ shows that $\alpha$ is well-defined and injective. The compatibility with addition and composition are immediate. If $G$ splits, the endomorphisms of $T(G)$ extend to the whole $G$ and so $\alpha$ is also surjective. ■

## 6. Classification

**Walk** was constructed as a quotient category of **Ab** in order to neglect torsion. Similarly, $\mathcal{A}$ is a quotient category of **Ab** which neglects torsion-freeness. It is natural to ask to what extent these two quotient categories characterize classes $\mathcal{M}$ of abelian groups.

Using 2.17 we easily get

6.1. Proposition. *If* $G \overset{\mathbf{Walk}}{\cong} H$ *and* $G \overset{\mathcal{A}}{\cong} H$ *then* $T(G) \cong T(H)$ *and* $G/T(G) \cong H/T(H)$.

Proof. If there are torsion groups $S, T$ and torsion-free groups $U, V$ such that $G \oplus S \cong H \oplus T$ and $G \oplus U \cong H \oplus V$ then $T(G) = T(G \oplus U) \cong T(H \oplus V) = T(H)$. Further, $G/T(G) \cong \frac{G \oplus S}{T(G) \oplus S} = \frac{G \oplus S}{T(G \oplus S)} \cong \frac{H \oplus T}{T(H \oplus T)} = \frac{H \oplus T}{T(H) \oplus T} \cong H/T(H)$ the second isomorphism being obtained as $G/ker\lambda \cong im(\lambda)$ for $\lambda = pr \circ inj : G \to G \oplus S \to \frac{G \oplus S}{T(G) \oplus S}$. ■

6.2. COROLLARY. *$G \overset{\mathbf{Walk}}{\cong} H$ and $G \overset{\mathcal{A}}{\cong} H$ characterize the class of all the splitting mixed groups.*

Finally, some open problems:

PROBLEM 1. Are the groups $G$ such that $T(G) \overset{\mathcal{A}}{\cong} G$ exactly the splitting (mixed) groups?
PROBLEM 2. Find classes $\mathcal{M}$ of abelian groups such that two groups $G$ and $H$ are isomorphic (in $\mathcal{M}$) iff $G$ and $H$ are isomorphic in **Walk** and in $\mathcal{A}$.

As for this last problem, following definitions from [9], the classes $\mathcal{M}_1$ and $\mathcal{M}_2$ of mixed abelian groups could be considered. Recall that

$G \in \mathcal{M}_1$ if $T(G)$ is a homomorphic image of $G$ and $G \in \mathcal{M}_2$ if $G/T(G)$ can be embedded in $G$.

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