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PASTING IN MULTIPLE CATEGORIES

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ABSTRACT. In the literature there are several kinds of concrete and abstract cell complexes representing composition in n-categories, ω-categories or ∞-categories, and the slightly more general partial ω-categories. Some examples are parity complexes, pasting schemes and directed complexes. In this paper we give an axiomatic treatment: that is to say, we study the class of 'ω-complexes' which consists of all complexes representing partial ω-categories. We show that ω-complexes can be given geometric structures and that in most important examples they become well-behaved CW complexes; we characterise ω-complexes by conditions on their cells; we show that a product of ω-complexes is again an ω-complex; and we describe some products in detail.

1. Introduction

In this paper we consider pasting diagrams representing compositions in multiple categories. To be specific, the multiple categories concerned are n-categories and their infinite-dimensional analogues, which are called ω-categories or ∞-categories; we also use the slightly more general partial ω-categories (see Section 2 for definitions).

Several authors have studied such diagrams; they have names such as parity complex, pasting scheme or directed complex. See Al-Agl and Steiner [2], Johnson [5], Kapranov and Voevodsky [6], Power [9], Steiner [10] and Street [12]. In each case the authors construct a class of more or less abstract cell complexes, subject to some apparently arbitrary axioms, and show that their cells generate partial ω-categories (often, in fact, genuine ω-categories). In each case the composites are represented by unions and the equality of complicated iterated composites can be tested by looking at the corresponding sets. It is not clear how the various constructions are related, and the aim of this paper is to clarify matters by giving an axiomatic treatment. That is to say, we define a class of 'ω-complexes' by requiring that their cells generate partial ω-categories; this means that all previous constructions are included. The definition is given in Section 2.

As an abstract cell complex, an ω-complex is a union of 'cells' or 'atoms' of various dimensions. In Section 3 we show that the entire structure is determined by a small amount of information on each cell; essentially we need its dimension and its boundary. The interiors of the cells (that is, the complements of the boundaries) are required to be non-empty disjoint sets, but so far as the algebra is concerned, it does otherwise matter what they are. They can be singletons, which gives a purely combinatorial theory,

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or they can be chosen so that the ω-complex has a geometric structure. In Section 4 we describe a natural geometric choice; in most important examples the cells are then genuine topological cells and the ω-complex is a well-behaved CW complex. In Section 5 we give a converse to Section 3: that is to say, we give necessary and sufficient conditions for dimension and boundary functions to produce an ω-complex.

The rest of the paper is concerned with products. We show that two ω-complexes have a natural product, which is also an ω-complex; the theory is described in Section 6, but most of the proofs are deferred to Section 8. In Section 7 we describe some interactions between products and geometric structures. As a by-product we give explicit formulae for tensor products of ω-categories (see [2], 12); in particular, we give very explicit descriptions for the product of two infinite-dimensional globes and some related products (see Example 6.19). This may help in understanding the work of Gray [4]; it may also help in understanding weak n-categories (compare the formulae of Baez and Neuchl [3] and Kapranov and Voevodsky [8]).

In principle, the work of Sections 3 and 5 gives an entirely satisfactory local theory for ω-complexes, although one of the conditions in Section 5 may be difficult to verify in practice. In effect, we produce a completely natural higher-dimensional version of a directed graph. There is also a global problem: to characterise the ω-complexes which correspond to genuine ω-categories, not just partial ω-categories. In graph-theoretic terms, we are seeking a higher-dimensional version of a loop-free directed graph. We discuss this problem at the end of Section 2, but we do not know a satisfactory answer.

2. Definitions

In this section we define ω-categories, partial ω-categories and ω-complexes, and discuss some examples.

An ω-category consists of a sequence of small categories C_0, C_1, ..., all of which have the same morphism set X. The category structures commute with one another, and for every $x \in X$ there is a non-negative integer p such that x is an identity in C_n if and only if $n \geq p$. In the category C_n we write $d_n^- x$ and $d_n^+ x$ for the identities of the target and source of a morphism x, and we write $\#_n$ for the composition operation. Thus $x \#_n y$ is defined if and only if $d_n^+ x = d_n^- y$. To get a partial ω-category, we require only that $d_n^+ x = d_n^- y$ if $x \#_n y$ is defined, but not necessarily vice versa.

In practice we avoid mentioning the objects, so we regard an ω-category or partial ω-category as a single set X with unary operations d_n^-, d_n^+ and partial binary operations $\#_n$. In this language, the definitions can be spelled out explicitly as follows.

2.1. Definition. *An* partial ω-category *is a set X together with unary operations d_0^-, d_0^+, d_1^-, d_1^+, ... and not everywhere defined binary operations $\#_0$, $\#_1$, ... such that the following conditions hold:*

(i) *if $x \#_n y$ is defined then $d_n^+ x = d_n^- y$;*

(ii) *for α, $\beta = \pm$ and for non-negative integers m, n,*

$$d_m^\beta d_n^\alpha x = \begin{cases} d_m^\beta x & \text{if } m < n, \\ d_n^\alpha x & \text{if } m \geq n; \end{cases}$$

(iii) $d_n^- x \#_n x = x \#_n d_n^+ x = x$ *for* $x \in X$ *and for all* n*;*

(iv) *if* $x \#_n y$ *is defined then*

$$d_m^\alpha (x \#_n y) = d_m^\alpha x = d_m^\alpha y \text{ for } m < n,$$
$$d_n^- (x \#_n y) = d_n^- x, \quad d_n^+ (x \#_n y) = d_n^+ y,$$
$$d_m^\alpha (x \#_n y) = d_m^\alpha x \#_n d_m^\alpha y \text{ for } m > n;$$

(v) $(x \#_n y) \#_n z = x \#_n (y \#_n z)$ *if either side is defined;*

(vi) $(x' \#_n y') \#_m (x'' \#_n y'') = (x' \#_m x'') \#_n (y' \#_m y'')$ *if* $m < n$ *and the left side is defined;*

(vii) *for every* $x \in X$ *there is a non-negative integer* p *such that* $d_n^\alpha x = x$ *if and only if* $n \geq p$*.*

2.2. Definition. *An* ω-category *is a partial* ω*-category in which* $x \#_n y$ *is defined if and only if* $d_n^+ x = d_n^- y$*.*

The term ∞-*category* is sometimes used instead of ω-category.

In some of the literature, condition (vii) of Definition 2.1 is omitted.

The integer p in condition (vii) of Definition 2.1 is called the *dimension* of x, denoted $\dim x$.

For $n = 0, 1, \ldots$, an n-*category* is an ω-category such that $\dim x \leq n$ for every element x.

We aim to represent partial ω-categories by 'pasting diagrams'. These should be cell complexes such that the elements of the partial ω-category are represented by appropriate subcomplexes, the operations d_n^α are represented by parts of boundaries, and composites are represented by unions.

As an example, consider the ω-category X with the following presentation: there are generators a, x, y and relations

$$\dim a = 1, \ \dim x = \dim y = 2, \ d_1^+ x = d_1^- y, \ d_0^+ a = d_0^- x = d_0^- y.$$

It turns out that X has 16 elements and that these can be represented by subcomplexes of the diagram in Figure 1: there are three cells a, x, y representing the generators; three additional 0-cells u, v, w representing $d_0^- a$, $d_0^+ a = d_0^- x = d_0^- y$ and $d_0^+ x = d_0^+ y$; three additional 1-cells b, c, d representing $d_1^- x$, $d_1^+ x = d_1^- y$ and $d_1^+ y$; and the seven subcomplexes

$$x \cup y, \ a \cup b, \ a \cup c, \ a \cup d, \ a \cup x, \ a \cup y, \ a \cup x \cup y$$

representing

$$x \#_1 y, \ a \#_0 b, \ a \#_0 c, \ a \#_0 d, \ a \#_0 x, \ a \#_0 y, \ a \#_0 (x \#_1 y).$$

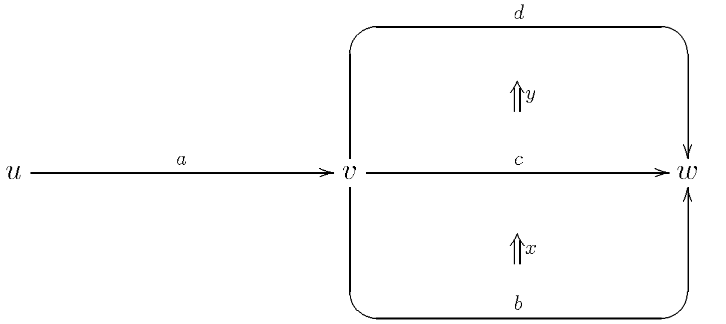

Figure 1

In this figure, d_0^-, d_0^+, d_1^-, d_1^+ are represented by left end, right end, bottom and top respectively; for example, $d_1^+ a = a$ because $\dim a = 1$, and

$$d_1^+[a \#_0 (x \#_1 y)] = d_1^+ a \#_0 d_1^+(x \#_1 y) = a \#_0 d_1^+ y = a \#_0 d.$$

Suppose that $\xi \#_n \eta$ is a composite in a partial ω-category, and suppose that ξ and η are represented by complexes x and y in a pasting diagram. We then have $d_n^+ \xi = d_n^- \eta = \zeta$, say, and ζ must be represented by a subcomplex z of the intersection $x \cap y$. In fact our intuition requires z to be the whole of $x \cap y$. For we want z to be at one extreme of x and at the opposite extreme of y, so $x \setminus z$ and $y \setminus z$ should be on opposite sides of z, and therefore disjoint.

For an example of what can go wrong if this requirement is not satisfied, let π, ρ, σ, τ be elements of a partial ω-category such that

$$d_0^+ \pi = d_0^- \rho = \sigma,\ d_0^+ \rho = d_0^- \pi = \tau,\ \sigma \neq \tau.$$

Let p, r, s, t be representatives for π, ρ, σ, τ in a pasting diagram; then the diagram must be as in Figure 2. Suppose also that the composites $\pi \#_0 \rho$ and $\rho \#_0 \pi$ both exist. They are distinct (because $d_0^-(\pi \#_0 \rho) \neq d_0^-(\rho \#_0 \pi)$), so it is not satisfactory to have them both represented by the union $p \cup r$. This unsatisfactory behaviour arises because $p \cap r$ strictly contains s and strictly contains t.

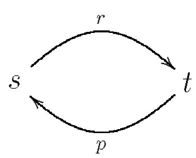

Figure 2

Taking these considerations into account, we now define ω-complexes as follows.

2.3. Definition. *An ω-*complex *is a set K together with two families of subsets called* atoms *and* molecules *subject to the following conditions.*

(i) *The molecules form a partial ω-category.*

(ii) *Let x and y be molecules. Then $x \#_n y$ exists if and only if $x \cap y = d_n^+ x = d_n^- y$; if $x \#_n y$ does exist, then $x \#_n y = x \cup y$.*

(iii) *Every atom is a molecule, and the molecules are the sets generated from the atoms by applying the composition operations $\#_0$, $\#_1$, ...*

(iv) *The set K is the union of its atoms.*

(v) *For each atom a, let $\partial^- a$ and $\partial^+ a$ be the sets given by*

$$\partial^\alpha a = \begin{cases} d_{p-1}^\alpha a & \text{if } \dim a = p > 0, \\ \emptyset & \text{if } \dim a = 0, \end{cases}$$

and let the interior *of a be the subset* $\operatorname{Int} a$ *given by*

$$\operatorname{Int} a = a \setminus (\partial^- a \cup \partial^+ a).$$

Then the interiors of the atoms are non-empty and disjoint.

Informally, the effect of Definition 2.3 is as follows: conditions (iv) and (v) say that K is a sort of cell complex with the atoms as cells; conditions (i)–(iii) say that the atoms generate a partial ω-category of molecules in which composites are represented by well-behaved unions.

2.4. EXAMPLE. An ω-complex u is called an *infinite-dimensional globe* if it has two n-dimensional atoms u_n^- and u_n^+ for each $n \geq 0$ and $d_m^\beta u_n^\alpha = u_m^\beta$ for $m < n$. It is easy to check that such ω-complexes exist; all the molecules are atoms, and they form an ω-category, not just a partial ω-category.

For a non-negative integer p, an ω-complex u is called a *p-dimensional globe* if its atoms can be listed in the form

$$u_p,\ d_{p-1}^- u_p,\ d_{p-1}^+ u_p,\ \ldots,\ d_0^- u_p,\ d_0^+ u_p$$

such that $\dim u_p = p$, $\dim d_n^\alpha u_p = n$ for $n < p$, and $d_n^- u_p \neq d_n^+ u_p$ for $n < p$. As before, p-dimensional globes exist, all the molecules are atoms, and and they form an ω-category.

For a non-negative integer p, one can check that there is also an ω-complex K_p as follows: K_p is the union of two infinite-dimensional globes u and v, and $u \cap v = u_p^+ = v_p^-$. For $n < p$ there are just two n-dimensional atoms, since $u_n^- = v_n^-$ and $u_n^+ = v_n^+$; there are three p-dimensional atoms u_p^-, $u_p^+ = v_p^-$ and v_p^+; for $n > p$ there are four n-dimensional atoms u_n^-, u_n^+, v_n^- and v_n^+ . The molecules are the atoms and the composites $u_m^\alpha \#_p v_n^\beta$ for $m > p$ and $n > p$, and they form an ω-category. By a slight abuse of language, K_p can be called a p-composite of u and v and denoted $u \#_p v$.

It is clear that the class of ω-complexes include the constructions of [2], [5], [6], [9], [10] and [12]. See also Aitchison [1] and Kapranov and Voevodsky [7] for the particular case of cubes, and Street [11] for simplexes. The characterisation of ω-complexes by local data in the next two section shows that the class of ω-complexes is not excessively large. We make two remarks, however.

First, one may ask for conditions under which the molecules in an ω-complex form a genuine ω-category, not just a partial ω-category. For complexes of dimension at most 1, it is necessary and sufficient that there should be no directed loops. For higher-dimensional complexes, the obvious generalisation of this condition is insufficient (see [9]). It is also unnecessary, as one can see from the example in Figure 3: here the molecules form a genuine ω-category but there is a 'higher-dimensional directed loop' a, x, b, y, a. There are various sufficient conditions (see [5], [10], [12]), but the example in Figure 3 shows that none of them is necessary.

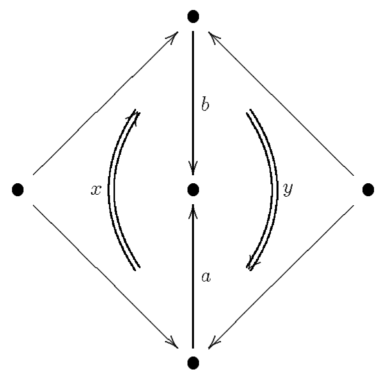

Figure 3

Second, most previous treatments make the following requirement: if a is a p-dimensional atom with $p > 0$, then the $d^{\alpha}_{p-1}a$ should be unions of $(p-1)$-dimensional atoms. This requirement is partially explained by Proposition 4.11 below. The requirement is not satisfied in all ω-complexes; for example, there is an ω-complex with two atoms x and y such that $\dim x = 2$, $\dim y = 0$, $d^{-}_{1}x = d^{+}_{1}x = d^{-}_{0}x = d^{+}_{0}x = y$. This ω-complex gives an economical representation for a 2-sphere, but is in some ways badly behaved. We have to exclude such ω-complexes from some of the geometrical work in Section 4.

3. A local description of an ω-complex

In this section we give a chain of results designed to show that an ω-complex is determined by its atoms, their dimensions, and the functions ∂^{-}, ∂^{+} (see Corollary 3.7). The main result (Theorem 3.6) shows how this local information determines the operations d^{α}_{n}. Propositions 3.1–3.5 are technical preliminaries.

3.1. Proposition. (i) *Let x be a molecule in an ω-complex. Then $d^{\alpha}_{n}x \subset x$ for $\alpha = \pm$ and $n \geq 0$.*

(ii) *Let a be an atom in an ω-complex. Then $\partial^{\alpha}a \subset a$ for $\alpha = \pm$; if $\partial^{\alpha}a \neq \emptyset$ then $\partial^{\alpha}a$ is a molecule such that $\dim \partial^{\alpha}a < \dim a$.*

Proof. (i) This holds because $x = d^{-}_{n}x \#_{n} x \#_{n} d^{+}_{n}x = d^{-}_{n}x \cup x \cup d^{+}_{n}x$.

(ii) Suppose that $\dim a = 0$. Then $\partial^{\alpha}a = \emptyset$ and the results hold trivially.

On the other hand, suppose that $\dim a = p > 0$. Then $\partial^{\alpha}a = d^{\alpha}_{p-1}a$, so $\partial^{\alpha}a \subset a$ by part (i). Also, $\partial^{\alpha}a$ is a molecule, and $\dim \partial^{\alpha}a \leq p-1 < \dim a$ because $d^{\alpha}_{p-1}d^{\alpha}_{p-1}a = d^{\alpha}_{p-1}a$. ∎

3.2. PROPOSITION. *Let x be a molecule in an ω-complex with a decomposition into factors $y_1, \ldots, y_k$. Then the following hold:*

(i) $x \setminus d_n^\alpha x = (y_1 \setminus d_n^\alpha y_1) \cup \ldots \cup (y_k \setminus d_n^\alpha y_k)$ *for all n and α;*

(ii) $\dim y_i \leq \dim x$ *for all i.*

PROOF. (i) It suffices to consider the case $x = y_1 \#_p y_2$.

For $n < p$ we have $x = y_1 \cup y_2$ and $d_n^\alpha x = d_n^\alpha y_1 = d_n^\alpha y_2$, and the result is obvious.

For $n = p$ and $\alpha = -$, we use Proposition 3.1(i) to get $d_p^- y_1 \subset y_1$, and we then get

$$x \setminus d_p^- x = (y_1 \cup y_2) \setminus d_p^- y_1 = (y_1 \setminus d_p^- y_1) \cup [y_2 \setminus (y_1 \cap y_2)] = (y_1 \setminus d_p^- y_1) \cup (y_2 \setminus d_p^- y_2).$$

For $n = p$ and $\alpha = +$ the proof is similar.

Finally, suppose that $n > p$. Using Proposition 3.1(i) we get

$$y_1 \cap d_n^\alpha y_2 \subset y_1 \cap y_2 = d_p^+ y_1 = d_p^+ d_n^\alpha y_1 \subset d_n^\alpha y_1,$$

and similarly $y_2 \cap d_n^\alpha y_1 \subset d_n^\alpha y_2$. It follows that

$$x \setminus d_n^\alpha x = (y_1 \cup y_2) \setminus (d_n^\alpha y_1 \cup d_n^\alpha y_2) = (y_1 \setminus d_n^\alpha y_1) \cup (y_2 \setminus d_n^\alpha y_2),$$

as required.

(ii) Let n be the dimension of x, so that $d_n^- x = x$. By part (i), $y_i \setminus d_n^- y_i \subset x \setminus d_n^- x = \emptyset$, so $y_i \subset d_n^- y_i$. It now follows from Proposition 3.1(i) that $d_n^- y_i = y_i$, so $\dim y_i \leq n = \dim x$ as required. ■

3.3. PROPOSITION. *Let x be a molecule and a be an atom in an ω-complex. Then the following are equivalent:*

(i) $a \subset x$;

(ii) $\operatorname{Int} a \cap x \neq \emptyset$;

(iii) *a is a factor in some decomposition of x.*

PROOF. It is obvious that (iii) implies (i). Since $\operatorname{Int} a \neq \emptyset$, (i) implies (ii). It remains to show that (ii) implies (iii). We use induction on $\dim x$.

Since x is a molecule, it has a decomposition into atoms $a_1, \ldots, a_k$. Since $\operatorname{Int} a \cap x \neq \emptyset$, we must have $\operatorname{Int} a \cap a_i \neq \emptyset$ for some i. Since $a_i \setminus (\partial^- a_i \cup \partial^+ a_i) = \operatorname{Int} a_i$, we must have $\operatorname{Int} a \cap \operatorname{Int} a_i \neq \emptyset$ or $\operatorname{Int} a \cap \partial^\alpha a_i \neq \emptyset$ for some α.

Suppose that $\operatorname{Int} a \cap \operatorname{Int} a_i \neq \emptyset$. Since distinct atoms have disjoint interiors, it follows that $a = a_i$, so a is certainly a factor of x.

Now suppose that $\operatorname{Int} a \cap \partial^\alpha a_i \neq \emptyset$. By Propositions 3.1(ii) and 3.2(ii), $\partial^\alpha a_i$ is a molecule and $\dim \partial^\alpha a_i < \dim a_i \leq \dim x$. By the inductive hypothesis, a is a factor in some decomposition of $\partial^\alpha a_i$. Since $\partial^\alpha a_i = d_{p-1}^\alpha a_i$, where $p = \dim a_i$, and since $a_i = d_{p-1}^- a_i \#_{p-1} a_i \#_{p-1} d_{p-1}^+ a_i$, it follows that a is a factor in a decomposition of x. This completes the proof. ■

3.4. Proposition. *Let ξ be a point in an ω-complex, and let a be an atom of minimal dimension such that $\xi \in a$. Then $\xi \in \operatorname{Int} a$.*

Proof. Suppose that $\xi \notin \operatorname{Int} a$; we must then have $\xi \in \partial^\alpha a$ for some α. By Proposition 3.1(i), $\partial^\alpha a$ is a molecule, so it is a composite of atoms $a_1, \ldots, a_k$, and we get $\xi \in a_i$ for some i. But $\dim a_i \leq \dim \partial^\alpha a < \dim a$ by Propositions 3.2(ii) and 3.1(ii), so this contradicts the minimality of $\dim a$. This contradiction shows that ξ must be in $\operatorname{Int} a$. ■

3.5. Proposition. *Let ξ be an element of a molecule x in an ω-complex. For given n and α, suppose that b is an atom in x of minimal dimension such that $\xi \in b \setminus d_n^\alpha b$. Then $\xi \in \operatorname{Int} b$ or $\dim b = n+1$.*

Proof. Suppose that $\xi \notin \operatorname{Int} b$ and $\dim b \neq n+1$. We get a contradiction as follows. Since $\xi \notin \operatorname{Int} b$ we have $\xi \in \partial^\beta b$ for some β. Since $b \setminus d_n^\alpha b \neq \emptyset$ we have $\dim b > n$, so in fact $\dim b > n+1$. Therefore $d_n^\alpha \partial^\beta b = d_n^\alpha b$, so $\xi \in \partial^\beta b \setminus d_n^\alpha \partial^\beta b$. By Proposition 3.2(i), $\xi \in b' \setminus d_n^\alpha b'$ for some atomic factor b' of $\partial^\beta b$. Using Proposition 3.1(ii), we get $b' \subset \partial^\beta b \subset b \subset x$. Using Propositions 3.2(ii) and 3.1(ii), we get $\dim b' \leq \dim \partial^\beta b < \dim b$, so we have a contradiction to the minimality of $\dim b$. This contradiction completes the proof. ■

3.6. Proposition. *Let x be a molecule in an ω-complex. Then*

$$d_n^\alpha x = \Big[\bigcup_{\substack{a \subset x \\ \dim a \leq n}} a \Big] \setminus \Big[\bigcup_{\substack{b \subset x \\ \dim b = n+1}} (b \setminus \partial^\alpha b) \Big],$$

where the unions are over atoms a and b.

A similar formula has been used as a definition ([10], 2.3), but the only justification was that it led to good results; it was not shown to be necessary.

Proof. First, let ξ be a member of $d_n^\alpha x$. We show that $\xi \in a \subset x$ for some atom a such that $\dim a \leq n$. Indeed, $d_n^\alpha x$ is a molecule of dimension at most n (because $d_n^\alpha d_n^\alpha x = d_n^\alpha x$), so $d_n^\alpha x$ is a composite of atoms of dimensions at most n by Proposition 3.2(ii). Therefore $\xi \in a \subset d_n^\alpha x$ for some atom a such that $\dim a \leq n$. Since $d_n^\alpha x \subset x$ (Proposition 3.1(i)), we get $\xi \in a \subset x$ as required.

Next, let ξ be a member of $d_n^\alpha x$ and b be an $(n+1)$-dimensional atom contained in x. We show that $\xi \notin b \setminus \partial^\alpha b$. Indeed, b is a factor of x by Proposition 3.3, so $b \setminus \partial^\alpha b = b \setminus d_n^\alpha b \subset x \setminus d_n^\alpha x$ by Proposition 3.2(i), and this gives the required result $\xi \notin b \setminus \partial^\alpha b$.

Finally, let ξ be a point not in $d_n^\alpha x$ such that $\xi \in a \subset x$ for some atom a of dimension at most n. We show that $\xi \in b \setminus \partial^\alpha b$ for some atom $b \subset x$ of dimension $n+1$. Indeed, $\xi \in x \setminus d_n^\alpha x$, so $\xi \in b \setminus d_n^\alpha b$ for some atomic factor b of x by Proposition 3.2(i). We can therefore choose an atom $b \subset x$ of minimal dimension such that $\xi \in b \setminus d_n^\alpha b$. By Proposition 3.5, $\xi \in \operatorname{Int} b$ or $\dim b = n+1$. But $\dim b > n$ (since $d_n^\alpha b \neq b$), so we have $\xi \in a \subset x$ with $\dim a < \dim b$. By Proposition 3.4, $\xi \notin \operatorname{Int} b$. Therefore $\dim b = n+1$ and we get $\xi \in b \setminus d_n^\alpha b = b \setminus \partial^\alpha b$ as required. ■

3.7. Corollary. *An ω-complex is determined by its atoms, their dimensions, and the functions ∂^-, ∂^+.*

Proof. According to Definition 2.3, an ω-complex is determined by its atoms, its molecules, and the ω-category operations d_n^α, $\#_n$. But the molecules are determined by the atoms and the $\#_n$ (Definition 2.3(iii)), the $\#_n$ are determined by the d_n^α (Definition 2.3(ii)), and the d_n^α are determined by the atoms, their dimensions, and the ∂^α (Theorem 3.6). The result follows. ∎

4. Geometric structures

Let K be an ω-complex. The interiors of the atoms must be non-empty and disjoint, but so far as the algebra is concerned they can otherwise be chosen arbitrarily. In this section we describe a natural choice which produces a topological space. The result is like a CW complex, and under mild restrictions we show that it really is a CW complex (Corollary 4.16).

The topological construction is based on *cones*; we recall the definition.

4.1. Definition. *A topological space C is a* cone *if it is homeomorphic to a quotient space of the form*

$$\frac{(X \times [0,1]) \sqcup P}{\sim},$$

where X is an arbitrary topological space, $[0,1]$ is the unit closed interval, P is a singleton space, $\sqcup$ denotes disjoint union, and the equivalence relation is got by collapsing the subspace $(X \times \{0\}) \sqcup P$ to a single point. The base *of C is the subspace of C corresponding to the image of $X \times \{1\}$ (thus the base is a closed subspace homeomorphic to X).*

4.2. Example. A Euclidean ball is a cone on its boundary sphere; in particular a 0-ball (a singleton space) is a cone on its boundary (the empty set).

It is also convenient to define skeleta for ω-complexes, as follows.

4.3. Definition. *Let K be an ω-complex and let n be an integer. Then the n-*skeleton *of K, denoted K^n, is the subcomplex consisting of the atoms of dimension at most n.*

The main definition is as follows.

4.4. Definition. *An ω-complex K is* geometric *if it has a topology satisfying the following conditions:*

(i) *each atom a is a cone with base $\partial^- a \cup \partial^+ a$;*

(ii) *a subset F of K^n is closed in K^n if and only if $F \cap a$ is closed in a for each atom a of dimension at most n;*

(iii) *a subset F of K is closed in K if and only if $F \cap a$ is closed in a for each atom a.*

Obviously a geometric ω-complex is determined up to homeomorphism by its partial ω-category structure.

4.5. Example. If one interprets the ω-complex of Figure 1 as a subset of the plane, then it becomes a geometric ω-complex.

There is a geometric p-dimensional globe (see Example 2.4) as follows: the generator u_p is the Euclidean p-ball

$$\{ (t_0, \ldots, t_{p-1}) \in \mathbf{R}^p : t_0^2 + \ldots + t_{p-1}^2 \leq 1 \};$$

for $0 \leq n < p$ the atom $d_n^\alpha u_p$ is the n-dimensional hemisphere

$$\{ (t_0, \ldots, t_n, 0, \ldots, 0) : t_0^2 + \ldots + t_n^2 = 1,\ \alpha t_n \geq 0 \}.$$

We shall need the following technical property.

4.6. Proposition. *If x is a subcomplex of a geometric ω-complex K, then x is closed in K.*

Proof. We must show that $x \cap a$ is closed in a for each atom a. Let n be the dimension of a; we shall use induction on n.

Suppose first that $x \cap \operatorname{Int} a \neq \emptyset$. By Proposition 3.3, $a \subset x$, so $x \cap a$ is certainly closed in a.

Now suppose that $x \cap \operatorname{Int} a = \emptyset$. This means that $x \cap a = x \cap (\partial^- a \cup \partial^+ a)$. Since a is a cone with base $\partial^- a \cup \partial^+ a$, it suffices to show that $x \cap (\partial^- a \cup \partial^+ a)$ is closed in $\partial^- a \cup \partial^+ a$. Since $\partial^- a \cup \partial^+ a \subset K^{n-1}$, it suffices to show that $x \cap K^{n-1}$ is closed in K^{n-1}. But $x \cap a'$ is closed in a' for each atom a' of dimension less than n by the inductive hypothesis, so $x \cap K^{n-1}$ is closed in K^{n-1} by Definition 4.4(ii).

This completes the proof. ■

We shall now show that every ω-complex is equivalent to a geometric ω-complex, where equivalence is formally defined as follows.

4.7. Definition. *Two ω-complexes K and L are* equivalent *if there is a bijection f from the atoms of K to the atoms of L such that, for each atom a in K, the following properties hold:*

(i) $\dim f(a) = \dim a$*;*

(ii) $\partial^\alpha f(a)$ *is the union of the atoms $f(b)$ such that $b \subset \partial^\alpha a$.*

4.8. Proposition. *Every ω-complex is equivalent to a geometric ω-complex.*

Proof. Let the given ω-complex be K; we must construct a geometric ω-complex L and a bijection f as in Definition 4.7.

We begin by constructing geometric ω-complexes L^0, L^1, ... such that L^{n-1} is a subcomplex of L^n and the L^n are compatibly equivalent to the skeleta K^n.

We take L^0 to be a discrete space whose points are in one-one correspondence with the 0-dimensional atoms of K; the atoms of L^0 are to be the singleton subsets, and all the atoms are to be 0-dimensional. It is clear that L^0 is a geometric ω-complex equivalent to K^0.

For $n > 0$, suppose that we have constructed L^{n-1} with the desired properties. Let f be the bijection from the atoms of K^{n-1} to the atoms of L^{n-1}. For a an n-dimensional atom of K, let $\delta^\alpha a$ be the union of the atoms $f(b)$ such that $b \subset \partial^\alpha a$ and let $C(a)$ be a cone with base $\delta^- a \cup \delta^+ a$. Let L^n be the quotient of the disjoint union $L^{n-1} \sqcup \bigsqcup_a C(a)$ got by identifying the base of each cone $C(a)$ with the corresponding subspace of L^{n-1}. Since the base of a cone is closed in that cone, we may identify L^{n-1} with a closed subspace of L^n. Since the sets $\delta^- a \cup \delta^+ a$ are closed in L^{n-1} (Proposition 4.6), we may also identify the $C(a)$ with closed subspaces of L^n. Clearly the geometric ω-complex structure of L^{n-1} can be extended to a geometric ω-complex structure on L^n: the additional atoms are the sets $C(a)$, they are n-dimensional, and $\partial^\alpha C(a) = \delta^\alpha a$. The equivalence f from K^{n-1} to L^{n-1} extends in the obvious way to give an equivalence from K^n to L^n.

We now have a chain of geometric ω-complexes $L^0 \subset L^1 \subset L^2 \subset \ldots$ compatibly equivalent to the K^n, and each L^{n-1} is a subcomplex of L^n. By Proposition 4.6, L^{n-1} is closed in L^n. To complete the proof, let L be the union of the L^n, topologised so that F is closed in L if and only if F is closed in L^n for all n. It is then clear how to make L into a geometric ω-complex equivalent to K. This completes the proof. ∎

In many ways, geometric ω-complexes are like CW complexes; in particular we have the following properties.

4.9. PROPOSITION. (i) *If x is a finite union of atoms in a geometric ω-complex, then x is compact.*

(ii) *A geometric ω-complex is compactly generated Hausdorff.*

PROOF. (i) It suffices to show that the atoms are compact. But an atom is a cone with base a finite union of atoms of lower dimension, so this holds by induction on dimension.

(ii) Let K be the geometric ω-complex. Because of part (i) and Definition 4.4(iii), it suffices to show that K is Hausdorff. Let ξ and η be distinct points in K; we must find disjoint open sets U and V in K such that $\xi \in U$ and $\eta \in V$.

By induction on n, it is straightforward to construct disjoint sets U^n and V^n open in K^n with the following properties: $U^n \cap K^{n-1} = U^{n-1}$; $V^n \cap K^{n-1} = V^{n-1}$; if $\xi \in K^n$ then $\xi \in U^n$; if $\eta \in K^n$ then $\eta \in V^n$.

Now let $U = \bigcup_n U_n$ and $V = \bigcup_n V_n$. We find that U and V are disjoint open sets in K with $\xi \in U$ and $\eta \in V$ as required. ∎

For a geometric ω-complex to be a genuine CW complex, it clearly suffices that the n-dimensional atoms should be homeomorphic to n-balls. To achieve this, we impose a restriction, which is usually satisfied in practice: we require the atoms to be *round* in the sense of the following definition.

4.10. DEFINITION. *A molecule x in an ω-complex is* round *provided that the following conditions hold:*

(i) *if* $\dim x > 0$ *then* $d_0^- x \cap d_0^+ x = \emptyset$;

(ii) *if* $0 < n < \dim x$ *then* $d_n^- x \cap d_n^+ x = d_{n-1}^- x \cup d_{n-1}^+ x$.

For example, the atoms in globes (see Example 2.4) are round, the atoms in Figure 1 are round, but the molecule $a \#_0 (x \#_1 y)$ in Figure 1 is not round. In the important examples such as cubes and simplexes, all the atoms are round, and the same is true for most examples in the literature. Also, if an ω-complex is in any sense loop-free, then one expects its atoms to be round.

We shall use the following properties of round molecules.

4.11. PROPOSITION. *Let x be a round molecule in an ω-complex and let n be such that $0 \le n \le \dim x$. Then $d_n^\alpha x$ is a union of n-dimensional atoms, $\dim d_n^\alpha x = n$, and $d_n^\alpha x$ is round.*

PROOF. Let a be an atom contained in $d_n^\alpha x$ such that $\dim a = m < n$; we claim that a is contained in some $(m+1)$-dimensional atom contained in $d_n^\alpha x$. Indeed, if $m = 0$ then $d_m^- x \cap d_m^+ x = \emptyset$, and if $m > 0$ then $d_m^- x \cap d_m^+ x = d_{m-1}^- x \cup d_{m-1}^+ x$, and in both cases we see that $d_m^- d_n^\alpha x \cap d_m^+ d_n^\alpha x = d_m^- x \cap d_m^+ x$ is disjoint from the interior of a. Let ξ be a point of $\operatorname{Int} a$; we must then have $\xi \in b$ for some $(m+1)$-dimensional atom $b \subset d_n^\alpha x$. By Proposition 3.3, $a \subset b$. This verifies the claim.

By iterating this argument, we see that $d_n^\alpha x$ is contained in a union of atoms of dimension at least n; on the other hand, by Propositions 3.2(ii) and 3.3, an atom contained in $d_n^\alpha x$ is at most n-dimensional. Therefore $d_n^\alpha x$ is a union of atoms of dimension exactly n.

It follows that $d_m^\beta d_n^\alpha x \neq d_m^\beta x$ for $m < n$; therefore $\dim d_n^\alpha x = n$.

Finally, since $d_m^\beta d_n^\alpha x = d_m^\beta x$ for $m < n$, it follows that $d_n^\alpha x$ is round. ∎

In earlier work, if a is a p-dimensional atom, then the $\partial^\alpha a$ have usually been required to be unions of $(p-1)$-dimensional atoms. Proposition 4.11 shows that this property holds whenever the atoms are round.

Let x be a molecule in a geometric ω-complex. Intuitively, one expects to construct x as follows: let the $d_0^\alpha x$ be points, let the $d_1^\alpha x$ be got by joining $d_0^- x$ to $d_0^+ x$, let the $d_2^\alpha x$ be got by joining $d_1^- x$ to $d_1^+ x$, and so on up to the dimension of x. We shall formalise the idea of joining by using *bands*, which we define as follows.

4.12. DEFINITION. *Let B be a topological space with subspaces H^- and H^+. Then B is a* band *from H^- to H^+ if there are a topological space X and continuous real-valued functions ϕ^-, ϕ^+ on X such that the following conditions hold:*

(i) *$\phi^-(\xi) \le \phi^+(\xi)$ for $\xi \in X$;*

(ii) *B is homeomorphic to the space*

$$\{\, (\xi, t) \in X \times \mathbf{R} : \phi^-(\xi) \le \phi^+(\xi) \,\}$$

such that H^- and H^+ correspond to the subspaces $\{(\xi, \phi^-(\xi))\}$ and $\{(\xi, \phi^+(\xi))\}$.

Roughly speaking, in Definition 4.12, B is the product of X with a closed interval, but the interval varies from point to point in X and may be reduced to a single point. Note that H^- and H^+ are both homeomorphic to X. Note also that they need not be disjoint; their intersection corresponds to points ξ in X such that $\phi^-(\xi) = \phi^+(\xi)$.

As an example, if B is a closed ball of positive dimension and its boundary is divided into hemispheres H^- and H^+ in the standard way, then B is a band from H^- to H^+.

We can describe geometric ω-complexes with round atoms in terms of bands as follows.

4.13. Theorem. *Let x be a molecule in a geometric ω-complex all of whose atoms are round. Then the following conditions hold:*

(i) *the $d_0^\alpha x$ are single point spaces;*

(ii) *for $n > 0$ the $d_n^\alpha x$ are bands from $d_{n-1}^- x$ to $d_{n-1}^+ x$;*

(iii) *if x is round and $\dim x = p$ then x is homeomorphic to a p-ball; if also $p > 0$ then $d_{p-1}^- x \cup d_{p-1}^+ x$ corresponds to the boundary $(p-1)$-sphere.*

Theorem 4.13 may be verified for the geometric ω-categories in Example 4.5.

Molecules with the properties described in Theorem 4.13(i)–(ii) have been considered in [2] and [6]; they are homeomorphic in an obvious way to subspaces of Euclidean spaces.

Before proving Theorem 4.13, we give two lemmas.

4.14. Lemma. *Let x be a p-dimensional round molecule in a geometric ω-complex such that the $d_n^\alpha x$ are bands from $d_{n-1}^- x$ to $d_{n-1}^+ x$ for $0 < n \le p$. Then x is homeomorphic to a p-ball and, if $p > 0$, then $d_{p-1}^- x \cup d_{p-1}^+ x$ corresponds to the boundary $(p-1)$-sphere.*

Proof. We use induction on p.

Suppose that $p = 0$. Then x is a composite of 0-dimensional atoms (Proposition 3.2(ii)), so it is a 0-dimensional atom, and is therefore a single point (see Example 4.2). The result follows.

Suppose that $p > 0$. Here x is a band from $d_{p-1}^- x$ to $d_{p-1}^+ x$. By Proposition 4.11, $d_{p-1}^- x$ is round and $(p-1)$-dimensional, so $d_{p-1}^- x$ is homeomorphic to a $(p-1)$-ball B by the inductive hypothesis. If $p = 1$ then $d_{p-1}^- x \cap d_{p-1}^+ x$ is empty; if $p > 1$, then

$$d_{p-1}^- x \cap d_{p-1}^+ x = d_{p-2}^- x \cup d_{p-2}^+ x = d_{p-2}^- d_{p-1}^- x \cup d_{p-2}^+ d_{p-1}^- x,$$

which corresponds to the boundary of B. In both cases, it follows that x is homeomorphic to a p-ball and $d_{p-1}^- x \cup d_{p-1}^+ x$ corresponds to the boundary $(p-1)$-sphere. ∎

4.15. Lemma. *Let $x = x^- \#_q x^+$ be a composite of molecules in a geometric ω-complex such that for some $n > 0$ and some α the $d_n^\alpha x^\theta$ are bands from $d_{n-1}^- x^\theta$ to $d_{n-1}^+ x^\theta$. Then $d_n^\alpha x$ is a band from $d_{n-1}^- x$ to $d_{n-1}^+ x$.*

Proof. If $0 < n \le q$ then the result holds because $d_n^\alpha x = d_n^\alpha x^\alpha$, $d_{n-1}^- x = d_{n-1}^- x^\alpha$, and $d_{n-1}^+ x = d_{n-1}^+ x^\alpha$.

If $n = q + 1$ then the result holds because $d_n^\alpha x = d_n^\alpha x^- \cup d_n^\alpha x^+$, $d_{n-1}^- x = d_{n-1}^- x^-$, $d_{n-1}^+ x = d_{n-1}^+ x^+$, and $d_n^\alpha x^- \cap d_n^\alpha x^+ = d_{n-1}^+ x^- = d_{n-1}^- x^+$.

If $n > q+1$ then the result holds because $d_n^\alpha x = d_n^\alpha x^- \cup d_n^\alpha x^+$, $d_{n-1}^- x = d_{n-1}^- x^- \cup d_{n-1}^- x^+$, $d_{n-1}^+ x = d_{n-1}^+ x^- \cup d_{n-1}^+ x^+$, and $d_n^\alpha x^- \cap d_n^\alpha x^+ \subset d_{n-1}^- x \cap d_{n-1}^+ x$. ∎

Proof. Proof of Theorem 4.13 (i) Clearly $d_0^\alpha x$ is 0-dimensional and round, so this follows from Lemma 4.14.

(ii) Let the dimension of x be p; we shall use induction on p. By Proposition 3.2(ii), a p-dimensional molecule is a composite of atoms of dimension at most p; because of Lemma 4.15 we may therefore assume that x is an atom.

Suppose that $n < p$. Then $d_n^\alpha x = d_n^\alpha d_{p-1}^\alpha x$, $d_{n-1}^- x = d_{n-1}^- d_{p-1}^\alpha x$, $d_{n-1}^+ x = d_{n-1}^+ d_{p-1}^\alpha x$, and $\dim d_{p-1}^\alpha x < p$, so the result holds by the inductive hypothesis.

Suppose that $n > p$. Then $d_n^\alpha x = d_{n-1}^- x = d_{n-1}^+ x$, so the result holds trivially.

It remains to show that x is a band from $d_{p-1}^- x$ to $d_{p-1}^+ x$, assuming that $p > 0$. By hypothesis, x is round, so $d_{p-1}^- x$ and $d_{p-1}^+ x$ are round and $(p-1)$-dimensional by Proposition 4.11. By the inductive hypothesis and Lemma 4.14, $d_{p-1}^- x$ and $d_{p-1}^+ x$ are homeomorphic to $(p-1)$-balls H^- and H^+. If $p = 1$, then $d_{p-1}^- x \cap d_{p-1}^+ x = \emptyset$; if $p > 1$ then $d_{p-1}^- x \cap d_{p-1}^+ x = d_{p-2}^- x \cup d_{p-2}^+ x$, which corresponds to the boundaries of H^- and H^+ under the homeomorphisms. It follows that $d_{p-1}^- x \cup d_{p-1}^+ x$ is homeomorphic to a $(p-1)$-sphere, and that the $d_{p-1}^\beta x$ correspond to the two hemispheres in a standard subdivision. Since x is a cone with base $d_{p-1}^- x \cup d_{p-1}^+ x$, it follows that x is a band from $d_{p-1}^- x$ to $d_{p-1}^+ x$ as required.

(iii) This follows from part (ii) and Lemma 4.14. ∎

We now give two corollaries. The first is obvious; the second goes a little way towards characterising molecules.

4.16. COROLLARY. *If K is a geometric ω-complex all of whose atoms are round, then K is a CW complex in which the n-cells are the n-dimensional atoms and the characteristic maps are homeomorphisms from n-balls onto subcomplexes.*

4.17. COROLLARY. *If x is a molecule in a geometric ω-complex all of whose atoms are round, then x is contractible.*

PROOF. From Theorem 4.13, $d_0^- x$ is a point and $d_n^- x$ has $d_{n-1}^- x$ as a deformation retract for $n > 0$. By an inductive argument, each $d_n^- x$ is contractible. The result follows, since $x = d_n^- x$ for n sufficiently large. ∎

5. Atomic complexes

According to Corollary 3.7, ω-complexes may be regarded as families of sets, called atoms, together with functions dim, ∂^-, ∂^+. In this section we find necessary and sufficient conditions for such families and functions to produce ω-complexes. The main result is Theorem 5.14. The arguments are similar to those of [10], 3.

The first step is to show that an ω-complex is an *atomic complex* in the sense of the following definition.

5.1. DEFINITION. *An* atomic complex *is a set K together with a family of subsets called* atoms *and functions* dim, ∂^-, ∂^+ *defined on the family of atoms such that the following conditions hold.*

(i) *If a is an atom then* $\dim a$ *is a non-negative integer called the* dimension *of a.*

(ii) *If a is an atom then the $\partial^\alpha a$ are subsets of a and are unions of atoms of dimension less than that of a.*

(iii) *The set K is the union of the atoms.*

(iv) *For each atom a, let the* interior *of a be the subset* $\operatorname{Int} a$ *given by*

$$\operatorname{Int} a = a \setminus (\partial^- a \cup \partial^+ a).$$

Then the interiors of the atoms are non-empty and disjoint.

5.2. PROPOSITION. *An ω-complex is an atomic complex.*

PROOF. It is obvious that the atoms of an ω-complex satisfy conditions (i), (iii) and (iv) of Definition 5.1. By Propositions 3.1(ii) and 3.2(ii) they also satisfy condition (ii). ∎

The problem is now to find necessary and sufficient conditions for an atomic complex K to be an ω-complex. Our strategy is as follows: we define operations d_n^α on arbitrary subsets of K; we define induced operations $\#_n$ and an induced notion of molecule; we show that there is a family of subsets called 'finite-dimensional globelike subcomplexes' which forms a partial ω-category under these operations d_n^α and the induced operations $\#_n$; we find necessary and sufficient conditions for the molecules to form a sub-partial ω-category making K an ω-complex.

We define the d_n^α (and then the $\#_n$ and the molecules) by generalising Theorem 3.6.

5.3. DEFINITION. *Let K be an atomic complex.*

If x is a subset, $\alpha = \pm$ and $n \geq 0$, then $d_n^\alpha x$ is the subset of x given by

$$d_n^\alpha x = \Big[\bigcup_{\substack{a \subset x \\ \dim a \leq n}} a\Big] \setminus \Big[\bigcup_{\substack{b \subset x \\ \dim b = n+1}} (b \setminus \partial^\alpha b)\Big],$$

where the unions are over atoms a and b.

If x and y are subsets and $n \geq 0$, then the composite $x \#_n y$ is defined if and only if $x \cap y = d_n^+ x = d_n^- y$; if $x \#_n y$ is defined, then $x \#_n y = x \cup y$.

A molecule *is a subset generated from the atoms by applying the composition operations $\#_0$, $\#_1$, ...*

Next we define the terms 'subcomplex', 'finite-dimensional' and 'globelike'.

5.4. DEFINITION. *A* subcomplex *of an atomic complex is a subset which is a union of atoms.*

A subcomplex x is finite-dimensional *if $d_n^- x = d_n^+ x = x$ for some n.*

A subcomplex x is globelike *if it satisfies the following conditions:*

(i) *$d_n^\alpha x$ is a subcomplex for all $n \geq 0$ and for all α;*

(ii) *$d_m^\beta d_n^\alpha x = d_m^\beta x$ for $0 \leq m < n$ and for all α, β.*

Note that, if the molecules form a partial ω-category, then they must necessarily be finite-dimensional globelike subcomplexes.

We shall now show that the finite-dimensional globelike subcomplexes of an atomic complex form a partial ω-category (Theorem 5.12). We need several preliminary results (Propositions 5.5–5.11).

5.5. PROPOSITION. (i) *Let a and c be distinct atoms in an atomic complex such that* $\operatorname{Int} c \cap a \neq \emptyset$. *Then* $\dim c < \dim a$ *and $c \subset \partial^\alpha a$ for some α.*

(ii) *Let a be a p-dimensional atom in an atomic complex. Then*

$$d_n^\alpha a = \begin{cases} d_n^\alpha(\partial^- a \cup \partial^+ a) & \text{for } 0 \leq n < p-1, \\ \partial^\alpha a & \text{for } n = p-1 \geq 0, \\ a & \text{for } n \geq p. \end{cases}$$

(iii) *Let x be a subcomplex of an atomic complex contained in the union of a family of atoms of dimension at most n. Then $d_n^\alpha x = x$.*

PROOF. (i) We use induction on $\dim a$. Since a and c are distinct, $\operatorname{Int} c \cap \operatorname{Int} a = \emptyset$; therefore $\operatorname{Int} c \cap \partial^\alpha a \neq \emptyset$ for some α. Since $\partial^\alpha a$ is a union of atoms of dimension less than that of a, we have $\operatorname{Int} c \cap b \neq \emptyset$ for some atom $b \subset \partial^\alpha a$ such that $\dim b < \dim a$. If $c \neq b$ then the inductive hypothesis implies that $\dim c < \dim b$ and $c \subset \partial^\beta b$ for some β. Whether or not $c = b$, we then get $\dim c \leq \dim b < \dim a$ and $c \subset b \subset \partial^\alpha a$.

(ii) From part (i) the atoms contained in a are a itself and the atoms contained in $\partial^- a$ or $\partial^+ a$, and the atoms contained in $\partial^- a$ or $\partial^+ a$ all have dimension less than p. The result follows.

(iii) It suffices to show that $\dim c \leq n$ for every atom c contained in x. But $\operatorname{Int} c$ is a non-empty subset of c, so $\operatorname{Int} c \cap a \neq \emptyset$ for some atom a in the given family, and it follows from part (i) that $\dim c \leq \dim a \leq n$ as required. ∎

5.6. PROPOSITION. *If x is a globelike subcomplex of an atomic complex, then $d_m^\beta d_n^\alpha x = d_n^\alpha x$ for $m \geq n$.*

PROOF. Since x is globelike, $d_n^\alpha x$ is a union of atoms. By definition, $d_n^\alpha x$ is contained in a union of atoms of dimension at most n. The result now follows from Proposition 5.5(iii). ∎

5.7. PROPOSITION. *If x is a globelike subcomplex of an atomic complex then the $d_n^\alpha x$ are finite-dimensional globelike subcomplexes.*

PROOF. By Definition 5.4, $d_n^\alpha x$ is a subcomplex. By Proposition 5.6, $d_n^\alpha x$ is finite-dimensional. It remains to show that $d_n^\alpha x$ is globelike. We must show that the $d_m^\beta d_n^\alpha x$ are subcomplexes and that $d_p^\gamma d_m^\beta d_n^\alpha x = d_p^\gamma d_n^\alpha x$ for $p < m$.

Suppose that $m \geq n$. Then $d_m^\beta d_n^\alpha x = d_n^\alpha x$ by Proposition 5.6. Since x is globelike, $d_m^\beta d_n^\alpha x$ is a subcomplex, and for $p < m$ we trivially get $d_p^\gamma d_m^\beta d_n^\alpha x = d_p^\gamma d_n^\alpha x$.

Now suppose that $m < n$. Since x is globelike, $d_m^\beta d_n^\alpha x = d_m^\beta x$, which is a subcomplex, and for $p < m$ we get

$$d_p^\gamma d_m^\beta d_n^\alpha x = d_p^\gamma d_m^\beta x = d_p^\gamma x = d_p^\gamma d_n^\alpha x.$$

∎

5.8. PROPOSITION. *If x and y are subcomplexes of an atomic complex such that $y \subset x$, then $d_n^\alpha x \cap y \subset d_n^\alpha y$.*

PROOF. Let ξ be a point of $d_n^\alpha x \cap y$; we must show that $\xi \in d_n^\alpha y$. It suffices to show that $\xi \in c \subset y$ for some atom c of dimension at most n. To do this, let c be an atom containing ξ which is of minimal dimension. Since $\xi \in d_n^\alpha x$, we must have $\dim c \leq n$. From Definition 5.1(ii) we see that $\xi \in \operatorname{Int} c$. Since y is a subcomplex, $\operatorname{Int} c \cap a \neq \emptyset$ for some atom a such that $a \subset y$. By Proposition 5.5(i), $c \subset a$; therefore $c \subset y$. This completes the proof. ∎

5.9. PROPOSITION. *If y and z are subcomplexes of an atomic complex, then*

$$d_n^\alpha(y \cup z) = (d_n^\alpha y \cap d_n^\alpha z) \cup (d_n^\alpha y \setminus z) \cup (d_n^\alpha z \setminus y).$$

PROOF. By Proposition 5.8, $d_n^\alpha(y \cup z) \cap y \subset d_n^\alpha y$ and $d_n^\alpha(y \cup z) \cap z \subset d_n^\alpha z$. Since $d_n^\alpha(y \cup z) \subset y \cup z$, it follows that the set on the left is contained in the set on the right.

To show that the set on the right is contained in the set on the left, it suffices to show that every $(n+1)$-dimensional atom b contained in $y \cup z$ is contained in y or z. But $\operatorname{Int} b$ is a non-empty subset of b, so we must have $\operatorname{Int} b \cap a \neq \emptyset$ for some atom a contained in y or z. By Proposition 5.5(i), $b \subset a$, so $b \subset y$ or $b \subset z$ as required.

This completes the proof. ∎

5.10. PROPOSITION. *If y and z are globelike subcomplexes of an atomic complex such that $y \#_p z$ is defined, then*

$$\begin{gathered} d_n^\alpha(y \#_p z) = d_n^\alpha y = d_n^\alpha z \text{ for } 0 \leq n < p, \\ d_p^-(y \#_p z) = d_p^- y, \ d_p^+(y \#_p z) = d_p^+ z, \\ d_n^\alpha(y \#_p z) = d_n^\alpha y \#_p d_n^\alpha z \text{ for } n > p. \end{gathered}$$

PROOF. We use the formula for $d_n^\alpha(y \#_p z) = d_n^\alpha(y \cup z)$ given in Proposition 5.9.

For $0 \leq n < p$ we have $d_n^\alpha y = d_n^\alpha d_p^+ y = d_n^\alpha d_p^- z = d_n^\alpha z$, because y and z are globelike, and the result follows.

For $n = p$ we have

$$\begin{aligned} d_p^-(y \#_p z) &= (d_p^- y \cap d_p^- z) \cup (d_p^- y \setminus z) \cup (d_p^- z \setminus y) \\ &= [d_p^- y \cap (y \cap z)] \cup (d_p^- y \setminus z) \cup [(y \cap z) \setminus y] \\ &= (d_p^- y \cap z) \cup (d_p^- y \setminus z) \cup \emptyset \\ &= d_p^- y, \end{aligned}$$

and similarly $d_p^+(y \#_p z) = d_p^+ z$.

Now suppose that $n > p$. We have

$$y \cap z = d_p^+ y = d_p^+ d_n^\alpha y \subset d_n^\alpha y$$

and

$$y \cap z = d_p^- z = d_p^- d_n^\alpha z \subset d_n^\alpha z,$$

so

$$d_n^\alpha y \cap d_n^\alpha z = y \cap z = d_p^+ d_n^\alpha y = d_p^- d_n^\alpha z.$$

It follows that $d_n^\alpha y \#_p d_n^\alpha z$ is defined. It also follows that

$$\begin{aligned} d_n^\alpha(y \#_p z) &= (d_n^\alpha y \cap d_n^\alpha z) \cup (d_n^\alpha y \setminus z) \cup (d_n^\alpha z \setminus y) \\ &= [(d_n^\alpha y \cap z) \cup (d_n^\alpha z \cap y)] \cup (d_n^\alpha y \setminus z) \cup (d_n^\alpha z \setminus y) \\ &= d_n^\alpha y \cup d_n^\alpha z \\ &= d_n^\alpha y \#_p d_n^\alpha z. \end{aligned}$$

This completes the proof. ∎

5.11. Proposition. *Let y and z be globelike subcomplexes of an atomic complex such that $y \#_p z$ is defined. Then $y \#_p z$ is a globelike subcomplex, which is finite-dimensional if y and z are finite-dimensional.*

Proof. Obviously $y \#_p z$ is a subcomplex.

To show that $y \#_p z$ is globelike, we must show that the $d_n^\alpha(y \#_p z)$ are subcomplexes and that $d_m^\beta d_n^\alpha(y \#_p z) = d_m^\beta(y \#_p z)$ for $m < n$. But $d_n^\alpha y$ and $d_n^\alpha z$ are subcomplexes, so $d_n^\alpha(y \#_p z)$ is a subcomplex by Proposition 5.10. When $m < n \leq p$, it is straightforward to check that $d_m^\beta d_n^\alpha(y \#_p z) = d_m^\beta(y \#_p z)$. When $m < n$ and $n > p$, one can compute

$$d_m^\beta d_n^\alpha(y \#_p z) = d_m^\beta(d_n^\alpha y \#_p d_n^\alpha z)$$

by Proposition 5.10, since $d_n^\alpha y$ and $d_n^\alpha z$ are globelike by Proposition 5.7, and one finds that $d_m^\beta d_n^\alpha(y \#_p z) = d_m^\beta(y \#_p z)$ in all cases.

Finally, suppose that y and z are finite-dimensional. Then y and z are unions of atoms of bounded dimensions, so $y \#_p z$ is a union of atoms of bounded dimensions. It now follows from Proposition 5.5(iii) that $y \#_p z$ is finite-dimensional.

This completes the proof. ∎

5.12. Theorem. *The finite-dimensional globelike subcomplexes in an atomic complex form a partial ω-category which is closed under the composition operations.*

Proof. We must verify the conditions of Definition 2.1.

By Proposition 5.7, if x is a finite-dimensional globelike subcomplex then $d_n^\alpha x$ is a finite-dimensional globelike subcomplex.

By Proposition 5.11, the class of finite-dimensional globelike subcomplexes is closed under the composition operations (where defined).

Condition (i) holds by construction (see Definition 5.3).

Condition (ii) holds by Definition 5.4 and Proposition 5.6.

As to condition (iii), let x be a finite-dimensional globelike subcomplex. Then $d_n^- x \cap x = d_n^- x = d_n^+ d_n^- x$ by Proposition 5.6, so the composite $d_n^- x \#_n x$ exists, and $d_n^- x \#_n x = d_n^- x \cup x = x$. Similarly $x \#_n d_n^+ x = x$.

Condition (iv) holds by Proposition 5.10.

As to condition (v), suppose that $(x \#_n y) \#_n z$ exists. Then

$$y \cap z \subset (x \#_n y) \cap z = d_n^- z = d_n^+(x \#_n y) = d_n^+ y,$$

using condition (iv). Since $d_n^+ y \subset y$ and $d_n^- z \subset z$, it follows that $y \cap z = d_n^+ y = d_n^- z$; therefore $y \#_n z$ exists. We now get

$$x \cap z \subset (x \#_n y) \cap z = d_n^+ y \subset y,$$

so

$$x \cap (y \#_n z) = x \cap y = d_n^+ x = d_n^- y = d_n^-(y \#_n z),$$

using condition (iv) again. Therefore $x \#_n (y \#_n z)$ exists. Clearly

$$(x \#_n y) \#_n z = x \cup y \cup z = x \#_n (y \#_n z).$$

By a similar argument, this equation also holds if its right side is defined.

Condition (vi) is proved in a similar way. Suppose that $(x' \#_n y') \#_m (x'' \#_n y'')$ exists, where $m < n$. We then have

$$(x' \#_n y') \cap (x'' \#_n y'') = d_m^+(x' \#_n y') = d_m^+ x'$$

and

$$(x' \#_n y') \cap (x'' \#_n y'') = d_m^-(x'' \#_n y'') = d_m^- x''.$$

Since $x' \cap x'' \subset (x' \#_n y') \cap (x'' \#_n y'')$, it follows that $x' \cap x'' = d_m^+ x' = d_m^- x''$; therefore $x' \#_m x''$ exists. Similarly $y' \#_m y''$ exists. Using condition (iv), we get

$$d_n^+(x' \#_m x'') = d_n^+ x' \#_m d_n^+ x'' = d_n^- y' \#_m d_n^- y'' = d_n^-(y' \#_m y'').$$

We also have $x' \cap y' = d_n^+ x' \subset d_n^+(x' \#_m x'')$ and

$$x' \cap y'' \subset (x' \#_n y') \cap (x'' \#_n y'') = d_m^+(x' \#_n y') = d_m^+ x' = d_m^+ d_n^+ x' \subset d_n^+ x' \subset d_n^+(x' \#_m x''),$$

etc., so

$$(x' \#_m x'') \cap (y' \#_m y'') = d_n^+(x' \#_m x'') = d_n^-(y' \#_m y'').$$

Therefore $(x' \#_m x'') \#_n (y' \#_m y'')$ exists. It is clear that

$$(x' \#_n y') \#_m (x'' \#_n y'') = (x' \#_m x'') \#_n (y' \#_m y'').$$

Finally, for condition (vii), let x be a finite-dimensional globelike complex. By definition, there exists n such that $d_n^- x = d_n^+ x = x$. By Proposition 5.6, there exists p such that $d_n^\alpha x = x$ if and only if $n \geq p$.

This completes the proof. ∎

We now know that the finite-dimensional globelike subcomplexes of an atomic complex form a well-behaved partial ω-category. An atom belongs to this partial ω-category if and only if it is globelike, and we now give necessary and sufficient conditions for this to happen.

5.13. Proposition. *Let a be a p-dimensional atom in an atomic complex. If $p = 0$ or $p = 1$ then a is globelike. If $p \geq 2$ then the following conditions are necessary and sufficient for a to be globelike*:

(i) *the $d_{p-1}^\alpha a$ are globelike*;

(ii) *$d_{p-2}^\beta d_{p-1}^\alpha a = d_{p-2}^\beta a$ for all signs α and β.*

Proof. Suppose that $p = 0$. By Proposition 5.5(ii), $d_n^\alpha a = a$ for $n \geq 0$, and it easily follows that a is globelike.

Suppose that $p = 1$. By Proposition 5.5(ii), $d_0^\alpha a = \partial^\alpha a$, which is a subcomplex by Definition 5.1(ii), and $d_n^\alpha a = a$ for $n \geq 1$. Again it follows that a is globelike.

From now on, let p be at least 2.

Suppose that a is globelike. Then condition (i) holds by Proposition 5.7 and condition (ii) holds by Definition 5.4.

Conversely, suppose that conditions (i) and (ii) hold. We must show that $d_n^\alpha a$ is a subcomplex for all n, and that $d_m^\beta d_n^\alpha a = d_m^\beta a$ for $0 \le m < n$.

Suppose that $n \ge p$. Then $d_n^\alpha a = a$ by Proposition 5.5(ii), so $d_n^\alpha a$ is a subcomplex and $d_m^\beta d_n^\alpha a = d_m^\beta a$ trivially.

By Proposition 5.5(ii), $d_{p-1}^\alpha a = \partial^\alpha a$, which is a subcomplex by Definition 5.1(ii).

Let m be such that $0 \le m < p-1$. By conditions (i) and (ii),

$$d_m^\beta d_{p-1}^\alpha a = d_m^\beta d_{p-2}^\beta d_{p-1}^\alpha a = d_m^\beta d_{p-2}^\beta a$$

(in the case $m = p-2$ this uses Proposition 5.6). Similarly $d_m^\beta d_{p-1}^{-\alpha} a = d_m^\beta d_{p-2}^\beta a$. By Proposition 5.5(ii), $d_m^\beta a = d_m^\beta (d_{p-1}^\alpha a \cup d_{p-1}^{-\alpha} a)$. Since $d_m^\beta d_{p-1}^\alpha a = d_m^\beta d_{p-1}^{-\alpha} a$, it follows from Proposition 5.9 that $d_m^\beta d_{p-1}^\alpha a = d_m^\beta a$.

Finally, suppose that $n < p-1$. We have already seen that $d_n^\alpha a = d_n^\alpha d_{p-1}^- a$. Since $d_{p-1}^- a$ is globelike, $d_n^\alpha a$ is a subcomplex, and for $m < n$ we also get $d_m^\beta d_n^\alpha a = d_m^\beta d_n^\alpha d_{p-1}^- a = d_m^\beta d_{p-1}^- a = d_m^\beta a$.

This completes the proof. ■

The main theorem, characterising ω-complexes, is now as follows.

5.14. Theorem. *For an atomic complex to be an ω-complex the following conditions are necessary and sufficient.*

(i) *If a is an atom and* $\dim a = p > 0$*, then the $d_{p-1}^\alpha a$ are molecules.*

(ii) *If a is an atom and* $\dim a = p > 1$ *then $d_{p-2}^\beta d_{p-1}^\alpha a = d_{p-2}^\beta a$ for all signs α and β.*

This theorem is satisfactory, in that it gives necessary and sufficient conditions. The drawback is that condition (i) may be hard to verify, because it may be hard to recognise molecules; for an example of the work that can be involved, see Section 6, especially Example 6.19. In the most important cases one can use characterisations by well-formedness conditions ([5], [11], [12]); see also Corollary 4.17.

Proof. Clearly the conditions hold in an atomic complex which is an ω-complex.

Conversely, suppose that the conditions hold. Let x be a molecule; we first show that x is globelike and that the $d_n^\alpha x$ are molecules. We use induction on p, the maximum of the dimensions of the atoms in a decomposition of x. By Propositions 5.10 and 5.11, the results hold for x if they hold for each atomic factor of x; we may therefore assume that x is a p-dimensional atom.

If $p > 0$ then the $d_{p-1}^\alpha x$ are molecules by condition (i), and they are composites of atoms of dimension at most $p-1$ by Proposition 5.5(i), so they are globelike by the inductive hypothesis. If $p > 1$ then $d_{p-2}^\beta d_{p-1}^\alpha x = d_{p-2}^\beta x$ by condition (ii). For any value of p, it now follows from Proposition 5.13 that x is globelike. If $n \ge p$ then $d_n^\alpha x = x$ by Proposition 5.5(ii), so $d_n^\alpha x$ is a molecule. If $n < p$ then $d_n^\alpha x = d_n^\alpha d_{p-1}^\alpha x$ because x is globelike, so $d_n^\alpha x$ is a molecule by the inductive hypothesis.

We now see that the class of molecules consists of of finite-dimensional globelike subcomplexes and is closed under the operations d_n^α. Since the finite-dimensional globelike subcomplexes form a partial ω-category (Theorem 5.12), it follows that the molecules form a sub-partial ω-category, which clearly produces an ω-complex. By Proposition 5.5(ii), the functions dim, ∂^- and ∂^+ which are given by this ω-complex structure coincide with those given as part of the original atomic complex structure. Therefore the original atomic complex is an ω-complex, as required.

This completes the proof. ∎

6. Products

In this section we give the standard product construction for atomic complexes, and we outline a proof that the product of two ω-complexes is an ω-complex; most of the details are given in Section 8. For the most important special cases, there is an easier proof in [12], 5, and there is a more difficult proof covering more cases in [10], 7. The proof given here is completely general and easier than that of [10]; it amounts to an explicit algorithm for decomposing molecules in products of ω-complexes. The method can also be applied to joins; see [12] and [10].

Products of atomic complexes are modelled on products of cell complexes, with sign conventions taken from homological algebra, in the following way.

6.1. Definition. *Let K and L be atomic complexes; then the product set $K \times L$ is made into an atomic complex as follows. The atoms of $K \times L$ are the sets of the form $a \times b$ such that a is an atom in K and b is an atom in L. The structure functions are given by*

$$\dim(a \times b) = \dim a + \dim b$$

and

$$\partial^\gamma(a \times b) = (a \times \partial^\delta b) \cup (\partial^\gamma a \times b),$$

where $\delta = (-)^{\dim a}\gamma$.

It is straightforward to check that a product of atomic complexes is an atomic complex.

Up to coherent isomorphisms, the product construction is associative and has 0-dimensional globes (see Example 2.4) as identities.

We wish to prove that a product of ω-complexes is itself an ω-complex, by verifying the conditions of Theorem 5.14; in particular, we must show that $\partial^\gamma(a \times b)$ is a molecule when $a \times b$ is a product atom of positive dimension. Suppose that a and b have positive dimensions i and j; then

$$\partial^\gamma(a \times b) = (d_i^\alpha a \times d_{j-1}^\delta b) \cup (d_{i-1}^\gamma a \times d_j^\beta b),$$

where $\delta = (-)^i\gamma$ and where α and β are arbitrary. It is convenient to represent this subcomplex of $a \times b$ by the matrix

$$\begin{bmatrix} i & \alpha & j-1 & \delta \\ i-1 & \gamma & j & \beta \end{bmatrix},$$

and in general we make the following convention.

6.2. Notation. *Let x and y be molecules in ω-complexes and let λ be a matrix with rows of the form $[i\ \alpha\ j\ \beta]$ such that i and j are non-negative integers and α and β are signs. Then $\lambda(x,y)$ denotes the union of the corresponding complexes $d_i^\alpha x \times d_j^\beta y$.*

We aim to show that $\lambda(x,y)$ is a molecule when λ is a *molecular* matrix in the sense of the following definition.

6.3. Definition. *A* molecular *matrix is a non-empty matrix of the form*

$$\begin{bmatrix} i_0 & \alpha_0 & j_0 & \beta_0 \\ \vdots & \vdots & \vdots & \vdots \\ i_r & \alpha_r & j_r & \beta_r \end{bmatrix}$$

such that the i_q and j_q are non-negative integers, the α_q and β_q are signs, and

$$\begin{aligned} &i_0 > i_1 > \ldots > i_r, \\ &j_0 < j_1 < \ldots < j_r, \\ &\beta_{q-1} = -(-)^{i_q}\alpha_q \text{ for } 1 \leq q \leq r. \end{aligned}$$

For example, let

$$\lambda = \begin{bmatrix} i & \alpha & j-1 & \delta \\ i-1 & \gamma & j & \beta \end{bmatrix}$$

with $\delta = (-)^i\gamma$; then λ is a molecular matrix. Another important example appears in the next result.

6.4. Theorem. *Let x and y be molecules in ω-complexes. Then $d_n^\gamma(x \times y) = \lambda(x,y)$, where*

$$\lambda = \begin{bmatrix} n & \gamma & 0 & (-)^n\gamma \\ n-1 & \gamma & 1 & (-)^{n-1}\gamma \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \gamma & n & \gamma \end{bmatrix}.$$

The proof is given in Section 8.

We now define operations d_n^γ on molecular matrices; eventually we will show that they represent the corresponding operations on complexes.

6.5. Definition. *Let λ be a molecular matrix, let n be a non-negative integer and let γ be a sign. Then $d_n^\gamma\lambda$ is the matrix defined as follows. Form a block decomposition*

$$\lambda = \begin{bmatrix} \lambda_0 \\ \vdots \\ \lambda_t \end{bmatrix}$$

such that consecutive rows $[i'\ \alpha'\ j\ \beta]$ and $[i\ \alpha\ j'\ \beta']$ lie in the same block if and only if $j + i \geq n$. Then

$$d_n^\gamma\lambda = \begin{bmatrix} \nu_0 \\ \vdots \\ \nu_t \end{bmatrix},$$

where ν_s is constructed from λ_s as follows. Let the top row of λ_s start with $[i\ \alpha]$ and the bottom row of λ_s end with $[j\ \beta]$. If $i > n$, let $[i^*\ \alpha^*] = [n\ \gamma]$; otherwise, let $[i^*\ \alpha^*] = [i\ \alpha]$. If $j > n$, let $[j^*\ \beta^*] = [n\ \gamma]$; otherwise, let $[j^*\ \beta^*] = [j\ \beta]$. If $i^* + j^* \le n$, let $\nu_s = [i^*\ \alpha^*\ j^*\ \beta^*]$; otherwise, let

$$\nu_s = \begin{bmatrix} i^* & \alpha^* & n-i^* & (-)^{i^*}\gamma \\ i^*-1 & \gamma & n-i^*+1 & (-)^{i^*-1}\gamma \\ i^*-2 & \gamma & n-i^*+2 & (-)^{i^*-2}\gamma \\ \vdots & \vdots & \vdots & \vdots \\ n-j^*+1 & \gamma & j^*-1 & (-)^{n-j^*+1}\gamma \\ n-j^* & \gamma & j^* & \beta^* \end{bmatrix}.$$

For example, if

$$\lambda = \begin{bmatrix} 5 & - & 2 & - \\ 4 & + & 3 & - \\ 1 & - & 4 & - \\ 0 & + & 7 & + \end{bmatrix}$$

then

$$d_6^-\lambda = \begin{bmatrix} 5 & - & 1 & + \\ 4 & - & 2 & - \\ 3 & - & 3 & - \\ 1 & - & 4 & - \\ 0 & + & 6 & - \end{bmatrix}.$$

In Definition 6.5, note that a row $[i\ \alpha\ j\ \beta]$ in λ with $i + j \le n$ must form an entire block λ_s by itself, and that $\nu_s = \lambda_s$ in such cases. Note also that $[i^*\ \alpha^*] = [i\ \alpha]$ except possibly for the top block λ_0, and that $[j^*\ \beta^*] = [j\ \beta]$ except possibly for the bottom block λ_t.

The operations d_n^γ have the following properties.

6.6. Theorem. *Let λ be a molecular matrix, let n be a non-negative integer, and let γ be a sign. Then the following hold:*

(i) *$d_n^\gamma\lambda$ is a molecular matrix;*

(ii) *if x and y are molecules in ω-complexes, then $d_n^\gamma[\lambda(x, y)] = (d_n^\gamma\lambda)(x, y)$.*

The proof is given in Section 8.

Next we define operations $\#_n$ on molecular matrices. Let μ^- and μ^+ be molecular matrices such that $d_n^+\mu^- = d_n^-\mu^+$. We define $\mu^- \#_n \mu^+$ to be a molecular matrix consisting of certain rows from μ^- and μ^+. For this to make sense, we must of course show that the required rows really do form a molecular matrix if arranged in the right order; we shall do this in Section 8.

6.7. Definition. *Let μ^- and μ^+ be molecular matrices such that $d_n^+\mu^- = d_n^-\mu^+$. Then $\mu^- \#_n \mu^+$ is the molecular matrix consisting of the rows $[i\ \alpha\ j\ \beta]$ with $i + j > n$ which appear in μ^- or μ^+, and of the rows $[i\ \alpha\ j\ \beta]$ with $i + j \le n$ which are common to both factors.*

For example, let

$$\mu^- = \begin{bmatrix} 5 & - & 0 & + \\ 4 & - & 2 & + \\ 2 & - & 3 & - \\ 1 & - & 4 & + \\ 0 & - & 5 & + \end{bmatrix}, \quad \mu^+ = \begin{bmatrix} 6 & + & 0 & - \\ 5 & - & 1 & + \\ 3 & + & 2 & + \\ 2 & - & 4 & + \\ 0 & - & 5 & + \end{bmatrix};$$

then

$$d_5^+\mu^- = d_5^-\mu^+ = \begin{bmatrix} 5 & - & 0 & + \\ 4 & - & 1 & + \\ 3 & + & 2 & + \\ 2 & - & 3 & - \\ 1 & - & 4 & + \\ 0 & - & 5 & + \end{bmatrix}$$

and

$$\mu^- \#_5 \mu^+ = \begin{bmatrix} 6 & + & 0 & - \\ 5 & - & 1 & + \\ 4 & - & 2 & + \\ 2 & - & 4 & + \\ 0 & - & 5 & + \end{bmatrix}.$$

We have the following result.

6.8. THEOREM. *Let x and y be molecules in ω-complexes, and let μ^- and μ^+ be molecular matrices such that $d_n^+\mu^- = d_n^-\mu^+$. Then*

$$\mu^-(x,y) \#_n \mu^+(x,y) = (\mu^- \#_n \mu^+)(x,y).$$

The proof is given in Section 8.

Conversely, we have the following result.

6.9. THEOREM. *A matrix is molecular if and only if it has a decomposition into single-rowed matrices.*

The proof is given in Section 8.

Theorems 6.8 and 6.9 enable us to decompose sets into factors which are products of molecules. We actually wish to decompose them into atoms. We therefore need to decompose $x \times y$ whenever x or y is a proper composite; in other words, we must consider subsets of products of the forms $(x_- \#_p x_+) \times y$ and $x \times (y_- \#_p y_+)$. The two cases are similar, and we shall concentrate on $(x_- \#_p x_+) \times y$. We index its subsets by pairs of molecular matrices according to the following convention, which is designed to interact well with Proposition 5.9.

6.10. NOTATION. *Let x_- and x_+ be molecules in an ω-complex, let y be a molecule in an ω-complex, and let $\Lambda = (\Lambda_-, \Lambda_+)$ be a pair of molecular matrices. Then we write*

$$\Lambda(x_-, x_+; y) = [\Lambda_-(x_-,y) \cap \Lambda_+(x_+,y)] \cup [\Lambda_-(x_-,y) \setminus (x_+ \times y)] \cup [\Lambda_+(x_+,y) \setminus (x_- \times y)].$$

The relevant pairs turn out to be the *left p-compatible* pairs, which are defined as follows.

6.11. Definition. *A pair of molecular matrices* $\Lambda = (\Lambda_-,\, \Lambda_+)$ *is* left p-compatible *if*

$$\Lambda_-(x,y) \cap (d_p^- x \times y) \subset \Lambda_+(x,y) \quad \text{and} \quad \Lambda_+(x,y) \cap (d_p^+ x \times y) \subset \Lambda_-(x,y)$$

for all molecules x and y in all ω-complexes.

For example, a pair of the form (λ, λ) is left p-compatible; we also have certain *basic* examples as follows.

6.12. Example. Let

$$\Lambda = ([i_- \; \alpha_- \; j \; \beta],\; [i_+ \; \alpha_+ \; j \; \beta])$$

such that $i_- = i_+ < p$ and $\alpha_- = \alpha_+$, or such that $i_- \geq p$ and $[i_+ \; \alpha_+] = [p \; -]$, or such that $i_+ \geq p$ and $[i_- \; \alpha_-] = [p \; +]$; then Λ is left p-compatible. Left p-compatible pairs of these forms will be called *basic.*

If $x_- \#_p x_+$ is a composite of molecules in an ω-complex, if y is a molecule in an ω-complex, and if Λ is a basic left p-compatible pair of molecular matrices, then $\Lambda(x_-, x_+; y)$ has the form $d_i^\alpha x_\theta \times d_j^\beta y$.

Left p-compatible pairs can be characterised more explicitly as follows.

6.13. Proposition. *A pair of molecular matrices $\Lambda = (\Lambda_-,\, \Lambda_+)$ is left p-compatible if and only if the following conditions hold*:

(i) Λ_- *and* Λ_+ *have the same rows* $[i \; \alpha \; j \; \beta]$ *such that* $i < p$;

(ii) *if one of the matrices* Λ_- *and* Λ_+ *has rows beginning with* $[i \; \alpha]$ *such that* $i \geq p$, *then so also does the other, and the last two such rows in* Λ_- *and* Λ_+ *end with the same pair* $[j \; \beta]$;

(iii) *if* Λ_θ *has top row* $[p \; \theta \; j \; \beta]$, *then* $\Lambda_{-\theta}$ *has top row* $[p \; \theta \; j \; \beta]$;

(iv) *if* Λ_θ *has a row* $[p \; \theta \; j \; \beta]$ *preceded by a row ending with* $[l \; \delta]$, *then* $\Lambda_{-\theta}$ *has a row* $[p \; \theta \; j \; \beta]$ *either at the top or preceded by a row ending with* $[l' \; \delta]$ *such that* $l' \leq l$.

The proof is given in Section 8.

We define operations d_n^γ and $\#_n$ on pairs in the obvious way, as follows.

6.14. Definition. *Let $\Lambda = (\Lambda_-,\, \Lambda_+)$ be a pair of molecular matrices. Then $d_n^\gamma \Lambda$ is the pair of molecular matrices given by $d_n^\gamma \Lambda = (d_n^\gamma \Lambda_-,\, d_n^\gamma \Lambda_+)$.*

Let $\mathrm{M}^- = (\mathrm{M}_-^-,\, \mathrm{M}_+^-)$ and $\mathrm{M}^+ = (\mathrm{M}_-^+,\, \mathrm{M}_+^+)$ be pairs of molecular matrices such that $d_n^+ \mathrm{M}^- = d_n^- \mathrm{M}^+$. Then $\mathrm{M}^- \#_n \mathrm{M}^+$ is the pair of molecular matrices given by

$$\mathrm{M}^- \#_n \mathrm{M}^+ = (\mathrm{M}_-^- \#_n \mathrm{M}_-^+,\, \mathrm{M}_+^- \#_n \mathrm{M}_+^+).$$

We have the following results on left p-compatible pairs.

6.15. Theorem. *Let $x_- \#_p x_+$ be a composite of molecules in an ω-complex and let y be a molecule in an ω-complex.*

(i) *If Λ is a left p-compatible pair of molecular matrices, then $\Lambda(x_-, x_+; y)$ is a subcomplex of $(x_- \#_p x_+) \times y$.*

(ii) *If Λ is a left p-compatible pair of molecular matrices then $d_n^\gamma \Lambda$ is left p-compatible and $d_n^\gamma[\Lambda(x_-, x_+; y)] = (d_n^\gamma \Lambda)(x_-, x_+; y)$.*

(iii) *If* M^- *and* M^+ *are left* p*-compatible pairs of molecular matrices such that* $d_n^+\mathrm{M}^- = d_n^-\mathrm{M}^+$*, then* $\mathrm{M}^- \#_n \mathrm{M}^+$ *is left* p*-compatible and*

$$\mathrm{M}^-(x_-, x_+; y) \#_n \mathrm{M}^+(x_-, x_+; y) = (\mathrm{M}^- \#_n \mathrm{M}^+)(x_-, x_+; y).$$

The proof is given in Section 8.

6.16. Theorem. *A pair of molecular matrices is left* p*-compatible if and only if it has a decomposition into basic left* p*-compatible pairs.*

The proof is given in Section 8.

There are of course analogous results for right p-compatible pairs applied to products of the form $x\times(y_- \#_p y_+)$. We omit all the details, even the definition of right p-compatibility.

We now use our results to show that various sets are molecules.

6.17. Theorem. *Let* K *and* L *be* ω*-complexes, and let* w *be a subset of* $K \times L$*. Then* w *is a molecule if* $w = \lambda(x, y)$ *such that* x *and* y *are molecules and* λ *is a molecular matrix, or if* $w = \Lambda(x_-, x_+; y)$ *such that* $x_- \#_p x_+$ *is a composite of molecules,* y *is a molecule, and* Λ *is a left* p*-compatible pair of molecular matrices, or if* $w = \Lambda(x; y_-, y_+)$ *such that* x *is a molecule,* $y_- \#_p y_+$ *is a composite of molecules, and* Λ *is a right* p*-compatible pair of molecular matrices.*

Proof. In the first case, by Theorems 6.8 and 6.9, w is a composite of sets of the form $d_i^\alpha x \times d_j^\beta y$; therefore w is a composite of sets of the form $x' \times y'$ such that x' and y' are molecules. In the second case, the same conclusion follows from Theorems 6.15(iii) and 6.16. In the third case, it follows similarly. It therefore suffices to decompose $x' \times y'$ into atoms when x' and y' are molecules. We shall use induction on the dimensions of x' and y'.

Suppose that x' is not an atom. There is then a proper decomposition $x' = x_- \#_p x_+$. By Proposition 3.2(ii), $\dim x_\theta \leq \dim x'$ for each θ. We can express $x' \times y'$ in the form $\Lambda(x_-, x_+; y')$ with Λ left p-compatible (take $\Lambda = (\lambda, \lambda)$ such that $\lambda = [k\ \varepsilon\ l\ \zeta]$ with k and l large). As before, we can decompose $x' \times y'$ into factors of the form $d_i^\alpha x_\theta \times d_j^\beta y'$. By repeating this process until x' is decomposed into atoms, we decompose $x' \times y'$ into factors of the form $x'' \times y''$ such that x'' and y'' are molecules, x'' is an atom or $\dim x'' < \dim x'$, and $y'' = y'$ or $\dim y'' < \dim y'$. By using a decomposition of y' in a similar way and iterating, we eventually decompose $x' \times y'$ into atoms as required.

This completes the proof. ■

We can now deduce the main theorem.

6.18. Theorem. *If* K *and* L *are* ω*-complexes, then* $K \times L$ *is an* ω*-complex.*

Proof. We must verify the conditions of Theorem 5.14. Let c be an atom in $K \times L$; thus $c = a \times b$ for some atoms a and b. It follows that $c = \lambda(a, b)$ for some single-rowed matrix λ. Trivially λ is molecular.

Suppose that $\dim c = q > 0$; we must show that the $d_{q-1}^\gamma c$ are molecules. But $d_{q-1}^\gamma c = (d_{q-1}^\gamma \lambda)(a, b)$ by Theorem 6.6(ii) and $d_{q-1}^\gamma \lambda$ is molecular by Proposition 6.6(i), so $d_{q-1}^\gamma c$ is a molecule by Theorem 6.17.

Suppose that $\dim c = q > 1$; we must show that $d^{\delta}_{q-2}d^{\gamma}_{q-1}c = d^{\delta}_{q-2}c$. But, by repeated applications of Theorem 6.6(ii), it suffices to show that $d^{\delta}_{q-2}d^{\gamma}_{q-1}\lambda = d^{\delta}_{q-2}\lambda$, and this is easily verified.

This completes the proof. ∎

6.19. Example. Let u and v be infinite-dimensional globes as in Example 2.4. Recall that u has two n-dimensional atoms u^-_n and u^+_n for each n, and that $d^{\beta}_m u^{\alpha}_n = u^{\beta}_m$ for $m < n$; the structure of v is similar. If λ is a molecular matrix, then the value of $\lambda(u^{\alpha}_m, v^{\beta}_n)$ is independent of m, α, n and β, provided that m and n are sufficiently large; we shall write $\lambda(u, v)$ for the common value. It is easy to see that the function $\lambda \mapsto \lambda(u, v)$ is injective. By Theorem 6.17, the $\lambda(u, v)$ are molecules in $u \times v$.

Suppose that μ^- and μ^+ are molecular matrices such that $d^+_n[\mu^-(u, v)] = d^-_n[\mu^+(u, v)]$. By Theorem 6.6(ii), $(d^+_n\mu^-)(u, v) = (d^-_n\mu^+)(u, v)$. Since the function $\lambda \mapsto \lambda(u, v)$ is injective, $d^+_n\mu^- = d^-_n\mu^+$. By Theorem 6.8, $\mu^-(u, v) \#_n \mu^+(u, v)$ is defined and is equal to $(\mu^- \#_n \mu^+)(u, v)$. Since the atoms have the form $\lambda(u, v)$, it follows that every molecule has the form $\lambda(u, v)$.

We have now shown that the molecules of $u \times v$ are in one-to-one correspondence with the molecular matrices. Since $\mu^-(u, v) \#_n \mu^+(u, v)$ is defined whenever $d^+_n[\mu^-(u, v)] = d^-_n[\mu^+(u, v)]$, it follows that the molecules of $u \times v$ form an ω-category, not just a partial ω-category. The operations d^{γ}_n and $\#_n$ are given by Definitions 6.5 and 6.7. This recovers results of Street [12], with more explicit formulae.

In a similar way, let $u_- \#_p u_+$ and $v_- \#_p v_+$ be composites of infinite-dimensional globes; then the molecules in $(u_- \#_p u_+) \times v$ and $u \times (v_- \#_p v_+)$ form ω-categories in one-to-one correspondence with left and right p-compatible pairs of molecular matrices.

7. Geometric structures and products

In this section we make two observations relating the geometric structures of Section 4 to the products of Section 6.

Recall that a product of CW complexes is again a CW complex, provided that one uses the compactly generated topology. For geometric ω-complexes there is an analogous result.

7.1. Proposition. *If K and L are geometric ω-complexes and $K \times L$ is given the compactly generated topology, then $K \times L$ is also a geometric ω-complex.*

Proof. By Theorem 6.18, $K \times L$ is an ω-complex; to show that $K \times L$ is a geometric ω-complex we must verify the conditions of Definition 4.4. Conditions (ii) and (iii) are proved in a standard way from Proposition 4.9 and the properties of compactly generated topologies, and it remains to verify condition (i). Let a and b be atoms in K and L, so that $a \times b$ is an atom in $K \times L$; we must show that $a \times b$ is a cone with base

$$\partial^-(a \times b) \cup \partial^+(a \times b).$$

Let $\partial a = \partial^- a \cup \partial^+ a$ and $\partial b = \partial^- b \cup \partial^+ b$. Since K and L are geometric ω-complexes, a and b are cones with bases ∂a and ∂b; also, ∂a and ∂b are compact Hausdorff by Proposition 4.9. By properties of cones, $a \times b$ is a cone with base

$$(a \times \partial b) \cup (\partial a \times b).$$

But

$$(a \times \partial b) \cup (\partial a \times b) = \partial^-(a \times b) \cup \partial^+(a \times b)$$

by Definition 6.1; therefore $a \times b$ is a cone on

$$\partial^-(a \times b) \cup \partial^+(a \times b)$$

as required.

This completes the proof. ∎

The other observation concerns roundness. The geometric theory works best for ω-complexes in which all the atoms are round (see Theorem 4.13, Corollary 4.16 and Corollary 4.17). We therefore want this property to be preserved when we form products; in other words we want a product of round atoms to be round. This is in fact true, as a special case of the following result.

7.2. Proposition. *If x and y are round molecules in ω-complexes, then $x \times y$ is also a round molecule.*

Proof. As in the proof of Theorem 6.18, $x \times y = \lambda(x, y)$ for some single-rowed molecular matrix λ. By Theorem 6.17, $x \times y$ is a molecule. It follows from Theorem 6.4 that $x \times y$ satisfies the conditions of Definition 4.10; therefore $x \times y$ is also round. This completes the proof. ∎

8. Proofs

In this section we give the proofs omitted from Section 6.

Proof of Theorem 6.4 Let x and y be molecules in ω-complexes; we must show that $d_n^\gamma(x \times y)$ is the union of the sets $d_i^\gamma x \times d_{n-i}^{(-)^i\gamma} y$.

Suppose first that $(\xi, \eta) \in d_n^\gamma(x \times y)$; we shall show that $(\xi, \eta) \in d_i^\gamma x \times d_{n-i}^{(-)^i\gamma} y$ for some i. Since (ξ, η) belongs to an atom of dimension at most n contained in $x \times y$, we must have $\xi \in \operatorname{Int} a$ and $\eta \in \operatorname{Int} b$ for some atoms a in x and b in y of dimensions k and l with $k + l \leq n$. By considering $(n+1)$-dimensional atoms in $x \times y$ of the form $a' \times b$ such that $\xi \in a'$, we see that $\xi \in d_{n-l}^\gamma x$. Let i be minimal such that $\xi \in d_i^\gamma x$; thus $k \leq i \leq n - l$. There is then an i-dimensional atom a'' in x such that $\xi \in a'' \setminus \partial^\gamma a''$ (if $i = k$, take $a'' = a$; if $i > k$, use the fact that $\xi \notin d_{i-1}^\gamma x$). Now $\eta \in b \subset y$ with $\dim b = l \leq n - i$, and, by considering $(n+1)$-dimensional atoms in $x \times y$ of the form $a'' \times b'$ such that $\eta \in b'$, we see that $\eta \in d_{n-i}^{(-)^i\gamma} y$. Therefore $(\xi, \eta) \in d_i^\gamma x \times d_{n-i}^{(-)^i\gamma} y$ as required.

Conversely, suppose that $(\xi,\eta)\in d_i^\gamma x\times d_{n-i}^{(-)^i\gamma}y$ for some i; we shall show that $(\xi,\eta)\in d_n^\gamma(x\times y)$. It is clear that (ξ,η) belongs to some atom in $x\times y$ of dimension at most n. Suppose that $a\times b$ is an $(n+1)$-dimensional atom in $x\times y$ containing (ξ,η); we must show that $(\xi,\eta)\in\partial^\gamma(a\times b)$. Let $\dim a=k$, so that $\dim b=n+1-k$. If $i<k$ then Proposition 5.8 gives us

$$\xi\in d_i^\gamma a\subset d_{k-1}^\gamma a=\partial^\gamma a;$$

if $i\geq k$ then Proposition 5.8 gives us

$$\eta\in d_{n-i}^{(-)^i\gamma}b\subset d_{n-k}^{(-)^k\gamma}b=\partial^{(-)^k\gamma}b;$$

in both cases, $(\xi,\eta)\in\partial^\gamma(a\times b)$ as required.

This completes the proof. ■

Proof of Theorem 6.6(i) Let λ be a molecular matrix; we must show that $d_n^\gamma\lambda$ is a molecular matrix. We use the notation of Definition 6.5. It is clear that the consecutive rows inside each block ν_s satisfy the conditions of Definition 6.3, and it remains to consider the passage from one block to the next.

Suppose that ν_{s-1} has bottom row $[i'\ \alpha'\ j\ \beta]$ and ν_s has top row $[i\ \alpha\ j'\ \beta']$; we must show that $i'>i$, $j<j'$, and $\beta=-(-)^i\alpha$. We note that the bottom row of λ_{s-1} and the top row of λ_s have the forms $[i''\ \alpha''\ j\ \beta]$ and $[i\ \alpha\ j''\ \beta'']$; we also note that $i'=i''$ or $i'=n-j$, etc. But $i''>i$ because λ is molecular, and $n-j>i$ because the rows $[i''\ \alpha''\ j\ \beta]$ and $[i\ \alpha\ j''\ \beta'']$ are in different blocks; therefore $i'>i$ in all cases. Similarly $j<j'$. Finally, $\beta=-(-)^i\alpha$ because λ is molecular. ■

Proof of Theorem 6.6(ii) Let λ be a molecular matrix and let x and y be molecules in ω-complexes; we must show that $d_n^\gamma[\lambda(x,y)]=(d_n^\gamma\lambda)(x,y)$.

First we show that $(d_n^\gamma\lambda)(x,y)\subset\lambda(x,y)$; in other words, we show that $d_k^\varepsilon x\times d_l^\zeta y\subset\lambda(x,y)$ for each row $[k\ \varepsilon\ l\ \zeta]$ in $d_n^\gamma\lambda$. It is clear that λ has a row $[i\ \alpha\ j\ \beta]$ with $i\geq k$ and $j\geq l$. There are now three cases.

1. Suppose that $[i\ \alpha]=[k\ \varepsilon]$ or $i>k$; suppose also that $[j\ \beta]=[l\ \zeta]$ or $l>j$. Then $d_k^\varepsilon x\times d_l^\zeta y\subset d_i^\alpha x\times d_j^\beta y\subset\lambda(x,y)$.

2. Suppose that $[i\ \alpha]=[k\ -\varepsilon]$. We must then have $[k\ \varepsilon\ l\ \zeta]=[k\ \gamma\ n-k\ (-)^k\gamma]$; also, the row $[i\ \alpha\ j\ \beta]=[k\ -\gamma\ j\ \beta]$ in λ must be preceded by $[i'\ \alpha'\ j'\ \beta']$ such that $i'>i=k$, $j'\geq n-i=n-k=l$, and $\beta'=(-)^k\gamma=\zeta$. It follows that $d_k^\varepsilon x\times d_l^\zeta y\subset d_{i'}^{\alpha'}x\times d_{j'}^{\beta'}y\subset\lambda(x,y)$.

3. Suppose that $[j\ \beta]=[l\ -\zeta]$. Then $d_k^\varepsilon x\times d_l^\zeta y\subset\lambda(x,y)$ by a similar argument.

This completes the proof that $(d_n^\gamma\lambda)(x,y)\subset\lambda(x,y)$.

To prove that $(d_n^\gamma\lambda)(x,y)\subset d_n^\gamma[\lambda(x,y)]$, it now suffices, by Proposition 5.9, to prove that

$$(d_k^\varepsilon x\times d_l^\zeta y)\cap(d_i^\alpha x\times d_j^\beta y)\subset d_n^\gamma(d_i^\alpha x\times d_j^\beta y)$$

for every row $[k\ \varepsilon\ l\ \zeta]$ in $d_n^\gamma\lambda$ and every row $[i\ \alpha\ j\ \beta]$ in λ. There are again three cases.

1. Suppose that $[i\ \alpha\ j\ \beta]$ is in a block λ_s as in Definition 6.5 and that $[k\ \varepsilon\ l\ \zeta]$ is in the corresponding block of $d_n^\gamma\lambda$. Let the top row of λ_s begin with $[i'\ \alpha']$ and let the bottom

row of λ_s end with $[j'\ \beta']$. We then have $d_i^\alpha x \times d_j^\beta y \subset d_{i'}^{\alpha'} x \times d_{j'}^{\beta'} y$, and from Theorem 6.4 we have $d_k^\varepsilon x \times d_l^\zeta y \subset d_n^\gamma (d_{i'}^{\alpha'} x \times d_{j'}^{\beta'} y)$. It now follows from Proposition 5.8 that

$$(d_k^\varepsilon x \times d_l^\zeta y) \cap (d_i^\alpha x \times d_j^\beta y) \subset d_n^\gamma (d_i^\alpha x \times d_j^\beta y).$$

2. Suppose that $[i\ \alpha\ j\ \beta]$ is in a block λ_s as in Definition 6.5, and suppose that $[k\ \varepsilon\ l\ \zeta]$ is in a block of $d_n^\gamma \lambda$ which comes from a higher block in λ. We then have $k < i$ and $j < n - k$, and it follows from Theorem 6.4 that

$$(d_k^\varepsilon x \times d_l^\zeta y) \cap (d_i^\alpha x \times d_j^\beta y) \subset d_{k+1}^\gamma d_i^\alpha x \times d_{n-k-1}^{(-)^{k+1}\gamma} d_j^\beta y \subset d_n^\gamma (d_i^\alpha x \times d_j^\beta y).$$

3. Suppose that $[i\ \alpha\ j\ \beta]$ is in a block λ_s as in Definition 6.5, and suppose that $[k\ \varepsilon\ l\ \zeta]$ is in a block of $d_n^\gamma \lambda$ which comes from a lower block in λ. We use an argument similar to that for Case 2.

This completes the proof that $(d_n^\gamma \lambda)(x, y) \subset d_n^\gamma [\lambda(x, y)]$.

It remains to show that every point (ξ, η) of $d_n^\gamma [\lambda(x, y)]$ is in $(d_n^\gamma \lambda)(x, y)$. By Proposition 5.9, $(\xi, \eta) \in d_n^\gamma (d_i^\alpha x \times d_j^\beta y)$ for some row $[i\ \alpha\ j\ \beta]$ in λ. It then follows from Theorem 6.4 that $(\xi, \eta) \in (d_n^\gamma \lambda)(x, y)$ except in the following two cases: $\alpha = -\gamma$, the row $[i\ \alpha\ j\ \beta]$ is preceded by $[i'\ \alpha'\ j'\ \beta']$ with $j' \geq n - i$, and $\xi \in d_i^{-\gamma} x \setminus d_i^\gamma x$; or $\beta = -(-)^{n-j}\gamma$, the row $[i\ \alpha\ j\ \beta]$ is followed by $[i'\ \alpha'\ j'\ \beta']$ with $i' \geq n - j$, and $\eta \in d_j^{-(-)^{n-j}\gamma} y \setminus d_j^{(-)^{n-j}\gamma} y$. The argument is similar in both of these cases; we consider the first.

Since $(\xi, \eta) \in d_n^\gamma (d_i^\alpha x \times d_j^\beta y)$ and $\xi \notin d_k^\gamma d_i^\alpha x$ for $k < i$, it follows from Theorem 6.4 that $\eta \in d_{n-i}^{(-)^i \gamma} y$. We automatically have $i' > i$ and $\beta' = (-)^i \gamma$; since also $j' \geq n - i$, it follows that $(\xi, \eta) \in d_{i'}^{\alpha'} x \times d_{j'}^{\beta'} y$. By Proposition 5.8, $(\xi, \eta) \in d_n^\gamma (d_{i'}^{\alpha'} x \times d_{j'}^{\beta'} y)$. But $\xi \notin d_i^\gamma d_{i'}^{\alpha'} x$, so it follows from Theorem 6.4 that $\eta \in d_{n-i-1}^{(-)^{i+1}\gamma} y$. Also $d_n^\gamma \lambda$ has a row $[i+1\ \varepsilon\ n-i-1\ (-)^{i+1}\gamma]$ and $\xi \in d_i^{-\gamma} x \subset d_{i+1}^\varepsilon x$. Therefore $(\xi, \eta) \in (d_n^\gamma \lambda)(x, y)$as required.

This completes the proof. ∎

Justification of Definition 6.7 Let μ^- and μ^+ be molecular matrices such that $d_n^+ \mu^- = d_n^- \mu^+$. Let L be the set of rows $[i\ \alpha\ j\ \beta]$ with $i + j > n$ which appear in μ^- or μ^+ and of the rows $[i\ \alpha\ j\ \beta]$ with $i + j \leq n$ which are common to both. We must show that the rows of L can be formed into a molecular matrix.

Consider the blocks of μ^- and μ^+ as used in Definition 6.5. There are two types of block: those consisting of one or more rows $[i\ \alpha\ j\ \beta]$ with $i + j > n$, and those consisting of a single row $[i\ \alpha\ j\ \beta]$ with $i + j \leq n$. Let K be the set consisting of all the blocks of the first type, and of the blocks of the second type which are common to both factors; then the rows of L are precisely the rows of the members of K. It therefore suffices to list the members of K as $\kappa_0, \ldots, \kappa_t$ such that the bottom row of κ_{s-1} and the top row of κ_s satisfy the conditions of Definition 6.3.

Let $\nu = d_n^+ \mu^- = d_n^- \mu^+$; then the members of K correspond to blocks in ν. Inspection of Definition 6.5 shows that these blocks form the whole of ν and that the blocks corresponding to distinct members of K can overlap by at most one row. If a row of ν lies in an overlap, then it comes from blocks κ^- and κ^+ in μ^- and μ^+ of the first type, the common row has the form $[k\ \gamma\ n-k\ \ -(-)^k\gamma]$, it is the bottom row of $d_n^\gamma \kappa^{-\gamma}$ and the top row of $d_n^{-\gamma}\kappa^\gamma$. We can therefore list the members of K as $\kappa_0, \dots, \kappa_t$ such that the corresponding blocks $\nu_0, \dots, \nu_t$ of ν satisfy the following conditions: the bottom row of ν_{s-1} is immediately above the top row of ν_s or coincides with it; in the case of coincidence, the common row has the form $[k\ \gamma\ n-k\ \ -(-)^k\gamma]$ such that κ_{s-1} and κ_s are blocks of the first type in $\mu^{-\gamma}$ and μ^γ respectively.

We now check that the bottom row of κ_{s-1} and the top row of κ_s satisfy the the conditions of Definition 6.3 in the two possible cases.

In the case of non-coincidence, let the bottom row of ν_{s-1} and the top row of ν_s be $[i'\ \alpha'\ j\ \beta]$ and $[i\ \alpha\ j'\ \beta']$; then the bottom row of κ_{s-1} and the top row of κ_s have the forms $[i''\ \alpha''\ j\ \beta]$ and $[i\ \alpha\ j''\ \beta'']$ with $i'' \geq i'$ and $j' \leq j''$. Since ν is a molecular matrix (Theorem 6.6(i)), we get $i'' \geq i' > i$, $j < j' \leq j''$ and $\beta = -(-)^i\alpha$.

In the case of coincidence, let the common row be $[k\ \gamma\ n-k\ \ -(-)^k\gamma]$. Then the bottom row of κ_{s-1} and the top row of κ_s have the forms $[i''\ \alpha''\ n-k\ \ -(-)^k\gamma]$ and $[k\ \gamma\ j''\ \beta'']$ with $i'' > k$ and $n-k < j''$, and again the conditions are satisfied. ∎

Proof of Theorem 6.8 Let μ^- and μ^+ be molecular matrices such that $d_n^+\mu^- = d_n^-\mu^+$, and let x and y be molecules in ω-complexes. We must show that $\mu^-(x,y) \#_n \mu^+(x,y) = (\mu^- \#_n \mu^+)(x,y)$.

Let $\nu = d_n^+\mu^- = d_n^-\mu^+$ and let $\lambda = \mu^- \#_n \mu^+$. By Theorem 6.6(ii), $d_n^+[\mu^-(x,y)] = d_n^-[\mu^+(x,y)] = \nu(x,y)$. We shall prove the result by showing that

$$\mu^-(x,y) \cap \mu^+(x,y) \subset \nu(x,y)$$

and

$$\mu^-(x,y) \cup \mu^+(x,y) = \lambda(x,y).$$

To show that $\mu^-(x,y) \cap \mu^+(x,y) \subset \nu(x,y)$, let $\sigma^- = [i^-\ \alpha^-\ j^-\ \beta^-]$ and $\sigma^+ = [i^+\ \alpha^+\ j^+\ \beta^+]$ be rows of μ^- and μ^+; we must show that $\sigma^-(x,y) \cap \sigma^+(x,y) \subset \nu(x,y)$. If $i^\gamma + j^\gamma \leq n$ for some γ, then σ^γ is a row in ν and the result follows. We may therefore assume that $i^- + j^- > n$ and $i^+ + j^+ > n$, so that σ^- and σ^+ are rows in λ. Choose γ such that σ^γ comes above $\sigma^{-\gamma}$. Let κ^- and κ^+ be the blocks of μ^- and μ^+ as in Definition 6.5 containing σ^- and σ^+, let ν^- and ν^+ be the corresponding blocks of ν, let $[i'\ \alpha'\ j\ \beta]$ be the bottom row of ν^γ, and let $[i\ \alpha\ j'\ \beta']$ be the top row of $\nu^{-\gamma}$. We then have $j^\gamma < j$ or $[j^\gamma\ \beta^\gamma] = [j\ \beta]$, and similarly $i^{-\gamma} < i$ or $[i^{-\gamma}\ \alpha^{-\gamma}] = [i\ \alpha]$, so that

$$\sigma^-(x,y) \cap \sigma^+(x,y) \subset d_i^\alpha x \times d_j^\beta y.$$

But, as in the justification of Definition 6.7, $[i'\ \alpha'\ j\ \beta]$ comes above $[i\ \alpha\ j'\ \beta']$ in ν or coincides with it, so that $i < i'$ or $[i\ \alpha] = [i'\ \alpha']$. It follows that

$$d_i^\alpha x \times d_j^\beta y \subset d_{i'}^{\alpha'} x \times d_j^\beta y \subset \nu(x,y);$$

therefore $\sigma^-(x,y) \cap \sigma^+(x,y) \subset \nu(x,y)$ as required.

It remains to show that $\mu^-(x,y) \cup \mu^+(x,y) = \lambda(x,y)$. Every row of λ is a row in μ^- or μ^+, so $\lambda(x,y) \subset \mu^-(x,y) \cup \mu^+(x,y)$. Let $\sigma = [i\ \alpha\ j\ \beta]$ be a row of μ^γ which is not a row of λ; we must show that $\sigma(x,y) \subset \lambda(x,y)$. We note that $i + j \leq n$ and that σ is not a row in $\mu^{-\gamma}$. From Definition 6.5, σ is a row in $d_n^{-\gamma}\mu^\gamma = d_n^\gamma\mu^{-\gamma}$. By the proof of Theorem 6.6(ii), $\sigma(x,y) \subset \tau(x,y)$ for some row τ in $\mu^{-\gamma}$; also, since σ is not a row in $\mu^{-\gamma}$, we must have $\tau = [i'\ \alpha'\ j'\ \beta']$ with $i' + j' > n$. It follows that τ is a row in λ; therefore $\sigma(x,y) \subset \lambda(x,y)$ as required.

This completes the proof. ∎

PROOF OF THEOREM 6.9 Obviously a single-rowed matrix is molecular, so a composite of single-rowed matrices is molecular by Definition 6.7. We must prove the converse, that any molecular matrix is a composite of single-rowed matrices.

We shall use the partial ordering of the class of molecular matrices given in the following way: $\mu \leq \lambda$ if $\mu(x,y) \subset \lambda(x,y)$ for all molecules x and y in all ω-complexes. By Theorem 6.8, $\mu^- \leq \mu^- \#_n \mu^+$ and $\mu^+ \leq \mu^- \#_n \mu^+$ whenever $\mu^- \#_n \mu^+$ is defined. For a given row $[i\ \alpha\ j\ \beta]$ and for sufficiently high-dimensional x and y, there are only finitely many rows $[k\ \varepsilon\ l\ \zeta]$ such that $d_k^\varepsilon x \times d_l^\zeta y \subset d_i^\alpha x \times d_j^\beta y$; therefore any given molecular matrix has only finitely many predecessors. By an inductive argument, it therefore suffices to construct a proper decomposition $\lambda = \mu^- \#_n \mu^+$ whenever λ is a molecular matrix with more than one row.

To do this, choose an arbitrary partition

$$\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

such that λ_1 and λ_2 are non-empty. Let the bottom row of λ_1 end with $[j\ \beta]$, let the top row of λ_2 start with $[i\ \alpha]$, and let $n = j + i$. We find that the bottom row of $d_n^\alpha\lambda_1$ and the top row of $d_n^{-\alpha}\lambda_2$ are both equal to $[i\ \alpha\ j\ \beta]$. Let π_1 and π_2 be the (possibly empty) matrices got from $d_n^\alpha\lambda_1$ and $d_n^{-\alpha}\lambda_2$ by removing the row $[i\ \alpha\ j\ \beta]$, and let

$$\mu^{-\alpha} = \begin{bmatrix} \lambda_1 \\ \pi_2 \end{bmatrix}, \quad \mu^\alpha = \begin{bmatrix} \pi_1 \\ \lambda_2 \end{bmatrix}.$$

We find that μ^- and μ^+ are molecular matrices distinct from λ such that $\lambda = \mu^- \#_n \mu^+$.

This completes the proof. ∎

PROOF OF PROPOSITION 6.13 We must show that a pair $\Lambda = (\Lambda_-, \Lambda_+)$ is left p-compatible if and only if it satisfies conditions (i)–(iv).

Suppose that Λ is left p-compatible. It clearly follows that $\Lambda_-(x,y) = \Lambda_+(x,y)$ for all molecules x and y such that $\dim x \leq p$. From this it follows that Λ satisfies conditions (i)–(ii).

The rest of the proof is a straightforward calculation. ∎

Proof of Theorem 6.15(i) We must show that $\Lambda(x_-, x_+; y)$ is a subcomplex if Λ is a left p-compatible pair and the composite $x_- \#_p x_+$ exists. Let $\Lambda = (\Lambda_-,\ \Lambda_+)$. Using Proposition 6.13, it is straightforward to check the following results: if Λ_- and Λ_+ do not have a common row beginning with $[p\ \theta]$, then

$$\Lambda(x_-, x_+; y) = \Lambda_-(x_-, y) \cup \Lambda_+(x_+, y);$$

if Λ_- and Λ_+ do have a common row $[p\ \theta\ j\ \beta]$, then

$$\Lambda(x_-, x_+; y) = \Lambda_\theta(x_\theta, y) \cup \Lambda'_{-\theta}(x_{-\theta}, y),$$

where $\Lambda'_{-\theta}$ is got from $\Lambda_{-\theta}$ by deleting the row $[p\ \theta\ j\ \beta]$. In both cases, $\Lambda(x_-, x_+; y)$ is a subcomplex as required. ∎

Proof of Theorem 6.15(ii) Let Λ be a left p-compatible pair. We must show that $d_n^\gamma \Lambda$ is left p-compatible, and that $d_n^\gamma[\Lambda(x_-, x_+; y)] = (d_n^\gamma \Lambda)(x_-, x_+; y)$ when $x_- \#_p x_+$ exists. Let $\Lambda = (\Lambda_-,\ \Lambda_+)$, so that $d_n^\gamma \Lambda = (d_n^\gamma \Lambda_-,\ d_n^\gamma \Lambda_+)$.

To show that $d_n^\gamma \Lambda$ is left p-compatible, we verify the conditions of Proposition 6.13. It is easy to see that $d_n^\gamma \Lambda$ satisfies conditions (i) and (ii), because Λ does so. We therefore concentrate on conditions (iii) and (iv). There are two cases.

1. Suppose that $d_n^\gamma \Lambda_\theta$ has a row $[p\ \theta\ j\ \beta]$ at the top of a block as in Definition 6.5. One finds that $d_n^\gamma \Lambda_{-\theta}$ also has the row $[p\ \theta\ j\ \beta]$ at the top of a block. Suppose further that this row is preceded in $d_n^\gamma \Lambda_{-\theta}$ by a row ending with $[l'\ \delta]$. Then Λ_- and Λ_+ have a common row of the form $[p\ \theta\ j'\ \beta']$. In $\Lambda_{-\theta}$ this row is preceded by a row ending with $[l'\ \delta]$; in Λ_θ it is therefore preceded by a row ending with $[l\ \delta]$ such that $l \geq l'$. In $d_n^\gamma \Lambda_\theta$, it now follows that $[p\ \theta\ j\ \beta]$ is preceded by a row ending with $[l\ \delta]$. Therefore $d_n^\gamma \Lambda$ satisfies conditions (iii) and (iv).

2. Suppose that $d_n^\gamma \Lambda_\theta$ has a row $[p\ \theta\ j\ \beta]$ not at the top of a block as in Definition 6.5. Then $\theta = \gamma$ and the preceding row in $d_n^\gamma \Lambda_\gamma$ ends with $[n-p-1\ \delta]$. One finds that $d_n^\gamma \Lambda_{-\gamma}$ also has a row beginning with $[p\ \gamma\ j\ \beta]$. By general properties of d_n^γ (see Definition 6.5), the preceding row in $d_n^\gamma \Lambda_{-\gamma}$ (if any) must end with $[l'\ \delta]$ such that $l' \leq n-p-1$. Again $d_n^\gamma \Lambda$ satisfies conditions (iii) and (iv).

This completes the proof that $d_n^\gamma \Lambda$ is left p-compatible.

We now show that $d_n^\gamma[\Lambda(x_-, x_+; y)] = (d_n^\gamma \Lambda)(x_-, x_+; y)$. Recall from Theorem 6.6(ii) that $d_n^\gamma[\Lambda_\theta(x_\theta, y)] = (d_n^\gamma \Lambda_\theta)(x_\theta, y)$; we shall use this result without comment.

For ζ a point of $(x_\theta \times y) \setminus (x_{-\theta} \times y)$, an atom containing ζ is contained in $\Lambda(x_-, x_+; y)$ if and only if it is contained in $\Lambda_\theta(x_\theta, y)$. It easily follows that

$$d_n^\gamma[\Lambda(x_-, x_+; y)] \setminus (x_{-\theta} \times y) = (d_n^\gamma \Lambda_\theta)(x_\theta, y) \setminus (x_{-\theta} \times y)$$

for each θ.

By Proposition 5.9, $(d_n^\gamma \Lambda_-)(x_-, y) \cap (d_n^\gamma \Lambda_+)(x_+, y) \subset d_n^\gamma[\Lambda_-(x_-, y) \cup \Lambda_+(x_+, y)]$. Obviously

$$(d_n^\gamma \Lambda_-)(x_-, y) \cap (d_n^\gamma \Lambda_+)(x_+, y) \subset \Lambda(x_-, x_+; y) \subset \Lambda_-(x_-, y) \cup \Lambda_+(x_+, y),$$

and with the use of Proposition 5.8 we get

$$(d_n^\gamma\Lambda_-)(x_-,y)\cap(d_n^\gamma\Lambda_+)(x_+,y)\subset d_n^\gamma[\Lambda(x_-,x_+;y)]\cap(x_-\times y)\cap(x_+\times y).$$

To complete the proof, it suffices to show that $\zeta\in(d_n^\gamma\Lambda_-)(x_-,y)\cap(d_n^\gamma\Lambda_+)(x_+,y)$ whenever $\zeta\in d_n^\gamma[\Lambda(x_-,x_+;y)]\cap(x_-\times y)\cap(x_+\times y)$. There are two cases, as follows.

1. Suppose that Λ_- and Λ_+ do not have a common row beginning with $[p\ \theta]$. By the proof of part (i), $\zeta\in\Lambda_-(x_-,y)\subset\Lambda(x_-,x_+;y)$ and $\zeta\in\Lambda_+(x_+,y)\subset\Lambda(x_-,x_+;y)$, and it follows from Proposition 5.8 that $\zeta\in(d_n^\gamma\Lambda_-)(x_-,y)\cap(d_n^\gamma\Lambda_+)(x_+,y)$.

2. Suppose that Λ_- and Λ_+ have a common row $[p\ \theta\ j\ \beta]$. Let $\Lambda'_{-\theta}$ be the matrix got from $\Lambda_{-\theta}$ by deleting this common row. By the proof of part (i), $\zeta\in\Lambda_\theta(x_\theta,y)\subset\Lambda(x_-,x_+;y)$ and $\zeta\in\Lambda'_{-\theta}(x_{-\theta},y)\subset\Lambda(x_-,x_+;y)$. As in Case 1, it follows that $\zeta\in(d_n^\gamma\Lambda_\theta)(x_\theta,y)\cap d_n^\gamma[\Lambda'_{-\theta}(x_{-\theta},y)]$. To complete the proof, by Proposition 5.9, it suffices to show that

$$(d_n^\gamma\Lambda_\theta)(x_\theta,y)\cap(d_p^\theta x_{-\theta}\times d_j^\beta y)\subset d_n^\gamma(d_p^\theta x_{-\theta}\times d_j^\beta y).$$

But $[p\ \theta\ j\ \beta]$ is a row in Λ_θ, so

$$(d_n^\gamma\Lambda_\theta)(x_\theta,y)\cap(d_p^\theta x_\theta\times d_j^\beta y)\subset d_n^\gamma(d_p^\theta x_\theta\times d_j^\beta y)$$

by Proposition 5.8. The result now follows from calculations with Theorem 6.4, since $d_i^\gamma d_p^\theta x_{-\theta}=d_i^\gamma d_p^\theta x_\theta$ for $i<p$ and since $d_i^\gamma d_p^\theta x_{-\theta}=d_p^\theta x_{-\theta}$ for $i\geq p$. ■

Proof of Theorem 6.15(iii) Let M^- and M^+ be left p-compatible pairs such that $d_n^+\mathrm{M}^-=d_n^-\mathrm{M}^+$. We must show that $\mathrm{M}^-\#_n\mathrm{M}^+$ is left p-compatible, and that

$$\mathrm{M}^-(x_-,x_+;y)\#_n\mathrm{M}^+(x_-,x_+;y)=(\mathrm{M}^-\#_n\mathrm{M}^+)(x_-,x_+;y)$$

when $x_-\#_p x_+$ exists.

Let $\mathrm{M}^\alpha=(\mathrm{M}_-^\alpha,\ \mathrm{M}_+^\alpha)$ for each α. By Theorem 6.8(ii),

$$(\mathrm{M}_\theta^-\#_n\mathrm{M}_\theta^+)(x,y)=\mathrm{M}_\theta^-(x,y)\#_n\mathrm{M}_\theta^+(x,y)$$

for each θ, where x and y are arbitrary molecules in ω-complexes. The left p-compatibility of $\mathrm{M}^-\#_n\mathrm{M}^+$ now follows from Definition 6.11, and the equality

$$\mathrm{M}^-(x_-,x_+;y)\#_n\mathrm{M}^+(x_-,x_+;y)=(\mathrm{M}^-\#_n\mathrm{M}^+)(x_-,x_+;y)$$

follows by comparing Notation 6.10 with Proposition 5.9. ■

Proof of Theorem 6.16 As in the proof of Theorem 6.9, it suffices to find a proper decomposition

$$\Lambda=\mathrm{M}^-\#_n\mathrm{M}^+$$

for each non-basic left p-compatible pair Λ. We write $\Lambda=(\Lambda_-,\ \Lambda_+)$ and $\mathrm{M}^\alpha=(\mathrm{M}_-^\alpha,\ \mathrm{M}_+^\alpha)$, and we use Proposition 6.13 throughout. There are various cases, as follows.

Suppose that Λ_- and Λ_+ have only one row each. Since Λ is not basic, we must have

$$\Lambda = ([i_- \ \alpha_- \ j \ \beta], \ [i_+ \ \alpha_+ \ j \ \beta])$$

with $i_- > p$ and $i_+ > p$. We take $n = p + j$ and $\mathrm{M}^- = (\Lambda_-, \, d_n^- \Lambda_+)$, $\mathrm{M}^+ = (d_n^+ \Lambda_-, \, \Lambda_+)$.

From now on suppose that Λ_- or Λ_+ has more than one row. To be specific, suppose that

$$\Lambda_\theta = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

such that the bottom row of λ_1 ends with $[j \ \beta]$ and the top row of λ_2 starts with $[i \ \alpha]$. As in the proof of Theorem 6.9, we take $n = j + i$ and

$$\mathrm{M}_\theta^{-\alpha} = \begin{bmatrix} \lambda_1 \\ \pi_2 \end{bmatrix}, \quad \mathrm{M}_\theta^{\alpha} = \begin{bmatrix} \pi_1 \\ \lambda_2 \end{bmatrix},$$

where π_1 and π_2 are got from $d_n^\alpha \lambda_1$ and $d_n^{-\alpha} \lambda_2$ by deleting the rows $[i \ \alpha \ j \ \beta]$. To get the values of $\mathrm{M}_{-\theta}^{-\alpha}$ and $\mathrm{M}_{-\theta}^{\alpha}$ we split cases further, as follows.

First, suppose that $i < p$. Then $\Lambda_{-\theta}$ also contains a row ending with $[j \ \beta]$ followed by a row beginning with $[i \ \alpha]$, and we decompose $\Lambda_{-\theta}$ in the same way as Λ_θ.

Next, suppose that $[i \ \alpha] = [p \ \theta]$, and suppose that the row in $\Lambda_{-\theta}$ beginning with $[p \ \theta]$ is also preceded by a row ending with $[j \ \beta]$. Again we decompose $\Lambda_{-\theta}$ in the same way as Λ_θ.

Next, suppose that $[i \ \alpha] = [p \ \theta]$, and suppose that the row in $\Lambda_{-\theta}$ beginning with $[p \ \theta]$ is preceded by a row ending with $[j' \ \beta]$ such that $j' < j$. We then have

$$\Lambda_{-\theta} = \begin{bmatrix} \lambda_1' \\ \lambda_2 \end{bmatrix}$$

such that the bottom row of λ_1' ends with $[j' \ \beta]$ and we take

$$\mathrm{M}_{-\theta}^{-\alpha} = \begin{bmatrix} \lambda_1' \\ d_n^{-\alpha} \lambda_2 \end{bmatrix}, \quad \mathrm{M}_{-\theta}^{\alpha} = \begin{bmatrix} d_n^\alpha \lambda_1' \\ \lambda_2 \end{bmatrix}.$$

Next, suppose that $[i \ \alpha] = [p \ \theta]$, and suppose that the top row of $\Lambda_{-\theta}$ begins with $[p \ \theta]$. We take $\mathrm{M}_{-\theta}^{-\alpha} = d_n^{-\alpha} \Lambda_{-\theta}$ and $\mathrm{M}_{-\theta}^{\alpha} = \Lambda_{-\theta}$.

In the remaining cases, $[i \ \alpha] = [p \ -\theta]$ or $i > p$. We again take $\mathrm{M}_{-\theta}^{-\alpha} = d_n^{-\alpha} \Lambda_{-\theta}$ and $\mathrm{M}_{-\theta}^{\alpha} = \Lambda_{-\theta}$.

This completes the proof. ∎

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