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ON SOME PROPERTIES OF PURE MORPHISMS OF COMMUTATIVE RINGS

BACHUKI MESABLISHVILI

ABSTRACT. We prove that pure morphisms of commutative rings are effective $\mathbb{A}$-descent morphisms where $\mathbb{A}$ is a $(\mathbf{COMMUTATIVE\ RINGS})^{\mathrm{op}}$-indexed category given by (i) finitely generated modules, or (ii) flat modules, or (iii) finitely generated flat modules, or (iv) finitely generated projective modules.

1. Introduction

Let $\mathcal{E}$ be a category with pullbacks, and let $\mathbb{A} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathbb{CAT}$ be an indexed category [PS]. If $\mathbf{C}$ is an internal category of $\mathcal{E}$, then one defines an ordinary category $\mathbb{A}^{\mathbf{C}}$ of $\mathbf{C}$-diagrams in $\mathbb{A}$, and the assignment $\mathbf{C} \mapsto \mathbb{A}^{\mathbf{C}}$ induces a pseudo-functor

$$\mathbb{A}^{(-)} : \mathrm{cat}(\mathcal{E})^{\mathrm{op}} \longrightarrow \mathbb{CAT}$$

of 2-categories (see [JT2]) where $\mathrm{cat}(\mathcal{E})$ denotes the 2-category of internal categories of $\mathcal{E}$.

Let $p : E \longrightarrow B$ be a morphism in $\mathcal{E}$. Then, p gives rise to an internal category $\mathbf{Eq}(p)$ of $\mathcal{E}$, namely the equivalence relation induced by p, and to a fully faithful (internal) functor

$$\overline{p} : \mathbf{Eq}(p) \longrightarrow \mathbf{B},$$

where $\mathbf{B}$ is the discrete internal category on B. The category $\mathbb{A}^{\mathbf{Eq}(p)}$ is called the category of $\mathbb{A}$-descent data relative to p, and denoted by $\mathrm{Des}_{\mathbb{A}}(p)$.

The pseudo-functor

$$\mathbb{A}^{(-)} : \mathrm{cat}(\mathcal{E})^{\mathrm{op}} \longrightarrow \mathbb{CAT}$$

carries $\overline{p}$ into an ordinary functor

$$\phi^p : \mathbb{A}^B \longrightarrow \mathrm{Des}_{\mathbb{A}}(p),$$

and one says that p is an effective $\mathbb{A}$-descent morphism if ϕ^p is an equivalence of categories.

More details on Descent Theory can be found in [JT1, JT2].

We consider the case where $\mathcal{E}$ is the opposite of the category of commutative rings with unit, and $\mathbb{A}$ is given by (i) finitely generated modules, or (ii) flat modules, or (iii) finitely generated flat modules, or (iv) finitely generated projective modules.

The aim of the paper is to prove,

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1.1. Theorem. *Let $f : R \longrightarrow S$ be a pure morphism of commutative rings with unit. Then f is an effective $\mathbb{A}$-descent morphism.*

2. Preliminaries

Let $\mathcal{E}$ be a category with pullbacks and let

$$\mathbb{A}, \mathbb{B} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathbb{CAT}$$

be $\mathcal{E}$-indexed categories. We say that $\mathbb{A}$ is an $\mathcal{E}$-indexed full subcategory of $\mathbb{B}$ if there exists an $\mathcal{E}$-indexed functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ such that the functor $F(e) : \mathbb{A}^e \longrightarrow \mathbb{B}^e$, $e \in \mathrm{Ob}(\mathcal{E})$, is a fully faithful embedding.

The key role in proving the main result will be Corollary 2.7 of [JT1] that we now state in its indexed version, and since the same sort of proof as given in [JT1] works, we do not give a proof here.

2.1. Theorem. [Janelidze-Tholen] *Let $\mathbb{A}, \mathbb{B} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathbb{CAT}$ be $\mathcal{E}$-indexed categories. Suppose $\mathbb{A}$ is a $\mathcal{E}$-indexed full subcategory of $\mathbb{B}$. A morphism $p : E \longrightarrow B$ of $\mathcal{E}$ which is an effective $\mathbb{B}$-descent morphism is also an effective $\mathbb{A}$-descent morphism if and only if the following condition is satisfied: for each $x \in \mathbb{B}^B$, $p^*(x) \in \mathbb{A}^E$, where $p^* : \mathbb{A}^B \longrightarrow \mathbb{A}^E$ is the change-of-base functor, implies $x \in \mathbb{A}^B$.*

Let R be a commutative ring with unit. A morphism $f : M \longrightarrow M'$ of R-modules is called pure if

$$1_L \otimes_R f : L \otimes_R M \longrightarrow L \otimes_R M'$$

is a monomorphism for every R-module L.

For any R-module M, the abelian group $\mathrm{Hom}_{\mathbb{A}b}(M, \mathbb{Q}/\mathbb{Z})$ (where $\mathbb{Q}/\mathbb{Z}$ denotes the abelian group of rationals *mod* 1) is an R-module via the action $(r.h)(m) = h(r.m)$ ($r \in R$, $m \in M$, $h \in \mathrm{Hom}_{\mathbb{A}b}(M, \mathbb{Q}/\mathbb{Z})$). So one can define a functor

$$C_R : (R\text{-mod}) \longrightarrow (R\text{-mod})^{\mathrm{op}}$$

given by $C_R(M) = \mathrm{Hom}_{\mathbb{A}b}(M, \mathbb{Q}/\mathbb{Z})$. C_R is clearly an exact and faithful functor since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator for the category of abelian groups (see, for example, [F]).

2.2. Proposition. [see [Me], *[L]] A morphism $f : M \longrightarrow M'$ of R-modules is pure if and only if $C_R(f)$ is a split epimorphism of R-modules.*

A morphism $f : R \longrightarrow S$ of commutative rings with unit gives rise to two functors

(i) $f^* : S\text{-mod} \longrightarrow R\text{-mod}$, where for any S-module L, $f^*(L)$ is the underlying abelian group L with the R-action given by $r.l = f(r).l$ ($r \in R, l \in L$);

(ii) $f_! : R\text{-mod} \longrightarrow S\text{-mod}$, where for any R-module M, $f_!(M) = S \otimes_R M$.

It is well known that the functor $f_!$ is left adjoint to the functor f^*. Each component $\eta_M : M \longrightarrow S \otimes_R M$ of the unit $\eta : 1 \longrightarrow f^* f_!$ of the adjunction is given by $\eta_M(m) = 1 \otimes_R m$.

We denote by C_R^S the composite

$$C_R \circ f^* \circ f_! : (R\text{-mod}) \longrightarrow (R\text{-mod})^{\mathrm{op}}$$

2.3. Proposition. *The following statements are equivalent for a morphism $f : R \longrightarrow S$ of commutative rings with unit:*

(i) $C_R(f)$ is a split epimorphism of R-modules;

(ii) f is pure (i.e., f is pure as an R-module morphism);

(iii) The natural transformation

$$C_R.\eta : C_R^S \longrightarrow C_R$$

splits; i.e., there exists a natural transformation

$$\tau : C_R \longrightarrow C_R^S$$

such that $(C_R.\eta).\tau \approx 1$.

Proof. $(i) \Leftrightarrow (ii)$: It follows from the definition of pure morphisms and Proposition 2.2.

$(i) \Leftrightarrow (iii)$: To say $C_R.\eta$ splits is to say $\mathrm{Hom}_R(-, C_R(f))$ (and hence, by the Yoneda lemma, $C_R(f)$) splits because we have the following commutative (up to an isomorphism) diagram:

$$\begin{tikzcd}
C_R^S \arrow[r, "C_R.\eta"] \arrow[d, "\approx"'] & C_R \arrow[d, "\approx"] \\
\mathrm{Hom}_R(-, C_R(S)) \arrow[r, "\mathrm{Hom}_R(-, C_R(R))"'] & \mathrm{Hom}_R(-, C_R(R))
\end{tikzcd}$$

where the vertical morphisms are the canonical isomorphisms. ∎

Recall that an exact sequence in R-mod

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$$

is called pure-exact iff the induced sequence

$$0 \longrightarrow L \otimes_R M \longrightarrow L \otimes_R M' \longrightarrow L \otimes_R M'' \longrightarrow 0$$

is an exact sequence for every R-module L.

Using Proposition 2.2, we get

2.4. Proposition. [see [L]*] The following properties are equivalent for an exact sequence*

$$0 \longrightarrow M \longrightarrow M' \longrightarrow M'' \longrightarrow 0$$

(i) It is pure-exact;

(ii) The exact sequence in R-mod

$$0 \longrightarrow C_R(M'') \longrightarrow C_R(M') \longrightarrow C_R(M) \longrightarrow 0$$

splits.

We close this section by recalling a result on flat modules (see, e.g., [S]).

2.5. Proposition. *The following properties of an R-module M are equivalent:*

(i) M is a flat R-module;

(ii) Every exact sequence

$$0 \longrightarrow M' \longrightarrow M'' \longrightarrow M \longrightarrow 0$$

is pure-exact.

3. Proof of the main theorem

In the sequel we always assume that $\mathcal{E}$ is the opposite of the category of commutative rings with unit.

We start by recalling from [Me] the following result:

3.1. Theorem. *Let $\mathbb{M} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathbb{CAT}$ be the $\mathcal{E}$-indexed category given by all modules. A morphism in $\mathcal{E}$ is an effective $\mathbb{M}$-descent morphism if and only if it is pure.*

It is clear that $\mathbb{A} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathbb{CAT}$ as in the Main Theorem, is an $\mathcal{E}$-indexed full subcategory of $\mathbb{M}$. Then, in order to prove the Main Theorem, by Theorem 2.1 and Theorem 3.1, it is sufficient to prove the following

3.2. Theorem. *Let $f : R \longrightarrow S$ be a pure morphism of commutative rings with unit, and let $M \in \mathrm{Ob}(R\text{-}mod)$. Then*

(i) If $S \otimes_R M$ is a finitely generated S-module, then M is a finitely generated R-module.

(ii) If $S \otimes_R M$ is a flat S-module, then M is a flat R-module.

(iii) If $S \otimes_R M$ is a finitely generated flat S-module, then M is a finitely generated flat R-module.

(iv) If $S \otimes_R M$ is a finitely generated projective S-module, then M is a finitely generated R-module.

3.3. Remark. The above theorem tells us that the properties of modules of being finitely generated, flat, finitely generated flat, finitely generated projective descend along pure morphisms of commutative rings.

Proof. (i) Suppose $S \otimes_R M$ is finitely generated over S. Then we can choose a finite family of elements $\{s_{i,k}, m_{i,k}\}$, where $s_{i,k} \in S$, $m_{i,k} \in M$, such that the finite family $\{\sum s_{i,k} \otimes_R m_{i,k}\}_{i,k}$ generates the S-module $S \otimes_R M$. Let M' denote the R-submodule of M generated by the finite family $\{m_{i,k}\}_{i,k}$. It is clear that the induced morphism $1_S \otimes_R i : S \otimes_R M' \longrightarrow S \otimes_R M$, where $i : M' \longrightarrow M$ is the canonical embedding, is an epimorphism. But since f is pure by assumption, the functor

$$f_! : R\text{-mod} \longrightarrow S\text{-mod}$$

is faithful (see, for example, [B]); and since any faithful functor between arbitrary categories reflects epimorphisms (see [Mt]), $i : M' \longrightarrow M$ is an epimorphism, and hence an isomorphism since i is a monomorphism. It means that M is finitely generated R-module.

(ii) Suppose $S \otimes_R M$ is a flat S-module. According to Proposition 2.5, it suffices to show that each exact sequence

$$0 \longrightarrow M' \xrightarrow{u} M'' \xrightarrow{v} M \longrightarrow 0 \tag{1}$$

is pure-exact. By Proposition 2.4 this sequence is pure-exact iff the exact sequence in R-mod

$$0 \longrightarrow C_R(M) \xrightarrow{C_R(v)} C_R(M'') \xrightarrow{C_R(u)} C_R(M') \longrightarrow 0 \tag{2}$$

splits. Since $v : M'' \longrightarrow M$ is an epimorphism, the sequence in R-mod

$$S \otimes_R M'' \xrightarrow{1_S \otimes_R v} S \otimes_R M \longrightarrow 0 \tag{3}$$

is exact. This sequence is also exact in S-mod since the functor

$$f^* : S\text{-mod} \longrightarrow R\text{-mod}$$

is faithful, and hence reflects epimorphisms (see [Mt]).

Therefore, (3) is an exact sequence in S-mod, and so is the sequence

$$0 \longrightarrow C_S(S \otimes_R M) \xrightarrow{C_S(1 \otimes_R v)} C_S(S \otimes_R M'')$$

since the functor

$$C_S : (S\text{-mod}) \longrightarrow (S\text{-mod})^{\mathrm{op}}$$

is exact.

$C_S(S \otimes_R M)$ is an injective S-module since, by assumption, $S \otimes_R M$ is a flat S-module (see, e.g., [F]). Hence the last sequence of S-modules splits. But it is clear that $f^* \circ C_S \circ f_! = C_R$. Hence the exact sequence in R-mod

$$0 \longrightarrow C_R(S \otimes_R M) \xrightarrow{C_R(1 \otimes_R v)} C_R(S \otimes_R M'')$$

also splits, i.e., there is an R-module morphism

$$h : C_R(S \otimes_R M'') \longrightarrow C_R(S \otimes_R M)$$

such that $h.C_R(1 \otimes_R v) = 1$.

By naturality of τ (see Proposition 2.3.), one has the following commutative diagram

$$\begin{tikzcd} C_R(M) \arrow[r, "C_R(v)"] \arrow[d, "\tau_M"'] & C_R(M'') \arrow[d, "\tau_{M''}"] \\ C_R(S \otimes_R M) \arrow[r, "C_R(1 \otimes_R v)"'] & C_R(1 \otimes_R v) \end{tikzcd}$$

Let us show that $C_R(v)$ splits by the composition

$$C_R(f \otimes_R 1).h.\tau_{M''} : C_R(M'') \longrightarrow C_R(M).$$

Indeed, we know that $h.C_R(1 \otimes_R v) = 1$, and, moreover, commutativity of the diagram implies $\tau_{M''}.C_R(v) = C_R(1 \otimes_R v).\tau_M$. Then

$$\begin{aligned} C_R(f \otimes_R 1).h.\tau_{M''}.C_R(v) &= C_R(f \otimes_R 1).h.C_R(1 \otimes_R v).\tau_M \\ &= C_R(f \otimes_R 1).\tau_M. \end{aligned}$$

Since the composite of $f \otimes_R 1$ with the canonical isomorphism $M \cong R \otimes_R M$ is the M-component μ_M of $\mu : 1 \longrightarrow f^* f_!$, Proposition 2.3 gives $C_R(f \otimes_R 1).\tau_M = 1$. So

$$(C_R(f \otimes_R 1).h.\tau_{M''}).C_R(v) = 1.$$

It means that the sequence (2) splits, and, by Proposition 2.4, the sequence (1) is pure-exact. Consequently, M is a flat R-module by Proposition 2.5.

(iii) It follows immediately from (i) and (ii).

(iv) If $S_R \otimes M$ is finitely generated , then, in particular, it is finitely generated over S , and since every projective module is flat, by (i) and (ii), M is finitely generated flat module over R.

Since every finitely generated projective module is finitely presented, there exists an exact sequence in S-mod

$$0 \longrightarrow X \longrightarrow S^n \longrightarrow S \otimes_R M \longrightarrow 0 \tag{4}$$

where X is a finitely generated S-module.

Moreover, since M is finitely generated over R, there exists an exact sequence in R-mod

$$0 \longrightarrow M' \longrightarrow R^m \longrightarrow M \longrightarrow 0,$$

and since M is a flat R-module, by Proposition 2.5, this sequence is pure-exact. Therefore the sequence in S-mod

$$0 \longrightarrow S \otimes_R M' \longrightarrow S \otimes_R R^m \longrightarrow S \otimes_R M \longrightarrow 0 \tag{5}$$

is exact.

Apply Schanuel's lemma to (4) and (5); we get

$$X \oplus (S \otimes_R R^m) \approx S^n \oplus (S \otimes_R M').$$

It follows that $S \otimes_R M'$ is finitely generated over S, and by (i), M' is finitely generated over R. Therefore, M is finitely presented over R. But, as we have seen, M is a flat R-module, and since every finitely presented flat module is projective (see, for example, [F]), M is a finitely generated projective R-module.

This completes the proof. ∎

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References

[B] F. Borceux, *Handbook of Categorical Algebra*, Vol. **1**, Cambridge University Press, 1994.

[F] C. Faith, *Algebra: Rings, Modules and Categories*, Vol. **1**, Springer-Verlag, 1973.

[JT1] G. Janelidze and W. Tholen, Facets of Descent, I, *Appl. Categorical Structures*, **2** (1994), 245–281.

[JT2] G. Janelidze and W. Tholen, Facets of Descent, II, *Appl. Categorical Structures*, **5** (1997), 229–248.

[L] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Math. Vol. 109, Springer-Verlag, 1999.

[Me] B. Mesablishvili, Pure Morphisms of Commutative Rings Are Effective Descent Morphisms for Modules – A New Proof, *Theory and Applications of Categories*, **7** (2000), 38–42.

[Mt] B. Mitchell, *Theory of Categories*, Academic Press, 1965.

[S] B. Stenstrőm, *Rings of Quotients*, Lecture Notes in Mathematics, **237**, Springer-Verlag, 1971.

[PS] R. Paré and D. Schumacher, *Abstract Families and the Adjoint Functor Theorems*, Lecture Notes in Mathematics, **661**, Springer-Verlag 1978, 1–125.

A. Razmadze Mathematical Institute of the Georgian Academy of Sciences,
M.Alexidze St. 1,
Tbilisi 380093, Georgia
Email: `bachi@rmi.acnet.ge`

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