Theory and Applications of Categories, Vol. 10, No. 1, 2002, pp. 1–70.

A SURVEY OF DEFINITIONS OF n-CATEGORY

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ABSTRACT. Many people have proposed definitions of 'weak n-category'. Ten of them are presented here. Each definition is given in two pages, with a further two pages on what happens when $n \leq 2$. The definitions can be read independently. Chatty bibliography follows.

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Received by the editors 2001 August 9 and, in revised form, 2001 December 22.
Transmitted by Ross Street. Published on 2002 January 14.
2000 Mathematics Subject Classification: 18D05, 18D50, 18F99, 18A99.
Key words and phrases: n-category, higher-dimensional category, higher categorical structure.

Introduction

> *Lévy … once remarked to me that reading other mathematicians' research gave him actual physical pain.*

—J. L. Doob on the probabilist Paul Lévy, *Statistical Science* **1**, no. 1, 1986.

> *Hell is other people.*

—Jean-Paul Sartre, *Huis Clos.*

The last five years have seen a vast increase in the literature on higher-dimensional categories. Yet one question of central concern remains resolutely unanswered: what exactly is a weak n-category? There have, notoriously, been many proposed definitions, but there seems to be a general perception that most of these definitions are obscure, difficult and long. I hope that the present work will persuade the reader that this is not the case, or at least does not *need* to be: that while no existing approach is without its mysteries, it is quite possible to state the definitions in a concise and straightforward way.

What's in here, and what's not. The sole purpose of this paper is to state several possible definitions of weak n-category. In particular, I have made no attempt to compare the proposed definitions with one another (although certainly I hope that this work will help with the task of comparison). So the definitions of weak n-category that follow may or may not be 'equivalent'; I make no comment. Moreover, I have not included any notions of weak functor or equivalence between weak n-categories, which would almost certainly be required before one could make any statement such as 'Professor Yin's definition of weak n-category is equivalent to Professor Yang's'.

I have also omitted any kind of motivational or introductory material. The 'Further Reading' section lists various texts which attempt to explain the relevance of n-categories and other higher categorical structures to mathematics at large (and to physics and computer science). I will just mention two points here for those new to the area. Firstly, it is easy to define *strict* n-categories (see 'Preliminaries'), and it is true that every weak 2-category is equivalent to a strict 2-category, but the analogous statement fails for n-categories when $n > 2$: so the difference between the theories of weak and strict n-categories is nontrivial. Secondly, the issue of comparing definitions of weak n-category is a slippery one, as it is hard to say what it even *means* for two such definitions to be equivalent. For instance, suppose you and I each have in mind a definition of algebraic variety and of morphism of varieties; then we might reasonably say that our definitions of variety are 'equivalent' if your category of varieties is equivalent to mine. This makes sense because the structure formed by varieties and their morphisms is a category. It is widely held that the structure formed by weak n-categories and the functors, transformations, … between them should be a weak $(n+1)$-category; and if this is the case then the question is whether your weak $(n+1)$-category of weak n-categories is equivalent to mine—but whose definition of weak $(n+1)$-category are we using here…?

This paper gives primary importance to n-categories, with other higher categorical structures only mentioned where they have to be. In writing it this way I do not mean to imply that n-categories are the only interesting structures in higher-dimensional category theory: on the contrary, I see the subject as including a whole range of interesting structures, such as operads and multicategories in their various forms, double and n-tuple categories, computads and string diagrams, homotopy-algebras, n-vector spaces, and structures appropriate for the study of braids, knots, graphs, cobordisms, proof nets, flowcharts, circuit diagrams, ... Moreover, consideration of n-categories seems inevitably to lead into consideration of some of these other structures, as is borne out by the definitions below. However, n-categories are here allowed to upstage the other structures in what is probably an unfair way.

Finally, I do not claim to have included *all* the definitions of weak n-category that have been proposed by people; in fact, I am aware that I have omitted a few. They are omitted purely because I am not familiar with them. More information can be found under 'Further Reading'.

Layout. The first section is 'Background'. This is mainly for reference, and it is not really recommended that anyone starts reading here. It begins with a page on ordinary category theory, recalling those concepts that will be used in the main text and fixing some terminology. Everything here is completely standard, and almost all of it can be found in any introductory book or course on the subject; but only a small portion of it is used in each definition of weak n-category. There is then a page each on strict n-categories and bicategories, again recalling widely-known material.

Next come the ten definitions of weak n-category. They are absolutely independent and self-contained, and can be read in any order. This means that there is a certain amount of redundancy: in particular, sizeable passages occur identically in definitions B and L, and similarly in definitions Si and Ta. (The reasons for the names are explained below.) No significance should be attached to the order in which they are presented; I tried to arrange them so that definitions with common themes were grouped together in the sequence, but that is all. (Some structures just don't fit naturally into a single dimension.)

Each definition of weak n-category is given in two pages, so that if this is printed double-sided then the whole definition will be visible on a double-page spread. This is followed, again in two pages, by an explanation of the cases $n = 0, 1, 2$. We expect weak 0-categories to be sets, weak 1-categories to be categories, and weak 2-categories to be bicategories—or at least, to resemble them to some reasonable degree—and this is indeed the case for all of the definitions as long as we interpret the word 'reasonable' generously. Each main definition is given in a formal, minimal style, but the analysis of $n \leq 2$ is less formal and more explanatory; partly the analysis of $n \leq 2$ is to show that the proposed definition of n-category is a reasonable one, but partly it is for illustrative purposes. The reader who gets stuck on a definition might therefore be helped by looking at $n \leq 2$.

Taking a definition of weak n-category and performing a rigorous comparison between the case $n = 2$ and bicategories is typically a long and tedious process. For this reason, I have not checked all the details in the $n \leq 2$ sections. The extent to which I feel confident in my assertions can be judged from the number of occurrences of phrases such as 'probably' and 'it appears that', and by the presence or absence of references under 'Further Reading'.

There are a few exceptions to this overall scheme. The section labelled B consists, in fact, of *two* definitions of weak n-category, but they are so similar in their presentation that it seemed wasteful to give them two different sections. The same goes for definition L, so we have definitions of weak n-category called B1, B2, L1 and L2. A variant for definition ST is also given (in the $n \leq 2$ section), but this goes nameless. However, definition X is not strictly speaking a mathematical definition at all: I was unable to find a way to present it in two pages, so instead I have given an informal version, with one sub-definition (opetopic set) done by example only. The cases $n \leq 2$ are clear enough to be analysed precisely.

Another complicating factor comes from those definitions which include a notion of weak ω-category (= weak ∞-category). There, the pattern is very often to define weak ω-category and then to define a weak n-category as a weak ω-category with only trivial cells in dimensions $> n$. This presents a problem when one comes to attempt a precise analysis of $n \leq 2$, as even to determine what a weak 0-category is involves considering an infinite-dimensional structure. For this reason it is more convenient to redefine weak n-category in a way which never mentions cells of dimension $> n$, by imitating the original definition of weak ω-category. Of course, one then has to show that the two different notions of weak n-category are equivalent, and again I have not always done this with full rigour (and there is certainly not the space to give proofs here). So, this paper actually contains significantly more than ten possible definitions of weak n-category.

'Further Reading' is the final section. To keep the definitions of n-category brief and self-contained, there are no citations at all in the main text; so this section is a combination of reference list, historical notes, and general comments, together with a few pointers to literature in related areas.

Overview of the definitions. Table 1 shows some of the main features of the definitions of weak n-category. Each definition is given a name such as A or Z, according to the name of the author from whom the definition is derived. (Definition X is a combination of the work of many people, principally Baez, Dolan, Hermida, Makkai and Power.) The point of these abbreviations is to put some distance between the definitions as proposed by those authors and the definitions as stated below. At the most basic level, I have in all cases changed some notation and terminology. Moreover, taking what is often a long paper and turning it into a two-page definition has seldom been just a matter of leaving out words; sometimes it has required a serious reshaping of the concepts involved. Whether the end result (the definition of weak n-category) is mathematically the same as that of the original author is not something I always know: on various occasions there have been passages in the source paper that have been opaque to me, so I have guessed at the author's intended meaning. Finally, in several cases only a definition of weak ω-category was explicitly given, leaving me to supply the definition of weak n-category for finite n.

Definition	*Author(s)*	*Shapes used*	*A/the*	*ω?*
Tr	Trimble	globular	the	×
P	Penon	globular	the	✓
B	Batanin	globular	the (B1), a (B2)	✓
L	Leinster	globular	the (L1), a (L2)	✓
L′	Leinster	globular	a	✓
Si	Simpson	simplicial/globular	a	×
Ta	Tamsamani	simplicial/globular	a	×
J	Joyal	globular/simplicial	a	✓
St	Street	simplicial	a	✓
X	see text	opetopic	a	×

Table 1: Some features of the definitions

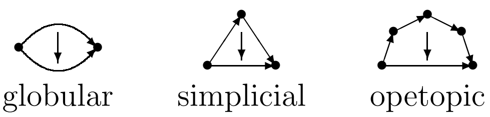


Figure 1: Shapes used in the definitions

In summary, then, I do believe that I have given ten reasonable definitions of weak n-category, but I do not guarantee that they are the same as those of the authors listed in Table 1; ultimately, the responsibility for them is mine.

The column headed 'shapes used' refers to the different shapes of m-cell (or 'm-arrow', or 'm-morphism') employed in the definitions. These are shown in Figure 1.

It has widely been observed that the various definitions of n-category fall into two groups, according to the attitude one takes to the status of composition. This distinction can be explained by analogy with products. Given two sets A and B, one can define *a product* of A and B to be a triple (P, p_1, p_2) where P is a set and $p_1 : P \longrightarrow A$, $p_2 : P \longrightarrow B$ are functions with the usual universal property. This is of course the standard thing to do in category theory, and in this context one can strictly speaking never refer to *the* product of A and B. On the other hand, one could define *the product* of A and B to be the set $A \times B$ of ordered pairs $(a, b) = \{\{a\}, \{a, b\}\}$ with $a \in A$ and $b \in B$; this has the virtue of being definite and allowing one to speak of *the* product in the customary way, but involves a wholly artificial construction. Similarly, in some of the proposed definitions of weak n-category, one can never speak of *the* composite of morphisms g and f, only of *a* composite (of which there may be many, all equally valid); but in some of the definitions one does have definite composites $g \circ f$, *the* composite of g and f. (The use of the word 'the' is not meant to imply strictness, e.g. the three-fold composite $h \circ (g \circ f)$ will in general be different from the three-fold composite $(h \circ g) \circ f$.) So this is the meaning of the column headed 'a/the'; it might also have been headed 'indefinite/definite', 'relational/functional', 'universal/coherent', or even 'geometric/algebraic'.

All of the sections include a definition of weak n-category for natural numbers n, but some also include a definition of weak ω-category (in which there are m-cells for all natural m). This is shown in the last column.

Finally, I warn the reader that the words 'contractible' and 'contraction' occur in many of the definitions, but mean different things from definition to definition. This is simply to save having to invent new words for concepts which are similar but not identical, and to draw attention to the common idea.

Acknowledgements. I first want to thank Eugenia Cheng and Martin Hyland. Their involvement in this project has both made it much more pleasurable for me and provided a powerful motivating force. Without them, I suspect it would still not be done.

I prepared for writing this by giving a series of seminars (one definition per week) in Cambridge in spring 2001, and am grateful to the participants: the two just mentioned, Mario Cáccamo, Marcelo Fiore, and Joe Templeton. I would also like to thank those who have contributed over the years to the many Cambridge Category Theory seminars on the subject of n-categories, especially Jeff Egger (who introduced me to Tamsamani's definition), Peter Johnstone, and Craig Snydal (with whom I have also had countless interesting conversations on the subject).

Todd Trimble was generous enough to let me publish his definition for the first time, and to cast his eye over a draft of what appears below as definition Tr—though all errors, naturally, are mine.

I am also grateful to the other people with whom I have had helpful communications, including Michael Batanin (who told me about Penon's definition), David Carlton, Jack Duskin, Keith Harbaugh, Anders Kock, Bill Lawvere, Peter May, Carlos Simpson, Ross Street, Bertrand Toen, Dominic Verity, Marek Zawadowski (who told me about Joyal's definition), and surely others, whose names I apologize for omitting.

Many of the diagrams were drawn using Paul Taylor's commutative diagrams package.

It is a pleasure to thank St John's College, Cambridge, where I hold the Laurence Goddard Fellowship, for their support.

Background

Category Theory. Here is a summary of the categorical background and terminology needed in order to read the entire paper. The reader who isn't familiar with everything below shouldn't be put off: each individual Definition only uses some of it.

I assume familiarity with *categories*, *functors*, *natural transformations*, *adjunctions*, *limits*, and *monads* and their *algebras*. Limits include *products*, *pullbacks* (with the pullback of a diagram $X \longrightarrow Z \longleftarrow Y$ sometimes written $X \times_Z Y$), and *terminal objects* (written 1, especially for the terminal set $\{*\}$); we also use *initial objects*. A monad (T, η, μ) is often abbreviated to T.

I make no mention of the difference between sets and classes ('small and large collections'). All the Definitions are really of *small* weak n-category.

Let $\mathcal{C}$ be a category. $X \in \mathcal{C}$ means that X is an object of $\mathcal{C}$, and $\mathcal{C}(X, Y)$ is the set of morphisms (or *maps*, or *arrows*) from X to Y in $\mathcal{C}$. If $f \in \mathcal{C}(X, Y)$ then X is the *domain* or *source* of f, and Y the *codomain* or *target*.

Set is the category (sets + functions), and **Cat** is (categories + functors). A set is just a *discrete category* (one in which the only maps are the identities).

$\mathcal{C}^{\mathrm{op}}$ is the *opposite* or *dual* of a category $\mathcal{C}$. $[\mathcal{C}, \mathcal{D}]$ is the category of functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations between them. Any object X of $\mathcal{C}$ induces a functor $\mathcal{C}(X, -) : \mathcal{C} \longrightarrow \mathbf{Set}$, and a natural transformation from $\mathcal{C}(X, -)$ to $F : \mathcal{C} \longrightarrow \mathbf{Set}$ is the same thing as an element of FX (the *Yoneda Lemma*); dually for $\mathcal{C}(-, X) : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is an *equivalence* if these equivalent conditions hold: (i) F is full, faithful and essentially surjective on objects; (ii) there exist a functor $G : \mathcal{D} \longrightarrow \mathcal{C}$ (a *pseudo-inverse* to F) and natural isomorphisms $\eta : 1 \longrightarrow GF$, $\varepsilon : FG \longrightarrow 1$; (iii) as (ii), but with $(F, G, \eta, \varepsilon)$ also being an adjunction.

Any set $\mathcal{D}_0$ of objects of a category $\mathcal{C}$ determines a *full subcategory* $\mathcal{D}$ of $\mathcal{C}$, with object-set $\mathcal{D}_0$ and $\mathcal{D}(X, Y) = \mathcal{C}(X, Y)$. Every category $\mathcal{C}$ has a *skeleton*: a subcategory whose inclusion into $\mathcal{C}$ is an equivalence and in which no two distinct objects are isomorphic. If $F, G : \mathcal{C} \longrightarrow \mathbf{Set}$, $GX \subseteq FX$ for each $X \in \mathcal{C}$, and F and G agree on morphisms of $\mathcal{C}$, then G is a *subfunctor* of F.

A *total order* on a set I is a reflexive transitive relation $\leq$ such that if $i \neq j$ then exactly one of $i \leq j$ and $j \leq i$ holds. $(I, \leq)$ can be seen as a category with object-set I in which each hom-set has at most one element. An *order-preserving map* $(I, \leq) \longrightarrow (I', \leq')$ is a function f such that $i \leq j \Rightarrow f(i) \leq' f(j)$.

Let Δ be the category with objects $[k] = \{0, \ldots, k\}$ for $k \geq 0$, and order-preserving functions as maps. A *simplicial set* is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$. Every category $\mathcal{C}$ has a *nerve* (the simplicial set $N\mathcal{C} : [k] \longmapsto \mathbf{Cat}([k], \mathcal{C})$), giving a full and faithful functor $N : \mathbf{Cat} \longrightarrow [\Delta^{\mathrm{op}}, \mathbf{Set}]$. So **Cat** is equivalent to the full subcategory of $[\Delta^{\mathrm{op}}, \mathbf{Set}]$ with objects $\{X \mid X \cong N\mathcal{C}$ for some $\mathcal{C}\}$; there are various characterizations of such X, but we come to that in the main text.

Leftovers: a *monoid* is a set (or more generally, an object of a monoidal category) with an associative binary operation and a two-sided unit. **Cat** is monadic over the category of directed graphs. The *natural numbers* start at 0.

Strict n-Categories. If $\mathcal{V}$ is a category with finite products then there is a category $\mathcal{V}$-**Cat** of $\mathcal{V}$-enriched categories and $\mathcal{V}$-enriched functors, and this itself has finite products. (A *$\mathcal{V}$-enriched category* is just like an ordinary category, except that the 'hom-sets' are now objects of $\mathcal{V}$.) Let 0-**Cat** = **Set** and, for $n \geq 0$, let $(n+1)$-**Cat** = (n-**Cat**)-**Cat**; a *strict n-category* is an object of n-**Cat**. Note that 1-**Cat** = **Cat**.

Any finite-product-preserving functor $U : \mathcal{V} \longrightarrow \mathcal{W}$ induces a finite-product-preserving functor $U_* : \mathcal{V}\text{-}\mathbf{Cat} \longrightarrow \mathcal{W}\text{-}\mathbf{Cat}$, so we can define functors $U_n : (n+1)\text{-}\mathbf{Cat} \longrightarrow n\text{-}\mathbf{Cat}$ by taking U_0 to be the objects functor and $U_{n+1} = (U_n)_*$. The category ω-**Cat** of *strict ω-categories* is the limit of the diagram

$$\cdots \xrightarrow{U_{n+1}} (n+1)\text{-}\mathbf{Cat} \xrightarrow{U_n} \cdots \xrightarrow{U_1} 1\text{-}\mathbf{Cat} = \mathbf{Cat} \xrightarrow{U_0} 0\text{-}\mathbf{Cat} = \mathbf{Set}.$$

Alternatively: a *globular set* (or *ω-graph*) A consists of sets and functions

$$\cdots \underset{t}{\overset{s}{\rightrightarrows}} A_m \underset{t}{\overset{s}{\rightrightarrows}} A_{m-1} \underset{t}{\overset{s}{\rightrightarrows}} \cdots \underset{t}{\overset{s}{\rightrightarrows}} A_0$$

such that for $m \geq 2$ and $\alpha \in A_m$, $ss(\alpha) = st(\alpha)$ and $ts(\alpha) = tt(\alpha)$. An element of A_m is called an *m-cell*, and we draw a 0-cell a as $\underset{a}{\bullet}$, a 1-cell f as $\underset{a}{\bullet} \xrightarrow{f} \underset{b}{\bullet}$ (where $a = s(f), b = t(f)$), a 2-cell α as $\underset{a}{\bullet} \underset{g}{\overset{f}{\Downarrow\alpha}} \underset{b}{\bullet}$, etc. For $m > p \geq 0$, write $A_m \times_{A_p} A_m = \{(\alpha', \alpha) \in A_m \times A_m \mid s^{m-p}(\alpha') = t^{m-p}(\alpha)\}$.

A *strict ω-category* is a globular set A together with a function $\circ_p : A_m \times_{A_p} A_m \longrightarrow A_m$ (*composition*) for each $m > p \geq 0$ and a function $i : A_m \longrightarrow A_{m+1}$ (*identities*, usually written $i(\alpha) = 1_\alpha$) for each $m \geq 0$, such that

i. if $m > p \geq 0$ and $(\alpha', \alpha) \in A_m \times_{A_p} A_m$ then

$$\begin{array}{lll} s(\alpha' \circ_p \alpha) = s(\alpha) & \text{and} \quad t(\alpha' \circ_p \alpha) = t(\alpha') & \text{for } m = p+1 \\ s(\alpha' \circ_p \alpha) = s(\alpha') \circ_p s(\alpha) & \text{and} \quad t(\alpha' \circ_p \alpha) = t(\alpha') \circ_p t(\alpha) & \text{for } m \geq p+2 \end{array}$$

ii. if $m \geq 0$ and $\alpha \in A_m$ then $s(i(\alpha)) = \alpha = t(i(\alpha))$

iii. if $m > p \geq 0$ and $\alpha \in A_m$ then $i^{m-p}(t^{m-p}(\alpha)) \circ_p \alpha = \alpha = \alpha \circ_p i^{m-p}(s^{m-p}(\alpha))$; if also α', α'' are such that $(\alpha'', \alpha'), (\alpha', \alpha) \in A_m \times_{A_p} A_m$, then $(\alpha'' \circ_p \alpha') \circ_p \alpha = \alpha'' \circ_p (\alpha' \circ_p \alpha)$

iv. if $p > q \geq 0$ and $(f', f) \in A_p \times_{A_q} A_p$ then $i(f') \circ_q i(f) = i(f' \circ_q f)$; if also $m > p$ and $\alpha, \alpha', \beta, \beta'$ are such that $(\beta', \beta), (\alpha', \alpha) \in A_m \times_{A_p} A_m$ and $(\beta', \alpha'), (\beta, \alpha) \in A_m \times_{A_q} A_m$, then $(\beta' \circ_p \beta) \circ_q (\alpha' \circ_p \alpha) = (\beta' \circ_q \alpha') \circ_p (\beta \circ_q \alpha)$.

The composition $\circ_p$ is 'composition of m-cells by gluing along p-cells'. Pictures for $(m, p) = (2, 1), (1, 0), (2, 0)$ are in the Bicategories section below.

Strict n-categories are defined similarly, but with the globular set only going up to A_n. *Strict n- and ω-functors* are maps of globular sets preserving composition and identities; the categories n-**Cat** and ω-**Cat** thus defined are equivalent to the ones defined above. The comments below on the two alternative definitions of bicategory give an impression of how this equivalence works.

Bicategories. Bicategories are the traditional and best-known formulation of 'weak 2-category'. A *bicategory* B consists of

- a set B_0, whose elements a are called *0-cells* or *objects* of B and drawn $\underset{a}{\bullet}$
- for each $a, b \in B_0$, a category $B(a,b)$, whose objects f are called *1-cells* and drawn $a \xrightarrow{f} b$, whose arrows $\alpha : f \longrightarrow g$ are called *2-cells* and drawn
$$\begin{tikzcd} a \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & b \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd},$$
and whose composition
$$\begin{tikzcd} a \arrow[r, bend left=60, "f"{name=U}] \arrow[r, "g"{name=M}] \arrow[r, bend right=60, "h"'{name=D}] & b \arrow[from=U, to=M, Rightarrow, "\alpha"] \arrow[from=M, to=D, Rightarrow, "\beta"] \end{tikzcd} \longmapsto \begin{tikzcd} a \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "h"'{name=D}] & b \arrow[from=U, to=D, Rightarrow, "\beta\circ\alpha"] \end{tikzcd}$$
is called *vertical composition* of 2-cells
- for each $a \in B_0$, an object $1_a \in B(a,a)$ (the *identity* on a); and for each $a, b, c \in B_0$, a functor $B(b,c) \times B(a,b) \longrightarrow B(a,c)$, which on objects is called *1-cell composition*, $a \xrightarrow{f} b \xrightarrow{g} c \longmapsto a \xrightarrow{g\circ f} c$, and on arrows is called *horizontal composition* of 2-cells,
$$\begin{tikzcd} a \arrow[r, bend left=50, "f"{name=U}] \arrow[r, bend right=50, "g"'{name=D}] & a' \arrow[r, bend left=50, "f'"{name=U2}] \arrow[r, bend right=50, "g'"'{name=D2}] & a'' \arrow[from=U, to=D, Rightarrow, "\alpha"] \arrow[from=U2, to=D2, Rightarrow, "\alpha'"] \end{tikzcd} \longmapsto \begin{tikzcd} a \arrow[r, bend left=50, "f'\circ f"{name=U}] \arrow[r, bend right=50, "g'\circ g"'{name=D}] & a'' \arrow[from=U, to=D, Rightarrow, "\alpha'*\alpha"] \end{tikzcd}$$
- *coherence 2-cells*: for each $f \in B(a,b), g \in B(b,c), h \in B(c,d)$, an *associativity isomorphism* $\xi_{h,g,f} : (h\circ g)\circ f \longrightarrow h\circ(g\circ f)$; and for each $f \in B(a,b)$, *unit isomorphisms* $\lambda_f : 1_b\circ f \longrightarrow f$ and $\rho_f : f\circ 1_a \longrightarrow f$

satisfying the following *coherence axioms*:

- $\xi_{h,g,f}$ is natural in h, g and f, and λ_f and ρ_f are natural in f
- if $f \in B(a,b), g \in B(b,c), h \in B(c,d), k \in B(d,e)$, then $\xi_{k,h,g\circ f} \circ \xi_{k\circ h,g,f} = (1_k * \xi_{h,g,f}) \circ \xi_{k,h\circ g,f} \circ (\xi_{k,h,g} * 1_f)$ (the *pentagon axiom*); and if $f \in B(a,b), g \in B(b,c)$, then $\rho_g * 1_f = (1_g * \lambda_f) \circ \xi_{g,1_b,f}$ (the *triangle axiom*).

An alternative definition is that a bicategory consists of a diagram of sets and functions $B_2 \overset{s}{\underset{t}{\rightrightarrows}} B_1 \overset{s}{\underset{t}{\rightrightarrows}} B_0$ satisfying $ss = st$ and $ts = tt$, together with functions determining composition, identities and coherence cells (in the style of the second definition of strict ω-category above). The idea is that B_m is the set of m-cells and that s and t give the source and target of a cell. Strict 2-categories can be identified with bicategories in which the coherence 2-cells are all identities.

A 1-cell $a \xrightarrow{f} b$ in a bicategory B is called an *equivalence* if there exists a 1-cell $b \xrightarrow{g} a$ such that $g\circ f \cong 1_a$ and $f\circ g \cong 1_b$.

A *monoidal category* can be defined as a bicategory with only one 0-cell, $\star$ say: for such a bicategory is just a category $B(\star,\star)$ equipped with an object I, a functor $\otimes : B(\star,\star)^2 \longrightarrow B(\star,\star)$, and associativity and unit isomorphisms satisfying coherence.

We can consider *strict functors* of bicategories, in which composition etc is preserved strictly; more interesting are *weak functors* F, in which there are isomorphisms $Fg\circ Ff \longrightarrow F(g\circ f)$ and $1_{Fa} \longrightarrow F(1_a)$ satisfying coherence axioms.

Definition Tr

Topological Background

Spaces. Let **Top** be the category of topological spaces and continuous maps. Recall that compact spaces are exponentiable in **Top**: that is, if K is compact then the set Z^K of continuous maps from K to a space Z can be given a topology (namely, the compact-open topology) in such a way that there is an isomorphism $\mathbf{Top}(Y, Z^K) \cong \mathbf{Top}(K \times Y, Z)$ natural in $Y, Z \in \mathbf{Top}$.

Operads. A *(non-symmetric, topological) operad* D is a sequence $(D(k))_{k \geq 0}$ of spaces together with an element (the *identity*) of $D(1)$ and for each $k, r_1, \ldots, r_k \geq 0$ a map

$$D(k) \times D(r_1) \times \cdots \times D(r_k) \longrightarrow D(r_1 + \cdots + r_k)$$

(*composition*), obeying unit and associativity laws. (Example: fix an object M of a monoidal category $\mathcal{M}$, and define $D(k) = \mathcal{M}(M^{\otimes k}, M)$.)

The All-Important Operad. There is an operad E in which $E(k)$ is the space of continuous endpoint-preserving maps from $[0, 1]$ to $[0, k]$. ('Endpoint-preserving' means that 0 maps to 0 and 1 to k.) The identity element of $E(1)$ is the identity map, and composition in the operad is by substitution.

Path Spaces. For any space X and $x, x' \in X$, a *path* from x to x' in X is a map $p : [0, 1] \longrightarrow X$ satisfying $p(0) = x$ and $p(1) = x'$. There is a space $X(x, x')$ of paths from x to x', a subspace of the exponential $X^{[0,1]}$.

Operad Action on Path Spaces. Fix a space X. For any $k \geq 0$ and $x_0, \ldots, x_k \in X$, there is a canonical map

$$act_{x_0,\ldots,x_k} : E(k) \times X(x_0, x_1) \times \cdots \times X(x_{k-1}, x_k) \longrightarrow X(x_0, x_k).$$

These maps are compatible with the composition and identity of the operad E, and the construction is functorial in X.

Path-Components. Let $\Pi_0 : \mathbf{Top} \longrightarrow \mathbf{Set}$ be the functor assigning to each space its set of path-components, and note that Π_0 preserves finite products.

The Definition

We will define inductively, for each $n \geq 0$, a category **Wk-n-Cat** with finite products and a functor $\Pi_n : \mathbf{Top} \longrightarrow$ **Wk-n-Cat** preserving finite products. A *weak n-category* is an object of **Wk-n-Cat**. (Maps in **Wk-n-Cat** are to be thought of as *strict* n-functors.)

Base Case. **Wk-0-Cat** = **Set**, and $\Pi_0 : \mathbf{Top} \longrightarrow \mathbf{Set}$ is as above.

Objects of Wk-$(n+1)$-Cat. Inductively, a *weak (n+1)-category* (A, γ) consists of

- a set A_0
- a family $(A(a, a'))_{a,a' \in A_0}$ of weak n-categories
- for each $k \geq 0$ and $a_0, \ldots, a_k \in A_0$, a map

  $$\gamma_{a_0,\ldots,a_k} : \Pi_n(E(k)) \times A(a_0, a_1) \times \cdots \times A(a_{k-1}, a_k) \longrightarrow A(a_0, a_k)$$

  in **Wk**-n-**Cat**,

such that the $\gamma_{a_0,\ldots,a_k}$'s satisfy compatibility axioms of the same form as those satisfied by the $act_{x_0,\ldots,x_k}$'s. (All this makes sense because Π_n preserves finite products and **Wk**-n-**Cat** has them.)

Maps in Wk-$(n+1)$-Cat. A *map* $(A, \gamma) \longrightarrow (B, \delta)$ in **Wk**-$(n+1)$-**Cat** consists of

- a function $F_0 : A_0 \longrightarrow B_0$
- for each $a, a' \in A_0$, a map $F_{a,a'} : A(a, a') \longrightarrow B(F_0 a, F_0 a')$ of weak n-categories,

satisfying the axiom

$$F_{a_0,a_k} \circ \gamma_{a_0,\ldots,a_k} = \delta_{F_0 a_0,\ldots,F_0 a_k} \circ (1_{\Pi_n(E(k))} \times F_{a_0,a_1} \times \cdots \times F_{a_{k-1},a_k})$$

for all $k \geq 0$ and $a_0, \ldots, a_k \in A_0$.

Composition and Identities in Wk-$(n+1)$-Cat. Obvious.

Π_{n+1} **on Objects.** For a space X we define $\Pi_{n+1}(X) = (A, \gamma)$, where

- A_0 is the underlying set of X
- $A(x, x') = \Pi_n(X(x, x'))$
- for $x_0, \ldots, x_k \in X$, the map $\gamma_{x_0,\ldots,x_k}$ is the composite

$$\begin{aligned} &\Pi_n(E(k)) \times \Pi_n(X(x_0, x_1)) \times \cdots \times \Pi_n(X(x_{k-1}, x_k)) \\ &\xrightarrow{\sim} \quad \Pi_n(E(k) \times X(x_0, x_1) \times \cdots \times X(x_{k-1}, x_k)) \\ &\xrightarrow{\Pi_n(act_{x_0,\ldots,x_k})} \quad \Pi_n(X(x_0, x_k)). \end{aligned}$$

Π_{n+1} **on Maps.** The functor Π_{n+1} is defined on maps in the obvious way.

Finite Products Behave. It is easy to show that **Wk**-$(n+1)$-**Cat** has finite products and that Π_{n+1} preserves finite products: so the inductive definition goes through.

Definition Tr for $n \leq 2$

First observe that the space $E(k)$ is contractible for each k (being, in a suitable sense, convex). In particular this tells us that $E(k)$ is path-connected, and that the path space $E(k)(\theta, \theta')$ is path-connected for every $\theta, \theta' \in E(k)$.

$n = 0$

By definition, **Wk**-0-**Cat** = **Set** and $\Pi_0 : \mathbf{Top} \longrightarrow \mathbf{Set}$ is the path-components functor.

$n = 1$

The Category Wk-1-Cat. A weak 1-category (A, γ) consists of

- a set A_0
- a set $A(a, a')$ for each $a, a' \in A_0$
- for each $k \geq 0$ and $a_0, \ldots, a_k \in A_0$, a function

$$\gamma_{a_0,\ldots,a_k} : \Pi_0(E(k)) \times A(a_0, a_1) \times \cdots \times A(a_{k-1}, a_k) \longrightarrow A(a_0, a_k)$$

such that these functions satisfy certain axioms. So a weak 1-category looks something like a category: A_0 is the set of objects, $A(a, a')$ is the set of maps from a to a', and γ provides some kind of composition. Since $E(k)$ is path-connected, we may strike out $\Pi_0(E(k))$ from the product above; and then we may suggestively write

$$(f_k \circ \cdots \circ f_1) = \gamma_{a_0,\ldots,a_k}(f_1, \ldots, f_k).$$

The axioms on these 'k-fold composition functions' mean that a weak 1-category is, in fact, exactly a category. Maps in **Wk**-1-**Cat** are just functors, and so **Wk**-1-**Cat** is equivalent to **Cat**.

The Functor Π_1. For a space X, the (weak 1-)category $\Pi_1(X) = (A, \gamma)$ is given by

- A_0 is the underlying set of X
- $A(x, x')$ is the set of path-components of the path-space $X(x, x')$: that is, the set of homotopy classes of paths from x to x'
- Let $x_0 \xrightarrow{p_1} \cdots \xrightarrow{p_k} x_k$ be a sequence of paths in X, and write $[p]$ for the homotopy class of a path p. Then

$$([p_k] \circ \cdots \circ [p_1]) = [act_{x_0,\ldots,x_k}(\theta, p_1, \ldots p_k)]$$

  where θ is any member of $E(k)$—it doesn't matter which. In other words, composition of paths is by laying them end to end.

Hence $\Pi_1(X)$ is the usual fundamental groupoid of X, and indeed $\Pi_1 : \mathbf{Top} \longrightarrow \mathbf{Cat}$ is the usual fundamental groupoid functor.

$n = 2$

A weak 2-category (A, γ) consists of

- a set A_0
- a category $A(a, a')$ for each $a, a' \in A_0$
- for each $k \geq 0$ and $a_0, \ldots, a_k \in A_0$, a functor

$$\gamma_{a_0,\ldots,a_k} : \Pi_1(E(k)) \times A(a_0, a_1) \times \cdots \times A(a_{k-1}, a_k) \longrightarrow A(a_0, a_k)$$

such that these functors satisfy axioms expressing compatibility with the composition and identity of the operad E.

By the description of Π_1 and the initial observations of this section, the category $\Pi_1(E(k))$ is indiscrete (i.e. all hom-sets have one element) and its objects are the elements of $E(k)$. So γ assigns to each $\theta \in E(k)$ and $a_i \in A_0$ a functor

$$\overline{\theta} : A(a_0, a_1) \times \cdots \times A(a_{k-1}, a_k) \longrightarrow A(a_0, a_k),$$

and to each $\theta, \theta' \in E(k)$ and $a_i \in A_0$ a natural isomorphism

$$\omega_{\theta,\theta'} : \overline{\theta} \xrightarrow{\sim} \overline{\theta'}.$$

(Really we should add '$a_0, \ldots, a_k$' as a subscript to $\overline{\theta}$ and to $\omega_{\theta,\theta'}$.) Functoriality of $\gamma_{a_0,\ldots,a_k}$ says that

$$\omega_{\theta,\theta} = 1, \qquad \omega_{\theta,\theta''} = \omega_{\theta',\theta''} \circ \omega_{\theta,\theta'}.$$

The 'certain axioms' say firstly that

$$\overline{\theta \circ (\theta_1, \ldots, \theta_k)} = \overline{\theta} \circ (\overline{\theta_1} \times \cdots \times \overline{\theta_k}), \qquad \overline{1} = 1$$

for $\theta \in E(k)$ and $\theta_i \in E(r_i)$, where the left-hand sides of the two equations refer respectively to composition and identity in the operad E; and secondly that the natural isomorphisms $\omega_{\theta,\theta'}$ fit together in a coherent way.

So a weak 2-category is probably not a structure with which we are already familiar. However, it nearly is. For define $\mathbf{tr}(k)$ to be the set of k-leafed rooted trees which are 'unitrivalent' (each vertex has either 0 or 2 edges coming up out of it); and suppose we replaced $\Pi_1(E(k))$ by the indiscrete category with object-set $\mathbf{tr}(k)$, so that the θ's above would be trees. A weak 2-category would then be exactly a bicategory: e.g. if $\theta =$  then $\overline{\theta}$ is binary composition, and if $(\theta, \theta') = ($, $)$ then $\omega_{\theta,\theta'}$ is the associativity isomorphism. And in some sense, a k-leafed tree might be thought of as a discrete version of an endpoint-preserving map $[0, 1] \longrightarrow [0, k]$.

With this in mind, any weak 2-category (A, γ) gives rise to a bicategory B (although the converse process seems less straightforward). First pick at random an element θ_2 of $E(2)$, and let θ_0 be the unique element of $E(0)$. Then take $B_0 = A_0$, $B(a, a') = A(a, a')$, binary composition to be $\overline{\theta_2}$, identities to be $\overline{\theta_0}$, the associativity isomorphism to be $\omega_{\theta_2 \circ (1,\theta_2),\theta_2 \circ (\theta_2,1)}$, and similarly units. The coherence axioms on B follow from the coherence axioms on ω: and so we have a bicategory.

Definition P

Some Globular Structures

Reflexive Globular Sets. Let $\mathbb{R}$ be the category whose objects are the natural numbers $0, 1, \ldots$, and whose arrows are generated by

$$\begin{tikzcd} \cdots & m+1 \arrow[r, shift left, "\sigma_{m+1}"] \arrow[r, shift right, "\tau_{m+1}"'] & m \arrow[l, bend left=60, "\iota_{m+1}"] \arrow[r, shift left, "\sigma_m"] \arrow[r, shift right, "\tau_m"'] & \cdots \arrow[l, bend left=60, "\iota_m"] & \cdots \arrow[r, shift left, "\sigma_1"] \arrow[r, shift right, "\tau_1"'] & 0 \arrow[l, bend left=60, "\iota_1"] \end{tikzcd}$$

subject to the equations

$$\sigma_m \circ \sigma_{m+1} = \sigma_m \circ \tau_{m+1}, \qquad \tau_m \circ \sigma_{m+1} = \tau_m \circ \tau_{m+1}, \qquad \sigma_m \circ \iota_m = 1 = \tau_m \circ \iota_m$$

($m \geq 1$). A functor $A : \mathbb{R} \longrightarrow \mathbf{Set}$ is called a *reflexive globular set*. I will write s for $A(\sigma_m)$, and t for $A(\tau_m)$, and 1_a for $(A(\iota_m))(a)$ when $a \in A(m-1)$.

Strict ω-Categories, and ω-Magmas. A *strict ω-category* is a reflexive globular set S together with a function (*composition*) $\circ_p : S(m) \times_{S(p)} S(m) \longrightarrow S(m)$ for each $m > p \geq 0$, satisfying

- axioms determining the source and target of a composite (part (i) in the Preliminary section 'Strict n-Categories')
- strict associativity, unit and interchange axioms (parts (iii) and (iv)).

An *ω-magma* is like a strict ω-category, but only satisfying the first group of axioms ((i)) and not necessarily the second ((iii), (iv)). A *map of ω-magmas* is a map of reflexive globular sets which commutes with all the composition operations. (A strict ω-functor between strict ω-categories is, therefore, just a map of the underlying ω-magmas.)

Contractions

Let $\phi : A \longrightarrow B$ be a map of reflexive globular sets. For $m \geq 1$, define

$$V_\phi(m) = \{(f_0, f_1) \in A(m) \times A(m) \mid s(f_0) = s(f_1), t(f_0) = t(f_1), \phi(f_0) = \phi(f_1)\},$$

and define

$$V_\phi(0) = \{(f_0, f_1) \in A(0) \times A(0) \mid \phi(f_0) = \phi(f_1)\}.$$

A *contraction* γ on ϕ is a family of functions

$$(\gamma_m : V_\phi(m) \longrightarrow A(m+1))_{m \geq 0}$$

such that for all $m \geq 0$ and $(f_0, f_1) \in V_\phi(m)$,

$$s(\gamma_m(f_0, f_1)) = f_0, \quad t(\gamma_m(f_0, f_1)) = f_1, \quad \phi(\gamma_m(f_0, f_1)) = 1_{\phi(f_0)} (= 1_{\phi(f_1)}),$$

and for all $m \geq 0$ and $f \in A(m)$,

$$\gamma_m(f, f) = 1_f.$$

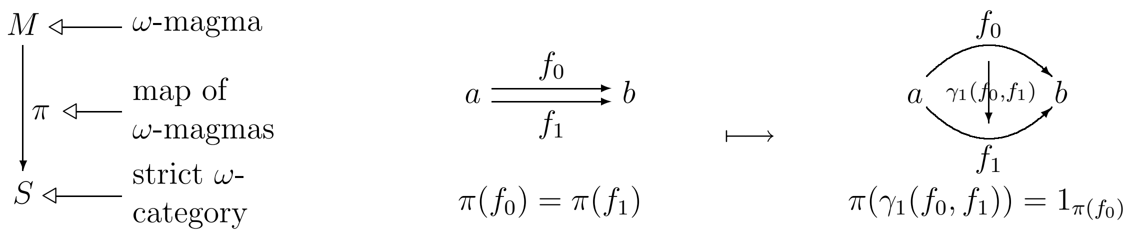


Figure 2: An object of $\mathcal{Q}$, with γ shown for $m = 1$

The Mysterious Category $\mathcal{Q}$

Objects. An object of $\mathcal{Q}$ (see Fig. 2) is a quadruple (M, S, π, γ) in which

- M is an ω-magma
- S is a strict ω-category
- π is a map of ω-magmas from M to (the underlying ω-magma of) S
- γ is a contraction on π.

Maps. A map $(M, S, \pi, \gamma) \longrightarrow (M', S', \pi', \gamma')$ in $\mathcal{Q}$ is a pair $(M \xrightarrow{\chi} M',\ S \xrightarrow{\zeta} S')$ commuting with everything in sight. That is, χ is a map of ω-magmas, ζ is a strict ω-functor, $\pi' \circ \chi = \zeta \circ \pi$, and $\gamma'_m(\chi(f_0), \chi(f_1)) = \chi(\gamma_m(f_0, f_1))$ for all $(f_0, f_1) \in V_M(m)$.

Composition and Identities. These are defined in the obvious way.

The Definition

An Adjunction. Let $U : \mathcal{Q} \longrightarrow [\mathbb{R}, \mathbf{Set}]$ be the functor sending (M, S, π, γ) to the underlying reflexive globular set of the ω-magma M. It can be shown that U has a left adjoint: so there is an induced monad T on $[\mathbb{R}, \mathbf{Set}]$.

Weak ω-Categories. A *weak ω-category* is a T-algebra.

Weak n-Categories. Let $n \geq 0$. A reflexive globular set A is *n-dimensional* if for all $m \geq n$, the map $A(\iota_{m+1}) : A(m) \longrightarrow A(m+1)$ is an isomorphism (and so $s = t = (A(\iota_{m+1}))^{-1}$). A *weak n-category* is a weak ω-category whose underlying reflexive globular set is n-dimensional.

Definition P for $n \leq 2$

Direct Interpretation

The Left Adjoint in Low Dimensions. Here is a description of what the left adjoint F to U does in dimensions ≤ 2. It is perhaps not obvious that F as described does form the left adjoint; we come to that later. For a reflexive globular set A, write

$$F(A) = \left(\begin{array}{l} A^{\#} \\ \big\downarrow \pi_A \,, \ \gamma_A \\ A^* \end{array} \right).$$

A^* is, in fact, relatively easy to describe: it is the free strict ω-category on A, in which an m-cell is a formal pasting-together of cells of A of dimension $\leq m$.

Dimension 0 We have $A^{\#}(0) = A^*(0) = A(0)$ and $(\pi_A)_0 = \mathrm{id}$.

Dimension 1 Next, $A^*(1)$ is the set of formal paths of 1-cells in A, where we identify each identity cell 1_a with the identity path on a. The set $A^{\#}(1)$ and the functions $s, t : A^{\#}(1) \longrightarrow A(0)$ are generated by the following recursive clauses:

- if $a_0 \xrightarrow{f} a_1$ is a 1-cell in A then $A^{\#}(1)$ contains an element called f, with $s(f) = a_0$ and $t(f) = a_1$
- if $w, w' \in A^{\#}(1)$ with $t(w) = s(w')$ then $A^{\#}(1)$ contains an element called $(w' \bullet_0 w)$, with $s(w' \bullet_0 w) = s(w)$ and $t(w' \bullet_0 w) = t(w')$.

The identities map $A(0) \longrightarrow A^{\#}(1)$ sends a to $1_a \in A(1) \subseteq A^{\#}(1)$; the map π_A removes parentheses and sends $\bullet_0$ to $\circ_0$; the contraction γ_A is given by $\gamma_A(a, a) = 1_a$.

Dimension 2 $A^*(1)$ is the set of formal pastings of 2-cells in A, again respecting the identities. $A^{\#}(2)$ and $s, t : A^{\#}(2) \longrightarrow A^{\#}(1)$ are generated by:

- if α is a 2-cell in A then $A^{\#}(2)$ has an element called α, with the evident source and target
- if $a \underset{w_1}{\overset{w_0}{\rightrightarrows}} b$ in $A^{\#}(1)$ with $\pi_A(w_0) = \pi_A(w_1)$ then $A^{\#}(2)$ has an element called $\gamma_A(w_0, w_1)$, with source w_0 and target w_1
- if $x, x' \in A^{\#}(2)$ with $t(x) = s(x')$ then $A^{\#}(2)$ has an element called $(x' \bullet_1 x)$, with source $s(x)$ and target $t(x')$
- if $x, x' \in A^{\#}(2)$ with $tt(x) = ss(x')$ then $A^{\#}(2)$ has an element called $(x' \bullet_0 x)$, with source $s(x') \bullet_0 s(x)$ and target $t(x') \bullet_0 t(x)$;

furthermore, if $f \in A(1)$ then 1_f (from the first clause) is to be identified with $\gamma_A(f, f)$ (from the second). The identity map $A^{\#}(1) \longrightarrow A^{\#}(2)$ sends w to $\gamma_A(w, w)$. The map π_A sends cells of the form $\gamma_A(w_0, w_1)$ to identity cells, and otherwise acts as in dimension 1. The contraction γ_A is defined in the way suggested by the notation.

Adjointness. We now have to see that this F is indeed left adjoint to U. First observe that there is a natural embedding of $A(m)$ into $A^\#(m)$ (for $m \leq 2$); this gives the unit of the adjunction. Adjointness then says: given $(M,S,\pi,\gamma) \in \mathcal{Q}$ and a map $A \xrightarrow{\phi} M$ of reflexive globular sets, there's a unique map

$$(\chi,\zeta) : \begin{pmatrix} A^\# \\ \downarrow \pi_A \\ A^* \end{pmatrix} \longrightarrow \begin{pmatrix} M \\ \downarrow \pi \\ S \end{pmatrix}$$

in $\mathcal{Q}$ such that χ extends ϕ. This can be seen from the description above.

Weak 2-Categories. A weak 2-category consists of a 2-dimensional reflexive globular set A together with:

- (a map $A^\#(0) \longrightarrow A(0)$ obeying axioms—which force it to be the identity)
- a map $A^\#(1) \longrightarrow A(1)$ obeying axioms, which amounts to a binary composition on the 1-cells of A (*not* obeying any axioms)
- similarly, vertical and horizontal binary compositions of 2-cells, not obeying any axioms 'yet'
- for each string $\cdot \xrightarrow{f_1} \cdots \xrightarrow{f_k} \cdot$ of 1-cells, and each pair τ, τ' of k-leafed binary trees, a 2-cell $\omega_{\tau,\tau'} : \circ_\tau(f_1,\ldots,f_k) \longrightarrow \circ_{\tau'}(f_1,\ldots,f_k)$, where $\circ_\tau$ indicates the iterated composition dictated by the shape of τ
- amongst other things in dimension 3: whenever we have some 2-cells (α_i), and two different ways of composing all the α_i's and some $\omega_{\tau,\tau'}$'s to obtain new 2-cells β and β' respectively, and these satisfy $s(\beta) = s(\beta')$ and $t(\beta) = t(\beta')$, then there is assigned a 3-cell $\beta \longrightarrow \beta'$. Since 'the only 3-cells of A are equalities', we get $\beta = \beta'$.

Analysing this precisely, we find that the category of weak 2-categories is equivalent to the category of bicategories and strict functors. And more easily, a weak 1-category is just a category and a weak 0-category is just a set.

Indirect Interpretation

An alternative way of handling weak n-categories is to work only with n-dimensional (not infinite-dimensional) structures throughout: e.g. reflexive globular sets A in which $A(m)$ is only defined for $m \leq n$. We then only speak of contractions on a map ϕ if $(f_0, f_1) \in V_\phi(n) \Rightarrow f_0 = f_1$ (and in particular, the map π must satisfy this in order for (M,S,π,γ) to qualify as an object of $\mathcal{Q}$). Our new category of weak n-categories appears to be equivalent to the old one, taking algebra maps as the morphisms in both cases.

The analysis of $n = 2$ is easier now: we can write down the left adjoint F explicitly, and so get an explicit description of the monad T on the category of 'reflexive 2-globular sets'. This monad is presumably the free bicategory monad.

Definitions B

Globular Operads and their Algebras

Globular Sets. Let $\mathbb{G}$ be the category whose objects are the natural numbers $0, 1, \ldots$, and whose arrows are generated by $\sigma_m, \tau_m : m \longrightarrow m-1$ for each $m \geq 1$, subject to equations

$$\sigma_{m-1} \circ \sigma_m = \sigma_{m-1} \circ \tau_m, \qquad \tau_{m-1} \circ \sigma_m = \tau_{m-1} \circ \tau_m$$

($m \geq 2$). A functor $A : \mathbb{G} \longrightarrow \mathbf{Set}$ is called a *globular set*; I will write s for $A(\sigma_m)$, and t for $A(\tau_m)$.

The Free Strict ω-Category Monad. Any (small) strict ω-category has an underlying globular set A, in which $A(m)$ is the set of m-cells and s and t are the source and target maps. We thus obtain a forgetful functor U from the category of strict ω-categories and strict ω-functors to the category $[\mathbb{G}, \mathbf{Set}]$ of globular sets. U has a left adjoint, so there is an induced monad $(T, \mathrm{id} \xrightarrow{\eta} T, T^2 \xrightarrow{\mu} T)$ on $[\mathbb{G}, \mathbf{Set}]$.

Collections. We define a monoidal category $\mathbf{Coll}$ of collections. Let 1 be the terminal globular set. A *(globular) collection* is a map $C \xrightarrow{d} T1$ into $T1$ in $[\mathbb{G}, \mathbf{Set}]$; a *map of collections* is a commutative triangle. The *tensor product* of collections $C \xrightarrow{d} T1$, $C' \xrightarrow{d'} T1$ is the composite along the top row of

$$\begin{tikzcd}
C \otimes C' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TC' \arrow[r, "Td'"] \arrow[d, "T!"] & T^2 1 \arrow[r, "\mu_1"] & T1 \\
C \arrow[r, "d"] & T1, & &
\end{tikzcd}$$

where the right-angle symbol means that the square containing it is a pullback, and ! denotes the unique map to 1. The *unit* for the tensor is $1 \xrightarrow{\eta_1} T1$.

Globular Operads. A *(globular) operad* is a monoid in the monoidal category $\mathbf{Coll}$; a *map of operads* is a map of monoids.

Algebras. Any operad C induces a monad $C \cdot -$ on $[\mathbb{G}, \mathbf{Set}]$. For an object A of $[\mathbb{G}, \mathbf{Set}]$, this is defined by pullback:

$$\begin{tikzcd}
C \cdot A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TA \arrow[d, "T!"] \\
C \arrow[r, "d"] & T1.
\end{tikzcd}$$

The multiplication and unit of the monad come from the multiplication and unit of the operad. A *C-algebra* is an algebra for the monad $C \cdot -$. Note that every C-algebra has an underlying globular set.

Contractions and Systems of Composition

Contractions. Let $C \xrightarrow{d} T1$ be a collection. For $m \geq 0$ and $\nu \in (T1)(m)$, write $C(\nu) = \{\theta \in C(m) \mid d(\theta) = \nu\}$. For $m \geq 1$ and $\nu \in (T1)(m)$, define

$$Q_C(\nu) = \{(\theta_0, \theta_1) \in C(\nu) \times C(\nu) \mid s(\theta_0) = s(\theta_1) \text{ and } t(\theta_0) = t(\theta_1)\},$$

and for $\nu \in (T1)(0)$, define $Q_C(\nu) = C(\nu) \times C(\nu)$. Part of the strict ω-category structure on $T1$ is that each element $\nu \in (T1)(m)$ gives rise to an element $1_\nu \in (T1)(m+1)$. A *contraction* on C is a family of functions

$$(\gamma_\nu : Q_C(\nu) \longrightarrow C(1_\nu))_{m \geq 0, \nu \in (T1)(m)}$$

satisfying

$$s(\gamma_\nu(\theta_0, \theta_1)) = \theta_0, \qquad t(\gamma_\nu(\theta_0, \theta_1)) = \theta_1$$

for every $m \geq 0$, $\nu \in (T1)(m)$ and $(\theta_0, \theta_1) \in Q_C(\nu)$.

Systems of Compositions. The map $\eta_1 : 1 \longrightarrow T1$ picks out an element $\eta_{1,m}$ of $(T1)(m)$ for each $m \geq 0$. The strict ω-category structure on $T1$ then gives an element

$$\beta_p^m = \eta_{1,m} \circ_p \eta_{1,m} \in (T1)(m)$$

for each $m > p \geq 0$; also put $\beta_m^m = \eta_{1,m}$. Defining $B(m) = \{\beta_p^m \mid m \geq p \geq 0\} \subseteq (T1)(m)$, we obtain a collection $B \hookrightarrow T1$.

Also, the elements $\beta_m^m = \eta_{1,m} \in (T1)(m)$ determine a map $1 \longrightarrow B$.

A *system of compositions* in an operad C is a map $B \longrightarrow C$ of collections such that the composite $1 \longrightarrow B \longrightarrow C$ is the unit of the operad C.

Initial Object. Let **OCS** be the category in which an object is an operad equipped with both a contraction and a system of compositions, and in which a map is a map of operads preserving both the specified contraction and the specified system of compositions. Then **OCS** can be shown to have an initial object, whose underlying operad will be written K.

The Definitions

Definition B1. A *weak ω-category* is a K-algebra.

Definition B2. A *weak ω-category* is a pair (C, A), where C is an operad satisfying $C(0) \cong 1$ and on which there exist a contraction and a system of compositions, and A is a C-algebra.

Weak n-Categories. Let $n \geq 0$. A globular set A is *n-dimensional* if for all $m \geq n$,

$$s = t : A(m+1) \longrightarrow A(m)$$

and this map is an isomorphism. A *weak n-category* is a weak ω-category whose underlying globular set is n-dimensional. This can be interpreted according to either B1 or B2.

Definitions B for $n \leq 2$

Definition B1

An alternative way of handling weak n-categories is to work with only n-dimensional (not infinite-dimensional) structures throughout. So we replace $\mathbb{G}$ by its full subcategory $\mathbb{G}_n$ with objects $0, \ldots, n$, and T by the free strict n-category monad T_n, to obtain definitions of *n-collection*, *n-operads*, and their *algebras*. *Contractions* are defined as before, except that we only speak of contractions on C if

$$\forall \nu \in (T_n 1)(n), (\theta_0, \theta_1) \in Q_C(\nu) \Rightarrow \theta_0 = \theta_1. \tag{$*$}$$

There is an initial n-operad K_n equipped with a contraction and a system of compositions, and the category of weak n-categories turns out to be equivalent to the category of K_n-algebras. The latter is easier to analyse.

$n = 0$. We have $[\mathbb{G}_0, \mathbf{Set}] \cong \mathbf{Set}$, $T_0 = \mathrm{id}$, and 0-$\mathbf{Coll} \cong \mathbf{Set}$; a 0-operad C is a monoid, and a C-algebra is a set with a C-action. By $(*)$, the only 0-operad with a contraction is the one-element monoid, so a weak 0-category is just a set.

$n = 1$. $[\mathbb{G}_1, \mathbf{Set}]$ is the category of directed graphs and T_1 is the free category monad. K_1 is the terminal 1-operad, by arguments similar to those under '$n = 2$' below. It follows that $K_1 \cdot -$ is just T_1, and so a weak 1-category is just a T_1-algebra, that is, a category.

$n = 2$. A functor $A : \mathbb{G}_2 \longrightarrow \mathbf{Set}$ consists of a set of 0-cells (drawn $\underset{a}{\bullet}$), a set of 1-cells ($\begin{tikzcd} a \arrow[r, "f"] & b \end{tikzcd}$), and a set of 2-cells ($\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd}$). A 2-collection C consists of a set $C(0)$, a set $C(\nu_k)$ for each $k \geq 0$ (where ν_k indicates the '1-pasting diagram' $\bullet \longrightarrow \cdots \longrightarrow \bullet$ with k arrows), and a set $C(\pi)$ for each '2-pasting diagram' π such as the π_i in Fig. 3, together with source and target functions.

A 2-operad is a 2-collection C together with 'composition' functions such as

$$\begin{array}{rcl} C(\nu_3) \times [C(\nu_2) \times_{C(0)} C(\nu_1) \times_{C(0)} C(\nu_2)] & \longrightarrow & C(\nu_5), \\ C(\pi_1) \times [C(\pi_2) \times_{C(\nu_2)} C(\pi_3)] & \longrightarrow & C(\pi_4). \end{array}$$

In the first, the point is that there are 3 terms $2, 1, 2$ and their sum is 5. This makes sense if an element of $C(\nu_k)$ is regarded as an operation which turns a string of k 1-cells into a single 1-cell. (The $\times_{C(0)}$'s denote pullbacks.) Similarly for the second; see Fig. 3. There are also identities. A C-algebra is a functor $A : \mathbb{G}_2 \longrightarrow \mathbf{Set}$ together with functions

$$\overline{\psi} : A(0) \longrightarrow A(0) \text{ for each } \psi \in C(0),$$

$$\overline{\phi} : \{\text{diagrams } \begin{tikzcd} a_0 \arrow[r, "f_1"] & \cdots \arrow[r, "f_k"] & a_k \end{tikzcd} \text{ in } A\} \longrightarrow A(1) \text{ for each } \phi \in C(\nu_k),$$

$$\overline{\theta} : \{\text{diagrams } \begin{tikzcd} a \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2ex, "\Downarrow\beta"] & b \arrow[r, bend left, "l"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow\gamma"] & c \end{tikzcd} \text{ in } A\} \longrightarrow A(2) \text{ for each } \theta \in C\left(\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2ex, "\Downarrow"] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \right)$$

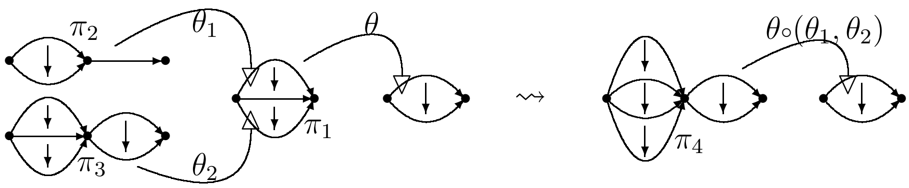


Figure 3: Composition of operations in a globular operad

(etc), all compatible with the source, target, composition and identities in C.

K_2 is generated from the empty collection by adding in the minimal amount to obtain a 2-operad with contraction and system of compositions. We have the identity $1 \in K_2(0)$. Then, contraction gives an element ('1-cell identities') of $K_2(\nu_0)$, the system of compositions gives an element ('1-cell composition') of $K_2(\nu_2)$, and composition in K_2 gives one element of $K_2(\nu_k)$ for each k-leafed tree in which every vertex has 0 or 2 edges coming up out of it. Contraction at the next level gives associativity and unit isomorphisms and identity 2-cells; the system of compositions gives vertical and horizontal 2-cell composition. Condition $(*)$ gives coherence axioms. Thus a weak 2-category is exactly a bicategory.

Definition B2

Definition B2 of weak n-category refers to infinite-dimensional globular operads. So in order to do a concrete analysis of $n \leq 2$, we redefine a *weak n-category* as a pair (C, A) where C is a n-operad admitting a contraction and a system of compositions and with $C(0) \cong 1$, and A is a C-algebra. (Temporarily, call such a C *good*.) I do not know to what extent this is equivalent to B2, but the spirit, at least, is the same.

$n = 0$. From '$n = 0$' above we see that a weak 0-category is just a set.

$n = 1$. The only good 1-operad is the terminal 1-operad, so by '$n = 1$' above, a weak 1-category is just a category.

$n = 2$. A *(non-symmetric) classical operad* D is a sequence $(D(k))_{k\geq 0}$ of sets together with an element (the *identity*) of $D(1)$ and for each $k, r_1, \ldots, r_k \geq 0$ a map $D(k) \times D(r_1) \times \cdots \times D(r_k) \longrightarrow D(r_1 + \cdots + r_k)$ (*composition*), obeying unit and associativity laws. It turns out that good 2-operads C correspond one-to-one with classical operads D such that $D(k) \neq \emptyset$ for each k, via $D(k) = C(\nu_k)$. A C-algebra is then something like a 2-category or bicategory, with one way of composing a string of k 1-cells for each element of $D(k)$, and all the appropriate coherence 2-cells. E.g. if $D = 1$ then a C-algebra is a 2-category; if $D(k)$ is the set of k-leafed trees in which each vertex has either 0 or 2 edges coming up out of it then a C-algebra is a bicategory. C can, therefore, be regarded as a theory of (more or less weak) 2-categories, and A as a model for such a theory.

Definitions L

Globular Operads and their Algebras

Globular Sets. Let $\mathbb{G}$ be the category whose objects are the natural numbers $0, 1, \ldots$, and whose arrows are generated by $\sigma_m, \tau_m : m \longrightarrow m-1$ for each $m \geq 1$, subject to equations

$$\sigma_{m-1} \circ \sigma_m = \sigma_{m-1} \circ \tau_m, \qquad \tau_{m-1} \circ \sigma_m = \tau_{m-1} \circ \tau_m$$

($m \geq 2$). A functor $A : \mathbb{G} \longrightarrow \mathbf{Set}$ is called a *globular set*; I will write s for $A(\sigma_m)$, and t for $A(\tau_m)$.

The Free Strict ω-Category Monad. Any (small) strict ω-category has an underlying globular set A, in which $A(m)$ is the set of m-cells and s and t are the source and target maps. We thus obtain a forgetful functor U from the category of strict ω-categories and strict ω-functors to the category $[\mathbb{G}, \mathbf{Set}]$ of globular sets. U has a left adjoint, so there is an induced monad $(T, \mathrm{id} \xrightarrow{\eta} T, T^2 \xrightarrow{\mu} T)$ on $[\mathbb{G}, \mathbf{Set}]$.

Collections. We define a monoidal category $\mathbf{Coll}$ of collections. Let 1 be the terminal globular set. A *(globular) collection* is a map $C \xrightarrow{d} T1$ into $T1$ in $[\mathbb{G}, \mathbf{Set}]$; a *map of collections* is a commutative triangle. The *tensor product* of collections $C \xrightarrow{d} T1$, $C' \xrightarrow{d'} T1$ is the composite along the top row of

$$\begin{tikzcd} C \otimes C' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TC' \arrow[r, "Td'"] \arrow[d, "T!"] & T^2 1 \arrow[r, "\mu_1"] & T1 \\ C \arrow[r, "d"] & T1, & & \end{tikzcd}$$

where the right-angle symbol means that the square containing it is a pullback, and $!$ denotes the unique map to 1. The *unit* for the tensor is $1 \xrightarrow{\eta_1} T1$.

Globular Operads. A *(globular) operad* is a monoid in the monoidal category $\mathbf{Coll}$; a *map of operads* is a map of monoids.

Algebras. Any operad C induces a monad $C \cdot -$ on $[\mathbb{G}, \mathbf{Set}]$. For an object A of $[\mathbb{G}, \mathbf{Set}]$, this is defined by pullback:

$$\begin{tikzcd} C \cdot A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & TA \arrow[d, "T!"] \\ C \arrow[r, "d"] & T1. \end{tikzcd}$$

The multiplication and unit of the monad come from the multiplication and unit of the operad. A *C-algebra* is an algebra for the monad $C \cdot -$. Note that every C-algebra has an underlying globular set.

Contractions

Contractions. Let $C \xrightarrow{d} T1$ be a collection. For $m \geq 0$ and $\nu \in (T1)(m)$, write $C(\nu) = \{\theta \in C(m) \mid d(\theta) = \nu\}$. For $m \geq 2$ and $\pi \in (T1)(m)$, define

$$P_C(\pi) = \{(\theta_0, \theta_1) \in C(s(\pi)) \times C(t(\pi)) \mid s(\theta_0) = s(\theta_1) \text{ and } t(\theta_0) = t(\theta_1)\},$$

and for $\pi \in (T1)(1)$, define $P_C(\pi) = C(s(\pi)) \times C(t(\pi))$. A *contraction* γ on C is a family of functions

$$(\gamma_\pi : P_C(\pi) \longrightarrow C(\pi))_{m \geq 1, \pi \in (T1)(m)}$$

satisfying

$$s(\gamma_\pi(\theta_0, \theta_1)) = \theta_0, \qquad t(\gamma_\pi(\theta_0, \theta_1)) = \theta_1$$

for every $m \geq 1$, $\pi \in (T1)(m)$ and $(\theta_0, \theta_1) \in P_C(\pi)$.

Initial Object. Let **OC** be the category in which an object is an operad equipped with a contraction and a map is a map of operads preserving the specified contraction. Then **OC** can be shown to have an initial object, whose underlying operad will be written L.

The Definitions

Definition L1. A *weak ω-category* is an L-algebra. (Maps of L-algebras should be regarded as *strict* ω-functors.)

Definition L2. A *weak ω-category* is a pair (C, A), where C is an operad on which there exists a contraction and satisfying $C(0) \cong 1$, and A is a C-algebra.

Weak n-Categories. Let $n \geq 0$. A globular set A is *n-dimensional* if for all $m \geq n$,

$$s = t : A(m+1) \longrightarrow A(m)$$

and this map is an isomorphism. A *weak n-category* is a weak ω-category whose underlying globular set is n-dimensional. This can be interpreted according to either L1 or L2.

Definitions L for $n \leq 2$

Definition L1

An alternative way of handling weak n-categories is to work with only n- (not infinite-) dimensional structures throughout. So we replace $\mathbb{G}$ by its full subcategory $\mathbb{G}_n$ with objects $0, \ldots, n$, and T by the free strict n-category monad T_n, to obtain definitions of *n-collection*, *n-operads*, and their *algebras*. *Contractions* are defined as before, except that we only speak of contractions on C if

$$\forall \nu \in (T_n 1)(n), \forall \theta_0, \theta_1 \in C(\nu),\ s(\theta_0) = s(\theta_1)\ \&\ t(\theta_0) = t(\theta_1) \Rightarrow \theta_0 = \theta_1 \qquad (\dagger)$$

(taking $C(-1) = 1$ to understand this when $n = 0$). There is an initial n-operad L_n equipped with a contraction, and the category of weak n-categories turns out to be equivalent to the category of L_n-algebras. The latter is easier to analyse.

$n = 0$. We have $[\mathbb{G}_0, \mathbf{Set}] \cong \mathbf{Set}$, $T_0 = \mathrm{id}$, and 0-$\mathbf{Coll} \cong \mathbf{Set}$; a 0-operad C is a monoid, and a C-algebra is a set with a C-action. By ($\dagger$), the only 0-operad with a contraction is the one-element monoid, so a weak 0-category is just a set.

$n = 1$. $[\mathbb{G}_1, \mathbf{Set}]$ is the category of directed graphs and T_1 is the free category monad. L_1 is the terminal 1-operad, by arguments similar to those under '$n = 2$' below. It follows that $L_1 \cdot -$ is just T_1, and so a weak 1-category is just a T_1-algebra, that is, a category.

$n = 2$. A functor $A : \mathbb{G}_2 \longrightarrow \mathbf{Set}$ consists of a set of 0-cells (drawn $\underset{a}{\bullet}$), a set of 1-cells ($a \xrightarrow{f} b$), and a set of 2-cells ($\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd}$). A 2-collection C consists of a set $C(0)$, a set $C(\nu_k)$ for each $k \geq 0$ (where ν_k indicates the '1-pasting diagram' $\bullet \longrightarrow \cdots \longrightarrow \bullet$ with k arrows), and a set $C(\pi)$ for each '2-pasting diagram' π such as the π_i in Fig. 4, together with source and target functions.

A 2-operad is a 2-collection C together with 'composition' functions such as

$$\begin{array}{rcl} C(\nu_3) \times [C(\nu_2) \times_{C(0)} C(\nu_1) \times_{C(0)} C(\nu_2)] & \longrightarrow & C(\nu_5), \\ C(\pi_1) \times [C(\pi_2) \times_{C(\nu_2)} C(\pi_3)] & \longrightarrow & C(\pi_4). \end{array}$$

In the first, the point is that there are 3 terms $2, 1, 2$ and their sum is 5. This makes sense if an element of $C(\nu_k)$ is regarded as an operation which turns a string of k 1-cells into a single 1-cell. (The $\times_{C(0)}$'s denote pullbacks.) Similarly for the second; see Fig. 4. There are also identities. A C-algebra is a functor $A : \mathbb{G}_2 \longrightarrow \mathbf{Set}$ together with functions

$\overline{\psi} : A(0) \longrightarrow A(0)$ for each $\psi \in C(0)$,
$\overline{\phi} : \{\text{diagrams } a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_k} a_k \text{ in } A\} \longrightarrow A(1)$ for each $\phi \in C(\nu_k)$,
$\overline{\theta} : \{\text{diagrams } \begin{tikzcd} a \arrow[r, bend left=60, "f"] \arrow[r, "g" description] \arrow[r, bend right=60, "h"'] \arrow[r, phantom, shift left=2ex, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2ex, "\Downarrow\beta"] & b \arrow[r, bend left, "l"] \arrow[r, bend right, "m"'] \arrow[r, phantom, "\Downarrow\gamma"] & c \end{tikzcd} \text{ in } A\} \longrightarrow A(2)$ for each $\theta \in C\left(\begin{tikzcd} \bullet \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2ex, "\Downarrow"] \arrow[r, phantom, shift right=2ex, "\Downarrow"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}\right)$

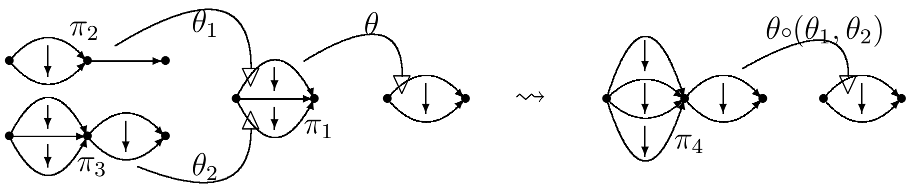


Figure 4: Composition of operations in a globular operad

(etc), all compatible with the source, target, composition and identities in C.

L_2 is generated from the empty collection by adding in the minimal amount to obtain a 2-operad-with-contraction. We have the identity $1 \in L_2(0)$. Then, contraction gives an element of $L_2(\nu_k)$ for each k, so that composition in L_2 gives an element of $L_2(\nu_k)$ for each k-leafed tree. Contraction at the next level (with (†)) says that if π is a 2-pasting diagram of width k then $L_2(\pi) = L_2(\nu_k) \times L_2(\nu_k)$. So $L_2(0) = \{1\}$, $L_2(\nu_k) = \{k\text{-leafed trees}\}$, $L_2(\pi) = \{k\text{-leafed trees}\}^2$. An L_2-algebra is, then, an 'unbiased bicategory': that is, just like a bicategory except that there is specified k-fold composition for every $k \geq 0$ rather than just $k = 2$ (binary composition) and $k = 0$ (identities). Since these are essentially the same as ordinary bicategories, so too are weak 2-categories.

Definition L2

Definition L2 of weak n-category refers to infinite-dimensional globular operads. So in order to do a concrete analysis of $n \leq 2$, we redefine a *weak n-category* as a pair (C, A) where C is a n-operad admitting a contraction and with $C(0) \cong 1$, and A is a C-algebra. (Temporarily, call such a C *good*.) I do not know to what extent this is equivalent to L2, but the spirit, at least, is the same.

$n = 0$. From '$n = 0$' above we see that a weak 0-category is just a set.

$n = 1$. The only good 1-operad is the terminal 1-operad, so by '$n = 1$' above, a weak 1-category is just a category.

$n = 2$. A *(non-symmetric) classical operad* D is a sequence $(D(k))_{k\geq 0}$ of sets together with an element (the *identity*) of $D(1)$ and for each $k, r_1, \ldots, r_k \geq 0$ a map $D(k) \times D(r_1) \times \cdots \times D(r_k) \longrightarrow D(r_1 + \cdots + r_k)$ (*composition*), obeying unit and associativity laws. It turns out that good 2-operads C correspond one-to-one with classical operads D such that $D(k) \neq \emptyset$ for each k, via $D(k) = C(\nu_k)$. A C-algebra is then something like a 2-category or bicategory, with one way of composing a string of k 1-cells for each element of $D(k)$, and all the appropriate coherence 2-cells. E.g. if $D = 1$ then a C-algebra is a 2-category; if $D(k)$ is the set of k-leafed trees in which each vertex has either 0 or 2 edges coming up out of it then a C-algebra is a bicategory. C can, therefore, be regarded as a theory of (more or less weak) 2-categories, and A as a model for such a theory.

Definition L′

Globular Multicategories

Globular Sets. Let $\mathbb{G}$ be the category whose objects are the natural numbers $0, 1, \ldots$, and whose arrows are generated by $\sigma_m, \tau_m : m \longrightarrow m-1$ for each $m \geq 1$, subject to equations

$$\sigma_{m-1} \circ \sigma_m = \sigma_{m-1} \circ \tau_m, \qquad \tau_{m-1} \circ \sigma_m = \tau_{m-1} \circ \tau_m$$

$(m \geq 2)$. A functor $A : \mathbb{G} \longrightarrow \mathbf{Set}$ is called a *globular set*; I will write s for $A(\sigma_m)$, and t for $A(\tau_m)$.

The Free Strict ω-Category Monad. Any (small) strict ω-category has an underlying globular set A, in which $A(m)$ is the set of m-cells and s and t are the source and target maps. We thus obtain a forgetful functor U from the category of strict ω-categories and strict ω-functors to the category $[\mathbb{G}, \mathbf{Set}]$ of globular sets. U has a left adjoint, so there is an induced monad $(T, \mathrm{id} \xrightarrow{\eta} T, T^2 \xrightarrow{\mu} T)$ on $[\mathbb{G}, \mathbf{Set}]$.

Globular Graphs. For each globular set A, we define a monoidal category $\mathbf{Graph}_A$. An object of $\mathbf{Graph}_A$ is a *(globular) graph on A*: that is, a globular set R together with maps of globular sets

$$\begin{tikzcd}
& R \arrow[dl, "\mathrm{dom}"'] \arrow[dr, "\mathrm{cod}"] & \\
TA & & A.
\end{tikzcd}$$

A *map* $(R, \mathrm{dom}, \mathrm{cod}) \longrightarrow (R', \mathrm{dom}', \mathrm{cod}')$ of graphs on A is a map $R \longrightarrow R'$ making the evident triangles commute. The *tensor product* of graphs $(R, \mathrm{dom}, \mathrm{cod})$, $(R', \mathrm{dom}', \mathrm{cod}')$ is given by composing along the upper edges of the following diagram, in which the right-angle symbol means that the square containing it is a pullback:

$$\begin{tikzcd}
& & & R \otimes R' \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
& & TR' \arrow[dl, "T\mathrm{dom}'"'] \arrow[dr, "T\mathrm{cod}'"'] & & R \arrow[dl, "\mathrm{dom}"] \arrow[dr, "\mathrm{cod}"] & \\
& T^2A \arrow[dl, "\mu_A"'] & & TA & & A. \\
TA & & & & &
\end{tikzcd}$$

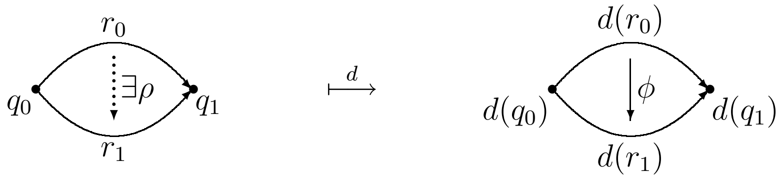


Figure 5: Part (ii) of the definition of contractibility, shown for $m = 1$

The *unit* for the tensor is the graph

$$\begin{tikzcd} & A \arrow[dl, "\eta_A"'] \arrow[dr, "1"] & \\ TA & & A. \end{tikzcd}$$

Globular Multicategories. A *globular multicategory* is a globular set A together with a monoid in $\mathbf{Graph}_A$. A globular multicategory $\mathcal{A}$ therefore consists of a globular set A, a graph $(R, \mathrm{dom}, \mathrm{cod})$ on A, and maps $\mathrm{comp} : R \otimes R \longrightarrow R$ and $\mathrm{ids} : A \longrightarrow R$ compatible with dom and cod and obeying associativity and identity laws.

Contractible Maps

A map $d : R \longrightarrow S$ of globular sets is *contractible* (Figure 5) if

i. the function $d_0 : R(0) \longrightarrow S(0)$ is bijective, and

ii. for every

$m \geq 0$,

$r_0, r_1 \in R(m)$ with $s(r_0) = s(r_1)$ and $t(r_0) = t(r_1)$,

$\phi \in S(m+1)$ with $s(\phi) = d_m(r_0)$ and $t(\phi) = d_m(r_1)$,

there exists $\rho \in R(m+1)$ with $s(\rho) = r_0$, $t(\rho) = r_1$, and $d_{m+1}(\rho) = \phi$. In the case $m = 0$ we drop the (nonsensical) conditions that $s(r_0) = s(r_1)$ and $t(r_0) = t(r_1)$.

The Definition

Weak ω-Categories. A *weak ω-category* is a globular multicategory $\mathcal{A} = (A, R, \mathrm{dom}, \mathrm{cod}, \mathrm{comp}, \mathrm{ids})$ such that $\mathrm{dom} : R \longrightarrow TA$ is contractible.

Weak n-Categories. Let $n \geq 0$. A globular set A is *n-dimensional* if for all $m \geq n$,

$$s = t : A(m+1) \longrightarrow A(m)$$

and this map is an isomorphism. A *weak n-category* is a weak ω-category $\mathcal{A}$ such that the globular sets A and R are n-dimensional.

Definition L′ for $n \leq 2$

An alternative way of handling weak n-categories is to work with only n-dimensional (not infinite-dimensional) structures throughout. Thus we replace $\mathbb{G}$ by its full subcategory $\mathbb{G}_n$ with objects $0, \ldots, n$, replace T by the free strict n-category monad T_n, and so obtain a definition of *n-globular multicategory.* We also modify part (ii) of the definition of contractibility by changing '$m \geq 0$' to '$n-1 \geq m \geq 0$', and 'there exists ρ' to 'there exists a unique ρ' in the case $m = n-1$. From these ingredients we get a new definition of weak n-category.

The new and old definitions give two different, but equivalent, categories of weak n-categories (with maps of multicategories as the morphisms); the analysis of $n \leq 2$ is more convenient with the new definition.

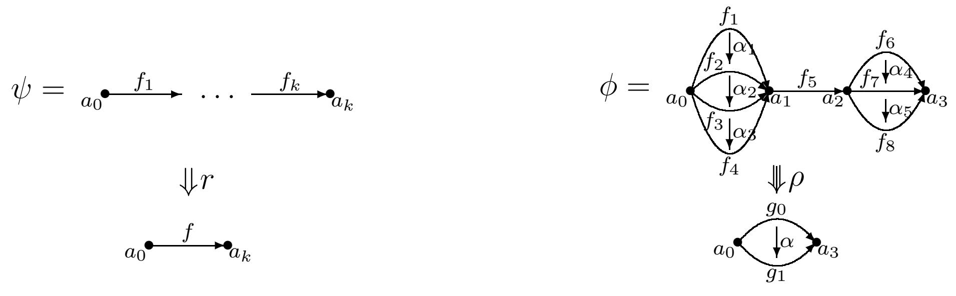
$n = 0$**.** We have $[\mathbb{G}_0, \mathbf{Set}] \cong \mathbf{Set}$ and $T_0 = \mathrm{id}$, and the contractible maps are the bijections. So a weak 0-category is a category whose domain map is a bijection; that is, a discrete category; that is, a set.

$n = 1$**.** $[\mathbb{G}_1, \mathbf{Set}]$ is the category of directed graphs, T_1 is the free category monad on it, and a map of graphs is contractible if and only if it is an isomorphism. So a weak 1-category is essentially a 1-globular multicategory whose underlying 1-globular graph looks like $T_1 A \xleftarrow{1} T_1 A \xrightarrow{\mathrm{cod}} A$. Such a graph has at most one multicategory structure, and it has one if and only if cod is a T_1-algebra structure on A. So a weak 1-category is just a T_1-algebra, i.e., a category.

$n = 2$**.** The free 2-category $T_2 A$ on a 2-globular set $A \in [\mathbb{G}_2, \mathbf{Set}]$ has the same 0-cells as A; 1-cells of $T_2 A$ are formal paths ψ in A as in Fig. 6(a); and a typical 2-cell of $T_2 A$ is the diagram ϕ in Fig. 6(b).

Next, what is a 2-globular multicategory $\mathcal{A} = (A, R, \mathrm{dom}, \mathrm{cod}, \mathrm{comp}, \mathrm{ids})$? Since we ultimately want to consider just those $\mathcal{A}$ in which dom is contractible, let us assume immediately that $R(0) = A(0)$. Then $\mathcal{A}$ consists of:

- a 2-globular set $A \in [\mathbb{G}_2, \mathbf{Set}]$
- for each ψ and f as in Fig. 6(a), a set of cells $r : \psi \Rightarrow f$; such an r is a 1-cell of R, and can be regarded as a 'reason why f is a composite of ψ'
- for each ϕ and g_0, g_1, α as in Fig. 6(b), a set of cells $\rho : \phi \Rrightarrow \alpha$; such a ρ is a 2-cell of R, and can be regarded as a 'reason why α is a composite of ϕ'
- source and target functions $R(2) \rightrightarrows R(1)$, which, for instance, assign to ρ a reason $s(\rho)$ why g_0 is a composite of $a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_5} a_2 \xrightarrow{f_6} a_3$
- composition and identities: given r as in Fig. 6(a) and similarly $r_i : (f_i^1, \ldots, f_i^{p_i}) \Rightarrow f_i$ for each $i = 1, \ldots, k$, there is a composite $r \circ (r_1, \ldots, r_k) : (f_1^1, \ldots, f_k^{p_k}) \Rightarrow f$; and similarly for 2-cells and for identities,



(a) (b)

Figure 6: (a) A 1-cell, and (b) a typical 2-cell, of R. Here $a_i, f_i, f, g_i, \alpha_i$ and α are all cells of A

such that the composition and identities satisfy associativity and identity axioms and are compatible with source and target.

Contractibility says that for each ψ as in Fig. 6(a) there is at least one pair (r, f) as in Fig. 6(a), and that for each ϕ as in Fig. 6(b) and each $r_0 : (f_1, f_5, f_6) \Rightarrow g_0$ and $r_1 : (f_4, f_5, f_8) \Rightarrow g_1$, there is exactly one pair (ρ, α) as in Fig. 6(b) satisfying $s(\rho) = r_0$ and $t(\rho) = r_1$. That is: every diagram of 1-cells has at least one composite, and every diagram of 2-cells has exactly one composite once a way of composing the 1-cells along its boundary has been chosen.

When $\phi = \begin{tikzcd} a_0 \arrow[r, bend left=60, "f_0"] \arrow[r, "f_1" description] \arrow[r, bend right=60, "f_2"'] & a_1 \end{tikzcd}$ (with $\Downarrow\alpha_1$ above and $\Downarrow\alpha_2$ below f_1), the identity reasons for f_0 and f_2 give via contractibility a composite $\alpha_2 \circ \alpha_1 : f_0 \longrightarrow f_2$, and in this way the 1- and 2-cells between a_0 and a_1 form a category $\mathcal{A}(a_0, a_1)$. Now suppose that ψ is as in Fig. 6(a) and $r : \psi \Rightarrow f$, $r' : \psi \Rightarrow f'$. Applying contractibility to the degenerate 2-cell diagram ϕ which looks exactly like ψ, we obtain a 2-cell $\begin{tikzcd} a_0 \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow"] & a_k \end{tikzcd}$; and similarly the other way round; so by the uniqueness property of the ρ's, $f \cong f'$ in $\mathcal{A}(a_0, a_k)$. Thus any two composites of a string of 1-cells are canonically isomorphic.

A weak 2-category is essentially what is known as an 'anabicategory'. To see how one of these gives rise to a bicategory, choose for each $a_0 \xrightarrow{f} a_1 \xrightarrow{g} a_2$ in A a reason $r_{f,g} : (f, g) \Rightarrow h$ and write $h = (g \circ f)$; and similarly for identities. Then, for instance, the horizontal composite of 2-cells $\begin{tikzcd} a_0 \arrow[r, bend left, "f_0"] \arrow[r, bend right, "f_1"'] \arrow[r, phantom, "\Downarrow"] & a_1 \arrow[r, bend left, "g_0"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow"] & a_2 \end{tikzcd}$ comes via contractibility from r_{f_0,g_0} and r_{f_1,g_1}, the associativity cells arise from the degenerate 2-cell diagram $\phi = a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2 \xrightarrow{f_3} a_3$, and the coherence axioms come from the uniqueness of the ρ's.

Definition SI

Simplicial Objects

The Simplicial Category. Let Δ be a skeleton of the category of nonempty finite totally ordered sets: that is, Δ has objects $[k] = \{0, \ldots, k\}$ for $k \geq 0$, and maps are order-preserving functions (with respect to the usual ordering $\leq$).

Some Maps in Δ. Let $\sigma, \tau : [0] \longrightarrow [1]$ be the maps in Δ with respective values 0 and 1. Given $k \geq 0$, let $\iota_1, \ldots, \iota_k : [1] \longrightarrow [k]$ denote the 'embeddings' of [1] into $[k]$, defined by $\iota_j(0) = j - 1$ and $\iota_j(1) = j$.

The Segal Maps. Let $k \geq 0$. Then the following diagram in Δ commutes:

$$\begin{tikzcd}
 & & & {[k]} & & & \\
{[1]} \arrow[urrr, "\iota_1"] & & {[1]} \arrow[ur, "\iota_2"] & \cdots & \cdots & & {[1]} \arrow[ulll, "\iota_k"'] \\
 & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & \cdots & {[0].} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

Let $X : \Delta^{\mathrm{op}} \longrightarrow \mathcal{E}$ be a functor into a category $\mathcal{E}$ possessing finite limits, and write $X[1] \times_{X[0]} \cdots \times_{X[0]} X[1]$ (with k occurrences of $X[1]$) for the limit of the diagram

$$\begin{tikzcd}
X[1] \arrow[dr, "X\tau"'] & & X[1] \arrow[dl, "X\sigma"] \arrow[dr, "X\tau"'] & & \cdots \arrow[dl, "X\sigma"] \arrow[dr, "X\tau"'] & & X[1] \arrow[dl, "X\sigma"] \\
 & X[0] & & X[0] & \cdots & X[0] &
\end{tikzcd}$$

(with, again, k occurrences of $X[1]$) in $\mathcal{E}$. Then by commutativity of the first diagram, there is an induced map in $\mathcal{E}$—a *Segal map*—

$$X[k] \longrightarrow X[1] \times_{X[0]} \cdots \times_{X[0]} X[1]. \tag{‡}$$

Contractibility

Sources and Targets. If $0 \leq p \leq r$, write I_p for the object $(\underbrace{[1], \ldots, [1]}_{p}, \underbrace{[0], \ldots, [0]}_{r-p})$ of Δ^r. Let $X : (\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$, $0 \leq p \leq r$, and $x, x' \in X(I_p)$. Then x, x' are *parallel* if $p = 0$ or if $p \geq 1$ and $s(x) = s(x')$ and $t(x) = t(x')$; here s and t are the maps

$$X(I_p) \underset{X(\mathrm{id}, \ldots, \mathrm{id}, \tau, \mathrm{id}, \ldots, \mathrm{id})}{\overset{X(\mathrm{id}, \ldots, \mathrm{id}, \sigma, \mathrm{id}, \ldots, \mathrm{id})}{\rightrightarrows}} X(I_{p-1}).$$

$$\begin{tikzcd} x \arrow[r, dotted, "g"] & x' \\ \phi(x) \arrow[r, "h"] & \phi(x') \end{tikzcd} \qquad \Big\downarrow \phi$$

Figure 7: Contractibility, shown for $p = 0$

Contractible Maps. Let $r \geq 1$ and let $X, Y : (\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$. A natural transformation $\phi : X \longrightarrow Y$ is *contractible* (Figure 7) if

- the function $\phi_{I_0} : X(I_0) \longrightarrow Y(I_0)$ is surjective
- given $p \in \{0, \ldots, r-1\}$, parallel $x, x' \in X(I_p)$, and $h \in Y(I_{p+1})$ satisfying
$$s(h) = \phi_{I_p}(x), \qquad t(h) = \phi_{I_p}(x'),$$
there exists $g \in X(I_{p+1})$ satisfying
$$s(g) = x, \qquad t(g) = x', \qquad \phi_{I_{p+1}}(g) = h$$
- given parallel $x, x' \in X(I_r)$ satisfying $\phi_{I_r}(x) = \phi_{I_r}(x')$, then $x = x'$.

If $r = 0$ then X and Y are just sets and ϕ is just a function $X \longrightarrow Y$; call ϕ *contractible* if it is bijective.

The Definition

Let $n \geq 0$. A *weak n-category* is a functor $A : (\Delta^n)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ such that for each $m \in \{0, \ldots, n-1\}$ and $K = ([k_1], \ldots, [k_m]) \in \Delta^m$,

i. the functor $A(K, [0], -) : (\Delta^{n-m-1})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is constant, and

ii. for each $[k] \in \Delta$, the Segal map
$$A(K, [k], -) \longrightarrow A(K, [1], -) \times_{A(K,[0],-)} \cdots \times_{A(K,[0],-)} A(K, [1], -)$$
is contractible. (We are taking $\mathcal{E} = [(\Delta^{n-m-1})^{\mathrm{op}}, \mathbf{Set}]$ and $X[j] = A(K, [j], -)$ in the definition of Segal map.)

Definition $\mathrm{S}\textsc{i}$ for $n \leq 2$

$n = 0$**.** Parts (i) and (ii) of the definition are vacuous, so a weak 0-category is just a functor $(\Delta^0)^{\mathrm{op}} \longrightarrow \mathbf{Set}$, that is, a set.

$n = 1$**.** A weak 1-category is a functor $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ (that is, a simplicial set) satisfying (i) and (ii). Part (i) is always true, and (ii) says that for each $k \geq 0$ the Segal map (‡) (with $X = A$) is a bijection—in other words, that A is a *nerve*. The category of nerves and natural transformations between them is equivalent to **Cat**, where a nerve A corresponds to a certain category with object-set $A[0]$ and morphism-set $A[1]$. So a weak 1-category is essentially just a category.

$n = 2$**.** A weak 2-category is a functor $A : (\Delta^2)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ such that

i. the functor $A([0], -) : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is constant

ii. for each $k \geq 0$, the Segal map

$$A([k], -) \longrightarrow A([1], -) \times_{A([0],-)} \cdots \times_{A([0],-)} A([1], -)$$

is contractible, and for each $k_1, k \geq 0$, the Segal map

$$A([k_1], [k]) \longrightarrow A([k_1], [1]) \times_{A([k_1],[0])} \cdots \times_{A([k_1],[0])} A([k_1], [1])$$

is a bijection.

The second half of (ii) says that $A([k_1], -)$ is a nerve for each k_1, so we can regard A as a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$. Note that if X and Y are nerves then a natural transformation $\phi : X \longrightarrow Y$ is the same thing as a functor between the corresponding categories, and that ϕ is contractible if and only if this functor is full, faithful and surjective on objects. So a weak 2-category is a functor $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ such that

i. $A[0]$ is a discrete category (i.e. the only morphisms are the identities)

ii. for each $k \geq 0$, the Segal functor

$$A[k] \longrightarrow A[1] \times_{A[0]} \cdots \times_{A[0]} A[1]$$

is full, faithful and surjective on objects.

I will now argue that a weak 2-category is essentially the same thing as a bicategory.

First take a weak 2-category $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$, and let us construct a bicategory B. The object-set of B is $A[0]$. The two functors $s, t : A[1] \longrightarrow A[0]$ express the category $A[1]$ as a disjoint union $\coprod_{a,b \in A[0]} B(a, b)$ of categories; the 1-cells from a to b are the objects of $B(a, b)$, and the 2-cells are the morphisms. Vertical composition of 2-cells in B is composition in each $B(a, b)$. To define horizontal composition of 1- and 2-cells, first choose for each k a pseudo-inverse $A[1] \times_{A[0]} \cdots \times_{A[0]} A[1] \xrightarrow{\psi_k} A[k]$ to the Segal functor ϕ_k (an equivalence of categories), and natural isomorphisms $\eta_k : 1 \longrightarrow \psi_k \circ \phi_k$, $\varepsilon_k : \phi_k \circ \psi_k \longrightarrow 1$. Horizontal composition is given by

$$A[1] \times_{A[0]} A[1] \xrightarrow{\psi_2} A[2] \xrightarrow{A\delta} A[1],$$

where $\delta : [1] \longrightarrow [2]$ is the injection whose image omits $1 \in [2]$. The associativity isomorphisms are built up from η_k's and ε_k's, and the pentagon commutes just as long as the equivalence $(\phi_k, \psi_k, \eta_k, \varepsilon_k)$ was chosen to be an adjunction too (which is always possible). Identities work similarly.

Conversely, take a bicategory B and construct a weak 2-category $A : (\Delta^2)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ (its '2-nerve') as follows. An element of $A([j], [k])$ is a quadruple

$$((a_u)_{0 \leq u \leq j}, (f^z_{uv})_{\substack{0 \leq u < v \leq j \\ 0 \leq z \leq k}}, (\alpha^z_{uv})_{\substack{0 \leq u < v \leq j \\ 1 \leq z \leq k}}, (\iota^z_{uvw})_{\substack{0 \leq u < v < w \leq j \\ 0 \leq z \leq k}})$$

where

- a_u is an object of B
- $f^z_{uv} : a_u \longrightarrow a_v$ is a 1-cell of B
- $\alpha^z_{uv} : f^{z-1}_{uv} \longrightarrow f^z_{uv}$ is a 2-cell of B
- $\iota^z_{uvw} : f^z_{vw} \circ f^z_{uv} \xrightarrow{\sim} f^z_{uw}$ is an invertible 2-cell of B

such that

- $\iota^z_{uvw} \circ (\alpha^z_{vw} * \alpha^z_{uv}) = \alpha^z_{uw} \circ \iota^{z-1}_{uvw}$ whenever $0 \leq u < v < w \leq j$, $1 \leq z \leq k$
- $\iota^z_{uwx} \circ (1_{f^z_{wx}} * \iota^z_{uvw}) \circ (\text{associativity isomorphism}) = \iota^z_{uvx} \circ (\iota^z_{vwx} * 1_{f^z_{uv}})$ whenever $0 \leq u < v < w < x \leq j$, $0 \leq z \leq k$.

This defines the functor A on objects of Δ^2; it is defined on maps by a combination of inserting identities and forgetting data.

To get a rough picture of A, consider the analogous construction for strict 2-categories, in which we insist that the isomorphisms ι^z_{uvw} are actually equalities. Then an element of $A([j], [k])$ is a grid of jk 2-cells, of width j and height k. (When $j = 0$ this is just a single object of B, regardless of k.) The bicategorical version is a suitable weakening of this construction.

Finally, passing from a bicategory to a weak 2-category and back again gives a bicategory isomorphic (in the category of bicategories and weak functors) to the original one. Passing from a weak 2-category to a bicategory and back again gives a weak 2-category which is 'equivalent' to the original one in a sense which we do not have quite enough vocabulary to make precise here.

Definition Ta

Simplicial Objects

The Simplicial Category. Let Δ be a skeleton of the category of nonempty finite totally ordered sets: that is, Δ has objects $[k] = \{0, \ldots, k\}$ for $k \geq 0$, and maps are order-preserving functions (with respect to the usual ordering $\leq$).

Some Objects and Morphisms. Given $k \geq 0$, let $\iota_1, \ldots, \iota_k : [1] \longrightarrow [k]$ denote the 'embeddings' of $[1]$ into $[k]$, defined by $\iota_j(0) = j-1$ and $\iota_j(1) = j$. Let $\sigma, \tau : [0] \longrightarrow [1]$ be the maps in Δ with respective values 0 and 1. Given $p \geq 0$, write $0^p = ([0], \ldots, [0]) \in \Delta^p$ and $1^p = ([1], \ldots, [1]) \in \Delta^p$. Let $X : (\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$, $0 \leq p \leq r$, and $x, x' \in X(1^p, 0^{r-p})$. Then x, x' are *parallel* if $p = 0$ or if $p \geq 1$ and $s(x) = s(x')$ and $t(x) = t(x')$; here s and t are the maps

$$X(1^p, 0^{r-p}) \underset{X(\mathrm{id}, \ldots, \mathrm{id}, \tau, \mathrm{id}, \ldots, \mathrm{id})}{\overset{X(\mathrm{id}, \ldots, \mathrm{id}, \sigma, \mathrm{id}, \ldots, \mathrm{id})}{\rightrightarrows}} X(1^{p-1}, 0^{r-p+1}).$$

The Segal Maps. Let $k \geq 0$. Then the following diagram in Δ commutes:

$$\begin{tikzcd}[column sep=small]
 & & & & [k] & & & & & & \\
 & & & & \cdots & & & & & & \\
{[1]} \arrow[uurrrr, "\iota_1"] & & {[1]} \arrow[uurr, "\iota_2"] & & & & \cdots & & & {[1]} \arrow[uullllll, "\iota_k"'] & \\
 & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & {[0]} \arrow[ul, "\tau"] \arrow[ur, "\sigma"'] & & & \cdots & & {[0].} \arrow[ull, "\tau"] \arrow[ur, "\sigma"'] & &
\end{tikzcd}$$

Let $X : \Delta^{\mathrm{op}} \longrightarrow \mathcal{E}$ be a functor into a category $\mathcal{E}$ possessing finite limits, and write $X[1] \times_{X[0]} \cdots \times_{X[0]} X[1]$ (with k occurrences of $X[1]$) for the limit of the diagram

$$\begin{tikzcd}[column sep=small]
X[1] \arrow[dr, "X\tau"'] & & X[1] \arrow[dl, "X\sigma"] \arrow[dr, "X\tau"'] & & \cdots & & X[1] \arrow[dl, "X\sigma"] \\
 & X[0] & & X[0] & \cdots & X[0] &
\end{tikzcd}$$

(with, again, k occurrences of $X[1]$) in $\mathcal{E}$. Then by commutativity of the first diagram, there is an induced map in $\mathcal{E}$—a *Segal map*—

$$X[k] \longrightarrow X[1] \times_{X[0]} \cdots \times_{X[0]} X[1]. \qquad (\S)$$

Nerves. Call $X : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ a *nerve* if for each $k \geq 0$, the Segal map (§) is a bijection. The category of nerves and natural transformations is equivalent to **Cat**, where a nerve X corresponds to a category with object-set $X[0]$ and morphism-set $X[1]$. Let QX be the set of isomorphism classes of objects of the category corresponding to X, and let $\pi_X : X[0] \longrightarrow QX$ be the quotient map.

Truncatability

For each $r \geq 0$, we define what it means for a functor $X : (\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ to be *truncatable*, writing r-**Trunc** for the category of truncatable functors $(\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and natural transformations between them. We also define functors $\mathrm{ob}^{(r)}, Q^{(r)} : r\text{-}\mathbf{Trunc} \longrightarrow \mathbf{Set}$ and a natural transformation $\pi^{(r)} : \mathrm{ob}^{(r)} \longrightarrow Q^{(r)}$.

The functor $\mathrm{ob}^{(r)}$ is given by $\mathrm{ob}^{(r)} X = X(0^r)$. All functors $(\Delta^0)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ are truncatable, and $Q^{(0)}$ and $\pi^{(0)}$ are identities. Inductively, when $r \geq 1$, a functor $X : (\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is truncatable if

- for each $k \geq 0$, the functor $X([k], -) : (\Delta^{r-1})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is truncatable
- the functor $\widehat{X} : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ defined by $[k] \longmapsto Q^{(r-1)}(X([k], -))$ is a nerve.

If X is truncatable then we define $Q^{(r)}(X) = Q(\widehat{X})$ and $\pi^{(r)}_X = \pi_{\widehat{X}} \circ \pi^{(r-1)}_{X([0],-)}$.

Equivalence

Internal Equivalence. Let $0 \leq p \leq r$, let $X : (\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be truncatable, and let $x_1, x_2 \in X(1^p, 0^{r-p})$. We call x_1 and x_2 *equivalent*, and write $x_1 \sim x_2$, if x_1 and x_2 are parallel and $\pi^{(r-p)}_{X(1^p,-)}(x_1) = \pi^{(r-p)}_{X(1^p,-)}(x_2)$.

External Equivalence. Let $r \geq 0$. A natural transformation $\phi : X \longrightarrow Y$ of truncatable functors $X, Y : (\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is called an *equivalence* if

- for each $y \in Y(0^r)$ there is a unique-up-to-equivalence $x \in X(0^r)$ such that $\phi_{0^r}(x) \sim y$
- for all $0 \leq p \leq r-1$, parallel $x, x' \in X(1^p, 0^{r-p})$, and $h \in Y(1^{p+1}, 0^{r-p-1})$ satisfying
$$s(h) = \phi_{(1^p,0^{r-p})}(x), \qquad t(h) = \phi_{(1^p,0^{r-p})}(x'),$$
there is an element $g \in X(1^{p+1}, 0^{r-p-1})$, unique up to equivalence, satisfying
$$s(g) = x, \qquad t(g) = x', \qquad \phi_{(1^{p+1},0^{r-p-1})}(g) \sim h.$$

The Definition

Let $n \geq 0$. A *weak n-category* is a truncatable functor $A : (\Delta^n)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ such that for each $m \in \{0, \ldots, n-1\}$ and $K = ([k_1], \ldots, [k_m]) \in \Delta^m$,

i. the functor $A(K, [0], -) : (\Delta^{n-m-1})^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is constant, and

ii. for each $[k] \in \Delta$, the Segal map
$$A(K, [k], -) \longrightarrow A(K, [1], -) \times_{A(K,[0],-)} \cdots \times_{A(K,[0],-)} A(K, [1], -) \qquad (\P)$$
is an equivalence. (We are taking $\mathcal{E} = [(\Delta^{n-m-1})^{\mathrm{op}}, \mathbf{Set}]$ and $X[j] = A(K, [j], -)$ in the definition of Segal map, and we can check that both the domain and the codomain of (¶) are truncatable.)

Definition TA for $n \leq 2$

$n = 0$. Parts (i) and (ii) of the definition are vacuous, and truncatability is automatic, so a weak 0-category is just a functor $(\Delta^0)^{\mathrm{op}} \longrightarrow \mathbf{Set}$, that is, a set.

$n = 1$. Note that a functor $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is truncatable exactly when it is a nerve; that a functor $X : (\Delta^0)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is merely a set, and two elements of X are equivalent just when they are equal; and that a map $\phi : X \longrightarrow Y$ of functors $X, Y : (\Delta^0)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is an equivalence just when it is a bijection. A weak 1-category is a truncatable functor $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ satisfying (i) and (ii). Part (i) is trivially true, and both truncatability and (ii) say that A is a nerve. So a weak 1-category is just a nerve, that is, a category.

$n = 2$. First note that if $X : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a nerve then two elements of $X[0]$ are equivalent just when they are isomorphic (as objects of the category corresponding to X), and two elements of $X[1]$ are equivalent just when they are equal. Note also that a map $\phi : X \longrightarrow Y$ of nerves is an equivalence if and only if (regarded as a functor between the corresponding categories) it is full, faithful and essentially surjective on objects—that is, an equivalence of categories.

A weak 2-category is a truncatable functor $A : (\Delta^2)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ such that

i. the functor $A([0], -) : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is constant

ii. for each $k \geq 0$, the Segal map

$$A([k], -) \longrightarrow A([1], -) \times_{A([0],-)} \cdots \times_{A([0],-)} A([1], -)$$

is an equivalence, and for each $k_1, k \geq 0$, the Segal map

$$A([k_1], [k]) \longrightarrow A([k_1], [1]) \times_{A([k_1],[0])} \cdots \times_{A([k_1],[0])} A([k_1], [1])$$

is a bijection.

The second half of (ii) says that $A([k_1], -)$ is a nerve for each k_1, so we can regard A as a functor $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$; then the first half of (ii) says that the Segal map (§) (with $X = A$) is an equivalence of categories. Truncatability of A says that the functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ given by $[k] \longmapsto \{$isomorphism classes of objects of $A[k]\}$ is a nerve, which follows anyway from the other conditions. So a weak 2-category is a functor $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$ such that

i. $A[0]$ is a discrete category (i.e. the only morphisms are the identities)

ii. for each $k \geq 0$, the Segal functor $A[k] \longrightarrow A[1] \times_{A[0]} \cdots \times_{A[0]} A[1]$ is an equivalence of categories.

It seems that a weak 2-category is essentially just a bicategory.

First take a weak 2-category $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Cat}$, and let us construct a bicategory B. The object-set of B is $A[0]$. The two functors $s, t : A[1] \longrightarrow A[0]$ express the category $A[1]$ as a disjoint union $\coprod_{a,b \in A[0]} B(a, b)$ of categories; the 1-cells from a to b are the objects of $B(a,b)$, and the 2-cells are the morphisms. Vertical composition of 2-cells in B is composition in each $B(a,b)$. To define horizontal composition of 1- and 2-cells, first choose for each k a pseudo-inverse $A[1]\times_{A[0]}\cdots\times_{A[0]} A[1] \xrightarrow{\psi_k} A[k]$ to the Segal functor ϕ_k, and choose natural isomorphisms $\eta_k : 1 \longrightarrow \psi_k\circ\phi_k$, $\varepsilon_k : \phi_k\circ\psi_k \longrightarrow 1$. Horizontal composition is then given as

$$A[1]\times_{A[0]} A[1] \xrightarrow{\psi_2} A[2] \xrightarrow{A\delta} A[1],$$

where $\delta : [1] \longrightarrow [2]$ is the injection whose image omits $1\in[2]$. The associativity isomorphisms are built up from η_k's and ε_k's, and the pentagon commutes just as long as the equivalence $(\phi_k,\psi_k,\eta_k,\varepsilon_k)$ was chosen to be an adjunction too (which is always possible). Identities work similarly.

Conversely, take a bicategory B and construct a weak 2-category $A : (\Delta^2)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ (its '2-nerve') as follows. An element of $A([j],[k])$ is a quadruple

$$\left((a_u)_{0\leq u\leq j}, (f^z_{uv})_{\substack{0\leq u<v\leq j\\ 0\leq z\leq k}}, (\alpha^z_{uv})_{\substack{0\leq u<v\leq j\\ 1\leq z\leq k}}, (\iota^z_{uvw})_{\substack{0\leq u<v<w\leq j\\ 0\leq z\leq k}}\right)$$

where

- a_u is an object of B
- $f^z_{uv} : a_u \longrightarrow a_v$ is a 1-cell of B
- $\alpha^z_{uv} : f^{z-1}_{uv} \longrightarrow f^z_{uv}$ is a 2-cell of B
- $\iota^z_{uvw} : f^z_{vw}\circ f^z_{uv} \xrightarrow{\sim} f^z_{uw}$ is an invertible 2-cell of B

such that

- $\iota^z_{uvw}\circ(\alpha^z_{vw}*\alpha^z_{uv}) = \alpha^z_{uw}\circ\iota^{z-1}_{uvw}$ whenever $0\leq u<v<w\leq j$, $1\leq z\leq k$
- $\iota^z_{uwx}\circ(1_{f^z_{wx}}*\iota^z_{uvw})\circ(\text{associativity isomorphism}) = \iota^z_{uvx}\circ(\iota^z_{vwx}*1_{f^z_{uv}})$ whenever $0\leq u<v<w<x\leq j$, $0\leq z\leq k$.

This defines the functor A on objects of Δ^2; it is defined on maps by a combination of inserting identities and forgetting data.

To get a rough picture of A, consider the analogous construction for strict 2-categories, in which we insist that the isomorphisms ι^z_{uvw} are actually equalities. Then an element of $A([j],[k])$ is a grid of jk 2-cells, of width j and height k. (When $j=0$ this is just a single object of B, regardless of k.) The bicategorical version is a suitable weakening of this construction.

Finally, it appears that passing from a bicategory to a weak 2-category and back again gives a bicategory isomorphic (by weak functors) to the original one, and that passing from a weak 2-category to a bicategory and back again gives a weak 2-category equivalent to the original one.

Definition J

Disks

Disks. A *disk* D is a diagram of sets and functions

$$
\begin{tikzcd}
\cdots & D_m \arrow[r, "p_m"] & D_{m-1} \arrow[l, bend left, "u_m"'] \arrow[l, bend left=50, "v_m"] & \cdots \arrow[r, "p_2"] & D_1 \arrow[l, bend left, "u_2"'] \arrow[l, bend left=50, "v_2"] \arrow[r, "p_1"] & D_0 = 1 \arrow[l, bend left, "u_1"'] \arrow[l, bend left=50, "v_1"]
\end{tikzcd}
$$

equipped with a total order on the fibre $p_m^{-1}(d)$ for each $m \geq 1$ and $d \in D_{m-1}$, such that for each $m \geq 1$ and $d \in D_{m-1}$,

- $u_m(d)$ and $v_m(d)$ are respectively the least and greatest elements of $p_m^{-1}(d)$
- $u_m(d) = v_m(d) \iff d \in \mathrm{image}(u_{m-1}) \cup \mathrm{image}(v_{m-1})$.

When $m = 1$, the second condition is to be interpreted as saying that $u_1 \neq v_1$ (or equivalently, that D_1 has at least two elements).

A *map* $D \xrightarrow{\psi} D'$ *of disks* is a family of functions $(D_m \xrightarrow{\psi_m} D'_m)_{m\geq 0}$ commuting with the p's, u's and v's and preserving the order in each fibre. (The last condition means that if $d \in D_{m-1}$ and $b, c \in p_m^{-1}(d)$ with $b \leq c$, then $\psi_m(b) \leq \psi_m(c) \in p'^{-1}_m(\psi_{m-1}(d))$.) Call ψ a *surjection* if each ψ_m is a surjection.

Interiors, Volume, Dimension. Let D be a disk. For $m \geq 1$, define

$$\iota D_m = D_m \backslash (\mathrm{image}(u_m) \cup \mathrm{image}(v_m)),$$

(the *interior* of D_m), and define $\iota D_0 = D_0$. If the set $\coprod_{m\geq 1} \iota D_m$ is finite then we call D *finite* and define the *volume* $|D|$ of D to be its cardinality. In this case we may also define the *dimension* of D to be the largest $m \geq 0$ for which $\iota D_m \neq \emptyset$.

Finite Disks. Write $\mathbb{D}$ for a skeleton of the category of finite disks and maps between them. In other words, take the category of all finite disks and choose one object in each isomorphism class; the objects of $\mathbb{D}$ are all these chosen objects, and the morphisms in $\mathbb{D}$ are all disk maps between them. Thus $\mathbb{D}$ is equivalent to the category of finite disks and no two distinct objects of $\mathbb{D}$ are isomorphic.

Faces and Horns, Cofaces and Cohorns

Cofaces. Let $D \in \mathbb{D}$. A *(covolume* 1*) coface* of D is a surjection $D \xrightarrow{\phi} E$ in $\mathbb{D}$ where $|E| = |D| - 1$. We call ϕ an *inner coface* of D if $\phi_m(\iota D_m) \subseteq \iota E_m$ for all $m \geq 0$.

Cohorns. For each $D \in \mathbb{D}$ and coface $D \xrightarrow{\phi} E$ of $\mathbb{D}$, define the *cohorn*

$$\Lambda^D_\phi : \mathbb{D} \longrightarrow \mathbf{Set}$$

by

$$\Lambda^D_\phi(C) = \{\psi \in \mathbb{D}(D, C) \mid \psi \text{ factors through some coface of } D \text{ other than } \phi\}.$$

That is, a map $\psi : D \longrightarrow C$ is a member of $\Lambda^D_\phi(C)$ if and only if there is a coface $(D \xrightarrow{\phi'} E') \neq (D \xrightarrow{\phi} E)$ of D and a map $\chi : E' \longrightarrow C$ such that

$$\begin{tikzcd} D \arrow[r, "\phi'"] \arrow[dr, "\psi"'] & E' \arrow[d, "\chi"] \\ & C \end{tikzcd}$$

commutes. There is an inclusion $\Lambda^D_\phi(C) \hookrightarrow \mathbb{D}(D, C)$ for each C, and Λ^D_ϕ is thus a subfunctor of $\mathbb{D}(D, -)$. Write $i^D_\phi : \Lambda^D_\phi \hookrightarrow \mathbb{D}(D, -)$ for the inclusion.

Fillers. Let $A : \mathbb{D} \longrightarrow \mathbf{Set}$, let $D \in \mathbb{D}$, and let ϕ be a coface of D. A (D, ϕ)-*cohorn in* A is a natural transformation $h : \Lambda^D_\phi \longrightarrow A$; if ϕ is an inner coface then h is an *inner cohorn in* A.

A *filler* for a (D, ϕ)-cohorn h in A is a natural transformation $\overline{h} : \mathbb{D}(D, -) \longrightarrow A$ making the following diagram commute:

$$\begin{tikzcd} \Lambda^D_\phi \arrow[r, hook, "i^D_\phi"] \arrow[dr, "h"'] & \mathbb{D}(D, -) \arrow[d, "\overline{h}"] \\ & A. \end{tikzcd}$$

The Definition

Weak ω-Categories. A *weak ω-category* is a functor $A : \mathbb{D} \longrightarrow \mathbf{Set}$ such that there exists a filler for every inner cohorn in A.

Weak n-Categories. Let $n \geq 0$. A functor $A : \mathbb{D} \longrightarrow \mathbf{Set}$ is *n-dimensional* if, whenever $\psi : D \longrightarrow E$ is a map in $\mathbb{D}$ such that

- D has dimension n
- ψ_m is a bijection for every $m \leq n$,

then $A(\psi)$ is a bijection. A *weak n-category* is an n-dimensional weak ω-category.

Definition J for $n \leq 2$

Let $n \geq 0$. An *n-disk* is defined in the same way as a disk, except that D_m, p_m, u_m and v_m are now only defined for $m \leq n$: so an n-disk is essentially the same thing as a disk of dimension $\leq n$. Write $\mathbb{D}_n$ for a skeleton of the category of finite n-disks. An n-dimensional functor $\mathbb{D} \longrightarrow \mathbf{Set}$ is determined by its effect on disks of dimension $\leq n$, and conversely any functor $\mathbb{D}_n \longrightarrow \mathbf{Set}$ extends uniquely to become an n-dimensional functor $\mathbb{D} \longrightarrow \mathbf{Set}$. So the category of n-dimensional functors $\mathbb{D} \longrightarrow \mathbf{Set}$ is equivalent to $[\mathbb{D}_n, \mathbf{Set}]$.

Take an n-dimensional functor $A : \mathbb{D} \longrightarrow \mathbf{Set}$ and its restriction $\widetilde{A} : \mathbb{D}_n \longrightarrow \mathbf{Set}$. Then there is automatically a unique filler for every cohorn of dimension $\geq n+2$ in A (that is, cohorn $\Lambda^D_\phi \longrightarrow A$ where D has dimension $\geq n+2$). Moreover, there exists a filler for every inner cohorn of dimension $n+1$ in A if and only if there is at most one filler for every inner cohorn of dimension n in $\widetilde{A}$. (We do not prove this, but the idea of the method is in the last sentence of '$n = 2$'.) So: a weak n-category is a functor $\mathbb{D}_n \longrightarrow \mathbf{Set}$ such that every inner cohorn has a filler, unique when the cohorn is of dimension n.

$n = 0$**.** $\mathbb{D}_0$ is the terminal category $\mathbf{1}$. The unique 0-disk has no cofaces, so a weak 0-category is merely a functor $\mathbf{1} \longrightarrow \mathbf{Set}$, that is, a set.

$n = 1$**.** An *interval* is a totally ordered set with a least and a greatest element, and is called *strict* if these elements are distinct. $\mathbb{D}_1$ is (a skeleton of) the category of finite strict intervals, so we can take its objects to be the intervals $\langle k \rangle = \{0, \ldots, k+1\}$ for $k \geq 0$ and its morphisms to be the interval maps.

The cofaces of $\langle k \rangle$ are the surjections $\langle k \rangle \longrightarrow \langle k-1 \rangle$ (assuming $k \geq 1$; if $k = 0$ then there are none). They are $\phi_0, \ldots, \phi_k$, where ϕ_i identifies i and $i+1$; of these, $\phi_1, \ldots, \phi_{k-1}$ are inner. The cohorn $\Lambda^{\langle k \rangle}_{\phi_i} : \mathbb{D}_1 \longrightarrow \mathbf{Set}$ sends $\langle l \rangle$ to

$$\{\psi : \langle k \rangle \longrightarrow \langle l \rangle \mid \psi \text{ factors through } \phi_{i'} \text{ for some } i' \in \{0, \ldots, i-1, i+1, \ldots, k\}\}.$$

Now, let Δ be a skeleton of the category of nonempty finite totally ordered sets, with objects $[k] = \{0, \ldots, k\}$ ($k \geq 0$). Then $\mathbb{D}_1 \cong \Delta^{\mathrm{op}}$, with $\langle k \rangle$ corresponding to $[k]$, the cofaces $\phi_i : \langle k \rangle \longrightarrow \langle k-1 \rangle$ to the usual face maps $[k-1] \longrightarrow [k]$, and the inner cofaces to the inner faces (i.e. all but the first and last). Trivially, cohorns $\Lambda^{\langle k \rangle}_{\phi_i}$ correspond to horns in the standard sense, and fillers to fillers. Hence a weak 1-category is a functor $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ in which every inner horn has a unique filler—exactly the condition that A is the nerve of a category. So a weak 1-category is just a category.

$n = 2$**.** Again we use a duality. Given natural numbers $l_1, \ldots, l_k$, let $T_{l_1, \ldots, l_k}$ be the strict 2-category generated by objects $x_0, \ldots, x_k$, 1-cells $p^j_i : x_{i-1} \longrightarrow x_i$ ($1 \leq i \leq k$, $0 \leq j \leq l_i$), and 2-cells $\xi^j_i : p^{j-1}_i \longrightarrow p^j_i$ ($1 \leq i \leq k$, $1 \leq j \leq l_i$). (E.g. the lower half of Fig. 8(a) shows $T_{1,0}$.) Let Δ_2 be the category whose objects are sequences $(l_1, \ldots, l_k)$ with $k, l_i \geq 0$, and whose maps $(l_1, \ldots, l_k) \longrightarrow (l'_1, \ldots, l'_{k'})$ are the strict 2-functors $T_{l_1, \ldots, l_k} \longrightarrow T_{l'_1, \ldots, l'_{k'}}$. Then $\mathbb{D}_2 \cong \Delta_2^{\mathrm{op}}$. On objects, this says that a finite 2-disk is just a finite sequence of numbers, e.g. $(1, 0)$ in Fig. 8(a).

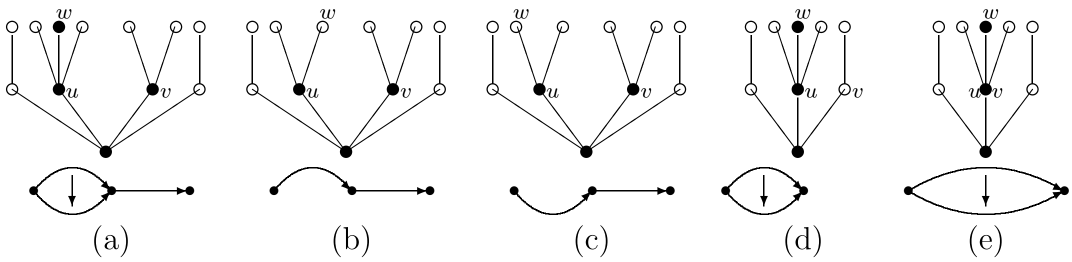


Figure 8: The duality. In the upper row, $\bullet$ denotes an interior element and $\circ$ an endpoint of a fibre, and the labels u, v, w show what the coface maps 'ϕ' do

Any bicategory B has a 'nerve' $A : \Delta_2^{\mathrm{op}} \longrightarrow \mathbf{Set}$, where $A(l_1, \ldots, l_k) = \{$weak functors $T_{l_1,\ldots,l_k} \longrightarrow B$ strictly preserving identities$\}$. We can recover B from A, so weak 2-categories are the same as bicategories just as long as the definition gives the right conditions on functors $\mathbb{D}_2 \cong \Delta_2^{\mathrm{op}} \longrightarrow \mathbf{Set}$. I do not have a full proof that this is so, hence there are gaps in what follows.

Defining *faces* of an object of Δ_2 as cofaces of the corresponding object of $\mathbb{D}_2$, and similarly *horns*, a weak 2-category is a functor $\Delta_2^{\mathrm{op}} \longrightarrow \mathbf{Set}$ in which every 1- (respectively, 2-) dimensional horn has a filler (respectively, unique filler). Faces are certain subcategories: e.g. Fig. 8(b)–(e) shows the 4 cofaces of a disk and correspondingly the 4 faces of $T_{1,0}$, of which only (e) is inner.

For the converse of the nerve construction, we take a weak 2-category A and form a bicategory B. Its graph $B_2 \rightrightarrows B_1 \rightrightarrows B_0$ is the image under A of the diagram $\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd} \leftleftarrows \bullet \longrightarrow \bullet \leftleftarrows \bullet$ in Δ_2. A diagram $a \xrightarrow{f} b \xrightarrow{g} c$ in B is a horn in A for the unique inner face of $T_{0,0} = \bullet \longrightarrow \bullet \longrightarrow \bullet$; choose a filler $K_{f,g}$ and write $g \circ f$ for its third face. This gives 1-cell composition; vertical 2-cell composition works similarly but without choice. Next, a diagram $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \arrow[r, "g"] & c \end{tikzcd}$, with $K_{f,g}$ and $K_{f',g}$, forms a horn for the unique inner face of $T_{1,0}$ (Fig. 8), so has a unique filler $K_{\alpha,g}$; write $g \circ \alpha : g \circ f \longrightarrow g \circ f'$ for face (e) of $K_{\alpha,g}$. Horizontal 2-cell composition is defined via this construction, its dual, and vertical composition. Next, $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$, together with $K_{f,g}, K_{g,h}, K_{g \circ f,h}$, gives an inner horn for $T_{0,0,0}$. There's a (unique?) filler, whose final face $L_{f,g,h}$ is itself a filler of $a \xrightarrow{f} b \xrightarrow{h \circ g} d$ with third face $h \circ (g \circ f)$. Considering $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\Downarrow 1"] & b \arrow[r, "h \circ g"] & d \end{tikzcd}$ with $K_{f,h \circ g}$ and $L_{f,g,h}$ gives an invertible 2-cell $(h \circ g) \circ f \longrightarrow h \circ (g \circ f)$.

Definition St

Simplicial Sets

The Simplicial Category. Let Δ be a skeleton of the category of nonempty finite totally ordered sets: that is, Δ has objects $[m] = \{0, \ldots, m\}$ for $m \geq 0$, and maps are order-preserving functions (with respect to the usual ordering $\leq$). A *simplicial set* is a functor $\Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$.

Maps in Δ. Let $m \geq 1$: then there are injections $\delta_0, \ldots, \delta_m : [m-1] \longrightarrow [m]$ in Δ, determined by saying that the image of δ_i is $[m] \backslash \{i\}$.

Let $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ and $m \geq 0$. An element $a \in A[m]$ is called *degenerate* if there exist a natural number $m' < m$, a surjection $\sigma : [m] \longrightarrow [m']$, and an element $a' \in A[m']$ such that $a = (A\sigma)a'$.

Horns. Given $0 \leq k \leq m$, we define the *horn* $\Lambda^k_m : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ by

$$\Lambda^k_m[m'] = \{\psi \in \Delta([m'], [m]) \mid \mathrm{image}(\psi) \not\supseteq [m] \backslash \{k\}\}.$$

That is, $\Lambda^k_m[m']$ is the set of all maps $\psi : [m'] \longrightarrow [m]$ in Δ except for the surjections and the maps with image $\{0, \ldots, k-1, k+1, \ldots, m\}$. So for each m' we have an inclusion $\Lambda^k_m[m'] \hookrightarrow \Delta([m'], [m])$, and Λ^k_m is thus a subfunctor of $\Delta(-, [m])$. Write $i^k_m : \Lambda^k_m \hookrightarrow \Delta(-, [m])$ for the inclusion.

Let A be a simplicial set. A *horn in A* is a natural transformation $h : \Lambda^k_m \longrightarrow A$, for some $0 \leq k \leq m$. A *filler* for the horn h is a natural transformation $\overline{h} : \Delta(-, [m]) \longrightarrow A$ making the following diagram commute:

$$\begin{tikzcd}
\Lambda^k_m \arrow[r, hook, "i^k_m"] \arrow[dr, "h"'] & \Delta(-, [m]) \arrow[d, "\overline{h}"] \\
& A.
\end{tikzcd}$$

Orientation

Alternating Sets. A set of natural numbers is *alternating* if its elements, when written in ascending order, alternate in parity.

Let $0 \leq k \leq m$, and write $k^{\pm} = \{k-1, k, k+1\} \cap [m]$. A subset $S \subseteq [m]$ is *k-alternating* if

- $k^{\pm} \subseteq S$
- the set $k^{\pm} \cup ([m] \backslash S)$ is alternating.

Admissible Horns. A *simplicial set with hollowness* is a simplicial set A together with a subset $H_m \subseteq A[m]$ for each $m \geq 1$, whose elements are called the *hollow* elements of $A[m]$ (and may also be thought of as 'thin' or 'universal').

Let (A, H) be a simplicial set with hollowness, and $0 \leq k \leq m$. A horn $h : \Lambda^k_m \longrightarrow A$ is *admissible* if for every $m' \geq 1$ and $\psi \in \Lambda^k_m[m']$,

$$\text{image}(\psi) \text{ is a } k\text{-alternating subset of } [m] \;\;\Rightarrow\;\; h_{[m']}(\psi) \text{ is hollow.}$$

The Definition

Weak ω-Categories. A *weak ω-category* is a simplicial set with hollowness (A, H) such that

i. for $m \geq 1$, $H_m \supseteq \{\text{degenerate elements of } A[m]\}$

ii. for $m \geq 1$ and $0 \leq k \leq m$, every admissible horn $h : \Lambda^k_m \longrightarrow A$ has a filler $\overline{h}$ satisfying $\overline{h}_{[m]}(1_{[m]}) \in H_m$ ('every admissible horn has a hollow filler')

iii. for $m \geq 2$ and $0 \leq k \leq m$, if $a \in H_m$ has the property that $(A\delta_i)a \in H_{m-1}$ for each $i \in [m]\backslash\{k\}$ then also $(A\delta_k)a \in H_{m-1}$.

Weak n-Categories. Let $n \geq 0$. A *weak n-category* is a weak ω-category (A, H) such that

i′. for $m > n$, $H_m = A[m]$

ii′. in condition (ii) above, when $m > n$ there is a *unique* filler $\overline{h}$ for h (which necessarily satisfies $\overline{h}_{[m]}(1_{[m]}) \in H_m$).

Definition St for $n \leq 2$

Let $m \geq 0$ and let S be a nonempty subset of $[m]$; then in Δ there is a unique injection ϕ into $[m]$ with image S. Given a simplicial set A and an element $a \in A[m]$, the *S-face* of a is the element $(A\phi)a$ of $A[l]$, where $l+1$ is the cardinality of S. Similarly, the *S-face* of a horn $h : \Lambda^k_m \longrightarrow A$ is $h_{[l]}(\phi) \in A[l]$ (which makes sense as long as $S \not\supseteq [m]\backslash\{k\}$).

To compare weak 1-(2-)categories with (bi)categories, we need to interpret elements $a \in A[m]$ as arrows pointing in some direction. Our convention is: if S is an m-element subset of the $(m+1)$-element set $[m]$ and the missing element is odd, then we regard the S-face of a as a source; if even, a target. See Fig. 9.

Suppose (A, H) is a simplicial set with hollowness satisfying (i), and let $h : \Lambda^k_m \longrightarrow A$ be a horn satisfying the defining condition for admissibility for just the *injective* $\psi \in \Lambda^k_m[m']$. Then, in fact, h is admissible. So h is admissible if and only if: *for every k-alternating subset S of $[m]$, the S-face of h is hollow.* Table 2 shows the k-alternating subsets of $[m]$ in the cases we need.

$n = 0$

A weak 0-category is a simplicial set A in which every horn has a unique filler—including those of shape Λ^k_1. It follows that the functor $A : \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is constant, so a weak 0-category is just a set.

$n = 1$

A horn of shape Λ^k_m is called *inner* if $0 < k < m$; a simplicial set is the nerve of a category if and only if every inner horn has a unique filler. If (A, H) is a simplicial set with hollowness satisfying (i′) for $n = 1$ then every inner horn is admissible, hence, if (ii) and (ii′) also hold, has a unique filler: so A is (the nerve of) a category. Working out the other conditions, we find that a weak 1-category is a category equipped with a set H_1 of isomorphisms containing all the identity maps and closed under composition and inverses. So given a weak 1-category we obtain a category by forgetting H; conversely, given a category we can take $H_1 = \{$all isomorphisms$\}$ (or $\{$all identities$\}$) to obtain a weak 1-category.

k	k-alternating subsets of $[m]$ of cardinality ≤ 3
0	$\{0,1\}, \{0,1,m\}$
$1, \ldots, m-1$	$\{k-1, k, k+1\}$
m	$\{m-1, m\}, \{0, m-1, m\}$

Table 2: k-alternating subsets of $[m]$ of cardinality ≤ 3, for $m \geq 1$

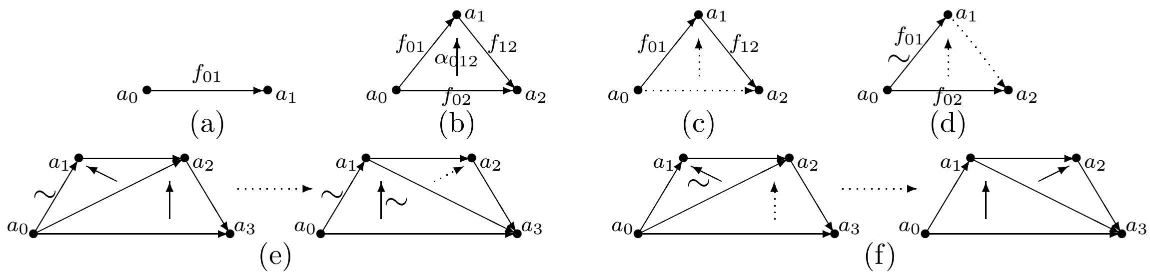
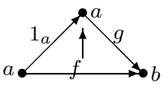


Figure 9: (a) Element of $A[1]$, with, for instance, the $\{0\}$-face labelled a_0; (b) element of $A[2]$; (c) (admissible) horn $\Lambda^1_2 \longrightarrow A$; (d) admissible horn $\Lambda^0_2 \longrightarrow A$, with $\sim$ indicating a hollow face; (e) admissible horn $\Lambda^0_3 \longrightarrow A$, with labels f_{ij}, α_{ijk} omitted; (f) as (e), but for Λ^1_3.

$n = 2$

A weak 2-category is a simplicial set A equipped with subsets $H_1 \subseteq A[1]$ and $H_2 \subseteq A[2]$, satisfying certain axioms. It appears that this is the same as a bicategory equipped with a set H_1 of 1-cells which are equivalences and a set H_2 of 2-cells which are isomorphisms, satisfying closure conditions similar to those under '$n = 1$' above.

So, let (A, H) be a weak 2-category. We construct a bicategory whose 0- and 1-cells are the elements of $A[0]$ and $A[1]$; a 2-cell $f \longrightarrow g$ is an element of $A[2]$ of the form



, where 1_a indicates a degenerate 1-cell. Composition of 1-cells is defined by making a random choice of hollow filler for each horn of shape Λ^1_2; composition of 2-cells is defined by filling in 3-dimensional horns Λ^k_3; identities are got from degeneracies. See Fig. 9. The associativity and unit isomorphisms are certain hollow cells, and the coherence axioms hold because of the uniqueness of certain fillers.

Conversely, let B be a bicategory, and construct a weak 2-category (A, H) as follows. $A[0]$ and $A[1]$ are the sets of 0- and 1-cells of B; an element of $A[2]$ as in Fig 9(b) is a 2-cell $\alpha_{012} : f_{02} \longrightarrow f_{12} \circ f_{01}$ in B. (In general, an element of $A[m]$ is a 'strictly unitary colax morphism' $[m] \longrightarrow B$, where $[m]$ is regarded as a 2-category whose only 2-cells are identities.) $H_1 \subseteq A[1]$ is the set of 1-cells which are equivalences, and H_2 is the set of 2-cells which are isomorphisms. Then all the axioms for a weak 2-category are satisfied.

Variant

We could add to conditions (i)–(iii) on (A, H) the further condition that H is maximal with respect to (i)–(iii): that is, if (A, H') also satisfies (i)–(iii) and $H'_m \supseteq H_m$ for all m then $H' = H$. (Compare the issue of maximal atlases in the definition of smooth manifold.) With this addition, a weak 1-category is essentially just a category, and a weak 2-category a bicategory. This is in contrast to the original St (as analysed above), where the flexibility in the choice of H means that the correspondence between weak 1-(2-)categories and (bi)categories is inexact.

Definition X

This definition is not intended to be rigorous, although it can be made so.

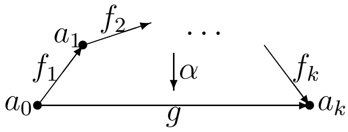
Opetopic Sets

An *opetopic set* A is a commutative diagram of sets and functions

$$\begin{tikzcd}
\cdots \arrow[r, "s"] \arrow[ddr, "t"] & A'_2 \arrow[r, "s"] \arrow[ddr, "t"] & A'_1 \arrow[r, "s"] \arrow[ddr, "t"] & A'_0 = A_0 \\
\cdots & & & \\
\cdots \arrow[r, "t"'] \arrow[uur, "s"] & A_2 \arrow[r, "t"'] \arrow[uur, "s"] & A_1 \arrow[r, "t"'] \arrow[uur, "s"] & A_0
\end{tikzcd}$$

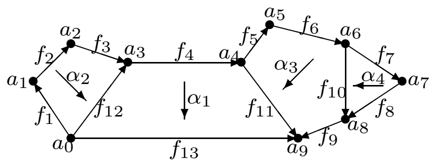
where for each $m \geq 1$, the set A'_m and the functions $s : A'_m \longrightarrow A'_{m-1}$ and $t : A'_m \longrightarrow A_{m-1}$ are defined from the sets $A_m, A'_{m-1}, A_{m-1}, \ldots, A'_1, A_1, A_0$ and the functions s, t between them in the following way.

An element $a \in A_0$ is regarded as a 0-cell, and drawn $\underset{a}{\bullet}$. An element $f \in A_1$ is regarded as a 1-cell $a \xrightarrow{f} b$, where $a = s(f)$ and $b = t(f)$. A'_1 is the set of '1-pasting diagrams' in A, that is, diagrams of 1-cells pasted together, that is, paths $a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} a_k$ $(k \geq 0)$ in A. An element $\alpha \in A_2$ has a source $s(\alpha)$ of this form and a target $t(\alpha)$ of the form $a_0 \xrightarrow{g} a_k$, and is drawn as



(‖)

Next, A'_2 is the set of '2-pasting diagrams', that is, diagrams of cells of the form (‖) pasted together, such as



$(**)$

Note that the arrows go in compatible directions: e.g. the target or 'output' edge f_{11} of α_3 is a source or 'input' edge of α_1. The source of this element of A'_2 is $a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_9} a_9 \in A'_1$, and the target is $f_{13} \in A_1$. Next, if $\gamma \in A_3$ and $s(\gamma)$ is $(**)$ then $t(\gamma)$ is of the form (‖) with $k = 9$ and $g = f_{13}$, and we picture γ as a 3-dimensional cell with a flat bottom face (labelled α) and four curved faces on top (labelled $\alpha_1, \alpha_2, \alpha_3, \alpha_4$). Carrying on, A'_3 is the set of 3-pasting diagrams, A_4 is the set of 4-cells, etc.

We need some terminology concerning cells. Let $\Phi \xrightarrow{\alpha} g$ be an m-cell: that is, let $\alpha \in A_m$ with $s(\alpha) = \Phi \in A'_{m-1}$ and $t(\alpha) = g \in A_{m-1}$.

For any p-cell e, there is a p-pasting diagram $\langle e \rangle$ consisting of e alone. If $\langle g \rangle \xrightarrow{\beta} h$ then α and β can be pasted to obtain $(\Phi \xrightarrow{\beta_*(\alpha)} h) \in A'_m$.

The *faces* of Φ are the $(m-1)$-cells which have been pasted together to form it, e.g. if α is as in ($\|$) then Φ has faces $f_1, \ldots, f_k$, and the Φ of ($**$) has faces $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. If f is a face of Φ and e is a cell *parallel* to f (i.e. $e \in A_{m-1}$ with $s(e) = s(f)$, $t(e) = t(f)$) then we obtain a new pasting diagram $\Phi(e/f) \in A'_{m-1}$ by replacing f with e in Φ. (Read $\Phi(e/f)$ as 'Φ with e replacing f'.) If also $\langle e \rangle \xrightarrow{\beta} f$ then α and β can be pasted to obtain $(\Phi(e/f) \xrightarrow{\beta^*(\alpha)} g) \in A'_m$.

Universal Cells

Let A be an opetopic set and fix $n \geq 0$. We define what it means for a cell $\Phi \xrightarrow{\varepsilon} g$ of A to be 'universal', and, when f is a face of Φ, what it means for f to be 'liminal' in the cell. The definitions depend on n, so I should really say 'n-universal' rather than just 'universal', and similarly 'n-liminal'; but I will drop the 'n' since it is regarded as fixed.

The two definitions are given inductively in an interdependent way.

Universality. For $m \geq n+1$, a cell $(\Phi \xrightarrow{\varepsilon} g) \in A_m$ is *universal* if whenever $(\Phi \xrightarrow{\varepsilon'} g') \in A_m$, then $\varepsilon' = \varepsilon$. Now let $1 \leq m \leq n$. A cell $(\Phi \xrightarrow{\varepsilon} g) \in A_m$ is *universal* if both

i. for every $\alpha : \Phi \longrightarrow h$, there exist $\overline{\alpha} : \langle g \rangle \longrightarrow h$ and a universal $U : \overline{\alpha}_*(\varepsilon) \longrightarrow \alpha$,

ii. for every $\alpha : \Phi \longrightarrow h$, $\overline{\alpha} : \langle g \rangle \longrightarrow h$ and universal $U : \overline{\alpha}_*(\varepsilon) \longrightarrow \alpha$, $\overline{\alpha}$ is liminal in U.

Liminality. Let $m \geq 1$, let $(\Phi \xrightarrow{\varepsilon} g) \in A_m$, and let f be a face of Φ. Then f is *liminal in ε* if $m \geq n+2$ or

i. for every cell e parallel to f and $\beta : \Phi(e/f) \longrightarrow g$, there exist $\overline{\beta} : \langle e \rangle \longrightarrow f$ and a universal cell $U : \overline{\beta}^*(\varepsilon) \longrightarrow \beta$, and

ii. for every e parallel to f, $\beta : \Phi(e/f) \longrightarrow g$, $\overline{\beta} : \langle e \rangle \longrightarrow f$ and universal $U : \overline{\beta}^*(\varepsilon) \longrightarrow \beta$, $\overline{\beta}$ is liminal in U.

The Definition

Let $n \geq 0$. A *weak n-category* is an opetopic set A satisfying

existence of universal fillers: for every $m \geq 0$ and $\Phi \in A'_m$, there exists a universal cell of the form $\Phi \xrightarrow{\varepsilon} g$

closure of universals under composition: if $m \geq 2$, $(\Phi \xrightarrow{\varepsilon} g) \in A_m$, each face of Φ is universal, and ε is universal, then g is universal.

Definition X for $n \leq 2$

First consider high-dimensional cells in an n-category A. For every $m \geq n+1$ and $\Phi \in A'_{m-1}$, there is a unique $\varepsilon \in A_m$ whose source is Φ: in other words, $s : A_m \longrightarrow A'_{m-1}$ is a bijection. This means that the entire opetopic set is determined by the part of dimension $\leq n$ and the map $t : A_{n+1} \longrightarrow A_n$. Identifying A_{n+1} with A'_n, this map t assigns to each n-pasting diagram Φ the target g of the unique $(n+1)$-cell with source Φ, and we regard g as the composite of Φ. (In general, we may regard the target of a universal cell as a composite of its source; note that all cells of dimension $> n$ are universal.) This composition of n-cells is strictly associative and unital. A weak n-category therefore consists of a commutative diagram

$$\begin{tikzcd}
A'_n \arrow[r, "s"] \arrow[d, "\circ"'] \arrow[dr, "t"] & \cdots \arrow[r, "s"] \arrow[dr, "t"] & A'_1 \arrow[r, "s"] \arrow[dr, "t"] & A'_0 = A_0 \\
A_n \arrow[r, "t"'] \arrow[ur, "s"] & \cdots \arrow[r, "t"'] \arrow[ur, "s"] & A_1 \arrow[r, "t"'] \arrow[ur, "s"] & A_0
\end{tikzcd} \qquad (\dagger\dagger)$$

of sets and functions such that n-dimensional composition $\circ$ obeys strict laws and the defining conditions on existence and closure of universals hold in lower dimensions.

$n = 0$. A weak 0-category is a set A_0 together with a function $\circ : A_0 \longrightarrow A_0$ obeying strict laws—which say that $\circ$ is the identity. So a weak 0-category is just a set.

$n = 1$. When $n = 1$, diagram $(\dagger\dagger)$ is a directed graph $A_1 \overset{s}{\underset{t}{\rightrightarrows}} A_0$ together with a map $\circ : A'_1 \longrightarrow A_1$ compatible with the source and target maps: in other words, assigning to each string of edges

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \cdots \xrightarrow{f_k} a_k$$

in A a new edge $a_0 \xrightarrow{(f_k \circ \cdots \circ f_1)} a_k$. The axioms on $\circ$ say that

$$((f_k^{r_k} \circ \cdots \circ f_k^1) \circ \cdots \circ (f_1^{r_1} \circ \cdots \circ f_1^1)) = (f_k^{r_k} \circ \cdots \circ f_1^1), \qquad (f) = f$$

—in other words, that A forms a category. A weak 1-category is therefore a category satisfying the extra conditions that every object is the domain of some universal morphism and that the composite of universal morphisms is universal. I claim that the universal morphisms are the isomorphisms, so that these conditions hold automatically and a weak 1-category is just a category.

So: a morphism $\varepsilon : a \longrightarrow b$ is universal if (i) every morphism $\alpha : a \longrightarrow c$ factors as $\alpha = \overline{\alpha} \circ \varepsilon$, and (ii) such an $\overline{\alpha}$ is always liminal in the unique 2-cell $\begin{tikzcd}[column sep=small] & b \arrow[dr, "\overline{\alpha}"] & \\ a \arrow[ur, "\varepsilon"] \arrow[rr, "\alpha"'] & {} \arrow[u, phantom, "\Downarrow U"] & c \end{tikzcd}$.

Liminality of $\overline{\alpha}$ in U means, in turn, that if $\widetilde{\alpha} : b \longrightarrow c$ satisfies $\widetilde{\alpha} \circ \varepsilon = \alpha$ then $\widetilde{\alpha} = \overline{\alpha}$. (This is part (i) of the definition of liminality; part (ii) holds trivially.) So $\varepsilon : a \longrightarrow b$ is universal if and only if every morphism out of a factors uniquely through ε, which holds if and only if ε is an isomorphism.

$n = 2$**.** A weak 2-category is essentially the same thing as a bicategory. More precisely, the category of bicategories and weak functors is equivalent to the category whose objects are weak 2-categories and whose morphisms are those maps of opetopic sets which send universal cells to universal cells. The equivalence works as follows.

Given a bicategory B, define a weak 2-category A by taking A_0 and A_1 to be the sets of 0- and 1-cells in B, and a 2-cell $(\|)$ in A to be a 2-cell $(f_k \circ \cdots \circ f_1) \longrightarrow g$ in B. Here $(f_k \circ \cdots \circ f_1)$ is defined inductively as $f_k \circ (f_{k-1} \circ \cdots \circ f_1)$ if $k \geq 1$, or as 1 if $k = 0$; any other iterated composite would do just as well. Composition $\circ : A'_2 \longrightarrow A_2$ is pasting of 2-cells in B. Then a 1-cell (respectively, 2-cell) in A turns out to be universal if and only if the corresponding 1-cell (2-cell) in B is an equivalence (isomorphism), and it follows that A is a weak 2-category.

Conversely, take a weak 2-category A and construct a bicategory B as follows. The 0- and 1-cells of B are just those of A, and a 2-cell of B is an element of A_2 of the form $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & b \end{tikzcd}$ (i.e. $\alpha : \langle f \rangle \longrightarrow g$). For each diagram $a \xrightarrow{f} b \xrightarrow{g} c$ of 1-cells, choose at random a universal filler $\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g \circ f"'] \arrow[rr, phantom, shift left=3, "\Downarrow\varepsilon_{f,g}"] & & c \end{tikzcd}$, where by definition $g \circ f = t(\varepsilon_{f,g})$; this defines composition of 1-cells. Vertical composition of 2-cells comes from $\circ : A'_2 \longrightarrow A_2$. To define the horizontal composite of $\begin{tikzcd} a \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & a' \arrow[r, bend left, "f'"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & a'' \end{tikzcd}$, consider pasting $\varepsilon_{g,g'}$ to α and α', and then use the universality of $\varepsilon_{f,f'}$. Next observe that given two universal fillers $\Phi \xrightarrow{\varepsilon} g$, $\Phi \xrightarrow{\varepsilon'} g'$ for a 1-pasting diagram $\Phi = (a_0 \xrightarrow{f_1} \cdots \xrightarrow{f_k} a_k)$, there is a unique 2-cell $\langle g \rangle \xrightarrow{\delta} g'$ such that the composite $\circ(\delta_*(\varepsilon))$ is ε'. Applying this observation to a certain pair of universal fillers for $(\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet)$ gives the associativity isomorphism, and the word 'unique' in the observation gives the pentagon axiom. Identities work similarly, where this time we choose a random universal filler for each degenerate 1-pasting diagram $\underset{a}{\bullet}$.

Further Reading

This section contains the references and historical notes missing in the main text. It is not meant to be a survey of the literature. Where I have omitted relevant references it is almost certainly a result of my own ignorance, and I hope that the authors will forgive me.

First are some references to introductory and general material, and a very brief account of the history of higher-dimensional category theory. Then there are references for each of the sections in turn: 'Background', followed by the ten definitions. Finally there are references to some proposed definitions of n-category which I didn't include, and a very few references to areas of mathematics related to n-categories.

Citations such as `math.CT/9810058` and `alg-geom/9708010` refer to the electronic mathematics archive at `http://arXiv.org`. Readers unfamiliar with the archive may find it easiest to go straight to the address of the form `http://arXiv.org/abs/math.CT/9810058` .

Introductory Texts

Introductions to n-categories come slanted towards various different audiences. One for theoretical computer scientists and logicians is

[1] A. J. Power, Why tricategories?, *Information and Computation* **120** (1995), no. 2, 251–262; also LFCS report ECS-LFCS-94-289, April 1994, available via `http://www.lfcs.informatics.ed.ac.uk`

and another with a logical slant, but this time with foundational concerns, is

[2] M. Makkai, Towards a categorical foundation of mathematics, in *Logic Colloquium '95 (Haifa)*, Lecture Notes in Logic **11**, Springer, 1998, pp. 153–190.

Moving to introductions for those more interested in topology, geometry and physics, one which starts at a very basic level (the definition of category) is

[3] John C. Baez, A tale of n-categories, available via `http://math.ucr.edu/home/baez/week73.html`, 1996–97.

With similar themes but at a more advanced level, there are

[4] John C. Baez, An introduction to n-categories, in *Category theory and computer science (Santa Margherita Ligure, 1997)*, Lecture Notes in Computer Science **1290**, Springer, 1997, pp. 1–33; also e-print `q-alg/9705009`, 1997

(especially sections 1–3) and

[5] Tom Leinster, Topology and higher-dimensional category theory: the rough idea, e-print `math.CT/0106240`, 2001, 15 pages.

The ambitious might, if they can find a copy, like to look at the highly discursive 600-page letter of Grothendieck to Quillen,

[6] A. Grothendieck, Pursuing stacks, manuscript, 1983,

in which (amongst many other things) the idea is put that tame topology is really the study of weak ω-groupoids. A more accessible discussion of what higher-dimensional algebra might 'do', especially in the context of topology, is

[7] Ronald Brown, Higher dimensional group theory, available via `http://www.bangor.ac.uk/~mas010`.

General Comments and History

The easiest way to begin a history of n-categories is as follows.

0-categories—sets or classes—came into the mathematical consciousness around the end of the 19th century. 1-categories—categories—arrived in the middle of the 20th century. Strict 2-categories and, implicitly, strict n-categories, made their presence felt around the late 1950s and early 1960s, with the work of Ehresmann [19]. Weak 2-categories were first introduced by Bénabou [23] in 1967, under the name of bicategories, and thereafter the question was in the air: 'what might a weak n-category be?' The first precise proposal for a definition was given by Street [66] in 1987. This was followed by three more proposals around 1995: Baez and Dolan's [73], Batanin's [31], and Tamsamani's [46]. A constant stream of further proposed definitions has issued forth since then, and will doubtless continue for a while. Work on low values of n was also going on at the same time: an axiomatic definition of weak 3-category was proposed in

[8] R. Gordon, A. J. Power, Ross Street, *Coherence for Tricategories*, Memoirs of the American Mathematical Society **117**, no. 558, 1995,

and a proposal in similar vein for $n = 4$ was made in

[9] Todd Trimble, The definition of tetracategory, manuscript, 1995.

Crucially, it was shown in [8] that not every tricategory is equivalent to a strict 3-category (in contrast to the situation for $n = 2$), from which it follows that the theory of weak n-categories is genuinely different from that of strict ones.

But this is far too simplistic. A realistic history must take account of categorical structures other than n-categories *per se*: for instance, the various kinds of monoidal category (plain, symmetric, braided, tortile/ribbon, ...), and of monoidal 2-categories and monoidal bicategories. The direct importance of these is that a monoidal category is a bicategory with only one 0-cell, and similarly a braided monoidal category is a tricategory with only one 0-cell and one 1-cell. The basic reference for braided monoidal categories is

[10] André Joyal, Ross Street, Braided tensor categories, *Advances in Mathematics* **102** (1993), no. 1, 20–78,

and they can also be found in the new edition of Mac Lane's book [18].

Moreover, around the same time as the theory of n-categories was starting to develop, another theory was emerging with which it was later to converge: the theory of multicategories and operads. Multicategories first appeared in

[11] Joachim Lambek, Deductive systems and categories II: standard constructions and closed categories, in *Category Theory, Homology Theory and their Applications, I (Battelle Institute Conference, Seattle, 1968, Vol. One)*, ed. Hilton, Lecture Notes in Mathematics **86**, Springer, 1969, pp. 76–122.

(The definition is on page 103.) A multicategory is like a category, but each arrow has as its source or input a *sequence* of objects (and, as usual, as its target or output a single object). An operad is basically just a multicategory with only one object. For this reason, multicategories are sometimes called 'coloured operads', and the objects are then named after colours (black, white, etc.). The development of operads is generally attributed to Boardman, Vogt and May:

[12] J. M. Boardman, R. M. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces*, Lecture Notes in Mathematics **347**, Springer, 1973,

[13] J. P. May, *The Geometry of Iterated Loop Spaces*, Lectures Notes in Mathematics **271**, Springer, 1972,

although I am told that essentially the same idea was the subject of

[14] Michel Lazard, Lois de groupes et analyseurs, *Annales Scientifiques de l'École Normale Supérieure (3)* **72** (1955), 299–400

(where operads go by the name of 'analyseurs').

It seems to have taken a long time before it was realized that operads and multicategories were so closely related; I do not know of any pre-1995 text which mentions both Lambek and Boardman-Vogt or May in its bibliography. This can perhaps be explained by the different fields in which they were being studied: multicategories were introduced in the context of logic and found application in linguistics, whereas operads were used for the theory of loop spaces. Moreover, if one uses the terms in their original senses then it is not strictly true that an operad is the same thing as a one-object multicategory; operads are also equipped with a symmetric structure, and the 'hom-sets' (sets of operations) are topological spaces rather than just sets. (It is also very natural to consider multicategories with both these pieces of extra structure, but historically this is beside the point.)

Many short introductions to operads have appeared as section 1 of papers by topologists and quantum algebraists. The interested reader may also find useful the following texts dedicated to the subject:

[15] J. P. May, Definitions: operads, algebras and modules, in *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)*, Contemporary Mathematics **202**, AMS, 1997, pp. 1–7; also available via `http://www.math.uchicago.edu/~may`,

[16] J. P. May, Operads, algebras and modules, in *Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995)*, Contemporary Mathematics **202**, AMS, 1997, pp. 15–31; also available via `http://www.math.uchicago.edu/~may`,

[17] Martin Markl, Steve Shnider, Jim Stasheff, *Operads in Algebra, Topology and Physics*, book in preparation.

A glimpse of the role of operads and multicategories in higher-dimensional category theory can be seen in the definitions of weak n-category above. Often the 'operads' and 'multicategories' used are not the original kinds, but more general kinds adapted for the different shapes and dimensions which occur in the subject; for references, see 'Definitions B and L' below.

Background

Category Theory. Almost any book on the subject will provide the necessary background. The second edition of the classic book by Mac Lane,

[18] Saunders Mac Lane, *Categories for the Working Mathematician*, second edition, Graduate Texts in Mathematics **5**, Springer, 1998,

is especially useful, containing as it does two new chapters on such topics as bicategories and nerves of categories.

Strict n-Categories. I do not know of any good text introducing strict n-categories. Ehresmann's original book

[19] Charles Ehresmann, *Catégories et Structures*, Dunod, Paris, 1965

could be consulted, but is generally regarded as a very demanding read. Probably more useful is

[20] G. M. Kelly, Ross Street, Review of the elements of 2-categories, in *Category Seminar (Sydney, 1972/1973)*, Lecture Notes in Mathematics **420**, Springer, 1974, pp. 75–103,

which only covers strict 2-categories (traditionally just called '2-categories') but should give a good idea of strict n-categories for general n. This could usefully be supplemented by

[21] Samuel Eilenberg, G. Max Kelly, Closed categories, in *Proceedings of Conference on Categorical Algebra (La Jolla, California, 1965)*, Springer, 1966, pp. 421–562

(see e.g. page 552), which also covers enrichment. For another reference on enriched categories, see chapter 6 of

[22] Francis Borceux, *Handbook of Categorical Algebra 2: Categories and Structures*, Encyclopedia of Mathematics and its Applications **51**, Cambridge University Press, 1994.

Bicategories. Bicategories were first explained by Bénabou:

[23] Jean Bénabou, Introduction to bicategories, in *Reports of the Midwest Category Seminar*, ed. Bénabou et al, Lecture Notes in Mathematics **47**, Springer, 1967, pp. 1–77,

and further important work on them is in

[24] John W. Gray, *Formal Category Theory: Adjointness for 2-Categories*, Lecture Notes in Mathematics **391**, Springer, 1974.

(At least) two texts contain summaries of the 'basic theory' of bicategories: that is, the definitions of bicategory and of weak functor (homomorphism), transformation and modification between bicategories, together with the result that any bicategory is in a suitable sense equivalent to a strict 2-category. These are section 9 of

[25] Ross Street, Categorical structures, in *Handbook of Algebra* ***1***, ed. M. Hazewinkel, North-Holland, 1996, pp. 529–577

and the whole of

[26] Tom Leinster, Basic bicategories, e-print `math.CT/9810017`, 1998, 11 pages.

Definition Tr

The definition was given in a talk,

[27] Todd Trimble, What are 'fundamental n-groupoids'?, seminar at DPMMS, Cambridge, 24 August 1999,

and has not been written up previously. Trimble used the term 'flabby n-category' rather than 'weak n-category'.

As the title of the talk suggests, the idea was not to develop the weakest possible notion of n-category, but to provide (in his words) 'a sensible niche for discussing fundamental n-groupoids'. In a world where all the definitions have been settled, it may be that fundamental n-groupoids of topological spaces have certain special features (other than the invertibility of their cells) not shared by all weak n-categories. Thus it may be that the word 'weak' is less appropriate for definition Tr than the other definitions.

Evidence that this is the case comes from two directions. Firstly, the maps $\gamma_{a_0,\ldots,a_k}$ describing composition of hom-$(n-1)$-categories in an n-category are *strict* $(n-1)$-functors. This corresponds to having strict interchange laws. It therefore seems likely that a precise analysis of $n = 3$ would show that every weak 3-category gives rise to a tricategory (in a similar manner to $n = 2$) but that not every tricategory is triequivalent to one arising from a weak 3-category.

Secondly, forget Tr for the moment and consider in naive terms what the fundamental ω-groupoid A of a space X might look like. In the notation of the Background section on strict n-categories, we would probably imagine taking

$$\begin{aligned} A_0 &= \text{underlying set of } X, \\ A_1 &= \{\text{maps } f : [0,1] \longrightarrow X\}, \\ A_2 &= \{\text{maps } \alpha : [0,1]^2 \longrightarrow X \text{ such that } \alpha(0,-) \text{ and } \alpha(1,-) \text{ are constant}\}, \end{aligned}$$

with the obvious source and target functions. There are then composition functions

$$\circ_0 : A_1 \times_{A_0} A_1 \longrightarrow A_1, \quad \circ_0 : A_2 \times_{A_0} A_2 \longrightarrow A_2, \quad \circ_1 : A_2 \times_{A_1} A_2 \longrightarrow A_2$$

defined by the standard method of travelling at double speed whenever necessary, satisfying the usual source and target axioms. They are not strictly associative, nor are there strict identities; however, they *do* obey the strict interchange law,

$$(\beta' \circ_1 \alpha') \circ_0 (\beta \circ_1 \alpha) = (\beta' \circ_0 \beta) \circ_1 (\alpha' \circ_0 \alpha)$$

for any $\alpha, \alpha', \beta, \beta' \in A_2$ such that these composites make sense. More generally, one would expect the fundamental n-groupoid of a space to obey strict interchange in all dimensions. This provides the kind of 'special feature' of fundamental n-groupoids referred to above.

Other ideas on fundamental n-groupoids, n-categories, and how they tie together can be found in Grothendieck's letter [6]. More practical material on fundamental 1- and 2-groupoids is in

[28] K. H. Kamps, T. Porter, *Abstract Homotopy and Simple Homotopy Theory*, World Scientific Publishing Co., 1997.

For more on operads, see the references under 'General Comments and History' above. The name of the operad E was chosen not only to stand for 'endpoint-preserving', but also because it comes after D for 'disk'—the idea being that E is something like the one-dimensional little disks operad D (crucial in the theory of loop spaces). A touch more precisely, E seems to play the same kind of role for paths as D does for closed loops.

Prospects for comparing Tr with B and L look bright: it seems very likely that weak n-categories according to Tr are just the algebras for a certain globular n-operad (in the sense of B or L), namely, a truncated version of the globular operad $E^{\mathrm{op}}(D)$ of endomorphisms of the coglobular space of disks. $E^{\mathrm{op}}(D)$ is discussed in section 9 of Batanin's paper [31], and can be regarded as a higher-dimensional version of the operad E used in Tr.

More on how the theory of bicategories comes from the operad of trees can be found in Appendix A (and Chapter 1) of my thesis, [34], and in my [90].

Definition P

Definition P of weak ω-category is in

[29] Jacques Penon, Approche polygraphique des ∞-categories non strictes, *Cahiers de Topologie et Géométrie Différentielle* **40** (1999), no. 1, 31–80.

His chosen term for weak ω-category is 'prolixe', whose closest English translation is perhaps 'waffle'. As far as I can see there is no actual definition of weak n-category or n-dimensional prolixe in the paper, although he clearly has one in mind on page 48:

> Les prolixes de dimension ≤ 2 s'identifient exactement aux bicatégories [...] la preuve de ce résultat sera montré dans un article ultérieur

('waffles of dimension ≤ 2 correspond exactly to bicategories [...] the proof of this result will be given in a forthcoming paper').

Other translations: my category $[\mathbb{R}, \mathbf{Set}]$ of reflexive globular sets is his category ∞-$\mathbb{G}rr$ of reflexive ∞-graphs; my s and t are his s and b; my strict ω-categories are his ∞-categories; my category $\mathcal{Q}$ is called by him $\mathbb{E}tC$, the category of étirements catégoriques ('categorical stretchings'); my contractions γ are written $[-,-]$ (with the arguments reversed: $\gamma_m(f_0, f_1) = [f_1, f_0]$); and my adjunction $F \dashv U$ is called $\hat{\mathcal{E}} \dashv \hat{V}$.

The word 'magma' is borrowed from Bourbaki, who used it to mean a set equipped with a binary operation. It is a slightly inaccurate borrowing, in that ω-magmas are equipped with (nominal) identities as well as binary compositions; put another way, it would have been more suitable if Bourbaki had used the word to mean a set equipped with a binary operation and a distinguished basepoint.

It seems plausible that Penon's construction can be generalized to provide weak versions of structures other than ω-categories (e.g. up-to-homotopy topological monoids). Batanin has done something precise along the lines of generalizing Penon's definition and comparing it to his own:

[30] M. A. Batanin, On the Penon method of weakening algebraic structures, to appear in *Journal of Pure and Applied Algebra*; also available via `http://www.math.mq.edu.au/~mbatanin/papers.html`, 2001, 25 pages.

Definitions B and L

Batanin gave his definition, together with an examination of $n = 2$, in

[31] M. A. Batanin, Monoidal globular categories as a natural environment for the theory of weak n-categories, *Advances in Mathematics* **136** (1998), no. 1, 39–103; also available via `http://www.math.mq.edu.au/~mbatanin/papers.html`.

Another account of it is

[32] Ross Street, The role of Michael Batanin's monoidal globular categories, in *Higher category theory (Evanston, IL, 1997)*, Contemporary Mathematics **230**, AMS, 1998, pp. 99–116; also available via `http://www.math.mq.edu.au/~street`.

The definition of weak n-category which appears as 8.7 in [31] is (I believe) what is here called definition B1. More precisely, let $\mathcal{O}$ be the category whose objects are (globular) operads on which *there exist* a contraction and a system of compositions, and whose maps are just maps of operads. What Batanin does is to construct an operad K which is weakly initial in $\mathcal{O}$. 'Weakly initial' means that there is at least one map from K to any other object of $\mathcal{O}$, so this does not determine K up to isomorphism; one needs some further information. But in Remark 2 just before Definition 8.6, Batanin suggests that, once given the appropriate extra structure, K is initial in the category **OCS** of operads *equipped with* a contraction and a system of compositions, which does determine K. This is the approach taken in B1.

A weak n-category according to B2 is (I believe) almost exactly what Batanin calls a 'weak n-categorical object in *Span*' in his Definition 8.6. The only difference is my extra condition that the operad C is (in his terminology) *normalized*: $C(0) \cong 1$. Now $C(0)$ is the set of operations in the operad which take a 0-cell of an algebra and turn it into another 0-cell, so normality means that there are no such operations except, trivially, the identity. This seems reasonable in the context of n-categories, since one expects to have operations for composing m-cells only when $m \geq 1$. The lack of normality in Batanin's version ought to be harmless, since the contraction means that all the operations on 0-cells are in some sense equivalent to the identity operation, but it does make the analysis of $n \leq 2$ a good deal messier. (Note that the operad K is normalized, so any weak ω-/n-category in the sense of B1 is also one in the sense of B2; the same goes for L1 and L2.)

My modification L1 of Batanin's definition first appeared in

[33] Tom Leinster, Structures in higher-dimensional category theory, e-print `math.CT/0109021`, 1998, 81 pages,

but a more comprehensive and, I think, comprehensible account is in

[34] Tom Leinster, Operads in higher-dimensional category theory, Ph.D. thesis, University of Cambridge, 2000; also e-print `math.CT/0011106`, 2000, viii + 127 pages.

(There, $C \otimes C'$ is called $C \circ C'$, $C \cdot -$ is T_C, and $P_C(\pi)$ is $P_\pi(C)$.) [34] also contains a precise analysis of L1 for $n \leq 2$, including proofs of (a) the equivalence of the two different categories of weak n-categories (for *finite* n) mentioned at the start of the analysis of $n \leq 2$ above, and (b) the equivalence of the category of unbiased bicategories and weak functors with that of (classical) bicategories and weak functors. Definition L2 has not appeared before, and has just been added here for symmetry.

The globular operads in B and L arise as a special case of two different, more general theories of operads. Firstly, they are a special case of *generalized operads*, a family of higher-dimensional categorical structures which are perhaps as interesting and applicable as n-categories themselves. Briefly, the theory goes as follows. Given a monad T on a category $\mathcal{E}$, satisfying some natural conditions, one can define a category of T-*multicategories*. For example, when T is the identity monad on the category $\mathcal{E}$ of sets, a T-multicategory is just a category, and when T is the free-monoid monad on **Set**, a T-multicategory is just an ordinary multicategory (see 'General Comments and History' above). A *T-operad* is a one-object T-multicategory, so in the first of these examples it is a monoid and in the second it is an operad in the original sense (but without symmetric or topological structure). Now take T to be the free strict ω-category monad on the category $\mathcal{E}$ of globular sets, as in B and L: a T-operad is then exactly a globular operad. Algebras for T-multicategories can be defined in the general context, and again this notion specializes to the one in B and L.

Generalized (operads and) multicategories were first put forward in

[35] Albert Burroni, T-catégories (catégories dans un triple), *Cahiers de Topologie et Géométrie Différentielle* **12** (1971), 215–321

and were twice rediscovered independently:

[36] Claudio Hermida, Representable multicategories, *Advances in Mathematics* **151** (2000), no. 2, 164–225; also available via `http://www.cs.math.ist.utl.pt/s84.www/cs/claudio.html`,

[37] Tom Leinster, General operads and multicategories, e-print `math.CT/9810053`, 1997, 35 pages.

As far as I know, the notion of algebra for a T-multicategory only appears in the third of these. ([37] also appears, more or less, as Chapter I of [33] and Chapter 2 of [34].)

Secondly, globular operads arise as the principal special case of Batanin's theory of *operads in a monoidal globular category* (see [31] and [32]). A *strict* monoidal globular category is a strict ω-category in **Cat**; a monoidal globular category is a slightly relaxed version of this. There is a particular monoidal globular category *Span*, and an operad in *Span* is exactly what we call a globular operad. This general theory is capable of handling coglobular structures and coalgebras for operads, as, for instance, in the discussion above of Definition Tr, where the coglobular space (functor $\mathbb{G}^{\mathrm{op}} \longrightarrow \mathbf{Top}$) of disks was mentioned.

The difference between definitions L1 and B1 can be summarized by saying that L1 takes B1, dispenses with the notions of system of compositions and B-style contraction, and merges them into a single more powerful notion of contraction. A few more words on the difference are in section 4.5 of [34]. The operad L canonically carries a B-style contraction and a system of compositions, so there is a canonical map $K \longrightarrow L$ of operads, and this induces a functor in the opposite direction on the categories of algebras. Hence every weak ω-/n-category in the sense of L1 gives rise canonically to one in the sense of B1.

Definition L′

This is the first time in print for definition L′. Once we have the language of generalized multicategories (described in the previous section) and the theory of strict ω-categories, it is very quickly stated. My papers [34] and [33] (and to some extent [37]) cover generalized multicategories and globular operads, but not specifically globular multicategories. The 1-dimensional case, 1-globular multicategories, are the '**fc**-multicategories' described briefly in

[38] Tom Leinster, **fc**-multicategories, e-print `math.CT/9903004`, 1999, 8 pages,

at a little more length in

[39] Tom Leinster, Generalized enrichment of categories, to appear in *Journal of Pure and Applied Algebra*,

and in detail in

[40] Tom Leinster, Generalized enrichment for categories and multicategories, e-print `math.CT/9901139`, 1999, 79 pages.

Logicians might like to view L′ through proof-theoretic spectacles, substituting the word 'proof' for 'reason'. They (and others) might also be interested to read

[41] M. Makkai, Avoiding the axiom of choice in general category theory, *Journal of Pure and Applied Algebra* **108** (1996), no. 2, 109–173; also available via `http://www.math.mcgill.ca/makkai`

in which Makkai defines anafunctors and anabicategories and discusses the philosophical viewpoint which led him to them. In the same vein, see also Makkai's [2] and the remarks on 'a composite' *vs.* 'the composite' towards the end of the Introduction to the present paper.

A weak ω-/n-category in the sense of L2 (and so L1 too) gives rise to one in the sense of L′. For just as 'algebras' for a category C (functors $C \longrightarrow \mathbf{Set}$) correspond one-to-one with discrete opfibrations over C, via the so-called Grothendieck construction, so the same is true in a suitable sense for globular multicategories. This generalization is explained in section 4.2 of [37], section I.3 of [33], and section 3.4 of [34] (any one of which will do, but they are listed in increasing order of clarity). What this means is that an algebra for a globular operad gives rise to a globular multicategory (the domain of the opfibration), and if the operad admits a contraction in the sense of L then the resulting multicategory is a weak ω-category in the sense of L′.

Midway between L′ and J is another possible definition of weak ω-category, which for various reasons I have not included here. It was presented in a talk,

[42] Tom Leinster, Not quite Joyal's definition of n-category (a.k.a. 'algebraic nerves'), seminar at DPMMS, Cambridge, 22 February 2001,

notes from which, in the (2 + 2)-page format of this paper, are available on request. The idea behind it can be traced back to Segal's formalization of the notion of up-to-homotopy topological commutative monoid, *special* Γ*-spaces* (and their non-commutative counterparts, *special* Δ*-spaces*). The analogy is that just as Segal took the theory of honest topological commutative monoids and did something to it to obtain an up-to-homotopy version, so we take the theory of strict n-categories and do something similar to obtain a weak version. Segal's original paper is

[43] Graeme Segal, Categories and cohomology theories, *Topology* **13** (1974), 293–312.

A different generalization of his idea defines up-to-homotopy algebras for any (classical) operad. This is done at length in

[44] Tom Leinster, Homotopy algebras for operads, e-print `math.QA/0002180`, 2000, 101 pages,

or a much briefer explanation of the idea is

[45] Tom Leinster, Up-to-homotopy monoids, e-print `math.QA/9912084`, 1999, 8 pages.

Definitions Si and Ta

Tamsamani's original definition appeared in

[46] Zouhair Tamsamani, Sur des notions de n-catégorie et n-groupoïde non strictes via des ensembles multi-simpliciaux, *K-Theory* **16** (1999), no. 1, 51–99; also e-print `alg-geom/9512006`, 1995.

What I have called truncatability of a functor $(\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is called 'r-troncabilité' by Tamsamani. It is not immediately obvious that the two conditions are equivalent, but a thoroughly mundane induction shows that they are. Other translations: my 1^p is his I_p, my s and t are his s and b, my $Q^{(m)}$ is his T^m, my $\pi^{(m)}$ is his t^m, my internal equivalence of cells x_1, x_2 (as in the text of Ta) is his $(r-p)$-équivalence intérieure, and my external equivalence of functors $(\Delta^r)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is his r-équivalence extérieure. His term for a weak n-category is 'n-nerf' or 'n-catégorie large'. ('Large' has nothing to do with large and small categories: it means broad or generous, and can perhaps be translated here as 'lax'; compare the English word 'largesse'.)

Tamsamani also offers a proof that his weak 2-categories are essentially the same as bicategories, but I believe that it is slightly flawed, in that he has omitted a necessary axiom for the 2-nerve of a bicategory (the last bulleted item in 'Definition Ta for $n \leq 2$', starting 'ι^z_{uwx}'). Without this, the constructed functor $(\Delta^2)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ will not necessarily be a weak 2-category in the sense of Ta. (In this context, my a's are his x's, my α's are his λ's, and my ι's are his ε's.)

Working with him in Toulouse, Simpson produced a simplified version of Tamsamani's definition, which first appeared in

[47] Carlos Simpson, A closed model structure for n-categories, internal $\underline{Hom}$, n-stacks and generalized Seifert-Van Kampen, e-print `alg-geom/9704006`, 1997, 69 pages.

He used the term 'easy n-category' for his weak n-categories, and 'easy equivalence' for what is called a contractible map in Si.

The simplification lies in the treatment of equivalences. Weak 1-categories according to either Ta or Si are just categories, but whereas a Ta-style equivalence of weak 1-categories is a functor which is full, faithful and essentially surjective on objects (that is, an ordinary equivalence of categories), an easy equivalence is a functor which is full, faithful and *genuinely* surjective on objects. The latter property of functors is expressible at a significantly more primitive conceptual level than the former, since it is purely in terms of the underlying directed graphs and has nothing to do with the actual category structure. For this reason, Si is much shorter than Ta. (But to develop the theory of weak n-categories we still need Tamsamani's more general notion of equivalence; this is, for instance, the missing piece of vocabulary referred to at the very end of 'Definition Si for $n \leq 2$'.)

As one would expect from this description, any easy equivalence (contractible map) is an equivalence in the sense of Ta. So as long as it is true that any weak n-category $(\Delta^n)^{\mathrm{op}} \longrightarrow \mathbf{Set}$ in the sense of Si is truncatable (which I cannot claim to have proved), it follows that any weak n-category in the sense of Si is also one in the sense of Ta.

Following on from his definition, Tamsamani investigated homotopy n-groupoids of spaces:

[48] Zouhair Tamsamani, Equivalence de la théorie homotopique des n-groupoïdes et celle des espaces topologiques n-tronqués, e-print `alg-geom/9607010`, 1996, 24 pages.

Numerous papers by Simpson, using a mixture of his definition and Tamsamani's and largely in the language of Quillen model categories, push the theory of weak n-categories further along:

[49] Carlos Simpson, Limits in n-categories, e-print `alg-geom/9708010`, 1997, 92 pages,

[50] Carlos Simpson, Homotopy types of strict 3-groupoids, e-print `math.CT/9810059`, 1998, 29 pages,

[51] Carlos Simpson, On the Breen-Baez-Dolan stabilization hypothesis for Tamsamani's weak n-categories, e-print `math.CT/9810058`, 1998, 36 pages,

[52] Carlos Simpson, Calculating maps between n-categories, e-print `math.CT/0009107`, 2000, 13 pages,

[53] Carlos Simpson, Some properties of the theory of n-categories, e-print `math.CT/0110273`, 2001, 11 pages.

Toen has also applied Tamsamani's definition, as in

[54] B. Toen, Dualité de Tannaka supérieure I: structures monoidales, Max-Planck-Institut preprint MPI-2000-57, available via `http://www.mpim-bonn.mpg.de`, 2000, 71 pages

and

[55] B. Toen, Notes on higher categorical structures in topological quantum field theory, available via `http://guests.mpim-bonn.mpg.de/rosellen/etqft00.html`, 2000, 14 pages,

and the theory finds its way into some very grown-up mathematics in

[56] Carlos Simpson, Algebraic aspects of higher nonabelian Hodge theory, e-print `math.AG/9902067`, 1999, 186 pages.

The connection between categories and their nerves is covered briefly in one of the new chapters of Mac Lane's book [18]; two early sources are

[57] Graeme Segal, Classifying spaces and spectral sequences, *Institut des Hautes Études Scientifiques Publications Mathématiques* **34** (1968), 105–112,

[58] P. Gabriel, M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer, 1967.

Presumably the 'Segal maps' are so named because of the prominent role they play in Segal's paper [43] on loop spaces and homotopy-algebraic structures.

The basic method by which a Simpson or Tamsamani weak 2-category gives rise to a bicategory seems implicit in Segal's [43], is made explicit in section 3 of my [45], and is done in even more detail in section 3.3 of my [44]. (Actually, these last two papers only describe the method for monoidal categories rather than bicategories in general, but there is no substantial difference.) There is also a discussion of the converse process in section 4.4 of [44], and the idea behind this is once more implicit in the work of Segal.

Definition J

Joyal gave his definition in an unpublished note,

[59] A. Joyal, Disks, duality and Θ-categories, preprint, 1997, 6 pages.

There he defined a notion of weak ω-category, which he called 'θ-category'. He also wrote a few informal words about structures called θ^n-categories, and how one could derive from them a definition of weak n-category; but I was unable to interpret his meaning, and consequently definition J of weak n-category might not be what he envisaged.

The term 'disk' comes from the case where, in the notation of J, D_m is the closed m-dimensional unit disk (= ball) in $\mathbb{R}^m$, p_m is projection onto the first $(m-1)$ coordinates, and the order on the fibres is given by the usual order on the real numbers. The second bulleted condition in the paragraph headed 'Disks' holds at a point d of D_m if and only if d is on the boundary of D_m. From another point of view, this condition can be regarded as a form of exactness.

The handling of faces in J is not necessarily equivalent to that in [59]; again, I had trouble understanding the intended meaning and made my own path. In fact, Joyal works the duality discussed under $n \leq 2$ into the definition itself, putting $\Theta = \mathbb{D}^{\mathrm{op}}$ and calling

Θ the category of 'Batanin cells' (for reasons suggested by Figures 3 and 8). So he does not speak of cofaces and cohorns in $\mathbb{D}$, but rather of faces and horns in Θ.

Of the analyses of $n \leq 2$ for the ten definitions, that for J is probably the furthest from complete. It appears to be the case that in a weak n-category $A : \mathbb{D}_n \longrightarrow \mathbf{Set}$, any cohorn $\Lambda_\phi^D \longrightarrow A$ where D has volume $> n$ has a unique filler. (We know that this is true when the dimension of D is n.) If this conjecture holds then we can complete the proof (sketched in '$n \leq 2$') that any weak 2-category gives rise to a bicategory; for instance, applied to $T_{0,0,0}$ it tells us that there is a canonical choice of associativity isomorphism, and applied to $T_{0,0,0,0}$ it gives us the pentagon axiom. However, I have not been able to find a proof (or counterexample).

Introductory material on simplicial sets and horns can be found in, for instance, Kamps and Porter's book [28].

The duality between the skeletal category Δ of nonempty finite totally ordered sets and the skeletal category $\mathbb{I}$ of finite strict intervals has been well-known for a long time. Nevertheless, I have been unable to trace the original reference, or even a text where it is explained directly—except for Joyal's preprint [59], which the reader may have trouble obtaining. A slightly indirect explanation is in section VIII.7 of

[60] Saunders Mac Lane, Ieke Moerdijk, *Sheaves in Geometry and Logic*, Springer, 1992,

as part of a discussion of the related fact that the category of simplicial sets is a classifying topos for the theory of intervals. The duality is also a restricted version of the duality between finite distributive lattices and finite posets. Put briefly, it comes from mapping into the 2-element ordered set: if k is a natural number then the set $\Delta([k],[1])$ naturally has the structure of an interval (isomorphic to $\langle k\rangle$) and the set $\mathbb{I}(\langle k\rangle, \langle 0\rangle)$ naturally has the structure of a totally ordered set (isomorphic to $[k]$). This provides functors $\Delta(-,[1]) : \Delta^{\mathrm{op}} \longrightarrow \mathbb{I}$ and $\mathbb{I}(-,\langle 0\rangle) : \mathbb{I}^{\mathrm{op}} \longrightarrow \Delta$ which are mutually inverse, so $\Delta^{\mathrm{op}} \cong \mathbb{I}$.

Some of the fundamental facts on the higher duality were established in

[61] Clemens Berger, A cellular nerve for higher categories, to appear in *Advances in Mathematics*; Université de Nice—Sophia Antipolis Prépublication 602 (2000), 50 pages; also available via `http://math.unice.fr/~cberger`

(where a closed model category structure on $[\mathbb{D}, \mathbf{Set}]$ is also discussed), and more on this and on the relationship between definitions J and B is in

[62] Michael Batanin, Ross Street, The universal property of the multitude of trees, *Journal of Pure and Applied Algebra* **154** (2000), no. 1-3, 3–13; also available via `http://www.math.mq.edu.au/~mbatanin/papers.html`.

The higher duality has also been the subject of detailed investigation by Makkai and Zawadowski:

[63] Mihaly Makkai, Marek Zawadowski, Duality for simple ω-categories and disks, *Theory and Applications of Categories* **8** (2001), 114–243,

[64] Marek Zawadowski, Duality between disks and simple categories, talk at 70th Peripatetic Seminar on Sheaves and Logic, Cambridge, 1999,

[65] Marek Zawadowski, A duality theorem on disks and simple ω-categories, with applications to weak higher-dimensional categories, talk at CT2000, Como, Italy, 2000.

(Slides and notes from Zawadowski's talks have the virtue of containing some pictures absent in the published version.)

As mentioned above, there is another way to define weak n-category which has strong connections to both J and L′: [42].

Definition St

Street proposed his definition of weak ω-category in a very tentative manner, in the final sentence of

[66] Ross Street, The algebra of oriented simplexes, *Journal of Pure and Applied Algebra* **49** (1987), no. 3, 283–335.

He did not explicitly formulate a notion of weak n-category for finite n; this small addition is mine, as is the Variant at the end of the section on $n \leq 2$.

There is one minor but material difference, and a small number of cosmetic differences, between Street's definition and St. The material difference is that in a weak ω-category as proposed in [66], the only hollow 1-cells are the degenerate ones. One terminological difference is that a pair (A, H) is called a 'simplicial set with hollowness' in [66] only when (i) and the aforementioned condition on 1-cells hold: so the term has a narrower meaning there than here. (Street informs me that he and Verity have used the term 'stratified simplicial set' for the same purpose, either with the two conditions or without.) Another is that he uses 'ω-category' in a wider sense: his potentially have infinite-dimensional cells, and the category of strict ω-categories in the sense of the present paper is denoted ω-Cath. Further translations: I say that a subset $S \subseteq [m]$ is k-alternating where Street says that the set $[m]\backslash S$ is 'k-divided', and he calls a map $[l] \longrightarrow [m]$ 'k-monic' if it is monic and its image is a k-alternating subset of $[m]$.

The focus of [66] is actually on *strict* n- and ω-categories. To this end he considers the condition on 1-cells mentioned above, and conditions (i)–(iii) of St with 'unique' inserted before the word 'filler' in (ii). Having spent much of the paper constructing the nerve of a strict ω-category (this being a simplicial set with hollowness), he conjectures that a given simplicial set with hollowness is the nerve of some strict ω-category if and only if all the conditions just mentioned hold. (The conjecture was a result of joint work with John Roberts.) The necessity of these conditions was proved soon afterwards in

[67] Ross Street, Fillers for nerves, in *Categorical algebra and its applications (Louvain-La-Neuve, 1987)*, Lecture Notes in Mathematics **1348**, Springer, 1988, pp. 337–341.

A proof of their sufficiency was supplied by Dominic Verity; this has not appeared in print, but was presented at various seminars in Berkeley, Bangor and Sydney around 1993.

It is entirely possible that most of the detailed work for $n \leq 2$ has already been done by Duskin. A short account of his work on this was presented as

[68] John W. Duskin, A simplicial-matrix approach to higher dimensional category theory, talk at CT2000, Como, Italy, 2000,

and a full-length version is in preparation:

[69] John W. Duskin, A simplicial-matrix approach to higher dimensional category theory I: nerves of bicategories, preprint, 2001, 82 pages,

[70] John W. Duskin, A simplicial-matrix approach to higher dimensional category theory II: bicategory morphisms and simplicial maps, preprint, 2001, 53 pages.

What Duskin does is to construct the nerve of any bicategory (this being a simplicial set) and to give exact conditions saying which simplicial sets arise in this way. He moreover shows how to recover a bicategory from its nerve. Duskin does not deal explicitly with Street's conditions or his notion of hollowness (although he does mention them); indeed, the results just mentioned suggest that for $n = 2$, the hollow structure on the nerve of a bicategory is superfluous. The same point is addressed in a recent note,

[71] Ross Street, Weak omega-categories, to appear in *Contemporary Mathematics*; also available via `http://www.math.mq.edu.au/~street`, 2001, 7 pages,

where Street modifies definition St so that a weak ω-category is now a simplicial set with properties rather than structure.

The word 'thin' has been used for the same purpose as 'hollow', hence the name *T-complex*, as discussed in III.2.26 and onwards in Kamps and Porter's book [28] (which also contains basic information on simplicial sets and horn-filling). The original definition of T-complex was given by M. K. Dakin in his 1975 Ph.D. thesis, published as

[72] M. K. Dakin, Kan complexes and multiple groupoid structures, *Mathematical sketches (Esquisses Mathématiques)* **32** (1983), xi+92 pages, University of Amiens.

T-complexes are simplicial sets with hollowness satisfying conditions (i)–(iii), but with 'admissible' dropped and 'filler' changed to 'unique filler' in (ii). The dropping of 'admissible' means that the delicate orientation considerations of Street's paper are ignored and any direction is as good as any other—everything can be run backwards. Thus, T-complexes are meant to be like strict ω-groupoids rather than strict ω-categories.

Definition X

The story of X is complicated. Essentially it is a combination of the ideas of Baez, Dolan, Hermida, Makkai and Power. Baez and Dolan proposed a definition of weak n-category, drawing on that of Street, in

[73] John C. Baez, James Dolan, Higher-dimensional algebra III: n-categories and the algebra of opetopes, *Advances in Mathematics* **135** (1998), no. 2, 145–206; also e-print q-alg/9702014, 1997.

An informal account is in section 4 of Baez's [4], and another account, which also discusses relations between homotopy theory and n-category theory, is in

[74] John C. Baez, James Dolan, Categorification, in *Higher category theory (Evanston, IL, 1997)*, Contemporary Mathematics **230**, AMS, 1998, pp. 1–36; also e-print math.QA/9802029, 1998.

In turn, Hermida, Makkai and Power drew on the work of Baez and Dolan, producing a modified version of Baez and Dolan's opetopic sets, which they called multitopic sets. (My use in X of the former term rather than the latter should not be interpreted as significant.) Their original preprint still seems to be available somewhere on the web:

[75] Claudio Hermida, Michael Makkai, John Power, On weak higher dimensional categories, available via http://fcs.math.sci.hokudai.ac.jp/doc/info/ncat.html, 1997, 104 pages

and is currently enjoying a journal serialization:

[76] Claudio Hermida, Michael Makkai, John Power, On weak higher dimensional categories I: Part 1, *Journal of Pure and Applied Algebra* **154** (2000), no. 1-3, 221–246,

[77] Claudio Hermida, Michael Makkai, John Power, On weak higher-dimensional categories I—2, *Journal of Pure and Applied Algebra* **157** (2001), no. 2-3, 247–277,

[78] Claudio Hermida, Michael Makkai, John Power, On weak higher-dimensional categories I—3, *Journal of Pure and Applied Algebra* **166** (2002), no. 1-2, 83–104.

A related paper with a somewhat different slant and in a much more elementary style is

[79] Claudio Hermida, Michael Makkai, John Power, Higher-dimensional multigraphs, in *Thirteenth Annual IEEE Symposium on Logic in Computer Science (Indianapolis, IN, 1998)*, IEEE Computer Society, Los Alamitos, CA, 1998, pp. 199–206.

Hermida, Makkai and Power's original work did not go as far as an alternative definition of weak n-category, although see the description below of [84].

I learned something near to definition X from

[80] Martin Hyland, Definition of lax n-category, seminar at DPMMS, Cambridge, based on a conversation with John Power, 18 June 1997.

Whether this is closer to the approach of Baez and Dolan or of Hermida, Makkai and Power is hard to say. The Baez-Dolan definition falls into two parts: the definition of opetopic set, then the definition of universality. Certainly the universality in X is Baez and Dolan's, but the sketch of the definition of opetopic set is very elementary, in contrast to the highly involved definitions of opetopic/multitopic set given by both these groups of authors.

Opetopic sets are, it is claimed in [73], just presheaves on a certain category, the category of *opetopes*. (The situation can be compared with that of simplicial sets, which are just presheaves on the category Δ.) Multitopic sets are shown in [75] to be presheaves on a category of multitopes. A third notion of opetope, going (perhaps reprehensibly) by the same name, is given briefly in section 4.1 of my [37], and is laid out in more detail in Chapter IV of my [33]. Roughly speaking, it is shown that all three notions are equivalent in

[81] Eugenia Cheng, The relationship between the opetopic and multitopic approaches to weak n-categories, available via `http://www.dpmms.cam.ac.uk/~elgc2`, 2000, 36 pages

(which compares Baez-Dolan's notion with Hermida-Makkai-Power's) and

[82] Eugenia Cheng, Equivalence between approaches to the theory of opetopes, available via `http://www.dpmms.cam.ac.uk/~elgc2`, 2000, 36 pages

(which adds in my own). More accurately, Cheng begins [81] by modifying Baez and Dolan's notion of operad; the effect of this is that the symmetries present in Baez and Dolan's account are now handled much more cleanly and naturally, especially when it comes to the crucial process of 'slicing'. So this means that the Baez-Dolan opetopes are not necessarily the same as the three equivalent kinds of opetope involved in Cheng's result, and it remains to be seen whether they fit in.

Let us now turn from opetopic sets to universality. The notion of liminality does not appear in Baez and Dolan's paper, and is in some sense a substitute for their notion of 'balanced punctured niche'. I made this change in order to shorten the inductive definitions; it is just a rephrasing and has no effect on the definition of universal cell. The price to be paid is that in isolation, liminality is probably a less meaningful concept than that of balanced punctured niche.

More on the formulation of universality can be found in

[83] Eugenia Cheng, A notion of universality in the opetopic theory of n-categories, available via `http://www.dpmms.cam.ac.uk/~elgc2`, 2001, 12 pages.

Makkai appears to have hit upon a notion of 'ω-dimensional universal properties', and thereby developed the definition of multitopic set into a new definition of weak ω-category:

[84] M. Makkai, The multitopic ω-category of all multitopic ω-categories, available via `http://mystic.biomed.mcgill.ca/M_Makkai`, 1999, 67 pages.

I do not, unfortunately, know enough about this to include an account here. Nor have I included the definition of

[85] Tom Leinster, Batanin meets Baez and Dolan: yet more ways to define weak n-category, seminar at DPMMS, Cambridge, 6 February 2001,

which uses opetopic shapes but an algebraic approach like that of L. This definition can be repeated for various other shapes, such as globular (giving exactly L) and computads (which are like opetopes but with many outputs as well as many inputs), and perhaps simplicial and even cubical.

Finally, the analysis of $n \leq 2$ has been done in a very precise way, in

[86] Eugenia Cheng, Equivalence between the opetopic and classical approaches to bicategories, available via `http://www.dpmms.cam.ac.uk/~elgc2`, 2000, 68 pages.

This uses the notion of opetopic/multitopic set given by Cheng's modification of Baez and Dolan, or by my opetopes, or by Hermida, Makkai and Power (for by her equivalence result, all three notions are the same), together with the Baez-Dolan notion of universality. I am fairly confident that this gives the same definition of weak 2-category as is described in X above.

Other Definitions of n-Category

I have already mentioned several proposed definitions of weak n-category which are not presented here. My own [42] and [85] are missing, as is Street's recent modification [71] of St. The opetopic definitions—those related to definition X—are under-represented, as I have not given *any* such definition in precise terms; in particular, there is no exact presentation of Baez-Dolan's definition [73], of Cheng's modification of Baez-Dolan's definition ([81], [82]), or of Makkai's definition [84].

In the final stages of writing this I received a preprint,

[87] J. P. May, Operadic categories, A_∞-categories and n-categories, available via `http://www.math.uchicago.edu/~may`, 2001, 10 pages,

containing another definition of weak n-category. I have not had time to assimilate this; nor have I yet digested the approach to weak n-categories in

[88] Hiroyuki Miyoshi, Toru Tsujishita, Weak ω-categories as ω-hypergraphs, e-print `math.CT/0003137`, 2000, 26 pages,

[89] Akira Higuchi, Hiroyuki Miyoshi, Toru Tsujishita, Higher dimensional hypercategories, e-print `math.CT/9907150`, 1999, 25 pages.

Comparing Definitions

It seems that not a great deal of rigorous work has been done on comparing the proposed definitions, although there are plenty of informal ideas floating about. The papers that I know of are listed above under the appropriate definitions.

The '$n \leq 2$' sections show that there are many reasonable notions even of weak 2-category. This is not diminished by restricting to one-object weak 2-categories, that is, monoidal categories. So by examining and trying to compare various possible notions of monoidal category, one can hope to get some idea of what things will be like for weak n-categories in general. A proof of the equivalence of various 'algebraic' or 'definite' notions of monoidal category is in

[90] Tom Leinster, What's a monoidal category?, poster at CT2000, Como, Italy, 2000,

and a similar but less general result is in Chapter 1 of my [34] (actually stated for bicategories). Hermida compares the indefinite with the definite in his paper [36] on representable multicategories, and a different definite/indefinite comparison is in section 3 of my [45] or section 3.3 of my [44]. (I use the terms 'definite' and 'algebraic' in the sense of the Introduction.)

No-one who has seen the definition of tricategory given by Gordon, Power and Street in [8] will take lightly the prospect of analysing the case $n = 3$. However, it is worth pointing out an aspect of this definition less well-known than its complexity: that it is not quite algebraic. In precise terms, what I mean by this is that the category whose objects are tricategories and whose maps are strict maps of tricategories is not monadic over the category of 3-globular sets. (3-globular sets are globular sets as in the 'Strict n-Categories' section of 'Background', but with m only running from 0 up to 3. So the graph structure of a tricategory is a 3-globular set.) For whereas most of the definition of tricategory consists of some data subject to some equations, a small part does not: in items (TD5) and (TD6), it is stipulated that certain transformations of bicategories are equivalences. This is not an algebraic axiom; to make it into one, we would have to add in as data a pseudo-inverse for each of these equivalences, together with two invertible modifications witnessing the fact that it is a pseudo-inverse, and then we would want to add more coherence axioms (saying, amongst other things, that this data forms an *adjoint* equivalence). So there is little chance that the category of weak 3-categories and strict maps according to P, B1, L1, or any other algebraic definition is equivalent to the category of tricategories and strict maps, in contrast to the situation for $n = 2$.

Related Areas

I will be extremely brief here; as stated above, this is not meant to be a survey of the literature. However, there are two areas I feel it would be inappropriate to omit. Most of the references that follow are meant to function as 'meta-references', and are chosen for their comprehensive bibliographies.

The first area is the Australian school of 2-dimensional algebra, a representative of which is

[91] R. Blackwell, G. M. Kelly, A. J. Power, Two-dimensional monad theory, *Journal of Pure and Applied Algebra* **59** (1989), no. 1, 1–41.

The issues arising there merge into questions of coherence, one starting point for which is the paper 'On braidings, syllepses and symmetries' by Sjoerd Crans:

[92] Sjoed Crans, On braidings, syllapses and symmetries, *Cahiers de Topologie et Géométrie Différentielle* **41** (2000), no. 1, 2–74; also available via `http://math.unice.fr/∼crans`,

[93] S. Crans, Erratum: 'On braidings, syllapses and symmetries', *Cahiers de Topologie et Géométrie Différentielle* **41** (2000), no. 2, 156.

More references for work in this area are to be found in Street's [25].

The second area is from algebraic topology: where higher-dimensional category theorists want to take strict algebraic structures and weaken them, stable homotopy theorists like to take strict topological-algebraic structures and do them up to homotopy (in a more sensitive way than one might at first imagine). The two have much in common. Various systematic ways of doing the latter have been proposed, and some of these are listed on the last page of text in my [44]. Missing from that list is the method of

[94] Mikhail A. Batanin, Homotopy coherent category theory and A_∞-structures in monoidal categories, *Journal of Pure and Applied Algebra* **123** (1998), no. 1-3, 67–103; also available via `http://www.math.mq.edu.au/∼mbatanin/papers.html`.

Another connection with homotopy theory and loop spaces is in

[95] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, R. Vogt, Iterated monoidal categories, e-print `math.AT/9808082`, 1998, 55 pages.

Further references for these two areas and more can be found in the 'Introductory Texts' listed above.

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