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THE CYCLIC SPECTRUM OF A BOOLEAN FLOW

JOHN F. KENNISON

ABSTRACT. This paper defines flows (or discrete dynamical systems) and cyclic flows in a category and investigates how the trajectories of a point might approach a cycle. The paper considers cyclic flows in the categories of Sets and of Boolean algebras and their duals and characterizes the Stone representation of a cyclic flow in Boolean algebras. A cyclic spectrum is constructed for Boolean flows. Examples include attractive fixpoints, repulsive fixpoints, strange attractors and the logistic equation.

Introduction

Suppose we have a flow in **Sets** by which we mean a set S with a function $t : S \to S$. We might think of S as the set of all possible states of a system and $t(s)$ as the state the system will be in one time unit after being in state s. We are interested in the long-term future of a system that starts out in state s. If S has no further structure, then the successive states, $\{t^n(s)\}$, (called the "trajectory of s") either repeat so that the system eventually cycles or the system takes on a sequence of distinct states.

But suppose S has more structure, such as a topological structure. In this case, the trajectory of s might approach a limiting state, or a limiting cycle, or a strange attractor or become chaotic. To account for this possible structure on S, we define a flow in a category and define when a flow is cyclic.

After some general observations, we focus on flows in **Sets**, the category of Sets, and **Bool**, the category of Boolean algebras, and their duals. We show that cyclic flows in **Bool** have special properties and show how any flow in this category can be broken down into its cyclic parts by constructing the cyclic spectrum (which is an example of a Cole spectrum, see [2]).

There is a one-to-one correspondence between flows in **Bool** and in the dual category **Stone** of compact, Hausdorff, totally disconnected spaces. But the notion of a cyclic flow in a category is not self-dual, so there are two different ways a flow in these categories can be cyclic. For example, in **Stone** we define what is meant by a nearly cyclic flow and, if not finite, such a flow is cyclic as a flow in **Bool** but not in **Stone**. It essentially corresponds to a "strange attractor".

For applications we would like to examine flows on a set S which has a Stone topology. In many well-known examples, S has a topology but it is usually connected. However, as shown in Example 10 we can often find an associated Stone topology.

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Much of this paper should be of interest to dynamical system theorists who know a little category theory as found in [5] or even the first chapter of [1]. Further concepts are in [1, 2, 3, 6], while [7, 8] give information about dynamical systems.

1. Flows in a Category

In this section we define flows and cyclic flows in a category.

Definition 1. Let C be a category. Then (X, t) is a *flow* (or discrete dynamical system) in C if X is an object of C and $t : X \to X$ is a morphism.

If (A, s) and (B, t) are flows in C, then $h : A \to B$ is a *flow homomorphism* from (A, s) to (B, t) if $hs = th$.

Definition 2. A category C is *well-behaved* if it is complete and co-complete and if every morphism f can be written as $f = me$ where m is a mono and e is an extremal epi (meaning that whenever $e = gh$ with g mono then g is an isomorphism.)

About well-behaved categories. The following notation and elementary properties for well-behaved categories is either fairly standard or well-known or readily proven:

(WB1) If A and B are objects of C, then $A \subseteq B$ indicates that A is a *subobject* of B, meaning that there is an implicit mono from A to B.

(WB2) For $f, g \in \mathrm{Hom}(X, Y)$ we let $\mathrm{Equ}(f, g) \subseteq X$ denote the *equalizer* of f, g.

(WB3) If $\{A_i \subseteq B\}$ for all i, then $\bigvee A_i$ denotes the *supremum* of $\{A_i\}$ in the partially ordered family of all subobjects of B.

(WB4) If $f : X \to Y$ and if $A \subseteq X$ then then $f(A)$ denotes the smallest subobject of Y through which $A \to X \to Y$ factors.

(WB5) (Diagonal lemma) If $se = mr$ with m mono and e extremal epi, then there exists a "diagonal" map d with $de = r$ and $md = s$.

(WB6) Given $f : X \to Y$ and $B \subseteq Y$ and $\{A_i \subseteq X\}$ a set of subobjects of X with $f(A_i) \subseteq B$ for all i, then $f(\bigvee A_i) \subseteq B$.

Definition 3. Let (X, t) be a flow in a well-behaved category C. We say that (X, t) is *cyclic* if $\bigvee \mathrm{Equ}(t^n, \mathrm{Id}_X) = X$, the maximum subobject of X.

1.1. Lemma. *Let $h : (A, s) \to (B, t)$ be a flow homomorphism and let $E(n) = \mathrm{Equ}(s^n, \mathrm{Id}_A)$ and $F(n) = \mathrm{Equ}(t^n, \mathrm{Id}_B)$. Then $h(E(n)) \subseteq F(n)$.*

Proof. Straightforward. ∎

1.2. Lemma. *Let C be a well-behaved category and let $X = \mathrm{Colim}\,\{X_i\}$ be a colimit in C with coprojections $c_i : X_i \to X$. Let $B \subseteq X$ be such that $c_i(X_i) \subseteq B$ for all i. Then $B = X$, the maximum subobject of X.*

Proof. Let $m : B \to X$ be the mono which represents $B \subseteq X$. Let $d_i : X_i \to B$ be such that $md_i = c_i$ (which exist as $c_i(X_i) \subseteq B$). Because m is mono it is readily seen that the family $\{d_i\}$ has the compatibility property which implies the existence of $d : X \to B$ with $dc_i = d_i$ for all i.

It follows (using the colimit property) that $md = \mathrm{Id}_X$ and then (as m in mono) that $dm = \mathrm{Id}_B$ which shows that m is an isomorphism and so B represents the maximum subobject of X. ■

1.3. Proposition. *In a well-behaved category, colimits of cyclic flows are cyclic.*

Proof. Let $\{(X_i, t_i)\}$ be a small diagram of cyclic flows in the well-behaved category C. Let $X = \mathrm{Colim}\,\{X_i\}$ with coprojections $c_i : X_i \to X$. Then it is readily shown that there is a unique $t : X \to X$ for which each c_i is a flow homomorphism. Let $E_i(n) = \mathrm{Equ}(t_i^n, \mathrm{Id}_{X_i}) \subseteq X_i$ and let $E(n) = \mathrm{Equ}(t^n, \mathrm{Id}_X) \subseteq X$. Let $B = \bigvee E(n)$, It suffices to show that $B = X$.

By the Lemma 1.1, we see that $c_i(E_i(n)) \subseteq E(n)$ so $c_i(E_i(n)) \subseteq B$ for all i. By (WB6) and Lemma 1.2, $c_i(\bigvee\{E_i(n)\}) \subseteq B$ so $c_i(X_i) \subseteq B$ for all i and so $B = X$. ■

1.4. Proposition. *In a well-behaved category, extremal quotients of cyclic flows are cyclic.*

Proof. Let (A, s) be a cyclic flow and let the extremal epi $h : A \to B$ be a flow homomorphism. We must show that (B, t) is cyclic. Let $E(n) = \mathrm{Equ}(s^n, \mathrm{Id}_A)$ and $F(n) = \mathrm{Equ}(t^n, \mathrm{Id}_B)$. It suffices to show that $\bigvee F(n) = B$. But since $\bigvee E(n) = A$ it readily follows, using 1.1 and (WB6), that h factors through $\bigvee F(n)$ and an extremal epi can only factor through the maximum subobject. ■

Example 1. In **Sets**, the category of Sets, (X, t) is a cyclic flow iff X is a disjoint union of finite cycles, where a *finite cycle* is a set $\{x_0, x_1, \ldots, x_{n-1}\}$ where $t(x_i) = x_{i+1}$ where $i + 1$ is computed mod n. (We then say that x is a *periodic point* with period n.)

Example 2. Let $t : X \to X$ be a function on a set X. Then, regarded as a flow in $\mathbf{Sets}^{\mathrm{op}}$, (X, t) is cyclic iff t is one-to-one.

Proof. Let $Q(n)$ be the equivalence relation on X generated by all pairs of the form $(x, t^n(x))$. Then X is cyclic iff $(x, y) \in \bigcap Q_n$ implies $x = y$. Assume X is cyclic and that $t(x) = t(y)$. Then $(x, y) \in Q(n)$, for all n, as $(x, t^n(x)), (y, t^n(y)) \in Q(n)$ and $t^n(x) = t^n(y)$. So $x = y$ and t is one-to-one.

Conversely, assume the t is one-to-one and that $(x, y) \in Q(n)$ for all n. Since t is one-to-one, it can readily be shown that $(x, y) \in Q(n)$ iff $x = y$ or $x = t^k(y)$ with k a multiple of n or $y = t^m(x)$ with m a multiple of n. Now if $x = t^m(y)$ for a unique m. then (x, y) cannot be in $Q(n)$ once $n > m$. On the other hand, if $x = t^m(y)$ for more than one m, then x, y are in a cyclic orbit of some period p (as t is one-to-one) and $(x, y) \in Q(p)$ implies $x = y$. ■

NOTATION. The categories **Bool** and **Stone** were defined in the introduction. It is readily shown that both categories are well-behaved with the monos being the one-to-one maps and the extremal epis being just the epis which are the onto maps.

It is well-known that **Bool** and **Stone** are dual to each other, see [3]. We denote the contravariant equivalences between these categories by:

$$\mathrm{Clop} : \mathbf{Stone} \to \mathbf{Bool} \text{ and } \mathrm{Pts} : \mathbf{Bool} \to \mathbf{Stone}$$

So $\mathrm{Clop}(X)$ is the Boolean algebra of clopens of X and $\mathrm{Pts}(B)$ is the Stone space of all points of B and is equivalent to the prime ideal spectrum of B (the Zariski and Patch topologies coincide).

If (X, t) is a flow in **Stone** then (B, τ) is a flow in **Bool** where $B = \mathrm{Clop}(X)$ and $\tau = \mathrm{Clop}(t)$. We will often write $(B, \tau) = \mathrm{Clop}(X, t)$

DEFINITION 4. (X, t) is a *Boolean cyclic flow* in **Stone** if (X, t) is a flow in **Stone** such that $(B, \tau) = \mathrm{Clop}(X, t)$ is a cyclic flow in **Bool**.

EXAMPLE 3. Let (X, t) be a flow in **Stone**. Then (X, t) is cyclic in **Stone** iff the set of periodic points of X is dense, where $x \in X$ is periodic iff $t^n(x) = x$ for some $n \in \mathbf{N}$.

On the other hand, (X, t) is Boolean cyclic iff every $b \in B$ is periodic under τ where $(B, \tau) = \mathrm{Clop}(X, t)$.

EXAMPLE 4. The group of p-adic integers, $\widehat{\mathbf{Z}}_p = \mathrm{Lim}\, \mathbf{Z}_{p^n}$, with $t(z) = z + 1$, is Boolean cyclic but is not cyclic in **Stone**.

PROOF. It is not cyclic as a flow in **Stone** because it has no periodic points. In **Stone**, $\widehat{\mathbf{Z}}_p$ is the limit of cyclic flows. So $\mathrm{Clop}(\widehat{\mathbf{Z}})$ is the corresponding colimit in **Bool** which is cyclic by Proposition 1.3. ∎

REMARK. It is instructive to see why $\widehat{\mathbf{Z}}_p$ of the above example is Boolean cyclic. Let b be a clopen of $\widehat{\mathbf{Z}}_p$. Note that finite intersections of sets of the form $\pi_{p^n}^{-1}(a)$ for $n \in \mathbf{N}$, $a \in \mathbf{Z}_{p^n}$ form a base for the topology on X. Since b is open, b is a union of such basic open sets and since b is closed, and therefore compact, b is a finite such union. So b depends only on finitely many coordinates, which means there exist $k = p^n$ with $b = \pi_k^{-1}(\pi_k(b))$ and so $\tau^k(b) = b$. $\widehat{\mathbf{Z}}_p$ is an example of a "nearly cyclic" flow, see the remark following Definition 5 in the next section.

EXAMPLE 5. Let $X = \prod\{\mathbf{Z}_n : n \in \mathbf{N}\}$ where $\mathbf{Z}_n$ is the cyclic group of integers mod n. Then X has a compact product topology and (X, t) is a flow in **Stone** where $t(x) = x + 1$ (where 1 is the element of X which projects onto the image of 1 in each $\mathbf{Z}_n$). Then X is not at all cyclic in **Stone** (it has no finite cycles) but X is Boolean cyclic.

PROOF. By duality, $\mathrm{Clop}(X)$ is the coproduct of the cyclic flows $\mathrm{Clop}(\mathbf{Z}_n)$. The result follows from Proposition 1.3. ∎

EXAMPLE 6. Let X be the coproduct in **Stone** of $\mathbf{Z}_p$ for p prime and let $t(x) = x + 1$ on each component. Then X is cyclic in **Stone** but not Boolean cyclic. (But see Example 9.)

Proof. As a coproduct of Stone cyclic flows, X is Stone cyclic, by 1.3. Note that the coproduct is the Stone-Čech compactification of the discrete union of the $\mathbf{Z}_p$'s. But the clopens of X coincide with the subsets of this discrete union (as both represent maps to the two-point Stone space). From this it is easy to see that X is not Boolean cyclic. ∎

2. The Stone Space of a Cyclic Boolean Algebra.

In this section, we explore and characterize those flows (X,t) in **Stone** which are Boolean cyclic. Aside from the characterization result (Theorem 2.13) we show that, for a Boolean cyclic flow in **Stone**, the closure of the trajectory of a single point must be "nearly cyclic".

Notation and Remark. In what follows: (X,t) will be a flow in **Stone** with $(B,\tau) = \mathrm{Clop}(X,t)$. If B is (Boolean) cyclic, then τ is one-to-one and onto, so $t : X \to X$ must be one-to-one and onto also.

When (X,t) is a flow, we will often use just X, instead of (X,t), referring implicitly to the flow map t. Similarly, we may use B as short for (B,τ).

The Profinite Integers. We let $\widehat{\mathbf{Z}}$ denote the *profinite integers*, or the free profinite group on one generator, 1. We regard $\widehat{\mathbf{Z}}$ as a flow in **Stone** with $t(\zeta) = \zeta+1$ for all $\zeta \in \widehat{\mathbf{Z}}$. (Most of the profinite groups we work with also have a ring structure and we let 1 denote the unit. When there are obvious maps from the integers, $\mathbf{Z}$, then "1" denotes the image of 1 in $\mathbf{Z}$.) Clearly the ring of integers $\mathbf{Z}$ is a subring of $\widehat{\mathbf{Z}}$ and we will identify $\mathbf{Z}$ with its image in $\widehat{\mathbf{Z}}$.

By the Chinese Remainder Theorem, $\widehat{\mathbf{Z}} = \prod\{\widehat{\mathbf{Z}}_p\colon \text{p prime}\}$, the product of the p-adic integers as p varies over the primes of $\mathbf{N}$. For every $n \in \mathbf{N}$ let $q_n : \widehat{\mathbf{Z}} \to \mathbf{Z}_n$ be the continuous homomorphism which preserves the generator 1. It readily follows that if n divides m and if $f : \mathbf{Z}_m \to \mathbf{Z}_n$ is the homomorphism which preserves 1, then $fq_m = q_n$.

Note that $\widehat{\mathbf{Z}}$ is Boolean cyclic by Proposition 1.3 because it is a limit of cyclic flows in **Stone**, so $\mathrm{Clop}(\widehat{\mathbf{Z}})$ is a colimit of cyclic flows in **Bool**.

If $\zeta \in \widehat{\mathbf{Z}}$ is given, we will say that "$\zeta \cong k \pmod{n}$" if $q_n(\zeta) = q_n(k)$.

2.1. Theorem. *Let (B,τ) be a cyclic Boolean flow. Then there is a continuous group action of $\widehat{\mathbf{Z}}$ on B such that for $n \in \mathbf{N}$ and $b \in B$ the action maps (n,b) to $\tau^n(b)$. This determines the action and $\tau^\zeta(b)$ will denote the result of applying the action to (ζ,b).*

Proof. Let $\zeta \in \widehat{\mathbf{Z}}$ and $b \in B$ be given. Since B is cyclic, we can choose $n \in \mathbf{N}$ such that $\tau^n(b) = b$ and define $\tau^\zeta(b) = \tau^k(b)$ where $\zeta \cong k \pmod{n}$. We claim that this is well-defined. Suppose $\tau^n(b) = \tau^m(b) = b$. Let $s = \mathrm{lcm}(n,m)$ and let $f_n : \mathbf{Z}_s \to \mathbf{Z}_n$ and $f_m : \mathbf{Z}_s \to \mathbf{Z}_m$ be the maps which preserve 1. Then $f_nq_s = q_n$ and $f_mq_s = q_m$. Let $\zeta \cong k \pmod{n}$ and $\zeta \cong j \pmod{m}$.

We must show that $\tau^k(b) = \tau^j(b)$. But let $\zeta \cong h \pmod{s}$. Then (by considering the projections of ζ) we get $h \cong j \pmod{m}$ so $\tau^h(b) = \tau^j(b)$ (as $\tau^m(b) = b$) and, similarly, $\tau^h(b) = \tau^k(b)$ so $\tau^j(b) = \tau^k(b)$, proving that the action is well-defined.

We note that each map τ^ζ for $\zeta \in \widehat{\mathbf{Z}}$ is a Boolean homomorphism, because given any finite subset of B we can find n such that $\tau^n(b) = b$ for all b in the finite subset, so there is a k such that $\tau^\zeta = \tau^k$ on the entire finite subset. Since τ^k is a Boolean homomorphism, it follows that τ^ζ is too.

It is immediate that this is a group action on B so it remains to show that it is continuous (when B is given the discrete topology). Now suppose that $\tau^\zeta(b) = c$. Let n be chosen so that $\tau^n(b) = b$. Then $W = p_n^{-1}(p_n(\zeta))$ is open and $\tau^{\zeta'}(b) = c$ for all $\zeta' \in W$. This proves continuity. ∎

2.2. Theorem. *Let B be a cyclic Boolean flow and let X be the corresponding flow in* **Stone**. *Then $\widehat{\mathbf{Z}}$ acts continuously on X.*

Proof. For each $\zeta \in \widehat{\mathbf{Z}}$ we have a Boolean homomorphism τ^ζ on B. By duality, there is a corresponding continuous map t^ζ on X. It also easily follows that this defines a group action of $\widehat{\mathbf{Z}}$ on X. (A left group action on an object is the same as a right group action in the dual category, but since $\widehat{\mathbf{Z}}$ is abelian there is no real difference between left and right actions.) It remains to show that this action is continuous, using the given Stone topology on X.

Let $\zeta \in \widehat{\mathbf{Z}}$ and $x_0 \in X$ be such that $t^\zeta(x_0) = y_0$. Let b be a clopen neighborhood of y_0. We must find a neighborhood c of x_0 and a neighborhood W of ζ such that the action maps $W \times c$ into b. Since t^ζ is continuous, $(t^\zeta)^{-1}(b) = c$ is a clopen neighborhood of x_0. So $\tau^\zeta(b) = c$ (recall that $\tau = t^{-1}$). By the above proof, there exists a neighborhood W of ζ such that for $\zeta' \in W$ we have $\tau^{\zeta'}(b) = c$. Then W and c have the required properties. ∎

Definition 5. Let (X, t) be a flow in **Stone**. Then X is *nearly cyclic* if for every $p, q \in X$ and every neighborhood U of q, there exists a clopen b with $q \in b \subseteq U$ such that the sequence $\{t^m(p)\}$ is periodically in b, meaning that for some n there is a non-empty subset $\{m_1, \dots m_i \dots\} \subseteq \{0, 1, \dots n-1\}$ with $t^m(p) \in b$ iff $m \cong m_i \pmod{n}$ for some i.

Remark. Conceptually, a nearly cyclic flow is one which looks cyclic when we observe how often it visits a small neighborhood b. But when we observe more closely, using a smaller neighborhood, the period might change. Often it is doubled or multiplied by some prime p.

In effect, a nearly cyclic flow is a "strange attractor" because every trajectory comes close, infinitely often, to every point in the attractor and it is close to being cyclic (in the sense mentioned above) yet is not cyclic.

2.3. Lemma. *A nearly cyclic flow in* **Stone** *is Boolean cyclic.*

Proof. Let (X, t) be a nearly cyclic flow in **Stone** and let $(B, \tau) = \mathrm{Clop}(X, t)$. We say that b has period n with respect to $p \in X$ when $t^k(p) \in b$ iff $t^{k+n}(p) \in b$. We claim that this implies that $\tau^n(b) = b$. For let $U = (t^{-1})^n(b) + b$ (where "+" denotes the Boolean addition, or symmetric difference). We need to show that U is empty, so suppose that $q \in U$. Then by the definition of nearly cyclic, there exists $b' \in B$ with $q \in b' \subseteq U$ such that the sequence of points $t^k(p)$ frequently visits b'. But it is clear by the periodic property of b that no such power of p can be in U, let alone b'.

Now let W be any clopen of X. Clearly if $q \in W$ there exists b with $q \in b \subseteq W$ where b has the periodic property (with respect to any p) so $\tau^n(b) = b$ for some n. Thus W is a union of such periodic clopens b. But W is closed and therefore compact so W is a finite union of periodic elements and such a union is periodic, because the Boolean homomorphism τ, preserves finite unions. ∎

Definition 6. If (X, t) is a flow in **Stone** and $p \in X$, we let $T(p)$ denote the closure of the trajectory of p. We also call $T(p)$ the *closed trajectory* of p.

We further say that p is a *generating point* of X if $T(p) = X$.

2.4. Lemma. *Let (X, t) be a Boolean cyclic flow in* **Stone** *with a generating point p. If W is a clopen subset of X with $t^{-1}(W) = W$ then W is either empty or all of X.*

Proof. First we claim that $t(u) \in W$ iff $u \in W$. Note that $t(u) \in W$ iff $u \in t^{-1}(W)$ iff $u \in W$ (since $W = t^{-1}(W)$).

Now suppose W is non-empty. Since $T(p) = X$, the elements $\{t^n(p)\}$ are dense in X so they meet every non-empty open set. Therefore there exists n with $t^n(p) \in W$. It follows from the above claim and induction that $p \in W$ and then that every $t^k(p) \in W$. So W contains the trajectory of p and, since W is closed, the closed trajectory, $T(p)$ of p. But $T(p) = X$ as p is a generating point. ∎

2.5. Proposition. *A flow in* **Stone** *with a generating point is nearly cyclic iff it is Boolean cyclic.*

Proof. Lemma 2.3 gives one direction. Conversely, let (X, t) be a Boolean cyclic flow in **Stone** with a generating point. Let q, r be in X and let U be a clopen neighborhood of r. We claim there exist k with $t^k(q) \in U$. Since $\mathrm{Clop}(X, t)$ is Boolean cyclic, there exists $n \in \mathbf{N}$ with $\tau^n(U) = (t^{-1})^n(U) = U$. Let $U_k = \tau^k(U)$ for $k = 1, \ldots, n$. Let $W = \cup U_k$. So W is clopen. Also $\tau(W) = W$ as the Boolean homomorphism τ preserves finite unions and permutes the U_i's. Since $r \in W$ we see, by the above lemma that $W = X$. So $q \in W$ and so $q \in \tau^k(U)$ for some k and so $t^k(q) \in U$. Also q is periodically in U as $\tau^n(U) = U$. ∎

2.6. Lemma. *Boolean cyclic flows in* **Stone** *are closed under the operations of taking subflows and quotient flows.*

Proof. Cyclic flows in **Bool** are obviously closed under quotient flows and subflows so the result follows by duality. ∎

Definition 7. Let (X, t) be a flow in **Stone**. By the *t-action of $\mathbf{N}$ on (X, t)* we mean the mapping $\mathbf{N} \times X \to X$ which maps (n, x) to $t^n(x)$.

We say that (X, t) admits a *continuous t-action by $\widehat{\mathbf{Z}}$* if there is a continuous group action of $\widehat{\mathbf{Z}}$ on X which extends the t-action of $\mathbf{N}$. Clearly such an action is unique if it exists as $\mathbf{N}$ is dense in $\widehat{\mathbf{Z}}$. By Theorem 2.2, every Boolean cyclic flow in **Stone** has such a continuous action by $\widehat{\mathbf{Z}}$.

When (X, t) does admits a continuous t-action by $\widehat{\mathbf{Z}}$ we let $t^\zeta(x)$ denote the result of applying the action to (ζ, x).

2.7. Lemma. *Let (X,s) and (Y,t) be flows in* **Stone** *which admit continuous t-actions by $\widehat{\mathbf{Z}}$. If $h : X \to Y$ is a continuous homomorphism, then h preserves the actions of $\widehat{\mathbf{Z}}$ on X and Y.*

Proof. Given $p \in X$, we need to show that $h(s^\zeta(p)) = t^\zeta(h(p))$ for all $\zeta \in \widehat{\mathbf{Z}}$. But, as h is a flow homomorphism, this is true for $\zeta \in \mathbf{N}$ and the result follows as h is continuous and $\mathbf{N}$ is dense in $\widehat{\mathbf{Z}}$. ∎

2.8. Lemma. *Let (X,t), a flow in* **Stone**, *admit a continuous t-action by $\widehat{\mathbf{Z}}$. Then the orbit of a point under the action of $\widehat{\mathbf{Z}}$ is the closed trajectory of that point under the action of t. Also, each closed trajectory $T(p)$ is Boolean cyclic.*

Proof. Let us say that the "orbit" of p under $\widehat{\mathbf{Z}}$ is the set $\{t^\zeta(p) : \zeta \in \widehat{\mathbf{Z}}\}$ while the "trajectory" of p under t is the set $\{t^n(p) : n \in \mathbf{N}\}$. Then $T(p)$ is the closure of the trajectory of p under t.

The orbit of p under $\widehat{\mathbf{Z}}$ contains $T(p)$ because it contains the trajectory and is closed (as a compact set).

Conversely, since $\mathbf{N}$ is dense in $\widehat{\mathbf{Z}}$ we see that the orbit of p under $\widehat{\mathbf{Z}}$ is contained in $T(p)$, the closure of the trajectory under t.

Finally, $\widehat{\mathbf{Z}}$ is Boolean cyclic by Proposition 1.3 as it is a limit (in **Stone**) of cyclic flows (hence a colimit in **Bool**). Therefore $T(p)$ is Boolean cyclic by Proposition 1.4. ∎

2.9. Corollary. *Each trajectory $T(p)$ of a Boolean cyclic flow in* **Stone** *is a quotient of $\widehat{\mathbf{Z}}$ by a subgroup where the identity maps to p.*

2.10. Lemma. *Let (X,t), a flow in* **Stone**, *admit a continuous t-action by $\widehat{\mathbf{Z}}$. Then:*

(1) X has a generating point iff every point of X is generating.

(2) Two closed trajectories, $T(p)$ and $T(q)$ overlap iff they coincide.

Proof.

(1) If X has a generating point, then X is Boolean cyclic by 2.8. The result now follows from the proof of Proposition 2.5, which showed that every neighborhood of r contains points of the form $t^k(q)$ where q, r were arbitrary points of X.

(2) Suppose that $r \in T(p) \cap T(q)$. Then by (1), $T(p)$ and $T(q)$ are both generated by r so $T(p) = T(r) = T(q)$. ∎

Definition 8. Let (X,t) be a flow in **Stone**. We say that $p \sim q$ if $T(p) \cap T(q)$ is non-empty. We say that X is *regular* if $\sim$ is a closed equivalence relation on X (that is, an equivalence relation which is closed as subset of $X \times X$.)

If X is regular, we let $X/{\sim}$ denote the set of all $\sim$-equivalence classes and give it the quotient topology (induced by the obvious map $X \to X/{\sim}$).

Clearly $X/{\sim}$ is the set of closed trajectories of X and, as noted below, it has a compact Hausdorff topology.

Remark. For compact Hausdorff spaces, the quotient topology induced by a closed equivalence relation is always compact Hausdorff. While this result is well-known (it is part of the theorem that compact Hausdorff spaces are equational over **Sets**) the proof is often abbreviated or omitted in standard references. So, for the record, here is a proof, supplied to me by Michael Barr:

2.11. Proposition. *The quotient of a compact Hausdorff space by a closed equivalence relation is compact and Hausdorff.*

Proof. Let X be a compact Hausdorff space and let $E \subseteq X \times X$ be a closed equivalence relation. Let $q : X \to X/E$ be the quotient map and give X/E the quotient topology. Clearly X/E is compact, so it remains to show that it is Hausdorff.

We claim that q is a closed mapping. Let $A \subseteq X$ be a closed subset. To prove that $q(A)$ is closed in the quotient topology, we need to show that $q^{-1}(q(A))$ is closed. Regard $A \times X$ and E as closed subsets of $X \times X$ and let $F = E \cap (A \times X)$. Then F is clearly closed and so is compact, so $p_2(F)$ is a closed subset of X where $p_2 : X \times X \to X$ is the second projection. But $p_2(F) = q^{-1}(q(A))$.

Now let c, d be distinct points of X/E. Choose x, y in X with $q(x) = c, q(y) = d$. Then, since $\{x\}, \{y\}$ are closed subsets so are $\{c\}, \{d\}$ (as q is closed). We need to find disjoint open neighborhoods in X/E of c, d. But this is equivalent to finding closed subsets V, W of X/E such that $c \notin V$ and $d \notin W$ and $V \cup W = X/E$, as the complements of V, W are then the required neighborhoods. But $q^{-1}(c)$ and $q^{-1}(d)$ are disjoint closed subsets of X so, by normality, there exist disjoint open subsets E, F of X with $q^{-1}(c) \subseteq E$ and $q^{-1}(d) \subseteq F$. Then $V = q(X \setminus E)$ and $W = q(X \setminus F)$ have the required properties. ∎

2.12. Lemma. *Let (X, t), a flow in* **Stone**, *admit a continuous t-action by $\widehat{\mathbf{Z}}$. Then X is regular.*

Proof. That $\sim$ is an equivalence relation follows directly from Lemma 2.10. To show that $\sim$ is closed, let $\{p_i : i \in I\}$ be an indexed family of members of X and let $\mathcal{U}$ be an ultrafilter on I with $\operatorname{Lim}_{\mathcal{U}} p_i = p$. Choose q_i with $q_i \sim p_i$ for all i and let $\operatorname{Lim}_{\mathcal{U}} q_i = q$. Then we must show that $q \sim p$.

But by above results, if $q_i \sim p_i$ then there exists an $\zeta_i \in \widehat{\mathbf{Z}}$ such that $q_i = t^{\zeta_i}(p_i)$. Let $\operatorname{Lim}_{\mathcal{U}} \zeta_i = \zeta$ (in the compact topology on $\widehat{\mathbf{Z}}$). Then, in $X \times \widehat{\mathbf{Z}}$ we see that $\operatorname{Lim}_{\mathcal{U}} (\zeta_i, p_i) = (\zeta, p)$.

Since the action is continuous, it preserves this limit so $q = t^{\zeta}(p)$ which shows that $q \sim p$. ∎

2.13. Theorem. *The following statements are equivalent for (X, t) a flow in* **Stone**:

(1) (X, t) is Boolean cyclic

(2) (X, t) admits a continuous t-action by $\widehat{\mathbf{Z}}$.

(3) X is regular and every closed trajectory $T(p)$ is nearly cyclic.

PROOF.

(1) $\Rightarrow$ (2) By Theorem 2.2.

(2) $\Rightarrow$ (3) By Lemmas 2.8 and 2.12.

(3) $\Rightarrow$ (1) Let X be regular and let every $T(p)$ be nearly cyclic. Think of $\{T_i : i \in X/{\sim}\}$ as an indexing of the closed trajectories of X. (Strictly speaking, $T_i = i$ but this might be notationally confusing.)

Let $(B, \tau) = \mathrm{Clop}(X, t)$ be the corresponding Boolean flow. Let $b \in B$. Let $b_i = b \cap T_i$ and let n_i be the period of b_i in the cyclic Boolean algebra of all (relatively) clopen subsets of T_i. If the n_i are bounded, then let n be their least common multiple. It is clear that $\tau^n(b) = b$.

On the other hand, assume that $\{n_i\}$ is not bounded. Then we can find an ultrafilter $\mathcal{U}$ on the set $X/{\sim}$ such that for every $U \in \mathcal{U}$, the set $\{n_i : i \in U\}$ is unbounded. Let $T = \mathrm{Lim}_{\mathcal{U}} T_i$ in the quotient topology on $X/{\sim}$.

Let $b' = b \cap T$. Then, since $\mathrm{Clop}(T)$ is Boolean cyclic, there exists n such that $\tau^n(b') = b'$. Now, by choice of $\mathcal{U}$, we can choose $q_i \sim p_i$ such that $q_i \in \tau^n(b_i) + b_i$ for all $i \in U$ where $U \in \mathcal{U}$. (Note that $+$ denotes the symmetric difference.) Then let $q = \mathrm{Lim}_{\mathcal{U}} q_i$ so, by regularity, and because the map from X to $X/{\sim}$ is continuous, $q \in T$. But $q_i \in \tau^n(b) + b$ so (as $\tau^n(b) + b$ is closed, $q \in \tau^n(b) + b$) but this contradicts $q \in T$ as $T \cap (\tau^n(b) + b)$ is empty. ■

2.14. PROPOSITION. *Let (X, t) be a flow in* **Stone** *for which all closed trajectories are nearly cyclic and all but finitely many of them are actually cyclic. Further suppose there is a bound n such that every cyclic trajectory has period no bigger than n. Then (X, t) is Boolean cyclic.*

PROOF. The proof follows from the first two paragraphs of the above proof of (3) $\Rightarrow$ (1). ■

REMARK. The converse of 2.14 is false. Consider a product $X = \widehat{\mathbf{Z}}^m$ where m is a large cardinal (say at least as large as the cardinal of $\widehat{\mathbf{Z}}$). Then X is clearly Boolean cyclic as X is a product (and so $\mathrm{Clop}(X)$ is a coproduct) of Boolean cyclic spaces. But X can have no finite cyclic trajectories, as they would have to project onto finite cycles on each component $\widehat{\mathbf{Z}}$. Nor can X have only finitely many closed trajectories (by cardinality considerations).

3. Constructing the Spectrum

The previous section showed that Boolean cyclic spaces are fairly special. We would like to construct a cyclic reflection for every Boolean flow. Unfortunately, the subcategory of cyclic Boolean flows is not reflective (as, for example, it is not closed under products, see

Example 6). We can get a reflection if we work in a bigger category which might be called the "category of Boolean flows in Grothendieck toposes".

The objects of this category are pairs $(\mathcal{E}, B)$ where $\mathcal{E}$ is a Grothendieck topos and B is a Boolean flow in $\mathcal{E}$ (that is, a flow in the category $\mathbf{Bool}(\mathcal{E})$ of Boolean algebras in $\mathcal{E}$). By a morphism $(\lambda^*, f) : (\mathcal{E}, B) \to (\mathcal{F}, C)$ we mean an inverse image functor $\lambda^* : \mathcal{E} \to \mathcal{F}$ along with a flow homomorphism $f : \lambda^*(B) \to C$ in $\mathcal{F}$. Consider the subcategory of all $(\mathcal{E}, B)$ where B is cyclic and all morphisms (λ^*, f) where f is mono. Then this category is reflective and the reflection is an example of a *Cole spectrum*. See [2, pages 205-210].

Remark. The category of Boolean cyclic flows is not a reflective subcategory of all Boolean flows, but it is coreflective. The coreflection seems not very interesting as it just gives the subalgebra of all b for which $b = \tau^n(b)$ for some n. However, by duality, this means that the Boolean cyclic flows in **Stone** are reflective in the category of all flows in **Stone**. If we start with $\mathbf{N}$ with the map $t(n) = n + 1$ and take its Stone-Čech compactification, we get a flow in **Stone** whose Boolean cyclic reflection is $\widehat{\mathbf{Z}}$.

Before describing the Cole spectrum, we will establish some properties of Boolean cyclic flows in a Grothendieck topos.

3.1. Lemma. *If (C, τ) is a Boolean cyclic flow in a Grothendieck topos, then τ is an isomorphism.*

Proof. To prove that τ is one-to-one, let K be the kernel of τ. It suffices to prove that $K = \{0\}$. But let $E(n) = \mathrm{Equ}(\mathrm{Id}_C, \tau^n)$ then $\bigvee E(n) = C$. It is readily established that $K \cap E(n) = \{0\}$ for all n and so $K = \bigvee (E(n) \cap K) = \{0\}$.

A similar proof establishes that τ is onto as τ maps each $E(n)$ onto itself. ∎

3.2. Lemma. *Boolean cyclic flows in any topos are closed under the formation of subflows.*

Proof. Let (C, τ) be a cyclic flow in the topos $\mathcal{E}$. By the above lemma, τ is one-to-one and onto. Let A be a subflow. Then the maps Id_C and τ^n pull back along A to Id_A and τ_A^n. Since pullbacks preserve equalizers, $E(n) = \mathrm{Equ}(\mathrm{Id}_C, \tau^n)$ pulls back to $E_A(n) = \mathrm{Equ}(\mathrm{Id}_A, \tau_A^n)$ and the result follows because pulling back (or intersecting with the subobject A) preserves sups of subobjects. ∎

Remark. It can readily be seen that the above statement is not true for all categories. Even for the well-behaved category **Stone**, Example 6 shows a space that is **Stone** cyclic but with non-cyclic subobjects (for example, there are closed subsets of the Stone-Čech compactification which do not meet any $\mathbf{Z}_p$).

3.3. Proposition. *Let C be a Boolean flow in a Grothendieck topos $\mathcal{E}$. If C is cyclic (in $\mathcal{E}$) then there is a canonical action of $\widehat{\mathbf{Z}}$ on C which extends the action of $\mathbf{N}$.*

Proof. Let $E(n) = \mathrm{Equ}(\mathrm{Id}_C, \tau^n)$ so that $\bigvee E(n) = C$. Given $\zeta \in \widehat{\mathbf{Z}}$ we define τ^ζ on $E(n)$ as τ^k where $\zeta \cong k \pmod{n}$. We claim that these definitions are compatible on the overlaps, $E(n) \cap E(m)$. The claim follows by showing that $E(n) \cap E(m) = E(\gcd(n, m))$ the proof of which uses negative powers of τ, as τ^{-1} exists.

By the claim, the above maps $\{\tau^k\}$ patch together to give us a morphism $\tau^\zeta : C \to C$. ∎

3.4. Corollary. *Let C be a cyclic Boolean flow in a Grothendieck topos $\mathcal{E}$. Then there is a canonical action of $\widehat{\mathbf{Z}}$ on $\Gamma(C)$ which extends the action of $\mathbf{N}$. (where Γ is the global sections functor). Note that $\widehat{\mathbf{Z}}$ need not act continuously on $\Gamma(C)$.*

We now start to construct the spectrum for B, a Boolean flow in **Sets**. This means finding a map $(\gamma^*, \eta) : (\mathbf{Sets}, B) \to (\mathcal{E}, \mathrm{B}^\#)$ with the appropriate universal property (which is restated in Proposition 3.5 below) As a convenience, we will find the *non-trivial* cyclic spectrum, thus requiring that the cyclic object satisfy $0 \neq 1$.

3.5. Proposition. *Let B be a flow in Boolean algebras (in* **Sets***). Then the cyclic spectrum of B coincides with the topos which classifies non-trivial cyclic quotients of B.*

Proof. The classifying topos for cyclic quotients has a map $(\gamma^*, \eta) : (\mathbf{Sets}, B) \to (\mathcal{E}, \mathrm{B}^\#)$ with η epi and $\mathrm{B}^\#$ cyclic such that for any other map $(\lambda^*, f) : (\mathbf{Sets}, B) \to (\mathcal{F}, C)$, with f epi, there is (to within equivalence) a unique inverse image functor δ^* such that $\delta^*\gamma^* = \lambda^*$ and $\delta^*(\mathrm{B}^\#) = C$. To show that this is the Cole spectrum, let $(\lambda^*, f) : (\mathbf{Sets}, B) \to (\mathcal{F}, C)$ be given (without assuming that f is epi). Then use the mono/epi factorization of f to get a cyclic quotient of C and apply the above property (so that $\delta^*(\mathrm{B}^\#)$ maps via a mono to C).

The details that the two concepts (Cole spectrum and quotient classifier) coincide are now straightforward. For example, if $(\gamma^*, \eta) : (\mathbf{Sets}, B) \to (\mathcal{E}, \mathrm{B}^\#)$ is the Cole cyclic spectrum, then we need to prove that η is epi. But, if not, then η factors through a proper subobject of $\mathrm{B}^\#$ and, by the spectral property, $\mathrm{B}^\#$ must map monomorphically to that subobject which gives two ways to map $\mathrm{B}^\#$ monomorphically to itself. ∎

Notation.

(1) We will use $\gamma^*_{\mathcal{E}}$ or just γ^* denote the unique inverse image functor from **Sets** to any given Grothendieck topos $\mathcal{E}$. Its adjoint will be denoted by either γ_* or by Γ which is the usual notation for the global sections functor which maps E to the set $\mathrm{Hom}_{\mathcal{E}}(1, E)$.

(2) It follows that $\gamma^*(B)$ is just the coproduct of "B copies of 1" which we will denote by $\sum 1_b$ where the subscript b denotes the fact that the copies of 1 are indexed by B. We also let $b : 1_b \to \gamma^*(B)$ denote the coprojections associated with this coproduct. We may then drop the subscript b and simply write $b : 1 \to \gamma^*(B)$, particularly when we need to remember that $1_a = 1_b$ for all $a, b \in B$. The different sections labeled "b" are preserved by the functors and operations we are working with, so there is no danger of significant confusion.

First Step. We first construct the topos which classifies "Boolean monoflow quotients of B" (see below). Here the construction of [4] applies and it gives us a spatial topos which is a good starting point.

Definition 9. By a *Boolean monoflow* we mean a non-trivial flow (B, τ) in **Bool** for which τ is one-to-one. Note that both Boolean monoflows and Boolean cyclic flows make sense in any Grothendieck topos and by Lemma 1.1, every Boolean cyclic flow is a monoflow.

Construction of the monoflow quotient classifier. Let (B, τ) be a flow in **Bool** (in **Sets**). We follow the construction of [4].

Let $\{B_v : v \in V\}$ be an indexing of the non-trivial monoflow quotients of B and let $q_v : B \to B_v$ be the quotient map. For every $b \in B$ let $N(b) = \{v \in V : q_v(b) = 0\}$. Topologize V so that the sets $\{N(b)\}$ form a base. (Note that $N(a) \cap N(b) = N(a \vee b)$.) Define a set B^o and a function $q : B^o \to V$ so that $B_v = q^{-1}(v)$. Regard each $b \in B$ as a map $b : V \to B^o$ for which $b(v) = q_v(b)$. Topologize B^o with the largest topology making each map a continuous. Then:

3.6. Proposition. *With the notation of the above construction, B^o is a sheaf over V and it classifies non-trivial monoflow quotients of B in Grothendieck toposes.*

Proof. We sketch the proof, most of which follows from [4]. It is straightforward to show that B^o is the etale space of a sheaf over V and that B^o is a monoflow because the stalks $\{B_v\}$ are. It is clear that the canonical map from $\gamma_V^*(B)$ to B^o is an epi by considering its action on stalks.

Let $\mathcal{E}$ be a Grothendieck topos and let M be a Boolean monoflow in $\mathcal{E}$ along with an epi flow homomorphism $f : \gamma_{\mathcal{E}}^*(B) \to M$.

For $b \in B$ let $E(b)$ be the equalizer of $b, 0 : 1 \to M$. Let $\mathcal{O}(V)$ be the partially ordered set of open subsets of V and let $\mathrm{Sub}(1_{\mathcal{E}})$ be the partially ordered family of subobjects of 1 in $\mathcal{E}$. Define $\lambda : \mathcal{O}(V) \to \mathrm{Sub}(1_{\mathcal{E}})$ so that $\lambda(N(b)) = E(b)$ and extend to all of $\mathcal{O}(V)$ by taking unions. Then λ preserves all unions and finite intersections because each relevant property of $\mathcal{O}(V)$ follow from intuitionistic logic from a finite number of facts about B and axioms satisfied by non-trivial monoflows. As is well-known, this determines an inverse image functor $\lambda^* : \mathrm{Sh}(V) \to \mathcal{E}$.

Since M is a quotient of $\gamma^*(B)$, we see that M is $\gamma^*(B)$ modulo the ideal $\bigvee b(E(b))$.

So M is determined by the subobjects $E(b)$ and the coproduct and coequalizer structure on $\mathcal{E}$. In the same way, B^o is determined by the equalizers $N(b)$ and the coproduct and coequalizer structure on $\mathrm{Sh}(V)$ all of which is preserved by λ^*. So $\lambda^*(B^o) = M$. ■

Remark. In view of the above we need only to force the above generic monoflow quotient of B to be cyclic. This is done by using the smallest Lawvere-Tierney topology (or nucleus) j on $\mathrm{Sh}(V)$ for which the embedding of $\bigvee E(n)$ to B^o becomes epi. To that end, we make the following definition:

Definition 10. Use the above notation and for each $b \in B$ let $b_k = b + \tau^k(b)$. Then $N(b_k)$ is the largest open subset of V on which $b = \tau^k(b)$ (as sections of B^o). Let $\mathcal{O}(V)$ be the Heyting algebra of open subsets of V and let j be the smallest Lawvere-Tierney topology on $\mathcal{O}(V)$ such that for each $b \in B$, the set $\{N(b_k)\}$ covers V. (These form a subbase for the covers). We must then consider finite intersections of the $\{N(b_k)\}$ and intersect them with U to form a cover of U for each $U \in \mathcal{O}(V)$.)

Notation.

(1) We let j-$\mathrm{Sh}(V)$ denote the resulting topos obtained from $\mathcal{O}(V)$ and let "j-sheaves" refer to sheaves over $\mathcal{O}(V)$ which are also sheaves with respect to the topology j.

(2) We let $\mathrm{B}^{\#}$ denote the j-sheafification of B^o in the topos of j-Sh(V).

3.7. PROPOSITION. *The object* $\mathrm{B}^{\#}$ *in the topos* j-Sh(V) *classifies cyclic quotients of* B. *Equivalently this gives us the Cole cyclic spectrum.*

PROOF. We must show that j is the smallest topology making B^o cyclic. Clearly, by the definition of j, each $b \in B$ is in $\bigvee E(n)$ and, as $\gamma^*(B)$ maps onto B^o, this means that $\bigvee E(n)$ will cover B^o, and hence its j-sheafification, $\mathrm{B}^{\#}$.

Conversely, if $\bigvee E(n)$ is the maximum subobject, then, pulling this back down to 1_b, we find that the covers defining the j-topology must be epi families. ∎

3.8. PROPOSITION. *Let* $B, \mathrm{B}^{\#}, j$-Sh(V) *be as above. The points of* j-Sh(V) *correspond to cyclic quotients* $B \to C$ *in* **Sets** *and each such quotient extends to a flow homomorphism* $\Gamma(\mathrm{B}^{\#}) \to C$.

PROOF. A point of j-Sh(V) is an inverse image functor to **Sets**. Such a functor preserves epis and cyclicity and so maps η to a cyclic quotient of B. Conversely, any such quotient arises in this way because of the universal property of the spectrum.

Let $\lambda^* : j\text{-Sh}(V) \to \mathbf{Sets}$ be such a point and let $\lambda^*(\mathrm{B}^{\#}) = C$. By adjointness, there is a corresponding map $\Gamma(\mathrm{B}^{\#}) \to \Gamma(\lambda_*(C))$. But $\Gamma\lambda_*$ is a geometric functor from **Sets** to **Sets** so is equivalent to the identity and the result follows. ∎

NOTATION. Given a Boolean flow (B, τ), we let V and $\mathrm{B}^{\#}$ be constructed as above. We let $\Gamma(\mathrm{B}^{\#})$ be the global sections of $\mathrm{B}^{\#}$ and let $\eta_1 : B \to \Gamma(\mathrm{B}^{\#})$ be the adjunct of the canonical map $\eta : \gamma^*(B) \to \mathrm{B}^{\#}$ in j-Sh(V).

We say that "B has enough maps into cyclic flows in toposes" if η_1 is one-to-one. We note that for η_1 to be one-to-one it is sufficient, but not presumably necessary that "B has enough maps into cyclic flows in **Sets**" meaning that if $b \in B$ is non-zero then there exists a flow homomorphism $h : B \to C$ with C cyclic and $h(b)$ non-zero.

REMARK: TRANSFINITE TIME. Given a flow (X, t) in **Stone**, think of $t(x)$ as the state a system is in one unit of time after being in state x. Then we would like to define $t^\zeta(x)$ and think of it as the state ζ units of time after state x where ζ could be transfinite (i.e. in $\widehat{\mathbf{Z}}$). But if $\mathrm{X}^{\#}$ is defined as $\Gamma(\mathrm{B}^{\#})$ then, by 3.4, $t^\zeta(x)$ is defined only for $x \in \mathrm{X}^{\#}$ not for $x \in X$. However the map $\eta_1 : B \to \Gamma(\mathrm{B}^{\#})$ gives us a map from $\mathrm{X}^{\#} \to X$ so in some sense, $\mathrm{X}^{\#}$ is the space where transfinite time makes sense.

If B has "enough maps onto cyclic flows in **Sets**", then η_1 is one-to-one and $\mathrm{X}^{\#}$ maps onto X. In this case, we might think of $\mathrm{X}^{\#}$ as a sort of covering of X for which τ lifts to a one-to-one, onto map which comes from a cyclic flow in a topos.

REMARK: METHODS FOR COMPUTING THE SPECTRUM. The spectrum is further explored by the examples below. It is useful to first note the following:

Finding a Boolean quotient of B is equivalent to finding an ideal of B, by which we mean a subset I for which:

(1) $0 \in I$.

(2) $a, b \in I$ imply $a \vee b \in I$.

(3) $a \in I$ and $b \leq a$ imply $b \in I$.

If we want I to be a flow ideal (so that B/I has a natural flow) then we must add:

(4) $a \in I$ implies $\tau(a) \in I$.

If the quotient is to be non-trivial, then we require:

(5) $1 \notin I$

Finally, if we want the quotient to be a monoflow, then we need the kernel of τ to be $\{0\}$, so we add:

(6) $\tau(a) \in I$ implies $a \in I$.

Ideals satisfying (1)–(6) are the points of the space V. Another approach to finding quotients of B is to let $(X, t) = \mathrm{Pts}(B, \tau)$. Then quotients of B correspond, by duality, to closed subsets of X. Given such a closed subset A, the corresponding ideal I is the set of all clopens b for which $b \cap A$ is empty. Point (4), above, is equivalent to the condition:

(4′) $a \in A$ implies $t(a) \in A$. This makes sense as A is then a subflow in **Stone**.

Point (5) clearly translates into:

(5′) A is non-empty.

Point (6) is not so easy to restate as a condition about A but if t is one-to-one and onto, then point (6) is restated as:

(6′) (For t one-to-one and onto): $t(x) \in A$ implies $x \in A$.

Also, if t is not onto, point (6) implies that $A \subseteq t^n(X)$ for all n.

Example 7: An attractive fixpoint. Let $X = \mathbf{N} \cup \{\infty\}$ with $t(n) = n + 1$ and $t(\infty) = \infty$. Let $(B, \tau) = \mathrm{Clop}(X, t)$ (where X has the obvious Stone topology). We first find the monoflow quotients of B. They correspond to closed subsets A as discussed above. Since $A \subseteq \bigcap t^n(X)$ we see that the only admissible A is $\{\infty\}$. So V has a single point which contains those clopens b that do not contain ∞. The coverings add nothing as B^o is already cyclic (with a cyclic stalk of $\{0, 1\}$ over I).

In effect, the spectrum eliminated all but the point at infinity and $\Gamma(\mathrm{B}^{\#})$ is just $\mathrm{Clop}\{\infty\}$.

Example 8: An attractive and a repulsive fixpoint. Let $X = \mathbf{Z} \cup \{-\infty, \infty\}$ and let $t(z) = z + 1$, with $t(\infty) = \infty$ and $t(-\infty) = -\infty$. Give X the obvious Stone topology. Let $(B, \tau) = \mathrm{Clop}(X, t)$ Then t is one-to-one, onto, so we need only consider quotients by closed subsets A which satisfy $a \in A$ iff $t(a) \in A$.

This means that $A \cap \mathbf{Z}$ must be either empty or all of $\mathbf{Z}$. If it is all of $\mathbf{Z}$ then A must be X (as A is closed). If $A \cap \mathbf{Z}$ is empty then A may also contain any non-empty subset of $\{-\infty, \infty\}$. So V has four points. It is straightforward to verify that V has five open sets, namely:

$$\emptyset = N(X), \quad N(\infty), \quad N(-\infty), \quad N(-\infty, \infty), \quad N(0) = X$$

The j-coverings yield the fact that $N(-\infty, \infty)$ covers, so, in effect we get the two points $-\infty$ and ∞.

Remark. A goal for further work would be to extend this approach to try to distinguish repulsive from attractive fixpoints.

Example 9. Let $X = \sum\{\mathbf{Z}_p : p \in P\}$ where P is the set of primes, with $t(z) = z + 1$ on each component. Let $B = \mathrm{Clop}(X)$ We note that B is the algebra of clopens of X which is the Stone-Čech compactification of of $S = \bigcup\{\mathbf{Z}_p\}$ (the disjoint union in **Sets**). However B can more simply be viewed as the algebra of subsets of S (as subsets of S and clopens of X both correspond to maps from X to the two-point Stone space).

Since X is hard to work with, we will directly look for the right kind of ideals. We note that the zero ideal $I_0 = \{0\}$ is in V as t is onto (so τ is one-to-one). Also, the only open neighborhood in V of I_0 is V itself, so the stalk at I_0 will give us the set of global sections.

For any c the elements $c_k = c + \tau^k(c)$ will miss the $\mathbf{Z}_p \subseteq X$ once p divides k. So if $a = b$ modulo any of these j-coverings, they must agree when intersected with each $\mathbf{Z}_p$ which implies $a = b$. This means that the separation and the sheafification of B^o with respect to the j-topology do not affect the stalk at I_0. (The stalks at ∞ are undoubtedly affected by closed subsets A which are disjoint from every $\mathbf{Z}_p$.) But the upshot is that the set of global sections (= stalk at I_0) remains the same, so $\Gamma(\mathrm{B}^{\#}) = B$. Thus while B is not Boolean cyclic (see Example 6) B can be represented as the global sections of a cyclic flow in a topos. In particular, $\widehat{\mathbf{Z}}$ acts on B (although here the action is obvious). In view of Theorem 2.13, the corresponding action on X cannot be continuous.

Example 10: A brief look at the Logistic Equation. Consider the map $t(x) = rx(1 - x)$ which maps $[0, 1]$ to itself, where r is a parameter satisfying $0 \leq r \leq 4$. Unfortunately $[0, 1]$ is not a Stone space because it is connected, but we can readily disconnect it by replacing each $x \in [0, 1]$ with $x-$ and $x+$. We then discard $0-$ and $1+$ which are not needed. The resulting set, $\mathrm{St}[0, 1]$, has an obvious linear ordering and is a Stone space in the order topology (which is generated by the subbase $\{L(b) : b \in \mathrm{St}[0, 1]\} \cup \{U(a) : a \in \mathrm{St}[0, 1]\}$ where $L(b) = \{x : x < b\}$ and $U(a) = \{x : x > a\}$).

The clopens of $\mathrm{St}[0, 1]$ are essentially finite unions of intervals, with $b \vee c$ being the interior of the closure of $b \cup c$ as in the double negation topology. Note that there is a natural way to define t as a continuous flow on $\mathrm{St}[0, 1]$.

In examining the spectrum of $\mathrm{St}[0,1]$ we note that for small values of r, the methods used above show that the spectrum simply picks out the periodic points, without regard for whether they are attractive or repulsive. As r approaches a value called "r_∞" (which is about 3.569946) we pick up new periodic points whose period is twice the largest previous period. Eventually, at $r = r_\infty$, a strange attractor appears, see [8, pages 3-4]. This strange attractor is nearly cyclic and will therefore be a point in the spectrum.

Beyond r_∞, chaos begins.

For the case of $r = 4$, the set P of periodic points of $[0,1]$ (and so of $\mathrm{St}[0,1]$) is dense and two clopen subsets are the same iff they have the same intersection with P. Because of this, it can be shown that there are enough Boolean cyclic quotients in **Sets** for η_1 to be one-to-one. It follows that the dual map from $\mathrm{X}^\#$ to $\mathrm{St}[0,1]$ is onto. (r=4 is the only value for which $\mathrm{X}^\#$ maps onto $\mathrm{St}[0,1]$ as can be readily seen by the fact that t is not even onto when $r < 4$.)

Finally, there is the flow on $[0,4] \times [0,1]$ given by $t(r,x) = (r, rx(1-x))$. All I have on this is a rough beginning. We need to replace $[0,4] \times [0,1]$ by a Stone space. This is equivalent to describing a subalgebra of the algebra of double negation opens which is preserved by t^{-1}. One candidate is generated by the "vertical trapezoids" or sets of the form $\{(r,x) : a < r < b, L(r) < x < U(r)\}$ where L and U are linear functions. The trapezoid shape is needed in order for them to be preserved by t^{-1}.

References

[1] M. Barr, C. Wells. *Toposes, Triples and Theories* Springer-Verlag, New York, 1985.

[2] P.T. Johnstone. *Topos Theory* LMS Monograph 10, Academic Press, London, 1977.

[3] P.T. Johnstone. *Stone Spaces* Cambridge University Press, Cambridge, 1982.

[4] J.F. Kennison. Integral domain type representations in sheaves and other topoi, Mathematische Zeitschrift, 151, 35-56 (1976).

[5] S. Mac Lane. *Category Theory for the Working Mathematician* Springer-Verlag, New York, 1971.

[6] S. Mac Lane, I. Moerdijk. *Sheaves in Geometry and Logic* Springer-Verlag, New York, 1992.

[7] E. Ott. *Chaos in Dynamical Systems* Cambridge University Press, 1993.

[8] C. Preston. *Iterates of Maps on an Interval* Lecture Notes in Mathematics 999, Springer-Verlag, Berlin, 1983.

Clark University,
Worcester, MA, 01610,
USA
Email: `jkennison@clarku.edu`

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