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APPLICATIONS OF PEIFFER PAIRINGS IN THE MOORE COMPLEX OF A SIMPLICIAL GROUP

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ABSTRACT. Generalising a result of Brown and Loday, we give for $n = 3$ and 4, a decomposition of the group, d_nNG_n, of boundaries of a simplicial group $\mathbf{G}$ as a product of commutator subgroups. Partial results are given for higher dimensions. Applications to 2-crossed modules and quadratic modules are discussed.

Introduction

Simplicial groups occupy a place somewhere between homological group theory, homotopy theory, algebraic K-theory and algebraic geometry. In each sector they have played a significant part in developments over quite a lengthy period of time and there is an extensive literature on their homotopy theory. In homotopy theory itself, they model all connected homotopy types and allow analysis of features of such homotopy types by a combination of group theoretic methods and tools from combinatorial homotopy theory. Simplicial groups have a natural structure of Kan complexes and so are potentially models for weak infinity categories. They do however suffer from a lack of apparent linkage between their algebraic structure and the geometric structure they are used to model, so that modelling a space with a simplicial group, one can seem to be spreading the geometry so thinly around that it is no longer visible. In other words the algebra does not seem to reflect the geometry in any simple way.

Some interesting recent work in modelling geometry by algebra has tended towards the explicit use of weak infinity categories (Batanin, [2, 3], Baez and Dolan, [1], Leinster, [18], Tamsamani, [23]). These use globular, multisimplicial or operad algebra models, but not simplicial groups or groupoids as such, yet some of the structure of weak n-groupoids is already apparent in the related simplicial group models. For the transfer of simplicial homotopic technology to the weak infinity categoric models, it is clear that some *rapprochement* of the two theories would be for the mutual benefit of both.

The simplest and most obvious link between them is via the lemma of Brown and Loday, [7], which identifies the structure corresponding to the interchange law for categorical groups within the low dimensional group theoretic structure of a simplicial group or groupoid, as being a commutator of face-map kernels. At the next level results of Brown

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and Gilbert, [6], and related ideas of Conduché, [11], have been used by Carrasco and Cegarra, [10], to study braided categorical groups. The categorical braiding corresponding to a given simplicial group is a commutator of degenerate elements and the satisfaction of the axioms for a braiding corresponds, yet again, to the vanishing of certain commutators of intersections of face-map kernels. Thus 'higher order weak monoidal structures' and the corresponding weak 'interchange laws' would seem to be related to products of commutators of degeneracies and to the vanishing of commutators of intersections of face-map kernels in some subtle way. This thus turns our attention to the internal structure of simplicial groups as algebraic objects, however that internal structure has been studied relatively little.

The present article is one of a series in which we will study n-types of simplicial groups and the corresponding pairings or higher order braidings using algebraic methods, and will apply the results in various mainly homological settings. The pleasing, and we believe significant, result of this study is that simplicial groups lend themselves very easily to detailed general calculations of these structural maps and thus to a determination of a remarkably rich amount of internal structure. The calculations can be done by hand in low dimensions, but it seems likely that more general computations should be possible using computer aided calculations. A second pleasing conclusion is that these methods not only clarify the connection between simplicial groups and higher order pairings in the related infinity categories, but they also simplify certain calculations within simplicial group theory itself.

In a bit more detail, recall that R. Brown and J.-L. Loday, [7], noted that if the second dimension G_2 of a simplicial group, $\mathbf{G}$, is generated by degenerate elements, that is, elements coming from lower dimensions, then the image of the second term, $\partial_2 NG_2$, of the Moore complex, $(\mathbf{NG}, \partial)$, of $\mathbf{G}$ by the differential ∂ is

$$[\mathrm{Ker} d_1,\ \mathrm{Ker} d_0]$$

where the square brackets as usual denote the commutator subgroup. An easy argument then shows that this subgroup of NG_1 is generated by elements of the form $(s_0d_1(x)ys_0d_1(x)^{-1})(xyx^{-1})^{-1}$, that is the Peiffer elements of the precrossed module $\partial_1 : NG_1 \to NG_0$. Thus it is exactly the Peiffer subgroup of NG_1 the vanishing of which is equivalent to $\partial_1 : NG_1 \to NG_0$ being a crossed module. From the point of view of internal categories in *Groups*, this gives $[\mathrm{Ker} d_1,\ \mathrm{Ker} d_0]$ as the set of interchange law identities, whose vanishing would imply that the internal graph $G_1 \underset{s_0}{\overset{d_1}{\underset{d_0}{\rightrightarrows}}} G_0$ had an internal category structure. We seek higher dimensional analogues of these interchange identities giving the possibility of an interpretation of the structure of the NG_n, $n \geq 2$, and the corresponding categorical results on n-truncated Moore complexes.

In this paper, we generalise Peiffer elements to higher dimensions giving systematic ways of generating them. The methods we use are based on ideas of Conduché, [11] and techniques developed by Carrasco and Cegarra [9]. We recall from [21] and [22] the following:

Let $\mathbf{G}$ be a simplicial group with Moore complex $\mathbf{NG}$ and for $n \geq 0$, let D_n be the subgroup generated by the degenerate elements in dimension n, then

$$NG_n \cap D_n = N_n \text{ for all } n \geq 1.$$

where N_n is the normal subgroup in G_n generated by an explicitly given set of elements.

Alternative interpretations of this intersection and of its image in NG_{n-1} are often needed in applications. We aim to reveal decompositions of this image as a product of commutator subgroups with a hope that this will eventually shed further light on four problem areas:
(i) non-simply connected analogues of Curtis's convergence theorem [12],
(ii) information on and calculation with Samelson and Whitehead products (again see the survey [12] for the description of these for simplicial groups),
(iii) complete descriptions of algebraic models of the n-types of specific families of spaces for low values of n
and, as mentioned above,
(iv) weak interchange identities in higher dimensional weak n- groupoids.

For even reasonably small values of n, such as $n = 5$ or 6, the task is daunting and explicit calculations will almost certainly need the help of computer algebra packages, however for $n = 3$ and 4, a product decomposition generalising the $[\mathrm{Ker}d_0, \mathrm{Ker}d_1]$ description of Brown and Loday [7] is now known. Specifically in this article we show that

$$\partial_n(NG_n \cap D_n) = \prod_{I,J}[K_I, K_J]$$

for $n = 2, 3$ and 4, where $\emptyset \neq I, J \subset [n-1] = \{0, 1, \ldots, n-1\}$ with $I \cup J = [n-1]$ and

$$K_I = \bigcap_{i \in I} \mathrm{Ker}d_i \quad \text{and} \quad K_J = \bigcap_{j \in J} \mathrm{Ker}d_j,$$

and in general, for $n > 4$ we prove

$$\prod_{I,J}[K_I, K_J] \subseteq \partial_n(NG_n \cap D_n).$$

For $n = 2$, this reduces to the Brown-Loday result. We will also give some illustrative examples showing where for $n = 3$, the result relates to constructions of Conduché [11] and Baues [4] and note that these results have been applied by Inassaridze and Inassaridze, [14], in calculations of their non-abelian homology of groups.

(A word of caution needs adding here. As the groups considered are in general non-abelian, a general indexed product of a number of subgroups or of a set of elements is ill-defined unless the elements or subgroups are ordered in some way. In this paper the products of elements are ordered as in the semidirect product decomposition of the simplicial group or sometimes the order is specified explicitly in the formulae; the products of subgroups can easily be seen to be without ambiguity as the subgroups are normal in all instances. Group theoretic products of subgroups will be written by juxtaposition so if X and Y are subgroups of some group G then XY denotes the subgroup generated by the union of X and Y.)

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1. Definition and notation

A simplicial group $\mathbf{G}$ consists of a family of groups $\{G_n\}$ together with face and degeneracy maps $d_i = d_i^n : G_n \to G_{n-1}$, $0 \leq i \leq n$, $(n \neq 0)$ and $s_i = s_i^n : G_n \to G_{n+1}$, $0 \leq i \leq n$, satisfying the usual simplicial identities given in [12] and also [16], [17]. Another essential reference from our point of view is Carrasco's thesis, [8], where many of the basic techniques used here were developed systematically for the first time and the notion of hypercrossed complex was defined.

1.1. THE POSET OF SURJECTIVE MAPS. The following notation and terminology is derived from [8], where it is used both for simplicial groups and simplicial algebras, and the published version, [9], which handles just the group theoretic case.

For the ordered set $[\mathbf{n}] = \{0 < 1 < \cdots < n\}$, let $\alpha_i^n : [\mathbf{n+1}] \to [\mathbf{n}]$ be the increasing surjective map given by

$$\alpha_i^n(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

Let $S(n, n-l)$ be the set of all monotone increasing surjective maps from $[\mathbf{n}]$ to $[\mathbf{n-l}]$. This can be generated from the various α_i^n by composition. The composition of these generating maps satisfies the rule $\alpha_j\alpha_i = \alpha_{i-1}\alpha_j$ with $j < i$. This implies that every element $\alpha \in S(n, n-l)$ has a unique expression as $\alpha = \alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_l}$ with $0 \leq i_1 < i_2 < \cdots < i_l \leq n$, where the indices i_k are the elements of $[\mathbf{n}]$ at which $\alpha(i) = \alpha(i+1)$. We thus can identify $S(n, n-l)$ with the set $\{(i_l, ..., i_1) : 0 \leq i_1 < i_2 < \cdots < i_l \leq n-1\}$. In particular the single element of $S(n,n)$, defined by the identity map on $[\mathbf{n}]$, corresponds to the empty 0-tuple () denoted by $\emptyset_n$. Similarly the only element of $S(n, 0)$ is $(n-1, n-2, \ldots, 0)$. For all $n \geq 0$, let

$$S(n) = \bigcup_{0 \leq l \leq n} S(n, n-l).$$

We say that $\alpha = (i_l, ..., i_1) < \beta = (j_m, ..., j_1)$ in $S(n)$

$$\text{if } i_1 = j_1, \cdots, i_k = j_k \text{ but } i_{k+1} > j_{k+1}\ (k > 0)$$

or

$$\text{if } i_1 = j_1, \cdots, i_l = j_l \text{ and } l < m.$$

This makes $S(n)$ an ordered set. For instance, the orders of $S(2)$, $S(3)$ and $S(4)$ are respectively:

$S(2) = \{\emptyset_2 < (1) < (0) < (1,0)\}$,

$S(3) = \{\emptyset_3 < (2) < (1) < (2,1) < (0) < (2,0) < (1,0) < (2,1,0)\}$,

$S(4) = \{\emptyset_4 < (3) < (2) < (3,2) < (1) < (3,1) < (2,1) < (3,2,1) < (0) < (3,0) < (2,0) < (3,2,0) < (1,0) < (3,1,0) < (2,1,0) < (3,2,1,0)\}$.

If $\alpha, \beta \in S(n)$, we define $\alpha \cap \beta$ to be the set of indices which belong to both α and β.

1.2. The Moore complex. The Moore complex $\mathbf{NG}$ of a simplicial group $\mathbf{G}$ is defined to be the normal chain complex $(\mathbf{NG}, \partial)$ with

$$\mathbf{NG}_n = \bigcap_{i=0}^{n-1} \mathrm{Ker} d_i$$

and with differential $\partial_n : NG_n \to NG_{n-1}$ induced from d_n by restriction.

The Moore complex has the advantage of being smaller than the simplicial group itself and being a chain complex is of a better known form for manipulation. However being non-abelian in general, some new techniques do need developing for its study. Its homology gives the homotopy groups of the simplicial group and thus in specific cases, e.g. a truncated free simplicial resolution of a group, gives valuable higher dimensional information on elements.

The Moore complex, $\mathbf{NG}$, carries a hypercrossed complex structure (see [8] and [9]) which allows the original $\mathbf{G}$ to be rebuilt. We recall briefly some of those aspects of this reconstruction that we will need later.

1.3. The semidirect decomposition of a simplicial group. The fundamental idea behind this can be found in Conduché [11] . A detailed investigation of this for the case of simplicial groups is given in Carrasco and Cegarra [9].

1.4. Lemma. *Let* $\mathbf{G}$ *be a simplicial group. Then* G_n *can be decomposed as a semidirect product:*

$$G_n \cong \ Kerd_0^n \rtimes s_0^{n-1}(G_{n-1})$$

Proof. The isomorphism can be defined as follows:

$$\theta : G_n \to \mathrm{Ker} d_0^n \rtimes s_0^{n-1}(G_{n-1})$$

$$g \mapsto (g s_0 d_0 g^{-1}, s_0 d_0 g).$$

∎

Since we have the isomorphism $G_n \cong \mathrm{Ker}d_0 \rtimes s_0G_{n-1}$, we can repeat this process as often as necessary to get each of the G_n as a multiple semidirect product of degeneracies of terms in the Moore complex. In fact, let $\mathbf{K}$ be the simplicial group defined by

$$K_n = \mathrm{Ker}d_0^{n+1}, \quad d_i^n = d_{i+1}^{n+1}\,|_{\mathrm{Ker}d_0^{n+1}} \quad \text{and} \quad s_i^n = s_{i+1}^{n+1}\,|_{\mathrm{Ker}d_0^{n+1}}\ .$$

Applying Lemma 1.4 above, to G_{n-1} and to K_{n-1}, gives

$$\begin{array}{rcl} G_n & \cong & \mathrm{Ker}d_0 \rtimes s_0G_{n-1} \\ & = & \mathrm{Ker}d_0 \rtimes s_0(\mathrm{Ker}d_0 \rtimes s_0G_{n-2}) \\ & = & K_{n-1} \rtimes (s_0\mathrm{Ker}d_0 \rtimes s_0s_0G_{n-2}). \end{array}$$

Since $\mathbf{K}$ is a simplicial group, we have the following

$$\begin{array}{rcl} \mathrm{Ker}d_0 = K_{n-1} & \cong & \mathrm{Ker}d_0^K \rtimes s_0^K K_{n-2} \\ & = & (\mathrm{Ker}d_1 \cap \mathrm{Ker}d_0) \rtimes s_1\mathrm{Ker}d_0 \end{array}$$

and this enables us to write

$$G_n = ((\mathrm{Ker}d_1^n \cap \mathrm{Ker}d_0^n) \rtimes s_1(\mathrm{Ker}d_0^{n-1})) \rtimes (s_0(\mathrm{Ker}d_0^{n-1}) \rtimes s_0s_0(G_{n-2})).$$

We can thus decompose G_n as follows:

1.5. Proposition. *(cf. [11], p.158) If* $\mathbf{G}$ *is a simplicial group, then for any* $n \geq 0$

$$\begin{array}{rcl} G_n & \cong & (\ldots(NG_n \rtimes s_{n-1}NG_{n-1}) \rtimes \ldots \rtimes s_{n-2}\ldots s_1NG_1) \rtimes \\ & & (\ldots(s_0NG_{n-1} \rtimes s_1s_0NG_{n-2}) \rtimes \ldots \rtimes s_{n-1}s_{n-2}\ldots s_0NG_0). \end{array}$$

■

The bracketing and the order of terms in this multiple semidirect product are generated by the sequence:

$$\begin{array}{rcl} G_1 & \cong & NG_1 \rtimes s_0NG_0 \\ G_2 & \cong & (NG_2 \rtimes s_1NG_1) \rtimes (s_0NG_1 \rtimes s_1s_0NG_0) \\ G_3 & \cong & ((NG_3 \rtimes s_2NG_2) \rtimes (s_1NG_2 \rtimes s_2s_1NG_1)) \rtimes \\ & & ((s_0NG_2 \rtimes s_2s_0NG_1) \rtimes (s_1s_0NG_1 \rtimes s_2s_1s_0NG_0)) \end{array}$$

and

$$\begin{array}{rcl} G_4 & \cong & (((NG_4 \rtimes s_3NG_3) \rtimes (s_2NG_3 \rtimes s_3s_2NG_2)) \rtimes \\ & & ((s_1NG_3 \rtimes s_3s_1NG_2) \rtimes (s_2s_1NG_2 \rtimes s_3s_2s_1NG_1))) \rtimes \\ & & s_0(\text{decomposition of } G_3). \end{array}$$

and correspond to the order in $S(n)$ where the term corresponding to $\alpha = (i_l, \ldots, i_1) \in S(n)$ is $s_\alpha(NG_{n-\#\alpha}) = s_{i_l\ldots i_1}(NG_{n-\#\alpha}) = s_{i_l}\ldots s_{i_1}(NG_{n-\#\alpha})$, where $\#\alpha = l$. Hence any element $x \in G_n$ can be written in the form

$$x = y \prod_{\alpha \in S(n)} s_\alpha(x_\alpha) \qquad \text{with } y \in NG_n \text{ and } x_\alpha \in NG_{n-\#\alpha}.$$

1.6. Crossed modules. Recall that from [24], a crossed module, (M, P, ∂), is a group homomorphism $\partial : M \to P$, together with an action of P on M, written ${}^p m$ for $p \in P$ and $m \in M$, satisfying the following conditions: for all $m, m' \in M$, $p \in P$,

$$CM1: \quad \partial({}^p m) = p\partial(m)p^{-1}$$

$$CM2: \quad {}^{\partial m}m' = mm'm^{-1}.$$

The last condition is called the Peiffer identity.

If a group P acts on M and $\partial : M \longrightarrow P$ satisfies $CM1$ then it is sometimes convenient to refer to (M, P, ∂) as a *precrossed module.* For example making $P = NG_0$ act an $M = NG_1$ via conjugation using s_0 so ${}^m p = s_0(m)ps_0(m)^{-1}$, we get that

$$\partial_1 : NG_1 \longrightarrow NG_0$$

is a precrossed module. In such a context the element

$${}^{\partial m}m' \cdot (mm'm^{-1})^{-1}$$

is called the *Peiffer commutator* of m and m', or more briefly a *Peiffer element.* Of course the vanishing of these Peiffer elements is equivalent to (M, P, ∂) being a crossed module. The subgroup generated by such elements is known as the *Peiffer subgroup* of M for the given precrossed module structure on (M, P, ∂).

Given any precrossed module $\partial : M \to P$, one can form an internal directed graph in the category of groups simply by forming the semidirect product $M \rtimes P$ and taking the source, s, and target, t, to send an element (m, p) to p or $\partial m.p$ respectively. The Peiffer subgroup of M measures the obstruction to the directed graph having an internal category structure. It can easily be seen to be $[\mathrm{Ker}s, \mathrm{Ker}t]$.

2. Peiffer pairings and boundaries in the Moore complex of a simplicial group

The following lemma is noted by Carrasco [8].

2.1. Lemma. *For a simplicial group* $\mathbf{G}$, *if* $0 \leq l \leq n$, *let* $\overline{NG}_n^{(l)} = \bigcap\limits_{i \neq l} \mathrm{Ker} d_i$, *then the mapping* $\varphi : NG_n \longrightarrow \overline{NG}_n^{(l)}$ *in* G_n, *given by*

$$\varphi(g) = g\Big(\prod_{k=0}^{n-l-1} s_{l+k} d_n g^{(-1)^{k+1}}\Big)^{-1},$$

is a bijection. This mapping restricts to give a bijection between $NG_n \cap D_n$ *and* $\overline{NG}_n^{(l)} \cap D_n$.

■

Note that φ is not usually a homomorphism.

2.2. Lemma. *Given a simplicial group* $\mathbf{G}$*, then we have the following:*

$$d_n(NG_n) = d_l(\overline{NG}_n^{(l)}),$$

and

$$d_n(NG_n \cap D_n) = d_l(\overline{NG}_n^{(l)} \cap D_n),$$

Proof. It is easy to see that, for all elements of the form

$$g(\prod_{k=0}^{n-l-1} s_{l+k} d_n g^{(-1)^{k-1}})^{-1}$$

of $\overline{NG}_n^{(l)}$ with $g \in NG_n$, one gets

$$d_l(g(\prod_{k=0}^{n-l-1} s_{l+k} d_n g^{(-1)^{k+1}})^{-1}) = d_n g$$

as required, but by Lemma 2.1 all elements of $\overline{NG}_n^{(l)}$ have this form. ∎

2.3. Proposition. *Let* $\mathbf{G}$ *be a simplicial group. Then for* $n \geq 2$ *and* $I, J \subseteq [\mathbf{n-1}]$ *with* $I \cup J = [\mathbf{n-1}]$

$$[\ \bigcap_{i \in I} \mathrm{Ker} d_i, \bigcap_{j \in J} \mathrm{Ker} d_j\ ] \subseteq \partial_n(NG_n \cap D_n).$$

Proof. For any $I \subset [\mathbf{n-1}], I \neq \emptyset$, let l be the smallest element of I. If $l = 0$, then replace I by J and restart and if $0 \in I \cap J$, then redefine l to be the smallest nonzero element of I. Otherwise proceed as follows. Let $g_0 \in \bigcap_{i \in I} \mathrm{Ker} d_i$ and $g_1 \in \bigcap_{j \in J} \mathrm{Ker} d_j$. One obtains

$$d_i[s_{l-1}g_0, s_l g_1] = 1 \text{ for } i \neq l$$

and hence $[s_{l-1}g_0, s_l g_1] \in \overline{NG}_n^{(l)}$. It follows that

$$[g_0, g_1] = d_l[s_{l-1}g_0, s_l g_1] \in d_l(\overline{NG}_n^{(l)}) = d_n NG_n \quad \text{by the previous lemma,}$$

and this implies

$$[\ \bigcap_{i \in I} \mathrm{Ker} d_i, \bigcap_{j \in J} \mathrm{Ker} d_j\ ] \subseteq \partial_n(NG_n \cap D_n).$$

∎

Writing the abbreviations

$$K_I = \bigcap_{i\in I} \mathrm{Ker} d_i \quad \text{and} \quad K_J = \bigcap_{j\in J} \mathrm{Ker} d_j.$$

then Proposition 2.3 implies

$$\prod_{I,J} [\ K_I, K_J\ ] \subseteq \partial_n(NG_n \cap D_n)$$

for $\emptyset \neq I, J \subset [\mathbf{n-1}]$ and $I \cup J = [\mathbf{n-1}]$.

Example: We illustrate the inclusion of Proposition 2.3 for $n = 2$. We suppose that $x, y \in NG_1 = \mathrm{Ker} d_0$ so that $(x^{-1}(s_0 d_1(x)) \in \mathrm{Ker} d_1$. Note that

$$(s_0d_1(x)ys_0d_1(x)^{-1})(xyx^{-1})^{-1} \cong [x^{-1}s_0d_1(x), y] = d_2[s_1(x)^{-1}s_0(x), s_1(y)]$$

which corresponds to a Peiffer element. These elements vanish for all x, y if and only if $\partial_1 : NG_1 \to NG_0$ is a crossed module. Note that $[\mathrm{Ker} d_0, \mathrm{Ker} d_1] \subseteq \partial_2(NG_2 \cap D_2)$.

2.4. Truncated simplicial groups. A *$k-$truncated simplicial group* is given by similar data to that for a simplicial group, but with G_n given only for $n \leq k$. Given any simplicial group $\mathbf{G}$, one has a natural k-truncated simplicial group obtained by throwing away all elements of dimension strictly greater than k. We denote this by $Tr_k\mathbf{G}$. The following is a result of Conduché [11].

2.5. Corollary. *Let $\mathbf{G}'$ be $(n-1)$-truncated simplicial group. Then there is a simplicial group $\mathbf{G}$ with $Tr_k\mathbf{G} \cong \mathbf{G}'$ if and only if $\mathbf{G}'$ satisfies the following property:*
For all nonempty sets of indices $(I \neq J)$, $I, J \subset [n-1]$ with $I \cup J = [n-1]$,

$$[\ \bigcap_{i\in I} Kerd_i, \bigcap_{j\in J} Kerd_j\ ] = 1.$$

Proof. Since $\partial_n NG'_n = 1$, this follows from Proposition 2.3. ■

2.6. Peiffer pairings generate. In the following we will define a subgroup N_n of G_n. First of all we adapt ideas from Carrasco [8] to get the construction of a useful family of pairings. We define a set $P(n)$ consisting of pairs of elements (α, β) from $S(n)$ with $\alpha \cap \beta = \emptyset_n$ and $\alpha < \beta$, where $\alpha = (i_l, \ldots, i_1), \beta = (j_m, \ldots, j_1) \in S(n)$. It is immediate but important to note that if $(\alpha, \beta) \in P(n)$ then neither α nor β can be $\emptyset_n$. The pairings that we will need,

$$\{F_{\alpha,\beta} : NG_{n-\#\alpha} \times NG_{n-\#\beta} \longrightarrow NG_n : (\alpha, \beta) \in P(n), \quad n \geq 0\}$$

are given as composites by the diagram where

$$\begin{tikzcd}
NG_{n-\#\alpha} \times NG_{n-\#\beta} \arrow[r, "F_{\alpha,\beta}"] \arrow[d, "s_\alpha \times s_\beta"'] & NG_n \\
G_n \times G_n \arrow[r, "\mu"] & G_n \arrow[u, "p"']
\end{tikzcd}$$

where

$$s_\alpha = s_{i_l} \dots s_{i_1} : NG_{n-\#\alpha} \longrightarrow G_n, \quad s_\beta = s_{j_m} \dots s_{j_1} : NG_{n-\#\beta} \longrightarrow G_n,$$

$p : G_n \to NG_n$ is defined by the composite projections $p(z) = p_{n-1} \dots p_0(z)$, where

$$p_j(z) = z s_j d_j(z)^{-1} \qquad \text{with} \quad j = 0, 1, \dots, n-1$$

and $\mu : G_n \times G_n \to G_n$ is given by the commutator map. Thus

$$\begin{aligned} F_{\alpha,\beta}(x_\alpha, y_\beta) &= p\mu(s_\alpha \times s_\beta)(x_\alpha, y_\beta) \\ &= p[s_\alpha(x_\alpha), s_\beta(y_\beta)]. \end{aligned}$$

We now define the normal subgroup N_n of G_n to be that generated by elements of the form

$$F_{\alpha,\beta}(x_\alpha, y_\beta)$$

where $x_\alpha \in NG_{n-\#\alpha}$ and $y_\beta \in NG_{n-\#\beta}$, where $\#\alpha$ is the length of the string α. We illustrate this for $n = 2$ and $n = 3$ to show what this subgroup looks like.

Examples : (a) For $n = 2$, suppose $\alpha = (1)$, $\beta = (0)$ and $x, y \in NG_1 = \mathrm{Ker} d_0$. It follows that

$$\begin{aligned} F_{(0)(1)}(x, y) &= p_1 p_0 [s_0(x), s_1(y)] \\ &= p_1 [s_0(x), s_1(y)] \\ &= [s_0(x), s_1(y)][s_1(y), s_1(x)] \end{aligned}$$

which is thus a typical 'generating element' of the subgroup N_2.

(b) For $n = 3$, the pairings are the following

$$\begin{array}{lll} F_{(1,0)(2)}, & F_{(2,0)(1)}, & F_{(0)(2,1)}, \\ F_{(0)(2)}, & F_{(1)(2)}, & F_{(0)(1)}. \end{array}$$

For all $x \in NG_1$, $y \in NG_2$, the corresponding generators of N_3 are:

$$\begin{aligned} F_{(1,0)(2)}(x, y) &= [s_1 s_0(x), s_2(y)][s_2(y), s_2 s_0(x)] \\ F_{(2,0)(1)}(x, y) &= [s_2 s_0(x), s_1(y)][s_1(y), s_2 s_1(x)][s_2 s_1(x), s_2(y)][s_2(y), s_2 s_0(x)] \end{aligned}$$

and all $y \in NG_2, x \in NG_1$,

$$F_{(0)(2,1)}(x, y) = [s_0(x), s_2 s_1(y)][s_2 s_1(y), s_1(x)][s_2(x), s_2 s_1(y)],$$

whilst for all $x, y \in NG_2$,

$$\begin{aligned} F_{(0)(1)}(x, y) &= [s_0(x), s_1(y)][s_1(y), s_1(x)][s_2(x), s_2(y)], \\ F_{(0)(2)}(x, y) &= [s_0(x), s_2(y)], \\ F_{(1)(2)}(x, y) &= [s_1(x), s_2(y)][s_2(y), s_2(x)]. \end{aligned}$$

The following theorem is proved in [22].

2.7. THEOREM. [22] (THEOREM A) *Let* $\mathbf{G}$ *be a simplicial group and for* $n > 1$, *let* D_n *the subgroup of* G_n *generated by degenerate elements. Let* N_n *be the normal subgroup generated by elements of the form*

$$F_{\alpha,\beta}(x_\alpha, y_\beta) \qquad \text{with } (\alpha,\beta) \in P(n)$$

where $x_\alpha \in NG_{n-\#\alpha}$, $y_\beta \in NG_{n-\#\beta}$. *Then*

$$NG_n \cap D_n = N_n \cap D_n.$$

Proof. [22] ■

In what follows D_n will always be used to denote the subgroup of G_n generated by the degenerate elements.

3. Commutators of kernel elements and boundaries in the Moore complex of a simplicial group

By way of illustration of the potential of the above construction, we look at the case of $n = 2$, collecting up items from earlier discussions.

3.1. CASE $n = 2$. We know that any element g_2 of G_2 can be expressed in the form

$$g_2 = g s_1 y s_0 x s_0 u$$

with $g \in NG_2, x, y \in NG_1$ and $u \in s_0 G_0$. For simplicity we suppose $D_2 = G_2$. For $n = 1$, we take $\alpha = (0)$, $\beta = (1)$ and $x, y \in NG_1 = \mathrm{Ker} d_0$. By example (a), the subgroup N_2 is normally generated by elements of the form

$$F_{(0)(1)}(x, y) = [s_0 x, s_1 y][s_1 y, s_1 x].$$

The image of N_2 by ∂_2 is known to be $[\mathrm{Ker} d_1, \mathrm{Ker} d_0]$ by direct calculation. Indeed,

$$\begin{aligned} d_2[F_{(0)(1)}(x, y)] &= d_2([s_0 x, s_1 y][s_1 y, s_1 x]) \\ &= [s_0 d_1 x, y][y, x] \end{aligned}$$

where $x, y \in \mathrm{Ker} d_0$ and $(x^{-1} s_0 d_1(x)) \in \mathrm{Ker} d_1$ and all elements of $\mathrm{Ker} d_1$ have this form as is checked.

As $\partial_2 = d_2$ restricted to NG_2, this is precisely $d_2(F_{(0)(1)}(x, y))$. In other words the subgroup $\partial_2(N_2)$ is the 'Peiffer subgroup' of the pre-crossed module $\partial : NG_1 \to NG_0$, whose vanishing is equivalent to this being a crossed module and the internal directed graph, $\begin{tikzcd} G_1 \arrow[r, shift left=2, "d_1"] \arrow[r, "d_0"'] & G_0 \arrow[l, shift left=2, "s_0"] \end{tikzcd}$ having an internal category structure. The description of $\partial_2(N_2)$ as $[\mathrm{Ker} d_1, \mathrm{Ker} d_0]$ gives that its vanishing in this situation is module-like behaviour since a P-module, M, corresponds to the trivial case (M, P, ∂) where ∂ is itself the trivial homomorphism in which case this commutator is just $[M,\ M] = 1$. Thus if $(\mathbf{NG}, \partial)$ yields a crossed module, this fact will be reflected in the internal structure of $\mathbf{G}$ by the vanishing of $[\mathrm{Ker}d_1,\ \mathrm{Ker}d_0]$. Because the image of $F_{(0)(1)}(x,y)$ is the Peiffer element determined by x and y, we will call the $F_{\alpha,\beta}(x,y)$ in higher dimensions *higher dimensional Peiffer elements* and will seek similar internal conditions for their vanishing.

We have seen that in all dimensions

$$\prod_{I,J}[K_I, K_J] \subseteq \partial_n(NG_n \cap D_n) = \partial_n(N_n \cap D_n)$$

and we will show shortly that this inclusion is an equality, not only in dimension 2 (as above), but in dimensions 3 and 4 as well. The arguments are calculatory and do not seem to generalise in any obvious way to higher dimensions although similar arguments can be used to get partial results there. For completeness and to introduce notational conventions, we start with the case $n = 2$ in the form to be used later.

Here $\partial_2(N_2)) \subseteq [\mathrm{Ker}d_1,\ \mathrm{Ker}d_0]$. Using similar calculations to those in the example after Proposition 2.3, it is easy to obtain the converse of the equality and so $\partial_2(N_2 \cap D_2) = [\mathrm{Ker}d_1,\ \mathrm{Ker}d_0]$. We can summarise this in the following table

α	β	I, J
(0)	(1)	$\{1\}\ \{0\}$

We write $\partial_2(N_2) = [K_{\{1\}},\ K_{\{0\}}]$ where $K_{\{1\}}$ is $\mathrm{Ker}d_1$ and $K_{\{0\}}$ is $\mathrm{Ker}d_0$.

4. Case $n = 3$

This section provides analogues in dimension 3 of the Peiffer elements. The analogue of the '$[\mathrm{Ker}d_1,\ \mathrm{Ker}d_0]$' result here is:

4.1. Theorem. (Theorem B, case $n = 3$) *In any simplicial group,* $\mathbf{G}$,

$$\partial_3(NG_3 \cap D_3) = \Big(\prod_{I,J}[K_I, K_J]\Big)\ [K_{\{0,2\}}, K_{\{0,1\}}]\ [K_{\{1,2\}}, K_{\{0,1\}}]\ [K_{\{1,2\}}, K_{\{0,2\}}]$$

where $I \cup J = [2],\ I \cap J = \emptyset$ *and* $K_I = \bigcap_{i \in I} Kerd_i$ *for* $I \subsetneq [\mathbf{2}]$.

Proof. By example (b) and Theorem A, we know the generator elements of the normal subgroup N_3 and $\partial_3(N_3) = \partial_3(NG_3 \cap D_3)$. The image of all the listed generator elements of the subgroup N_3 can then be given as in the following table. The correspondence between the various $F_{\alpha,\beta}(x,y)$ and the corresponding parts of the decomposition is given in the second table.

	α	β	I, J
1	(1,0)	(2)	{2}{0,1}
2	(2,0)	(1)	{1}{0,2}
3	(0)	(2,1)	{0}{1,2} {0,2}{0,1}
4	(1)	(2)	{0,2}{0,1}
5	(0)	(2)	{1,2}{0,1} {0,2}{0,1}
6	(0)	(1)	{1,2}{0,1} {0,2}{0,1} {1,2}{0,2}

The explanation of this table is the following:
$\partial_3 F_{\alpha,\beta}(x_\alpha, y_\beta)$ is in $[K_I,\ K_J]$ in the simple cases corresponding to row 1, row 2 and row 4. In row 3, $F_{(0)(2,1)}(x_2, y_1) \in [K_{\{0\}},\ K_{\{1,2\}}]\ [K_{\{0,2\}},\ K_{\{0,1\}}]$ and similarly in row 5 and row 6, the higher Peiffer element is in the product of the indicated $[K_I,\ K_J]$.

To illustrate the sort of argument used we look at the case of $\alpha = (1,0)$ and $\beta = (2)$, i.e row 1, but will omit that for row 4, which is similar. For $x_1 \in NG_1$ and $y_2 \in NG_2$, the corresponding generator of $\partial_3 N_3$ is

$$\begin{aligned} d_3(F_{(1,0)(2)}(x_1,\ y_2)) &= d_3\{[s_1s_0x_1,\ s_2y_2]\ [s_2y_2,\ s_2s_0x_1]\}, \\ &= [s_1s_0d_1x_1,\ y_2]\ [y_2,\ s_0x_1]. \end{aligned}$$

The elements x, y give elements $(s_1s_0d_1(x_1)^{-1}s_0x_1) \in \mathrm{Ker}d_2$ and $y_2^{-1} \in NG_2$ and

$$[s_1s_0d_1(x_1)^{-1}s_0x_1,\ y_2^{-1}] \quad = \quad {}^{y_2^{-1}s_1s_0d_1(x_1)^{-1}}\{[s_1s_0d_1x_1,\ y_2]\ [y_2,\ s_0x_1]\},$$

that is,

$${}^{s_1s_0d_1(x_1)y_2}[s_1s_0d_1(x_1)^{-1}s_0x_1,\ y_2^{-1}] \quad = \quad d_3(F_{(1,0)(2)}(x_1,\ y_2)),$$

so we have

$$d_3(F_{(1,0)(2)}(x_1,\ y_2)) \in [\mathrm{Ker}d_2,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1] = [K_{\{2\}},\ K_{\{0,1\}}].$$

For row 2, $\alpha = (2,0)$, $\beta = (1)$ with $x_1 \in NG_1$, $y_2 \in NG_2$,

$$\begin{aligned} d_3(F_{(2,0)(1)}(x_1,\ y_2)) &= d_3\{[s_2s_0x_1,\ s_1y_2]\ [s_1y_2,\ s_2s_1x_1] \\ &\qquad [s_2s_1x_1,\ s_2y_2]\ [s_2y_2,\ s_2s_0x_1]\}, \\ &= [s_0x_1,\ s_1d_2y_2]\ [s_1d_2y_2,\ s_1x_1]\ [s_1x_1,\ y_2]\ [y_2,\ s_0x_1]. \end{aligned}$$

We have, for $x_1 \in NG_1$, $y_2 \in NG_2$, $(s_0(x_1)^{-1}s_1x_1) \in \mathrm{Ker}d_1$ and $(s_1d_2(y_2)^{-1}y_2) \in \mathrm{Ker}d_0 \cap \mathrm{Ker}d_2$, then

$$\begin{aligned} {}^{s_0x_1s_1d_2y_2}[s_0(x_1)^{-1}s_1x_1,\ s_1d_2(y_2)^{-1}y_2] &= [s_0x_1,\ s_1d_2y_2]\ [s_1d_2y_2,\ s_1x_1] \\ &\qquad [s_1x_1,\ y_2]\ [y_2,\ s_0x_1], \end{aligned}$$

so that

$$ {}^{s_0x_1s_1d_2y_2}[s_0(x_1)^{-1}s_1x_1,\ s_1d_2(y_2)^{-1}y_2] \ = \ d_3(F_{(2,0)(1)}(x_1,\ y_2)) $$

and
$d_3(F_{(2,0)(1)}(x_1,\ y_2)) \in [\mathrm{Ker}d_1,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_2] = [K_{\{1\}},\ K_{\{0,2\}}]$.
Row 3 and row 5 are similar.
For row 3, $\alpha = (0),\ \beta = (2,1)$ with $x_2 \in NG_2, y_1 \in NG_1$. It follows that

$$ \begin{aligned} d_3(F_{(0)(2,1)}(x_2,\ y_1)) &= d_3\{[s_0x_2,\ s_2s_1y_1]\ [s_2s_1y_1,\ s_1x_2]\ [s_2x_2,\ s_2s_1y_1]\}, \\ &= [s_0d_2x_2, s_1y_1]\ [s_1y_1,\ s_1d_2x_2]\ [x_2,\ s_1y_1]. \end{aligned} $$

We can take the following elements for $x_2 \in NG_2,\ y_1 \in NG_1$,
$(x_2s_1d_2(x_2)^{-1}s_0d_2x_2) \in \mathrm{Ker}d_1 \cap \mathrm{Ker}d_2,\ s_1y_1 \in \mathrm{Ker}d_0$ and $s_1d_2x_2(x_2)^{-1} \in \mathrm{Ker}d_0 \cap \mathrm{Ker}d_2$, then

$$ \begin{aligned} {}^{s_1d_2(x_2)x_2^{-1}}\underbrace{[x_2s_1d_2(x_2)^{-1}s_0d_2x_2,\ s_1y_1]}_{k} &= [s_0d_2x_2,\ s_1y_1]\ [s_1y_1,\ s_1d_2x_2] \\ & \quad\ [x_2,\ s_1y_1], \\ {}^{s_1d_2x_2(x_2)^{-1}}k\ [\ s_1d_2x_2(x_2)^{-1},\ [s_1y_1,\ x_2]\ ] &= d_3(F_{(0)(2,1)}(x_2,\ y_1)), \end{aligned} $$

and

$$ d_3(F_{(0)(2,1)}(x_2, y_1)) \in [\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2,\ \mathrm{Ker}d_0]\ [\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1]. $$

For row 6, $\alpha = (1),\ \beta = (0)$ and $x_2,\ y_2 \in NG_2 = \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1$,

$$ \begin{aligned} d_3(F_{(0)(1)}(x_2,\ y_2)) &= d_3\{[s_0x_2,\ s_1y_2]\ [s_1y_2,\ s_1x_2]\ [s_2x_2,\ s_2y_2]\}, \\ &= [s_0d_2x_2,\ s_1d_2y_2]\ [s_1d_2y_2,\ s_1d_2x_2]\ [x_2,\ y_2]. \end{aligned} $$

We can take the following elements $(x_2s_1d_2(x_2)^{-1}s_0d_2x_2) \in \mathrm{Ker}d_1 \cap \mathrm{Ker}d_2$ and $(s_1d_2y_2(y_2)^{-1}) \in \mathrm{Ker}d_0 \cap \mathrm{Ker}d_2$. Forming their commutator gives
$\underbrace{[x_2s_1d_2(x_2)^{-1}s_0d_2x_2,\ y_2^{-1}s_1d_2y_2]}_{m}$

$$ \begin{aligned} = & {}^{x_2s_1d_2(x_2)^{-1}(y_2)^{-1}}\{[y_2,\ s_0d_2x_2]\ [s_0d_2x_2,\ s_1d_2y_2]\ [s_1d_2y_2,\ s_1d_2x_2]\ [x_2,\ y_2]\} \\ & {}^{x_2s_1d_2(x_2)^{-1}}\{[y_2, x_2]\ [s_1d_2x_2, y_2]\}\ {}^{y_2^{-1}}\{[y_2, x_2]\ [x_2,\ s_1d_2y_2]\}, \end{aligned} $$

or

$$ \begin{aligned} & [s_0d_2x_2,\ y_2]\ {}^{y_2s_1d_2x_2x_2^{-1}}\{m\quad {}^{y_2^{-1}}\{[s_1d_2y_2,\ x_2]\ [x_2,\ y_2]\}\}\ [s_1d_2x_2,\ y_2]\ [y_2,\ x_2] \\ & = d_3(F_{(0)(1)}(x_2,\ y_2)), \end{aligned} $$

and

$$ \begin{aligned} d_3(F_{(0)(1)}(x_2,\ y_2)) \ \in\ & [\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1] \\ & [\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1] \\ & [\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_2] \\ & [\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1]. \end{aligned} $$

Thus we have shown

$$\partial_3(NG_3 \cap D_3) \subseteq (\prod_{I,J}[K_I,\ K_J])\ [K_{\{0,2\}},\ K_{\{0,1\}}]\ [K_{\{1,2\}},\ K_{\{0,1\}}]\ [K_{\{1,2\}},\ K_{\{0,2\}}].$$

The opposite inclusion is given by Proposition 2.3. Therefore

$$\begin{array}{rcl}\partial_3(NG_3 \cap D_3) & = & [\mathrm{Ker}d_2,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1]\ [\mathrm{Ker}d_1,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_2] \\ & & [\mathrm{Ker}d_0,\ \mathrm{Ker}d_1 \cap \mathrm{Ker}d_2]\ [\mathrm{Ker}d_0 \cap \mathrm{Ker}d_2,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1] \\ & & [\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1] \\ & & [\mathrm{Ker}d_1 \cap \mathrm{Ker}d_2,\ \mathrm{Ker}d_0 \cap \mathrm{Ker}d_2].\end{array}$$

This completes the proof of the result. ∎

Remark: This result has been used by Inassaridze and Inassaridze, [14] to aid calculation of their non-abelian homology group, $H_3(G, A)$.

5. Illustrative example: 2-crossed modules

The following definition of 2-crossed modules is equivalent to that given by D.Conduché, [11].

5.1. Definition. *A 2-crossed module consists of a complex of groups*

$$L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$$

together with an action of N on L and M, written ${}^n(\ )$, and on N itself by conjugation, so that ∂_2, ∂_1 are morphisms of N-groups, and a N-equivariant function

$$\{\quad ,\quad \} : M \times M \to L$$

called a Peiffer lifting, which satisfies the following axioms:

$$\begin{array}{llrcl} 2CM1: & & \partial_2\{m, m'\} & = & ({}^{\partial_1 m}m')\ (m(m')^{-1}(m)^{-1}), \\ 2CM2: & & \{\partial_2(l), \partial_2(l')\} & = & [l', l], \\ 2CM3: & (i) & \{mm', m''\} & = & {}^{\partial_1 m}\{m', m''\}\{m, m'm''(m')^{-1}\}, \\ & (ii) & \{m, m'm''\} & = & \{m, m'\}\ {}^{mm'(m)^{-1}}\{m, m''\}, \\ 2CM4: & & \{m, \partial_2 l\}\{\partial_2 l, m\} & = & {}^{\partial_1 m}l(l)^{-1}, \\ 2CM5 & & {}^n\{m, m'\} & = & \{{}^n m,\ {}^n m'\}, \end{array}$$

for all $l, l' \in L,\ m, m', m'' \in M$ and $n \in N$.

Here we have used ${}^m l$ as a shorthand for $\{\partial_2 l, m\}l$ in the condition 2CM3(ii) where l is $\{m, m''\}$ and m is $mm'(m)^{-1}$. This gives a new action of M on L. Using this notation, we can split 2CM4 into two pieces, the first of which is tautologous:

$$\begin{array}{llrcl} 2CM4: & (a) & \{\partial_2(l), m\} & = & {}^m(l).l^{-1}, \\ & (b) & \{m, \partial_2(l)\} & = & ({}^{\partial_1 m}l)({}^m(l)^{-1}), \end{array}$$

The old action of M on L via the $N-$action on L is in general distinct from this second action with $\{m, \partial_2(l)\}$ measuring the difference (by 2CM4(b)). An easy argument using 2CM2 and 2CM4(b) shows that with this action, ${}^m l$, of M on L, (L, M, ∂_2) becomes a crossed module.

We denote such a 2-crossed module by $\{L,\ M,\ N,\ \partial_2,\ \partial_1\}$. A morphism of 2-crossed modules is given by a diagram

$$\begin{tikzcd} L \arrow[r, "\partial_2"] \arrow[d, "f_2"'] & M \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & N \arrow[d, "f_0"'] \\ L' \arrow[r, "\partial'_2"'] & M' \arrow[r, "\partial'_1"'] & N' \end{tikzcd}$$

where $f_0\partial_1 = \partial'_1 f_1,\ f_1\partial_2 = \partial'_2 f_2,$

$$f_1({}^n m_1) = {}^{(f_0(n))}f_1(m_1), \quad f_2({}^n l) = {}^{(f_0(n))}f_2(l),$$

and

$$\{\ \ ,\ \ \}f_1 \times f_1 = f_2\{\ \ ,\ \ \},$$

for all $l \in L,\ m_1 \in M,\ n \in N$. These compose in an obvious way. We thus can consider the category of 2-crossed modules denoting it as $\mathfrak{X}_2\mathfrak{Mod}$. Conduché [11] proved that 2-crossed modules give algebraic models of homotopy 3-types.

5.2. Proposition. *Let* $\mathbf{G}$ *be a simplicial group with the Moore complex* $\mathbf{NG}$. *Then the complex of groups*

$$NG_2/\partial_3(NG_3 \cap D_3) \xrightarrow{\overline{\partial}_2} NG_1 \xrightarrow{\partial_1} NG_0$$

with NG_0 *acting via conjugation and the degeneracy maps, is a 2-crossed module, where the Peiffer map is defined as follows:*

$$\begin{array}{rlcl} \{\ \ ,\ \ \}: & NG_1 \times NG_1 & \longrightarrow & NG_2/\partial_3(NG_3 \cap D_3) \\ & (x_0,\ x_1) & \longmapsto & \overline{s_0x_0s_1x_1s_0(x_0)^{-1}s_1x_0s_1(x_1)^{-1}s_1(x_0)^{-1}}. \end{array}$$

Here the right hand side denotes a coset in $NG_2/\partial_3(NG_3 \cap D_3)$ *represented by an element in* NG_2. *The two actions of* NG_1 *on* $NG_2/\partial_3(NG_3 \cap D_3)$ *are given by*
(i) ${}^{\partial_1 m}l$ *corresponds to the action* $s_0(m)ls_0(m)^{-1}$ *via* s_0 *and conjugation;*
(ii) ml *corresponds to the action* $s_1(m)ls_1(m)^{-1}$ *via* s_1 *and conjugation.*

Proof. This is a reformulation of a result of Conduché [11]. Our aim is to show the role of the $F_{\alpha,\beta}$ in the structure. We will show that the axioms of a 2-crossed module are verified. It is readily checked that $s_0(m)ls_0(m)^{-1} = s_1s_0d_0(m)ls_1s_0d_0(m)^{-1}$ so the action "${}^{\partial_1 m}l$" is that via s_0.

In the following calculations we display the elements omitting the overlines:

2CM1:

$$\begin{aligned}
\overline{\partial}_2\{x_0,\ x_1\} &= \partial_2(s_0x_0s_1x_1s_0{x_0}^{-1}s_1x_0s_1{x_1}^{-1}s_1{x_0}^{-1}),\\
&= s_0d_1(x_0)x_1s_0d_1(x_0)^{-1}x_0(x_1)^{-1}(x_0)^{-1},\\
&= {}^{\partial_1(x_0)}x_1 \quad x_0(x_1)^{-1}(x_0)^{-1}.
\end{aligned}$$

2CM2:
From $\partial_3(F_{(1)(0)}(x,\ y)) = [s_0d_2x,\ s_1d_2y]\ [s_1d_2y,\ s_1d_2x]\ [x,\ y]$, one obtains

$$\begin{aligned}
\{\overline{\partial}_2(x),\ \overline{\partial}_2(y)\} &= s_0d_2(x)s_1d_2(y)s_0d_2(x)^{-1}s_1d_2(x)s_1d_2(y)^{-1}s_1d_2(x)^{-1},\\
&\equiv [y,\ x] \mod \partial_3(NG_3\cap D_3).
\end{aligned}$$

2CM3: (i)

$$\begin{aligned}
\{x_0x_1,\ x_2\} &= s_0(x_0)s_0(x_1)s_1(x_2)s_0(x_1)^{-1}s_0(x_0)^{-1}s_1(x_0)s_1(x_1)\\
&\quad s_1(x_2)^{-1}s_1(x_1)^{-1}s_1(x_0)^{-1},\\
&= (s_0(x_0)s_0(x_1)s_1(x_2)s_0(x_1)^{-1}s_1(x_1)s_1(x_2)^{-1}s_1(x_1)s_0(x_1)^{-1})\\
&\quad (s_0(x_0)s_1(x_1)s_1(x_2)s_1(x_1)^{-1}s_0(x_0)^{-1}s_1(x_0)s_1(x_1)\\
&\quad\quad s_1(x_2)^{-1}s_1(x_1)^{-1}s_1(x_0)^{-1}),\\
&\equiv {}^{\partial_1(x_0)}\{x_1,\ x_2\}\ \{x_0,\ x_1x_2(x_1)^{-1}\}.
\end{aligned}$$

(ii)

$$\begin{aligned}
\{x_0,\ x_1x_2\} &= s_0(x_0)s_1(x_1)s_1(x_2)s_0(x_0)^{-1}s_1(x_0)s_1(x_2)^{-1}s_1(x_1)^{-1}s_1(x_0)^{-1},\\
&= (s_0(x_0)s_1(x_1)s_0(x_0)^{-1}s_1(x_0)s_1(x_1)^{-1}s_1(x_0)^{-1})\\
&\quad s_1(x_0)s_1(x_1)s_1(x_0)^{-1}(s_0(x_0)s_1(x_2)s_0(x_0)^{-1}\\
&\quad\quad s_1(x_0)s_1(x_2)^{-1}s_1(x_0)^{-1})s_1(x_0)s_1(x_1)^{-1}s_1(x_0)^{-1},\\
&= \{x_0,\ x_1\}\ {}^{x_0x_1(x_0)^{-1}}\{x_0,\ x_2\}.
\end{aligned}$$

2CM4: (a)
From $\partial_3(F_{(0)(2,1)}(y,\ x)) = [s_0d_2y,\ s_1x_0]\ [s_1x_0,\ s_1d_2y]\ [y,\ s_1x_0] \in \partial_3(NG_3\cap D_3)$,

$$\begin{aligned}
\{\overline{\partial}_2(y),\ x_0\} &\equiv [s_1(x_0), y] \mod \partial_3(NG_3\cap D_3),\\
&= s_1(x_0)ys_1(x_0)^{-1}y^{-1},\\
&= {}^{x_0}y\ y^{-1}, \quad \text{by the definition of the action.}
\end{aligned}$$

This justifies claim (ii) of the statement of the proposition. (b) Since

$$\partial_3(F_{(2,0)(1)}(x_0,\ y)) = [s_0x_0,\ s_1d_2y]\ [s_1d_2y,\ s_1x_0]\ [s_1x_0,\ y]\ [y,\ s_0x_0],$$

$$\begin{aligned}
\{x_0,\ \overline{\partial}_2(y)\} &= [s_0(x_0),\ y]\ [y,\ s_1(x_0)],\\
&\equiv (x_0\cdot y)\ {}^{x_0}y^{-1} \mod \partial_3(NG_3\cap D_3),
\end{aligned}$$

or using the original form:

2CM4:

$$\begin{aligned} \{x_0,\ \overline{\partial}_2(y)\}\ \{\overline{\partial}_2(y),\ x_0\} &= (x_0\cdot y)\ {}^{x_0}y^{-1}\ {}^{x_0}y\ y^{-1},\\ &= {}^{\partial_1(x_0)}y\ y^{-1}. \end{aligned}$$

2CM5:

$$\begin{aligned} {}^{n}\{x_0,\ x_1\} &= s_1s_0(n)s_0x_0s_1x_1s_0x_0{}^{-1}s_1s_0(n)^{-1}\\ &\qquad s_1s_0(n)s_1x_0s_1x_1{}^{-1}s_1x_0{}^{-1}s_1s_0(n)^{-1},\\ &= \{\ {}^{n}x_0,\ \ {}^{n}x_1\}, \end{aligned}$$

here with $x,\ y \in NG_2/\partial_3(NG_3\cap D_3)$, $x_0,\ x_1,\ x_2 \in NG_1$ and $n \in NG_0$.

This completes the proof of the proposition. ∎

This only used the higher dimension Peiffer elements. A result in terms of the vanishing of the $[K_I,\ K_J]$ can also be given:

5.3. Proposition. *If in a simplicial group* $\mathbf{G}$*, one has* $[K_I,\ K_J] = 1$ *in dimension 2 for the following cases:* $I\cup J = [2]$*,* $I\cap J = \emptyset$*;* $I = \{0,1\}$*,* $J = \{0,2\}$ *or* $I = \{1,2\}$*; and* $I = \{0,2\}$*,* $J = \{1,2\}$*, then*

$$NG_2 \longrightarrow NG_1 \longrightarrow NG_0$$

can be given the structure of a 2-crossed module. ∎

Note that any 2-truncated simplicial group satisfies this condition, but many other simplicial groups may also satisfy it. Since Theorem A (that is 2.5) describes NG_2 in many instances, these results do provide a hope for calculating invariants of 3-types.

Remarks: (i) In [4] and [5] Baues introduces a related notion of quadratic module. As in the above, one has a complex

$$L \xrightarrow{\delta} M \xrightarrow{\partial} N$$

of groups with actions but the Peiffer lifting is replaced by a pairing

$$\omega : C\otimes C \longrightarrow L$$

where C is the abelianisation of the quotient group $M/P_2(\partial)$ where P_2 is the Peiffer subgroup of the precrossed module (M,N,∂). Baues, [4] and [5], gives a construction of a quadratic module from a simplicial group. Again we can use the $F_{\alpha,\beta}$'s in verifying the axioms. Quadratic modules have 'nilpotency degree two' in as much as the triple Peiffer commutators are trivial. Quadratic modules are thus 'nilpotent' algebraic modules of 3-types. In [5], Baues points out that a 'nilpotent' algebraic model for 4-types is not known. Preliminary attempts by various authors suggest that the structure of 3-truncated Moore complexes (3-crossed modules) goes some way towards that aim, but the nilpotency must allow for the 6 pairings used above, and any analogue of Baues' construction for 4-types will be likely to need information on d_4NG_4 to which we turn in the next section.

(ii) A braiding on a monoidal category relates the two tensor products $A\otimes B$ and $B\otimes A$, satisfying various compatibility conditions with the structure maps of the monoidal structure, (cf. Joyal and Street, [15]). Any crossed module determines an internal category in groups and hence a strict monoidal category, in fact a strict categorical group. The concept of a braiding allows one to weaken the strictness. Braided regular crossed modules were introduced by Brown and Gilbert, [6], whilst braided categorical groups were studied by Carrasco and Cegarra, [10]. The braiding in the strict categorical group associated to a simplicial group, G, is given in terms of the above Peiffer pairing, $\{\ ,\ \}$, (cf. p.4009 of [10]).

(iii) For comparison with Conduché's formulation in [11], we have written ${}^{m}l$ where he writes $m \cdot l$ and his Peiffer lifting is

$$\{m_1, m_2\} = [s_1m_1, s_1m_2][s_1m_2, s_0m_1],$$

so our formulae are often the reverse of his.

6. The case $n = 4$

With dimension 4, the situation is more complicated, but is still manageable.

6.1. Theorem. (Theorem B: case $n = 4$) *In a simplicial group,* $\mathbf{G}$,

$$\partial_4(NG_4 \cap D_4) = \prod_{I,J}[K_I,\ K_J]$$

where $I \cup J = [3],\ I = [3] - \{\alpha\},\ J = [3] - \{\beta\}$ *and* $(\alpha, \beta) \in P(4)$.

Proof. There is a natural isomorphism

$$\begin{aligned} G_4 \ \cong\ & NG_4 \rtimes s_3NG_3 \rtimes s_2NG_3 \rtimes s_3s_2NG_2 \rtimes s_1NG_3 \rtimes \\ & s_3s_1NG_2 \rtimes s_2s_1NG_2 \rtimes s_3s_2s_1NG_1 \rtimes s_0NG_3 \rtimes \\ & s_3s_0NG_2 \rtimes s_2s_0NG_2 \rtimes s_3s_2s_0NG_1 \rtimes \\ & s_1s_0NG_2 \rtimes s_3s_1s_0NG_1 \rtimes s_2s_1s_0NG_1 \rtimes s_3s_2s_1s_0NG_0. \end{aligned}$$

We firstly see what the generator elements of the normal subgroup N_4 look like. For $n = 4$, one gets

$$\begin{aligned} S(4) \ =\ & \{\emptyset_4 < (3) < (2) < (3,2) < (1) < (3,1) < (2,1) < (3,2,1) < (0) < \\ & (3,0) < (2,0) < (3,2,0) < (1,0) < (3,1,0) < (2,1,0) < (3,2,1,0)\}. \end{aligned}$$

The key pairings are thus the following:

$$\begin{array}{llll} F_{(0)(3,2,1)}, & F_{(3,2,0)(1)}, & F_{(3,1,0)(2)}, & F_{(2,1,0)(3)}, \\ F_{(3,0)(2,1)}, & F_{(2,0)(3,1)}, & F_{(1,0)(3,2)}, & F_{(1)(3,2)}, \\ F_{(0)(3,2)}, & F_{(0)(3,1)}, & F_{(0)(2,1)}, & F_{(3,1)(2)}, \\ F_{(2,1)(3)}, & F_{(3,0)(2)}, & F_{(3,0)(1)}, & F_{(2,0)(3)}, \\ F_{(2,0)(1)}, & F_{(1,0)(3)}, & F_{(1,0)(2)}, & F_{(2)(3)}, \\ F_{(1)(3)}, & F_{(0)(3)}, & F_{(1)(2)}, & F_{(0)(2)}, \\ F_{(0)(1)}. & & & \end{array}$$

For $x_1,\ y_1 \in NG_1,\ x_2,\ y_2 \in NG_2$ and $x_3,\ y_3 \in NG_3$, the generator elements of the normal subgroup N_4 are

$$
\begin{array}{rrcl}
1) & F_{(0)(3,2,1)}(x_3,\ y_1) & = & [s_0x_3,\ s_3s_2s_1y_1]\ [s_3s_2s_1y_1,\ s_1x_3] \\
 & & & [s_2x_3,\ s_3s_2s_1y_1]\ [s_3s_2s_1y_1,\ s_3x_3], \\
2) & F_{(3,2,0)(1)}(x_1,\ y_3) & = & [s_3s_2s_0x_1,\ s_1y_3]\ [s_1y_3,\ s_3s_2s_1x_1] \\
 & & & [s_3s_2s_1x_1,\ s_2y_3]\ [s_2y_3,\ s_3s_2s_0x_1] \\
 & & & [s_3s_2s_0x_1,\ s_3y_3]\ [s_3y_3,\ s_3s_2s_1x_1], \\
3) & F_{(3,1,0)(2)}(x_1,\ y_3) & = & [s_3s_1s_0x_1,\ s_2y_3]\ [s_2y_3,\ s_3s_2s_0x_1] \\
 & & & [s_3s_2s_0x_1,\ s_3y_3]\ [s_3y_3,\ s_3s_1s_0x_1], \\
4) & F_{(2,1,0)(3)}(x_1,\ y_3) & = & [s_2s_1s_0x_1,\ s_3y_3]\ [s_3y_3,\ s_3s_1s_0x_1], \\
5) & F_{(3,0)(2,1)}(x_2,\ y_2) & = & [s_3s_0x_2,\ s_2s_1y_2]\ [s_2s_1y_2,\ s_3s_1x_2] \\
 & & & [s_3s_2x_2,\ s_2s_1y_2]\ [s_3s_1y_2,\ s_3s_2x_2] \\
 & & & [s_3s_1x_2,\ s_3s_1y_2]\ [s_3s_1y_2,\ s_3s_0x_2], \\
6) & F_{(2,0)(3,1)}(x_2,\ y_2) & = & [s_2s_0x_2,\ s_3s_1y_2]\ [s_3s_1y_2,\ s_2s_1x_2] \\
 & & & [s_2s_1x_2,\ s_3s_2y_2]\ [s_3s_2y_2,\ s_2s_0x_2] \\
 & & & [s_3s_0x_2,\ s_3s_2y_2]\ [s_3s_2y_2,\ s_3s_1x_2] \\
 & & & [s_3s_1x_2,\ s_3s_1y_2]\ [s_3s_1y_2,\ s_3s_0x_2], \\
7) & F_{(1,0)(3,2)}(x_2,\ y_2) & = & [s_1s_0x_2,\ s_3s_2y_2]\ [s_3s_2y_2,\ s_2s_0x_2] \\
 & & & [s_3s_0x_2,\ s_3s_2y_2], \\
8) & F_{(1)(3,2)}(x_3,\ y_2) & = & [s_1x_3,\ s_3s_2y_2]\ [s_3s_2y_2,\ s_2x_3] \\
 & & & [s_3x_3,\ s_3s_2y_2], \\
9) & F_{(0)(3,2)}(x_3,\ y_2) & = & [s_0x_3,\ s_3s_2y_2], \\
10) & F_{(0)(3,1)}(x_3,\ y_2) & = & [s_0x_3,\ s_3s_1y_2]\ [s_3s_1y_2,\ s_1x_3] \\
 & & & [s_2x_3,\ s_3s_2y_2]\ [s_3s_2y_2,\ s_3x_2], \\
11) & F_{(0)(2,1)}(x_3,\ y_2) & = & [s_0x_3,\ s_2s_1y_2]\ [s_2s_1y_2,\ s_1x_3] \\
 & & & [s_2x_3,\ s_2s_1y_2]\ [s_3s_1y_2,\ s_3x_3], \\
12) & F_{(3,1)(2)}(x_2,\ y_3) & = & [s_3s_1x_2,\ s_2y_3]\ [s_2y_3,\ s_3s_2x_2] \\
 & & & [s_3s_2x_2,\ s_3y_3]\ [s_3y_3,\ s_3s_1x_2], \\
13) & F_{(2,1)(3)}(x_2,\ y_3) & = & [s_2s_1x_2,\ s_3y_3]\ [s_3y_3,\ s_3s_1x_2], \\
14) & F_{(3,0)(2)}(x_2,\ y_3) & = & [s_3s_0x_2,\ s_2y_3]\ [s_3y_3,\ s_3s_0x_2], \\
15) & F_{(3,0)(1)}(x_2,\ y_3) & = & [s_3s_0x_2,\ s_1y_3]\ [s_1y_3,\ s_3s_1x_2] \\
 & & & [s_3s_2x_2,\ s_2y_3]\ [s_3y_3,\ s_3s_2x_2], \\
16) & F_{(2,0)(3)}(x_2,\ y_3) & = & [s_2s_0x_2,\ s_3y_3]\ [s_3y_3,\ s_3s_0x_2], \\
17) & F_{(2,0)(1)}(x_2,\ y_3) & = & [s_2s_0x_2,\ s_1y_3]\ [s_1y_3,\ s_2s_1x_2] \\
 & & & [s_2s_1x_2,\ s_2y_3]\ [s_2y_3,\ s_2s_0x_2] \\
 & & & [s_3s_0x_2,\ s_3y_3]\ [s_3y_3,\ s_3s_1x_2], \\
18) & F_{(1,0)(3)}(x_2,\ y_3) & = & [s_1s_0x_2,\ s_3y_3], \\
19) & F_{(1,0)(2)}(x_2,\ y_3) & = & [s_1s_0x_2,\ s_2y_3]\ [s_2y_3,\ s_2s_0x_2] \\
 & & & [s_3s_0x_2,\ s_3y_3], \\
20) & F_{(2)(3)}(x_3,\ y_3) & = & [s_2x_3,\ s_3y_3]\ [s_3y_3,\ s_3x_3], \\
21) & F_{(1)(3)}(x_3,\ y_3) & = & [s_1x_3,\ s_3y_3], \\
22) & F_{(0)(3)}(x_3,\ y_3) & = & [s_0x_3,\ s_3y_3], \\
23) & F_{(1)(2)}(x_3,\ y_3) & = & [s_1x_3,\ s_2y_3]\ [s_2y_3,\ s_2x_3] \\
 & & & [s_3x_3,\ s_3y_3], \\
24) & F_{(0)(2)}(x_3,\ y_3) & = & [s_0x_3,\ s_2y_3], \\
25) & F_{(0)(1)}(x_3,\ y_3) & = & [s_0x_3,\ s_1y_3]\ [s_1y_3,\ s_1x_3] \\
 & & & [s_2x_3,\ s_2y_3]\ [s_3y_3,\ s_3x_3].
\end{array}
$$

By Theorem A, we have $\partial_4(NG_4 \cap D_4) = \partial_4(N_4)$. We take an image by ∂_4 of each $F_{\alpha,\beta}$, where $\alpha, \beta \in P(4)$. We summarise the image of all generator elements, which are listed early on, in the subsequent table.

	α	β	I, J
1	(0)	(3,2,1)	{1,2,3}{0} {0,2,3}{0,1,3}
2	(3,2,0)	(1)	{1}{0,2,3} {0,1,3}{0,1,2}
3	(3,1,0)	(2)	{2}{0,1,3}
4	(2,1,0)	(3)	{3}{0,1,2}
5	(3,0)	(2,1)	{0,3}{1,2} {0,2}{0,1,3}
6	(2,0)	(3,1)	{1,3}{0,2} {0,3}{0,1,2}
7	(1,0)	(3,2)	{2,3}{0,1} {3}{0,1,2}
8	(1)	(3,2)	{0,2,3}{0,1} {0,1,3}{0,1,2}
9	(0)	(3,2)	{0,1}{0,2,3} {1,2,3}{0,1}
10	(0)	(3,1)	{1,2,3}{0,2} {0,1}{1,2,3} {0,1,3}{0,2}
11	(0)	(2,1)	{1,2,3}{0,3} {1,2,3}{0} {0,2}{0,1,3} {0,1,2}{0,3}
12	(3,1)	(2)	{0,2}{0,1,3}
13	(2,1)	(3)	{0,3}{0,1,2}
14	(3,0)	(2)	{1,2}{0,1,3} {0,1,3}{0,2}
15	(3,0)	(1)	{1,2}{0,2,3} {0,1,3}{1,2} {0,2,3}{0,1} {0,1,3}{0,1,2}
16	(2,0)	(3)	{1,3}{0,1,2} {0,3}{0,1,2}
17	(2,0)	(1)	{1,3}{0,2,3} {0,1,2}{1,3} {0,2,3}{1} {3}{0,1,2}
18	(1,0)	(3)	{2,3}{0,1,2} {3}{0,1,2}
19	(1,0)	(2)	{2,3}{0,1,3} {0,1,2}{2,3} {3}{0,1,2} {0,2}{0,1,3} {0,1,3}{1,2}
20	(2)	(3)	{0,1,3}{0,1,2}
21	(1)	(3)	{0,2,3}{0,1,2} {0,1,3}{0,1,2}
22	(0)	(3)	{1,2,3}{0,1,2} {0,2,3}{0,1,2} {0,1,3}{0,1,2}
23	(1)	(2)	{0,2,3}{0,1,3} {0,1,2}{0,2,3} {0,1,3}{0,1,2}
24	(0)	(2)	{1,2,3}{0,1,3} {0,1,3}{0,2,3} {1,2,3}{0,1,2} {0,2,3}{0,1,2}
			{0,1,3}{0,1,2}
25	(0)	(1)	{1,2,3}{0,2,3} {0,1,3}{1,2,3} {0,2,3}{0,1,2} {0,1,3}{0,1,2}
			{0,2}{0,1,3} {0,2,3}{0,1,3} {0,1,2}{0,2,3}

As the proofs are largely similar to those for $n = 3$ we leave most to reader, limiting ourselves to one or two of the more complex cases by way of illustration.

Row: 1

$$\begin{aligned} d_4(F_{(0)(3,2,1)}(x_3,\ y_1)) &= [s_0d_3x_3,\ s_2s_1y_1]\,[s_2s_1y_1,\ s_1d_3x_3] \\ &\quad [s_2d_3x_3,\ s_2s_1y_1]\,[s_2s_1y_1,\ x_3]. \end{aligned}$$

Take elements $(x_2s_1d_2(x_2)^{-1}s_0d_2x_2) \in \mathrm{Ker}d_1 \cap \mathrm{Ker}d_2$ and
$(s_1d_2y_2(y_2)^{-1}) \in \mathrm{Ker}d_0 \cap \mathrm{Ker}d_2$,
so

$$\begin{aligned} d_4(F_{(0)(3,2,1)}(x_3,\ y_1)) &= {}^{a^{-1}}\{A\,[s_2s_1y_1,\ x_3^{-1}s_2d_3x_3]\} \\ &\quad [s_2d_3x_3,\ s_2s_1y_1][s_2s_1y_1,\ x_3] \end{aligned}$$

then we have

$$d_4(F_{(0)(3,2,1)}(x_3,\ y_1)) \ \in \ [K_{\{1,2,3\}},\ K_{\{0\}}]\,[K_{\{0,2,3\}},\ K_{\{0,1,3\}}],$$

where $[x_3^{-1}s_2d_3x_3s_1d_3(x_3)^{-1}s_0d_3x_3,\ s_2s_1y_1] = A$ and
$x_3^{-1}s_2d_3x_3s_1d_3(x_3)^{-1} = a$.

Row: 2

$$\begin{aligned} d_4(F_{(3,2,0)(1)}(x_1,\ y_3)) \ &= \ [s_2s_0x_1,\ s_1d_3y_3]\,[s_1d_3y_3,\ s_2s_1x_1] \\ &\quad [s_2s_1x_1,\ s_2d_3y_3]\,[s_2d_3y_3,\ s_2s_0x_1] \\ &\quad [s_2s_0x_1,\ y_3]\,[y_3,\ s_2s_1x_1]. \end{aligned}$$

Let $a,\ b,\ c,\ d,\ e$ be $s_2s_0x_1,\ s_1d_3y_3,\ s_2s_1x_1,\ s_2d_3y_3,\ y_3$ respectively and giving elements $(c^{-1}a) \in K_{\{1\}}$, $(ed^{-1}b) \in K_{\{0,2,3\}}$, $(ade^{-1}a^{-1}) \in K_{\{0,1,3\}}$ and
${}^{a}[a^{-1}c,\ e] \in K_{\{0,1,2\}}$. Then

$$\begin{aligned} d_4(F_{(3,2,0)(1)}(x_1,\ y_3)) \ &= \ {}^{ade^{-1}a^{-1}c}[c^{-1}a,\ ed^{-1}b]^{ade^{-1}}[a^{-1}c,\ e]\,[a,\ e]\,[e,\ c] \\ &= \ {}^{ade^{-1}a^{-1}c}[c^{-1}a,\ ed^{-1}b]\,[ade^{-1}a^{-1}, {}^{a}[a^{-1}c, e]\,], \end{aligned}$$

so it follows that

$$d_4(F_{(3,2,0)(1)}(x_1,\ y_3)) \ \in \ [K_{\{1\}},\ K_{\{0,2,3\}}]\,[K_{\{0,1,3\}},\ K_{\{0,1,2\}}].$$

Row: 10

$$\begin{aligned} d_4(F_{(0)(3,1)}(x_3,\ y_2)) \ &= \ [s_0d_3x_3,\ s_1y_2]\,[s_1y_2,\ s_1d_3x_3] \\ &\quad [s_2d_3x_3,\ s_2y_2]\,[s_2y_2,\ x_3]. \end{aligned}$$

Set $a = s_0d_3x_3,\ b = s_1y_2,\ c = s_1d_3x_3,\ d = s_2d_3x_3,\ e = s_2y_2,\ f = x_3$, and take elements $(f^{-1}dc^{-1}a) \in K_{\{1,2,3\}}$, $(e^{-1}b) \in K_{\{0,2\}}$, $e^{-1} \in K_{\{0,1\}}$,
$(f^{-1}dc^{-1}a) \in K_{\{1,2,3\}}$, $(f^{-1}d) \in K_{\{0,1,3\}}$, $(cd^{-1}) \in K_{\{0,2\}}$
and $[\ f^{-1}d,\ e\ ] \in [K_{\{0,1,3\}},\ K_{\{0,1\}}]$ then

$$\begin{aligned} d_4(F_{(0)(3,1)}(x_3,\ y_2)) \ &= \ {}^{cd^{-1}f}\{\ [\ f^{-1}dc^{-1}a,\ e^{-1}b\ ]\ {}^{b}[\ e^{-1},\ f^{-1}dc^{-1}a\ ]\ \} \\ &\quad {}^{cd^{-1}fb}[\ f^{-1}d,\ b^{-1}e\ ]\ [\ cd^{-1}\ [\ f^{-1}d,\ e\ ]\ ] \end{aligned}$$

so we get

$$\begin{aligned} d_4(F_{(0)(3,1)}(x_3,\ y_2)) \ &\in \ [K_{\{1,2,3\}},\ K_{\{0,2\}}]\,[K_{\{0,1\}},\ K_{\{1,2,3\}}] \\ &\quad [K_{\{0,1,3\}},\ K_{\{0,2\}}]\,[K_{\{0,2\}},\ K_{\{0,1,3\}}]. \end{aligned}$$

Row: 15

$$\begin{aligned} d_4(F_{(3,0)(1)}(x_2,\ y_3)) \ &= \ [\ s_0x_2,\ s_1d_3y_3\ ]\ [\ s_1d_3y_3,\ s_1x_2\ ] \\ &\quad [\ s_2x_2,\ s_2d_3y_2\ ]\ [\ y_3,\ s_2x_2\ ]. \end{aligned}$$

Let $a = s_0x_2, b = s_1d_3y_3, c = s_1x_2, d = s_2x_2, e = s_2d_3y_3, f = y_3$, and take elements $(dc^{-1}a) \in K_{\{1,2\}}$, $(be^{-1}f) \in K_{\{0,2,3\}}$, $(e^{-1}f) \in K_{\{0,1,3\}}$, $[d,\ b] \in K_{\{0,1,3\}}$, $d \in K_{\{0,1\}}$, $(ef^{-1}) \in K_{\{0,1,3\}}$ and $[d, f] \in K_{\{0,1,2\}}$ then

$$\begin{aligned} d_4(F_{(3,0)(1)}(x_2,\ y_3)) \ &= \ {}^{a^{-1}cd^{-1}}([dc^{-1}a,\ be^{-1}f\ ]\ {}^{b}[e^{-1}f,\ dc^{-1}a]) \\ &\quad [\ a^{-1}cd^{-1},\ [d,\ b]\ ]\ \ {}^{e^{-1}f}[\ fe^{-1}b,\ d]\ [ef^{-1},\ [d, f]\ ] \end{aligned}$$

and one gets

$$\begin{array}{rcl} d_4(F_{(3,0)(1)}(x_2,\ y_3)) & \in & [K_{\{1,2\}},\ K_{\{0,2,3\}}]\,[K_{\{0,1,3\}},\ K_{\{1,2\}}] \\ & & [K_{\{1,2\}},\ K_{\{0,1,3\}}]\,[K_{\{0,2,3\}},\ K_{\{0,1\}}] \\ & & [K_{\{0,1,3\}},\ K_{\{0,1,2\}}]. \end{array}$$

Row: 17

$$\begin{array}{rcl} d_4(F_{(2,0)(1)}(x_2,\ y_3)) & = & [\ s_2s_0d_2x_2,\ s_1d_3y_3\ ]\,[\ s_1d_3y_3,\ s_2s_1d_2x_2\ ] \\ & & [\ s_2s_1d_2x_2,\ s_2d_3y_3\ ]\,[\ s_2d_3y_3,\ s_2s_0d_2x_2\ ] \\ & & [\ s_0x_2,\ y_3\ ]\,[\ y_3,\ s_1x_2\ ]. \end{array}$$

Let $a = s_2s_0d_2x_2, b = s_1d_3y_3, c = s_2s_1d_2x_2, d = s_2d_3y_3, e = s_0x_2, f = y_3, g = s_1x_2$, giving elements $(g^{-1}ea^{-1}c) \in K_{\{1,3\}}$, $(b^{-1}df^{-1}) \in K_{\{0,2,3\}}$, $f^{-1} \in K_{\{0,1,2\}}$, $(g^{-1}e) \in K_{\{1\}}$, $(abe^{-1}) \in K_{\{3\}}$ and $([g,\ f]\ [f,e]) \in K_{\{0,1,2\}}$, then

$$\begin{array}{rcl} d_4(F_{(2,0)(1)}(x_2,\ y_3)) & = & {}^{abe^{-1}gf}(\ [g^{-1}ea^{-1}c,\ b^{-1}df^{-1}\ ]\ {}^{b^{-1}d}[\ f^{-1},\ g^{-1}ea^{-1}c\ ] \\ & & {}^{f}[b^{-1}df^{-1},\ g^{-1}e]\ )[\ abe^{-1},\ [g,\ f]\ [f,e]\ ] \end{array}$$

so we have

$$\begin{array}{rcl} d_4(F_{(2,0)(1)}(x_2,\ y_3)) & \in & [K_{\{1,3\}},\ K_{\{0,2,3\}}]\,[K_{\{0,1,2\}},\ K_{\{1,3\}}] \\ & & [K_{\{0,2,3\}},\ K_{\{1\}}]\,[K_{\{3\}},\ K_{\{0,1,2\}}]. \end{array}$$

Row: 19

$$\begin{array}{rcl} d_4(F_{(1,0)(2)}(x_2,\ y_3)) & = & [\ s_1s_0d_2x_2,\ s_2d_3y_3\ ]\,[\ s_2d_3y_3,\ s_2s_0d_2x_2\ ] \\ & & [\ s_0x_2,\ y_3\ ]. \end{array}$$

Let $a = s_1s_0d_2x_2, b = s_2d_3y_3, c = s_2s_0d_2x_2, d = s_0x_2, e = y_3, f = s_1x_2, g = s_2y_3$, and take elements $(dc^{-1}a) \in K_{\{2,3\}}$, $(be^{-1}) \in K_{\{0,1,3\}}$, $e^{-1} \in K_{\{0,1,2\}}$, $(cd^{-1}) \in K_{\{3\}}$, $[b,\ d] \in K_{\{0,1,2\}}$, $df^{-1}g \in K_{\{1,2\}}$ and $gf^{-1} \in K_{\{1,2\}}$, then

$$\begin{array}{rcl} d_4(F_{(1,0)(2)}(x_2,\ y_3)) & = & {}^{cd^{-1}}([dc^{-1}a,\ be^{-1}]\ {}^{b}[\ e^{-1},\ dc^{-1}a]) \\ & & [cd^{-1},\ [b,\ d]\ ]\ {}^{ded^{-1}g^{-1}f}\{[e^{-1}b,\ df^{-1}g]\ [gf^{-1},\ e^{-1}b]\} \end{array}$$

so

$$\begin{array}{rcl} d_4(F_{(1,0)(2)}(x_2,\ y_3)) & \in & [K_{\{2,3\}},\ K_{\{0,1,3\}}] \\ & & [K_{\{0,1,2\}},\ K_{\{2,3\}}]\,[K_{\{3\}},\ K_{\{0,1,2\}}] \\ & & [K_{\{0,2\}},\ K_{\{0,1,3\}}]\,[K_{\{0,1,3\}},\ K_{\{1,2\}}]. \end{array}$$

Row: 23

$$\begin{array}{rcl} d_4(F_{(1)(2)}(x_3,\ y_3)) & = & [\ s_1d_3x_3,\ s_2d_3y_3\ ]\,[\ s_2d_3y_3,\ s_2d_3x_3\ ] \\ & & [\ x_3,\ y_3\ ]. \end{array}$$

Let $a = s_1d_3x_3, b = s_2d_3y_3, c = s_2d_3x_3, d = x_3, e = y_3$, and taking elements $(dc^{-1}a) \in K_{\{0,2,3\}}$, $(be^{-1}) \in K_{\{0,1,3\}}$, $e^{-1} \in K_{\{0,1,2\}}$, $d \in K_{\{0,1,2\}}$,

$(cd^{-1}) \in K_{\{0,1,3\}}$ and $[d,\ e\ ] \in K_{\{0,1,2\}}$
we obtain

$$\begin{aligned} d_4(F_{(1)(2)}(x_3,\ y_3)) \ &= \ {}^{cd^{-1}}[dc^{-1}a,\ be^{-1}\ ]\ {}^{b}[\ e^{-1},\ dc^{-1}a\ ] \\ &\quad\ {}^{cd^{-1}e^{-1}}[e^{-1}b,\ d]\ [\ cd^{-1},\ [d,\ e\ ]\ ]. \end{aligned}$$

So

$$\begin{aligned} d_4(F_{(1)(2)}(x_3,\ y_3)) \ &\in \ [K_{\{0,2,3\}},\ K_{\{0,1,3\}}]\ [K_{\{0,1,2\}},\ K_{\{0,2,3\}}] \\ &\quad\ [K_{\{0,1,3\}},\ K_{\{0,1,2\}}]\ [K_{\{0,1,3\}},\ K_{\{0,1,2\}}]. \end{aligned}$$

Finally

Row: 25

$$\begin{aligned} d_4(F_{(0)(1)}(x_3,\ y_3)) \ &= \ [\ s_0d_3x_3,\ s_1d_3y_3\ ]\ [\ s_1d_3y_3,\ s_1d_3x_3\ ] \\ &\quad\ [\ s_2d_3x_3,\ s_2d_3y_3\ ]\ [\ y_3,\ x_3\ ]. \end{aligned}$$

Let $a = s_0d_3x_3$, $b = s_1d_2y_3$, $c = s_1d_3x_3$, $d = s_2d_3x_3$, $e = s_2d_3y_3$, $f = y_3$ and $g = x_3$, $A_1 = {}^{ef^{-1}}[fe^{-1}b,\ dg^{-1}]$ and $A_2 = {}^{ef^{-1}g^{-1}}[g^{-1}d,\ f]\ [e^{-1}f,\ [f,\ g]\ ]$. Considering elements $(g^{-1}dc^{-1}a) \in K_{\{1,2,3\}}$, $(be^{-1}f) \in K_{\{0,2,3\}}$, $(e^{-1}f) \in K_{\{0,1,3\}}$, $(cd^{-1}) \in K_{\{0,2\}}$, $(g^{-1}d) \in K_{\{0,1,3\}}$, $f \in K_{\{0,1,2\}}$, $(e^{-1}f) \in K_{\{0,1,3\}}$ and $[f,\ g] \in K_{\{0,1,2\}}$, we get

$$\begin{aligned} d_4(F_{(0)(1)}(x_3,\ y_3)) \ &= \ {}^{cd^{-1}g}(\ [\ g^{-1}dc^{-1}a,\ be^{-1}f\ ] \\ &\quad\ {}^{b}[\ e^{-1}f,\ g^{-1}dc^{-1}a\ ])^{cd^{-1}}(d_4F_{(3)(1)}(x_3,\ y_3)\ ) \\ &\quad\ [\ cd^{-1},\ [d\ ,b\ ]\ ]\ A_1\ A_2 \end{aligned}$$

and then one has

$$\begin{aligned} d_4(F_{(0)(1)}(x_3,\ y_3)) \ &\in \ [K_{\{1,2,3\}},\ K_{\{0,2,3\}}]\ [K_{\{0,1,3\}},\ K_{\{1,2,3\}}] \\ &\quad\ [K_{\{0,2,3\}},\ K_{\{0,1,2\}}]\ [K_{\{0,1,3\}},\ K_{\{0,1,2\}}] \\ &\quad\ [K_{\{0,2\}},\ K_{\{0,1,3\}}]\ [K_{\{0,2,3\}},\ K_{\{0,1,3\}}] \\ &\quad\ [K_{\{0,1,2\}},\ K_{\{0,2,3\}}]\ [K_{\{0,1,3\}},\ K_{\{0,1,2\}}] \\ &\quad\ [K_{\{0,1,3\}},\ K_{\{0,1,2\}}]. \end{aligned}$$

So we have shown that $\partial_4(N_4) \subseteq \prod_{I,J}[K_I,\ K_J]$. The opposite inclusion can be obtained from Proposition 2.3. ■

Collecting up these results, we have:

6.2. Theorem. (Theorem B) *For* $n = 2,3$ *or* 4, *let* $\mathbf{G}$ *be a simplicial group with Moore complex* $\mathbf{NG}$ *then*

$$\partial_n(NG_n \cap D_n) = \prod_{I,J}[\ K_I,\ K_J\ ]$$

for $I, J \subseteq [n-1]$ *with* $I \cup J = [n-1]$, $I = [n-1] - \{\alpha\}$ *and* $J = [n-1] - \{\beta\}$, *where* $(\alpha, \beta) \in P(n)$. ■

The categorical theory of tricategories is still relatively undeveloped and so we have not attempted to identify relationships between the above elements, the kernel-kernel commutators and the complicated conditions for a tricategory, cf. [13].

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