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τ-DISTANCE IN A GENERAL TOPOLOGICAL SPACE (X, τ) WITH APPLICATION TO FIXED POINT THEORY

M. AAMRI AND D. EL MOUTAWAKIL

ABSTRACT. The main purpose of this paper is to define the notion of a τ-distance function in a general topological space (X, τ). As application, we get a generalization of the well known Banach's fixed point theorem.

A.M.S. (MOS) Subject Classification Codes. 54A05, 47H10, 54H25, 54E70

Key Words and Phrases. Hausdorff topological spaces, Topological spaces of type F, symmetrizable topological spaces, Fixed points of contractive maps

1. INTRODUCTION

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended In many different directions ([2],[3],[4],[6],[9]). On the other hand, it has been observed ([3],[5]) that the distance function used in metric theorems proofs need not satisfy the triangular inequality nor $d(x, x) = 0$ for all x. Motivated by this fact, we define the concept of a τ-distance function in a general topological space (X, τ) and we prove that symmetrizable topological spaces ([5]) and F-type topological spaces introduced in 1996 by Fang [4] (recall that metric spaces, Hausdorff topological vector spaces and Menger probabilistic metric space are all a special case of F-type topological spaces) possess such functions. finally, we give a fixed point theorem for contractive maps in a general topological space (X, τ) with a τ-distance which gives the Banach's fixed point theorem in a new setting and also gives a generalization of jachymski's fixed point result [3] established in a semi-metric case.

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2. τ-distance

Let (X,τ) be a topological space and $p : X \times X \longrightarrow IR^+$ be a function. For any $\epsilon > 0$ and any $x \in X$, let $B_p(x,\epsilon) = \{y \in X : p(x,y) < \epsilon\}$.

Definition 2.1. The function p is said to be a τ-distance if for each $x \in X$ and any neighborhood V of x, there exists $\epsilon > 0$ with $B_p(x,\epsilon) \subset V$.

Example 2.1. Let $X = \{0;1;3\}$ and $\tau = \{\emptyset; X; \{0;1\}\}$. Consider the function $p : X \times X \longrightarrow IR^+$ defined by

$$p(x,y) = \begin{cases} y, & x \neq 1 \\ \frac{1}{2}y, & x = 1. \end{cases}$$

We have, $p(1;3) = \frac{3}{2} \neq p(3;1) = 1$. Thus, p us not symmetric. Moreover, we have

$$p(0;3) = 3 > p(0;1) + p(1;3) = \frac{5}{2}$$

which implies that p fails the triangular inequality. However, the function p is a τ-distance.

Example 2.2. Let $X = IR^+$ and $\tau = \{X, \emptyset\}$. It is well known that the space (X,τ) is not metrizable. Consider the function p defined on $X \times X$ by $p(x,y) = x$ for all $x, y \in X$. It is easy to see that the function p is a τ-distance.

Example 2.3. In [5], Hicks established several important common fixed point theorems for general contractive selfmappings of a symmetrizable (resp. semi-metrizable) topological spaces. Recall that a symmetric on a set X is a nonnegative real valued function d defined on $X \times X$ by

(1) $d(x,y) = 0$ if and only if $x = y$,
(2) $d(x,y) = d(y,x)$

A symmetric function d on a set X is a semi-metric if for each $x \in X$ and each $\epsilon > 0$, $B_d(x,\epsilon) = \{y \in X : d(x,y) \leq \epsilon\}$ is a neighborhood of x in the topology τ_d defined as follows

$$\tau_d = \{U \subseteq X / \ \forall x \in U,\ B_d(x,\epsilon) \subset U,\ for some\ \epsilon > 0\}$$

A topological space X is said to be symmetrizable (semi-metrizable) if its topology is induced by a symmetric (semi-metric) on X. Moreover, Hicks [5] proved that very general probabilistic structures admit a compatible symmetric or semi-metric. For further details on semi-metric spaces (resp. probabilistic metric spaces), see, for example, [8] (resp. [7]). Each symmetric function d on a nonempty set X is a τ_d-distance on X where the topology τ_d is defined as follows: $U \in \tau_d$ if $\forall x \in U$, $B_d(x,\epsilon) \subset U$, for some $\epsilon > 0$.

Example 2.4. Let $X = [0,+\infty[$ and $d(x,y) = |x-y|$ the usual metric. Consider the function $p : X \times X \longrightarrow IR^+$ defined by

$$p(x,y) = e^{|x-y|}, \ \ \forall x, y \in X$$

It is easy to see that the function p is a τ-distance on X where τ is the usual topology since $\forall x \in X$, $B_p(x,\epsilon) \subset B_d(x,\epsilon)$, $\epsilon > 0$. Moreover, (X,p) is not a symmetric space since for all $x \in X$, $p(x,x) = 1$.

Example 2.5 - Topological spaces of type (EL).

Definition 2.2. A topological space (X, τ) is said to be of type (EL) if for each $x \in X$, there exists a neighborhood base $F_x = \{U_x(\lambda, t)/\lambda \in D, t > 0\}$, where $D = (D, \prec)$ denotes a directed set, such that $X = \cup_{t>0} U_x(\lambda, t)$, $\forall \lambda \in D$, $\forall x \in X$.

remark 2.1. In [4], Fang introduced the concept of F-type topological space and gave a characterization of the kind of spaces. The usual metric spaces, Hausdorff topological vector spaces, and Menger probabilistic metric spaces are all the special cases of F-type topological Spaces. Furtheremore, Fang established a fixed point theorem in F-type topological spaces which extends Caristi's theorem [2]. We recall the concept of this space as given in [4]

Definition [4]. A topological space (X, θ) is said to be F-type topological space if it is Hausdorff and for each $x \in X$, there exists a neighborhood base $F_x = \{U_x(\lambda, t)/\lambda \in D, t > 0\}$, where $D = (D, \prec)$ denotes a directed set, such that

1. If $y \in U_x(\lambda, t)$, then $x \in U_y(\lambda, t)$,
2. $U_x(\lambda, t) \subset U_x(\mu, s)$ for $\mu \prec \lambda, t \leq s$,
3. $\forall \lambda \in D$, $\exists \mu \in D$ such that $\lambda \prec \mu$ and $U_x(\mu, t_1) \cap U_y(\mu, t_2) \neq \emptyset$, implies $y \in U_x(\lambda, t_1 + t_2)$,
4. $X = \cup_{t>0} U_x(\lambda, t)$, $\forall \lambda \in D$, $\forall x \in X$.

It is clear that a topological space of type F is a Hausdorff topological space of type (EL). Therefore The usual metric spaces, Hausdorff topological vector spaces, and Menger probabilistic metric spaces are special cases of a Hausdorff topological Space of type (EL).

proposition 2.1. *Let (X, τ) be a topological space of type (EL). Then, for each $\lambda \in D$, there exists a τ-distance function p_λ.*

Proof. Let $x \in X$ and $\lambda \in D$. Consider the set $E_x = \{U_x(\lambda, t) | \lambda \in D, t > 0\}$ of neighborhoods of x such that $X = \cup_{t>0} U_x(\lambda, t)$. Then for each $y \in X$, there exists $t^* > 0$ such that $y \in U_x(\lambda, t^*)$. Therefore, for each $\lambda \in D$, we can define a function $p_\lambda : X \times X \longrightarrow IR^+$ as follows

$$p_\lambda(x, y) = inf\{t > 0, y \in U_x(\lambda, t)\}.$$

set $B_\lambda(x, t) = \{y \in X | p_\lambda(x, y) < t\}$. let $x \in X$ and V_x a neighborhood of x. Then the exists $(\lambda, t) \in D \times IR^+$, such that $U_x(\lambda, t) \subset V_x$. We show that $B_\lambda(x, t) \subset U_x(\lambda, t)$. Indeed, consider $y \in B_\lambda(x, t)$ and suppose that $y \notin U_x(\lambda, t)$. It follows that $p_\lambda(x, y) \geq t$, which implies that $y \notin B_\lambda(x, t)$. A contradiction. Thus $B_\lambda(x, t) \subset V_x$. Therefore p_λ is a τ-distance function.

remark 2.2. As a consequence of proposition 3.1, we claim that each topological space of type (EL) has a familly of τ-distances $M = \{p_\lambda | \lambda \in D\}$.

3. Some properties of τ-distances

lemma 3.1. *Let (X, τ) be a topological space with a τ-distance p.*

1. *Let (x_n) be arbitrary sequence in X and (α_n) be a sequence in IR^+ converging to 0 such that $p(x, x_n) \leq \alpha_n$ for all $n \in IN$. Then (x_n) converges to x with respect to the topology τ.*

(2) *If* (X,τ) *is a Hausdorff topological space, then* (2.1) $p(x,y)=0$ *implies* $x=y$. (2.2) *Given* (x_n) *in* X,

$$\lim_{n\to\infty} p(x,x_n)=0 \text{ and } \lim_{n\to\infty} p(y,x_n)=0$$

imply $x=y$.

Proof.

(1) Let V be a neighborhood of x. Since $\lim p(x,x_n)=0$, there exists $N\in IN$ such that $\forall n\geq N,\ x_n\in V$. Therefore $\lim x_n=x$ with respect to τ.
(2) (2.1) Since $p(x,y)=0$, then $p(x,y)<\epsilon$ for all $\epsilon>0$. Let V be a neighborhood of x. Then there exists $\epsilon>0$ such that $B_p(x,\epsilon)\subset V$, which implies that $y\in V$. Since V is arbitrary, we conclude $y=x$. (2.2) From (2.1), $\lim p(x,x_n)=0$ and $\lim p(y,x_n)=0$ imply $\lim x_n=x$ and $\lim x_n=y$ with respect to the topology τ which is Hausdorff. Thus $x=y$.

Let (X,τ) be a topological space with a τ-distance p. A sequence in X is p-Cauchy if it satisfies the usual metric condition with respect to p. There are several concepts of completeness in this setting.

Definition 3.1. Let (X,τ) be a topological space with a τ-distance p.

(1) X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim p(x,x_n)=0$.
(2) X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim x_n=x$ with respect to τ.
(3) X is said to be p-bounded if $sup\{p(x,y)/x,y\in X\}<\infty$.

remark 3.1. Let (X,τ) be a topological space with a τ-distance p and let (x_n) be a p-Cauchy sequence. Suppose that X is S-complete, then there exists $x\in X$ such that $\lim p(x_n,x)=0$. Lemma 4.1(b) then gives $\lim x_n=x$ with respect to the topology τ. Therefore S-completeness implies p-Cauchy completeness.

4. Fixed point theorem

In what follows, we involve a function $\psi: IR^+\longrightarrow IR^+$ which satisfies the following conditions

(1) ψ is nondecreasing on IR^+ ,
(2) $\lim \psi^n(t)=0,\ \forall t\in]0,+\infty[$.

It is easy to see that under the above properties, ψ satisfies also the following condition

$$\psi(t)<t,\ foreach\ t\in]0,+\infty[$$

Theorem 4.1. *Let* (X,τ) *be a Hausdorff topological space with a* τ*-distance* p. *Suppose that* X *is p-bounded and S-complete. Let* f *be a selfmapping of* X *such that*

$$p(fx,fy)\leq\psi(p(x,y)),\quad \forall x,y\in X$$

Then f *has a unique fixed point.*

Proof. Let $x_0\in X$. Consider the sequence (x_n) defined by

$$\begin{cases} x_0 \in X, \\ x_{n+1} = fx_n \end{cases}$$

We have

$$\begin{aligned}p(x_n, x_{n+m}) =& p(fx_{n-1}, fx_{n+m-1})\\ \leq& \psi(p(x_{n-1}, x_{n+m-1})) = \psi(p(fx_{n-2}, fx_{n+m-2}))\\ \leq& \psi^2(p(x_{n-2}, x_{n+m-2}))\\ &....\\ \leq& \psi^n(p(x_0, x_m)) \leq \psi^n(M)\end{aligned}$$

where $M = sup\{p(x,y)/x, y \in X\}$. Since $\lim \psi^n(M) = 0$, we deduce that the sequence (x_n) is a p-cauchy sequence. X is S-complete, then $\lim p(u, x_n) = 0$, for some $u \in X$, and therefore $\lim p(u, x_{n+1}) = 0$ and $\lim p(fu, fx_n) = 0$. Now, we have $\lim p(fu, x_{n+1}) = 0$ and $\lim p(u, x_{n+1}) = 0$. Therefore, lemma 3.1(2.2) then gives $fu = u$. Suppose that there exists $u, v \in X$ such that $fu = u$ and $fv = v$. If $p(u,v) \neq 0$, then

$$p(u,v) = p(fu, fv) \leq \psi(p(u,v)) < p(u,v)$$

a contradiction. Therefore the fixed point is unique. Hence we have the theorem.

When $\psi(t) = kt,\ k \in [0,1[$, we get the following result, which gives a generalization of Banach's fixed point theorem in this new setting

Corollary 4.1. *Let (X,τ) be a Hausdorff topological space with a τ-distance p. Suppose that X is p-bounded and S-complete. Let f be a selfmapping of X such that*

$$p(fx, fy) \leq kp(x,y),\ k \in [0,1[,\ \forall x, y \in X$$

Then f has a unique fixed point.

Since a symmetric space (X,d) admits a τ_d-distance where τ_d is the topology defined earlier in example 2.3, corollary 4.1 gives a genaralization of the following known result (Theorem 1[5] for $f = Id_X$ which generalize Proposition 1[3]). Recall that $(W.3)$ denotes the following axiom given by Wilson [8] in a symmetric space (X,d): $(W.3)$ Given $\{x_n\}, x$ and y in X, $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$ imply $x = y$. It is clear that $(W.3)$ guarantees the uniqueness of limits of sequences.

corollary 4.2. *Let (X,d) be a d-bounded and S-complete symmetric space satisfying $(W.3)$ and f be a selfmapping of X such that*

$$d(fx, fy) \leq kd(x,y),\quad k \in [0,1[,\quad \forall x, y \in X$$

Then f has a fixed point.

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A SOLUTION TO AN "UNSOLVED PROBLEM IN NUMBER THEORY"

Allan J. MacLeod

Abstract. We discuss the problem of finding integer-sided triangles with the ratio base/altitude or altitude/base an integer. This problem is mentioned in Richard Guy's book "Unsolved Problems in Number Theory". The problem is shown to be equivalent to finding rational points on a family of elliptic curves. Various computational resources are used to find those integers in $[1, 99]$ which do appear, and also find the sides of example triangles.

A.M.S. (MOS) Subject Classification Codes. 11D25 , 11Y50

Key Words and Phrases. Triangle, Elliptic curve, Rank, Descent

1. Introduction

Richard Guy's book Unsolved Problems in Number Theory [5] is a rich source of fascinating problems. The final 3 paragraphs in section D19 of this book discuss the following problem:

Problem Which integers N occur as the ratios base/height in integer-sided triangles?

Also mentioned is the dual problem where height/base is integer. Some numerical examples are given together with some more analytical results, but no detailed analysis is presented.

Let BCD be a triangle with sides b,c,d using the standard naming convention. Let a be the height of B above the side CD. If one of the angles at C or D is obtuse then the height lies outside the triangle, otherwise it lies inside.

Assume, first, that we have the latter. Let E be the intersection of the height and CD, with $DE = z$ and $EC = b - z$. Then

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$$a^2 + z^2 = c^2$$
$$a^2 + (b - z)^2 = d^2 \tag{1}$$

Now, if base/height = N, the second equation is

$$a^2 + (z - Na)^2 = d^2$$

For altitudes outside the triangle the equations are the same, except for $z - Na$ replaced by $z + Na$. We thus consider the general system, with N positive or negative.

$$a^2 + z^2 = c^2$$
$$a^2 + (z - Na)^2 = d^2 \tag{2}$$

Clearly, we can assume that a and z have no common factors, so there exists integers p and q (of opposite parities) such that (1) $a = 2pq$,$z = p^2 - q^2$, or (2) $a = p^2 - q^2$, $z = 2pq$.

As a first stage, we can set up an easy search procedure. For a given pair (p, q), compute a and x using both the above possibilities. For N in a specified range test whether the resulting d value is an integer square.

This can be very simply done using the software package UBASIC, leading to the results in Table 1, which come from searching with $3 \leq p + q \leq 999$ and $-99 \leq N \leq 99$.

This table includes results for the formulae quoted in Guy, namely $N = 2m(2m^2+1)$ and $N = 8t^2 \pm 4t + 2$, and the individual values quoted except for $N = 19$. It also includes solutions from other values.

It is possible to extend the search but this will take considerably more time and there is no guarantee that we will find all possible values of N. We need alternative means of answering the following questions:

(1) can we say for a specified value of N whether a solution exists?

(2) if one exists, can we find it?

2. Elliptic Curve Formulation

In this section, we show that the problem can be considered in terms of elliptic curves.

Assuming $a = 2pq$ and $z = p^2 - q^2$, then the equation for d is

$$d^2 = p^4 - 4Np^3q + (4N^2 + 2)p^2q^2 + 4Npq^3 + q^4 \tag{3}$$

Define $j = d/q^2$ and $h = p/q$, so that

TABLE 1. Solutions for $2 \leq N \leq 99$

N	b	c	d	N	b	c	d
5	600	241	409	6	120	29	101
8	120	17	113	9	9360	1769	10841
13	291720	31849	315121	14	2184	685	1525
15	10920	2753	8297	18	6254640	439289	6532649
20	46800	8269	54781	26	15600	5641	10009
29	3480	169	3601	29	737760	31681	719329
29	706440	336841	371281	34	118320	4441	121129
36	4896	305	4625	40	24480	1237	23413
40	24360	3809	20609	40	741000	274853	1015397
42	24360	3389	21029	42	68880	26921	42041
42	2270520	262909	2528389	48	118320	4033	121537
61	133224	2305	132505	62	226920	93061	133981
68	4226880	90721	4293409	86	614040	260149	354061
94	3513720	42709	3493261	99	704880	198089	506969

$$j^2 = h^4 - 4Nh^3 + (4N^2+2)h^2 + 4Nh + 1 \tag{4}$$

This has an obvious rational point $h = 0, j = 1$, and so is birationally equivalent to an elliptic curve, see Mordell [7]. Using standard algebra, we can can link this equation to the curve

$$E_N : y^2 = x^3 + (N^2+2)x^2 + x \tag{5}$$

with the transformations $h = p/q = (Nx + y)/(x + 1)$.

If, however, $a = p^2 - q^2$ and $z = 2pq$, we have a different quartic for d^2, but leading to the same elliptic curve, with the relevant transformation $p/q = (Nx + x + y + 1)/(Nx - x + y - 1)$.

Thus the existence of solutions to the original problem is related to the rational points lying on the curve. There is the obvious point $(x, y) = (0, 0)$, which gives $p/q = 0$ or $p/q = -1$, neither of which give non-trivial solutions. A little thought shows the points $(-1, \pm N)$, giving $p/q = \infty$, $p/q = 0/0$, or $p/q = 1$, again failing to give non-trivial solutions.

We can, in fact, invert this argument and show the following

Lemma: If (x, y) is a rational point on the elliptic curve E_N with $x \neq 0$ or $x \neq -1$, then we get a non-trivial solution to the problem.

The proof of this is a straightforward consideration of the situations leading to $p^2 - q^2 = 0$ or $pq = 0$, and showing that the only rational points which can cause these are $x = 0$ or $x = -1$. It is also clear that if a or z become negative we can essentially ignore the negative sign.

3. Torsion Points

It is well known that the rational points on an elliptic curve form a finitely-generated group, which is isomorphic to the group $T \oplus \mathbb{Z}^r$, where $r \geq 0$ is the rank of the elliptic curve, and T is the torsion subgroup of points of finite order.

We first consider the torsion points. The point at infinity is considered the identity of the group. Points of order 2 have $y = 0$, so $(0,0)$ is one. The other roots of $y = 0$ are irrational for N integral, so there is only one point of order 2. Thus, by Mazur's theorem, the torsion subgroup is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, with the symmetry of the curve about $y = 0$ ensuring N one of $2, 4, 6, 8, 10, 12$.

For elliptic curves of the form $y^2 = x(x^2 + ax + b)$, a point $P = (x, y)$ leads to 2P having x-coordinate $(x^2 - b)^2/4y^2$. Thus, if P has order 4, then 2P has order 2, so 2P=(0,0) for the curves E_N. Thus $x^2 - 1 = 0$, so that $x = \pm 1$. The value $x = 1$ gives $y = \sqrt{N^2 + 4}$, which is irrational. $x = -1$ gives $y = \pm N$, so that $(-1, \pm N)$ are the only order 4 points. This reduces the possibilities for the torsion subgroup to $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/8\mathbb{Z}$, or $\mathbb{Z}/12\mathbb{Z}$.

For $\mathbb{Z}/8\mathbb{Z}$, we would have 4 points of order 8. Suppose Q is of order 8, giving 2Q of order 4. Thus the x-coordinate of 2Q must be -1, but as we stated previously, the x-coordinate of 2Q is a square. Thus there cannot be any points of order 8.

For $\mathbb{Z}/12\mathbb{Z}$, we would have 2 points of order 3, which correspond to any rational points of inflection of the elliptic curve. These are solutions to

$$3x^4 + 4(N^2 + 2)x^3 + 6x^2 - 1 = 0 \tag{6}$$

If $x = r/s$ is a rational solution to this, then $s|3$ and $r|1$, so the only possible rational roots are ± 1 and $\pm 1/3$. Testing each shows that they are not roots for any value of N.

Thus, the torsion subgroup consists of the point at infinity, (0,0), $(-1, \pm N)$. As we saw, in the previous section, these points all lead to trivial solutions. We thus have proven the following

Theorem: A non-trivial solution exists iff the rank of E_N is at least 1. If the rank is zero then no solution exists.

4. Parametric Solutions

As mentioned in the introduction, Guy quotes the fact that solutions exist for $N = 2m(2m^2 + 1)$ and $N = 8t^2 \pm 4t + 2$, though without any indication of how these forms were discovered. We show, in this section, how to use the elliptic curves E_N to determine new parametric solutions.

The simple approach used is based on the fact that rational points on elliptic curves of the form

$$y^2 = x^3 + ax^2 + bx$$

have $x = du^2/v^2$ with $d|b$. Thus, for E_N, we can only have $d = \pm 1$.

We look for integer points so $v = 1$, and searched over $1 \leq N \leq 999$ and $1 \leq u \leq 99999$ to find points on the curve. The data output is then analysed to search for patterns leading to parametric solutions.

For example, the above sequences have points P given by

1. $N = 2m(2m^2 + 1)$, $P = (4m^2, 2m(8m^4 + 4m^2 + 1))$,
2. $N = 8t^2 + 4t + 2$, $P = (-(8t^2 + 4t + 1)^2, 2(4t + 1)(4t^2 + 2t + 1)(8t^2 + 4t + 1))$,
3. $N = 8t^2 - 4t + 2$, $P = (-(8t^2 - 4t + 1)^2, 2(4t - 1)(4t^2 - 2t + 1)(8t^2 - 4t + 1))$.

These parametric solutions are reasonably easy to see in the output data. Slightly more difficult to find is the solution with $N = 4(s^2+2s+2)$, $x = (2s^3+6s^2+7s+3)^2$ and $y = (s + 1)(s^2 + 2s + 2)(2s^2 + 4s + 3)(4s^4 + 16s^3 + 32s^2 + 32s + 13)$.

Using $p/q = (Nx + y)/(x + 1)$ with $a = 2pq, z = p^2 - q^2$, we find the following formulae for the sides of the triangles:

$$b = 8(s + 1)(s^2 + 2s + 2)(2s^2 + 2s + 1)(2s^2 + 4s + 3)(2s^2 + 6s + 5)$$

$$\begin{aligned} c =&16s^{10} + 192s^9 + 1056s^8 + 3504s^7 + 7768s^6 + 12024s^5 \\ &+ 13168s^4 + 10076s^3 + 5157s^2 + 1594s + 226 \end{aligned}$$

$$\begin{aligned} d =&16s^{10} + 128s^9 + 480s^8 + 1104s^7 + 1720s^6 \\ &+ 1896s^5 + 1504s^4 + 868s^3 + 381s^2 + 138s + 34 \end{aligned}$$

Other parametric solutions can be found by adding the points on the curve to the torsion points.

5. Rank Calculations

We now describe a computational approach to the determination of the rank. This follows the approach of Zagier & Kramarcz [10] or Bremner & Jones [2] for example. The computations are based on the Birch and Swinnerton-Dyer (BSD) conjecture, which states (roughly) - if an elliptic curve has rank r, then the L-series of the curve has a zero of order r at the point 1. Smart [9] calls this the "conditional algorithm" for the rank.

The L-series of an elliptic curve can be defined formally as

$$L(s) = \sum_{k=1}^{\infty} \frac{a_k}{k^s}$$

where a_k are integers which depend on the algebraic properties of the curve. This form is useless for effective computation at $s = 1$, so we use the following form from Proposition 7.5.8. of Cohen [3]

$$L(1) = \sum_{k=1}^{\infty} \frac{a_k}{k} \left(\exp(-2\pi k A/\sqrt{N^*}) + \epsilon \exp(-2\pi k/(A\sqrt{N^*}))\right)$$

with $\epsilon = \pm 1$ - the sign of the functional equation, N^* - the conductor of the equation, and A **ANY** number.

N^* can be computed by Tate's algorithm - see Algorithm 7.5.3 of Cohen, while ϵ can be computed by computing the right-hand sum at two close values of A - say

1 and 1.1 - and seeing which choice of ϵ leads to agreement (within rounding and truncation error). If $\epsilon = 1$ then the curve has even rank, whilst if $\epsilon = -1$ the curve has odd rank.

We thus determine the value of ϵ. If $\epsilon = 1$, we compute

$$L(1) = 2\sum_{k=1}^{\infty} \frac{a_k}{k} \exp(-2\pi k/\sqrt{N^*})$$

and, if this is non-zero, then we assume $r = 0$, whilst, if zero, $r \geq 2$. For $\epsilon = -1$, we compute

$$L'(1) = 2\sum_{k=1}^{\infty} \frac{a_k}{k} E_1(2\pi k/\sqrt{N^*})$$

with E_1 the standard exponential integral special function. If this is non-zero, then we assume $r = 1$, whilst if zero, $r \geq 3$.

The most time-consuming aspect of these computations is the determination of the a_k values. Cohen gives a very simple algorithm which is easy to code, but takes a long time for k large. To achieve convergence in the above sums we clearly need $k = O(\sqrt{N^*})$. Even in the simple range we consider, N^* can be several million, so we might have to compute many thousands of a_k values.

6. Numerical Results

Using all the ideas of the previous section, we wrote a UBASIC program to estimate the rank of E_N for $1 \leq N \leq 99$. The results are given in the following table. We have no proof that these values are correct, but for every value of N with rank greater than 0 we have found a non-trivial solution to the original triangle problem.

TABLE 2. Rank of E_N for $1 \leq N \leq 99$

	0	1	2	3	4	5	6	7	8	9
00+		0	0	0	0	1	1	0	1	1
10+	0	0	0	1	1	1	0	1	1	1
20+	1	1	1	1	0	0	1	0	0	2
30+	0	1	1	0	1	1	1	1	1	0
40+	2	0	2	1	1	1	0	0	1	0
50+	0	0	1	2	0	0	0	0	0	0
60+	0	2	2	1	0	0	0	0	2	1
70+	0	1	1	1	1	0	1	1	0	1
80+	0	0	0	1	1	2	2	1	0	0
90+	0	0	1	1	1	1	0	1	1	2

To find an actual solution, we can assume that $x = du^2/v^2$ and $y = duw/v^3$, with $(u, v) = 1$ and d squarefree, and hence that

$$w^2 = du^4 + (N^2+2)u^2v^2 + v^4/d$$

implying that $d = \pm 1$.

For curves with rank 2, we found that a simple search quickly finds a solution. This also holds for a few rank 1 curves, but most curves did not produce an answer in a reasonable time.

A by-product of the L-series calculation is an estimate H of the height of a rational point on the curve. The height gives a rough idea of how many decimal digits will be involved in a point, and thus how difficult it will be to compute it. The following formula gives the height, see Silverman [8] for a more precise definition of the quantities involved.

$$H = \frac{L'(1)\, T^2}{2\, |\text{Ш}|\, \Omega\, c}$$

where T is the order of the torsion subgroup, Ш is the Tate-Safarevic group, Ω is the real period of the curve, and c is the Tamagawa number of the curve.

There is no known algorithm to determine $|\text{Ш}|$ and so we usually use the value 1 in the formula. Note that for this problem $T = 4$, and that this formula gives a value half that of an alternative height normalisation used in Cremona [4].

Unfortunately, this value is not always the height of the generator of the infinite subgroup, but sometimes of a multiple. An example comes from $N = 94$, where the height calculation gave a value $H = 55.1$, suggesting a point with tens of digits in the numerator and denominator. We actually found a point with $x = 4/441$.

To determine the values of (d, u, v, w), we used a standard descent procedure as described by Cremona or Bremner et al [1]. We consider equation (11) firstly as

$$w^2 = dz^2 + (N^2+2)zt + t^2/d$$

Since this is a quadratic, if we find a simple numerical solution, we can parameterise $z = f_1(r,s)$ and $t = f_2(r,s)$, with f_1 and f_2 homogeneous quadratics in r and s. We then look for solutions to $z = ku^2$, $t = kv^2$, with k squarefree.

Considering $q = kv^2$, if we find a simple numerical solution we can parameterise again for r and s as quadratics, which are substituted into $p = ku^2$, giving a quartic which needs to be square. We search this quartic to find a solution.

We wrote a UBASIC code which performs the entire process very efficiently. This enabled most solutions with heights up to about 16 to be found.

For larger heights we can sometimes use the fact that the curve E_N is 2-isogenous to the curve

$$f^2 = g^3 - 2(N^2+2)g^2 + N^2(N^2+4)g$$

with $x = f^2/4g^2$ and $y = f(g^2 - N^2(N^2+4))/8g^2$. This curve has the same rank as E_N and sometimes a point with estimated height half that of the equivalent point on E_N.

For points with height greater than about 20, however, we used a new descent method which involves trying to factorise the quartic which arises in the descent method discussed above. This method is described in the report [6]. This has enabled us to complete a table of solutions for all values in the range $1 < N \leq 99$.

The largest height solved is for $N = 79$ with E_{79} having equation $y^2 = x^3 + 6243x^2 + x$. The estimated height is roughly 40, but the 2-isogenous curve $f^2 = g^3 - 12486g^2 + 38975045g$ was indicated to have a point with height about 20.

We found a point with

$$g = \frac{2836\ 8499\ 3467\ 6319\ 5139\ 0020}{4689\ 8490\ 9449\ 9234\ 0041}$$

leading to a point on the original curve with

$$x = \frac{2654\ 7926\ 1289\ 1944\ 1996\ 8505\ 1867\ 1143\ 3025}{1705\ 4187\ 5947\ 7256\ 7676\ 9862\ 5643\ 5806\ 2336}$$

For interested readers, this point leads to the triangle with sides

$$\begin{aligned} b =& 146586997184778231835321971944006987 8\\ & 865747485658641082621328674163116 4960 \end{aligned}$$

$$\begin{aligned} c =& 89276765348874858876033629427095775 0\\ & 737827730811866599994108625538947124 9 \end{aligned}$$

$$\begin{aligned} d =& 57359536918230561955378662677931929 2\\ & 615973876797127975470731247711710820 9 \end{aligned}$$

7. Altitude/Base

If we wish altitude/base=M, then we can use the theory of section 2, with $N = 1/M$. If we define $s = M^3y$, $t = M^2x$, we get the system of elliptic curves F_M, given by

$$s^2 = t^3 + (2M^2 + 1)t^2 + M^4t$$

These curves have clearly the same torsion structure as E_N, with the point at infinity, $(0, 0)$, and $(-M^2, \pm M^2)$ being the torsion points. We can also search for parametric solutions, and we found that $M = s(s + 2)$ has the following points:

1. $(s^3(s + 2), \pm s^3(s + 2)(2s^2 + 4s + 1))$,
2. $(s(s + 2)^3, \pm s(s + 2)^3(2s^2 + 4s + 1))$,
3. $(-s(s + 2)(s + 1)^2, \pm s(s + 1)(s + 2))$

If we call the first point Q, then the second point comes from $Q + (0, 0)$ and the third from $Q + (-M^2, M^2)$.

Considering Q, we find

TABLE 3. Rank of F_M for $1 \leq M \leq 99$

	0	1	2	3	4	5	6	7	8	9
00+		0	0	1	0	1	0	1	1	0
10+	1	0	0	1	0	1	0	0	1	0
20+	0	1	0	0	1	0	0	1	0	0
30+	1	0	0	0	0	1	0	1	0	1
40+	1	2	2	2	1	0	0	1	1	1
50+	0	1	1	0	0	2	0	1	1	0
60+	1	1	1	2	1	0	0	1	1	0
70+	1	1	1	1	1	0	1	1	0	0
80+	2	1	0	0	0	0	1	0	2	0
90+	0	1	0	1	0	1	0	0	0	1

$$b = 2(s+1)\,,\; c = s(2s^2+6s+5)\,,\; d = (s+2)(2s^2+2s+1)$$

which always gives an obtuse angle.

The BSD conjecture gives rank calculations listed in Table 3.

As before, we used a variety of techniques to find non-torsion points on F_M. We must say that these curves proved much more testing than E_N. Several hours computation on a 200MHz PC were needed for $M = 47$, while we have not been able to find a point for $M = 67$, which has an estimated height of 45.7, though this is the only value in $[1, 99]$ for which we do not have a rational point.

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FIXED POINTS FOR NEAR-CONTRACTIVE TYPE MULTIVALUED MAPPINGS

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Abstract. In the present paper we prove some fixed point theorems for near-contractive type multivalued mappings in complete metric spaces. these theorems extend some results in [1], [5], [6] and others

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1 Basic Preliminaries

Let (X, d) be a metric space we put:

$$CB = \{A : \text{ A is a nonempty closed and bounded subset of X } \}$$

$$BN = \{A : \text{ A is a nonempty bounded subset of X } \}$$

If A, B are any nonempty subsets of X we put:

$$D(A, B) = \inf\{d(a, b) : a \in A, b \in B\},$$

$$\delta(A, B) = \sup\{d(a, b) : a \in A, b \in B\},$$

$$H(A, B) = \max\{ \{\sup\{D(a, B) : a \in A\}, \sup\{D(b, A) : b \in B\} \}.$$

If follows immediately from the definitoin that

$$\delta(A, B) = 0 \text{ iff } A = B = \{a\},$$

$$H(a, B) = \delta(a, B),$$

$$\delta(A, A) = diamA,$$

$$\delta(A, B) \leq \delta(A, C) + \delta(A, C),$$

$$D(a, A) = 0 \text{ if } a \in A,$$

for all A, B, C in $BN(X)$ and a in X.

In general both H and δ may be infinite. But on $BN(X)$ they are finite. Moreover, on $CB(X)$ H is actually a metric (the Hansdorff metric).

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Definition 1.1. [2] *A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if*

(*i*) *given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1, 2, ...,$ and $\{a_n\}$ converges to a*

(*ii*) *given $\varepsilon > 0$ there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centers in A and radius ε*

Lemma 1.1. [2,3].*If $\{A_n\}$ and $\{B_n\}$ are seqences in $BN(X)$ converging to A and B in $BN(X)$ respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 1.2. [3] *Let $\{A_n\}$ be a sequence in $BN(X)$ and x be a point of X such that $\delta(A_n, x) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{x\}$ in $BN(X)$.*

Definition 1.2. [3] *A set-valued mapping F of X into $BN(X)$ is said to be continuous at $x \in X$ if the sequence $\{Fx_n\}$ in $BN(X)$ converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X. F is said continuous on X if it is continuous at every point of X.*

The following Lemma was proved in [3]

Lemma 1.3. *Let $\{A_n\}$ be a sequence in $BN(X)$ and x be a point of X such that*

$$\lim_{n\to\infty} a_n = x,$$

x being independent of the particular choice of $a_n \in A_n$. If a selfmap I of X is continuous, then Ix is the limit of the sequence $\{IA_n\}$.

Definition 1.3. [4]. *The mappings $I : X \to X$ and $F : X \to BN(X)$ are δ-compatible if $\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in BN(X)$,*

$$Fx_n \to t \quad \text{and} \quad Ix_n \to t$$

for some t in X.

2. Our Results

We establish the following:

2. 1. A Coincidence Point Theorem

Theorem 2.1. *Let $I : X \to X$ and $T : X \to BN(X)$ be two mappings such that $FX \subset IX$ and*

$$(C.1)\quad \phi(\delta(Tx, Ty)) \le a\phi(d(Ix, Iy)) + b[\phi(H(Ix, Tx)) + \phi(H(Iy, Ty))] + c\min\{\phi(D(Iy, Tx)), \phi(D(Ix, Ty))\},$$

where $x, y \in X$, $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$. a, b, c are nonnegative, $a + 2b < 1$ and $a + c < 1$. Suppose in addition that $\{F, I\}$ are δ-compatible and F or I is continuous. Then I and T have a unique common fixed point z in X and further $Tz = \{z\}$.

Proof. Let $x_0 \in X$ be an arbitrary point in X. Since $TX \subset IX$ we choose a point x_1 in X such that $Ix_1 \in Tx_0 = Y_0$ and for this point x_1 there exists a point x_2 in X such that $Ix_2 \in Tx_1 = Y_1$, and so on. Continuing in this manner we can define a sequence $\{x_n\}$ as follows:

$$Ix_{n+1} \in Tx_n = Y_n$$

For sinplicity, we can put $V_n = \delta(Y_n, Y_{n+1})$, for $n = 0, 1, 2, ...$ By $(C, 1)$ we have

$$\begin{aligned}
\phi(V_n) &= \phi(\delta(Y_n, Y_{n+1})) = \phi(\delta(Tx_n, Tx_{n+1})) \\
&\leq a\phi(d(Ix_n, Ix_{n+1})) + b[\phi(H(Ix_n, Tx_n)) + \phi(H(Ix_{n+1}, Tx_{n+1}))] \\
&+ c\min\{\phi(D(Ix_{n+1}, Tx_n)), \phi(D(Ix_n, Tx_{n+1}))\} \\
&\leq A_1 + A_2 + A_3
\end{aligned}$$

Where

$$\begin{aligned}
A_1 &= a\phi(\delta(Y_{n-1}, Y_n)) \\
A_2 &= b[\phi(\delta(Y_{n-1}, Y_n)) + \phi(\delta(Y_n, Y_{n+1}))], \\
A_3 &= c\phi(D(Ix_{n+1}, Y_n)).
\end{aligned}$$

So

$$\phi(V_n) \leq a\phi(V_{n-1}) + b[\phi(V_{n-1}) + \phi(V_n)]$$

Hence we have

$$\phi(V_n) \leq \frac{a+b}{1-b}\phi(V_{n-1}) < \phi(V_{n-1}) \quad (1)$$

Since ϕ is increasing, $\{V_n\}$ is a decreasing sequence. Let $lim_n V_n = V$, assume that $V > 0$. By letting $n \to \infty$ in (1), Since ϕ is continuous , we have:

$$\phi(V) \leq \frac{a+b}{1-b}\phi(V) < \phi(V),$$

which is contradiction , hence $V = 0$.

Let y_n be an arbitrary point in Y_n for $n = 0, 1, 2, ...$ Then

$$\lim_{n\to\infty} d(y_n, y_{n+1}) \leq \lim_{n\to\infty} \delta(Y_n, Y_{n+1}) = 0.$$

Now, we wish to show that $\{y_n\}$ is a Cauchy sequence, we proceed by contradiction. Then there exist $\varepsilon > 0$ and two sequences of natural numbers $\{m(i)\}$, $\{n(i)\}$, $m(i) > n(i)$, $n(i) \to \infty$ as $i \to \infty$ such taht

$$\delta(Y_{n(i)}, Y_{m(i)}) > \varepsilon \quad \text{while} \quad \delta(Y_{n(i)}, Y_{m(i)-1}) \leq \varepsilon$$

Then we have

$$\begin{aligned}
\varepsilon < \delta(Y_{n(i)}, Y_{m(i)}) &\leq \delta(Y_{n(i)}, Y_{m(i)-1}) + \delta(Y_{m(i)-1}, Y_{m(i)}) \\
&\leq \varepsilon + V_{m(i)-1},
\end{aligned}$$

since $\{V_n\}$ converges to 0, $\delta(Y_{n(i)}, Y_{m(i)}) \to \varepsilon$. Futhermore, by triangular inequality, it follows that

$$| \ \delta(Y_{n(i)+1}, Y_{m(i)+1}) - \delta(Y_{n(i)}, Y_{m(i)}) \ | \leq V_{n(i)} + V_{m(i)},$$

and therefore the sequence $\{\delta(Y_{n(i)+1}, Y_{m(i)+1})\}$ converges to ε

¿From (C. 2), we also deduce:

$$\begin{aligned}\phi(\delta(Y_{n(i)+1}, Y_{m(i)+1})) &= \phi(\delta(Tx_{n(i)+1}, Tx_{m(i)+1}))\\ &\leq C_1 + C_2 + C_3\\ &\leq C_4 + C_5 + C_6 \quad (4)\end{aligned}$$

Where

$$\begin{aligned}&C_1 = a\phi(d(Ix_{n(i)+1}, Ix_{m(i)+1})),\\ &C_2 = b\Big\{\phi(\delta(Ix_{n(i)+1}, Tx_{n(i)+1})) + \phi(\delta(Ix_{m(i)+1}, Tx_{m(i)+1}))\Big\},\\ &C_3 = cmin\{\phi(D(Ix_{n(i)+1}, Y_{m(i)+1}), \phi(D(Ix_{n(i)+1}, Y_{m(i)+1})\},\\ &C_4 = a\phi(\delta(Y_{n(i)}, Y_{m(i)})),\\ &C_5 = [\phi(V_{n(i)}) + \phi(V_{m(i)}],\\ &C_6 = c\phi(\delta(Y_{n(i)}, Y_{m(i)}) + V_{m(i)}).\end{aligned}$$

Letting $i \to \infty$ in (4), we have

$$\phi(\varepsilon) \leq (a+c)\phi(\varepsilon) < \phi(\varepsilon)$$

This is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence in X and it has a limit y in X. So the sequence $\{Ix_n\}$ converge to y and further, the sequence $\{Tx_n\}$ converge to set $\{y\}$. Now supose that I is continuous. Then

$$I^2x_n \to Iy \quad \text{and} \quad ITx_n \to \{Iy\}$$

by Lemma 1.3. Since I and T are δ-compatible. Therefore $TIx_n \to \{Iy\}$. Using inequality $(C.1)$, we have

$$\begin{aligned}\phi(\delta(TIx_n, Tx_n)) \leq\ & a\phi(d(I^2x_n, Ix_n)) + b[\phi(H(Ix_n, Tx_n)) + \phi(H(I^2x_n, TIx_n))]\\ &+ cmin\{\phi(D(Ix_n, TIx_n)), \phi(D(I^2x_n, Tx_n))\},\end{aligned}$$

for $n \geq 0$. As $n \to \infty$ we obtain by Lemma 1.1

$$\phi(d(Iy, y)) \leq a\phi(d(Iy, y)) + c\phi(d(y, Iy)),$$

That is $\phi(d(Iy, y)) = 0$ which implies that $Iy = y$. Further

$$\begin{aligned}\phi(\delta(Ty, Tx_n)) \leq\ & a\phi(d(Iy, Ix_n)) + b[\phi(H(Iy, Ty)) + \phi(H(Ix_n, Tx_n))]\\ &+ cmin\{\phi(D(Ix_n, Ty)), \phi(D(Iy, Tx_n))\},\end{aligned}$$

for $n \geq 0$. As $n \to \infty$ we obtain by Lemma 1.1

$$\phi(\delta(Ty, y)) \leq b\phi(\delta(Ty, y)),$$

which implies that $Ty = y$. Thus y is a coincidence point for T and I. Now suppose that T and I have a second common fixed point z such that $Tz = \{z\} = \{Iz\}$. Then, using inequality $(C.1)$, we obtain

$$\phi(d(y, z)) = \phi(\delta(Ty, Tz)) \leq (a+c)\phi(d(z, y)) < \phi(d(z, y))$$

which is a contradiction. This completes the proof of the Theorem.

Corollary 2.1 ([6.$Theorem$2.1])**.** *Let (X,d) be a complete metric space, $T : X \longrightarrow CB(X)$ a multi-valued map satisfying the following condition :*

$$\phi(\delta(Tx,Ty)) \leq a\phi(d(x,y)) + b\Big[\phi(\delta(x,Tx)) + \phi(\delta(y,Ty))\Big] + \\ + c\min\Big\{\phi(d(x,Ty)), \phi(d(y,Tx))\Big\} \qquad \forall x,y \in X,$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$ and a, b, c are three positive constants such that $a + 2b < 1$ and $a + c < 1$, then T has a unique fixed point.

Note that the proof of Theorem 2.1 is another proof of Corollary 2.1 which is of interest in part because it avoids the use of Axiom of choice.

2. 2. A Fixed Point Theorem

Theorem 2.2. *Let (X,d) be a complete metric space. If $F : X \to CB(X)$ is a multi-valued mapping and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$. Furthermore, let a, b, c be three functions from $(0,\infty)$ into $[0,1)$ such that*
$a + 2b : \ (0,\infty) \to [0,1)$ and $a + c : \ (0,\infty) \to [0,1)$ are decreasing functions. Suppose that F satisfies the following condition:

$$(C.3) \quad \phi(\delta(Fx,Fy)) \leq a(d(x,y))\phi(d(x,y)) + b(d(x,y))[\phi(H(x,Fx)) + \phi(H(y,Fy))] \\ + c(d(x,y))\ min\{\phi(D(y,Fx)), \phi(D(x,Fy))\},$$

then F has a unique fixed point z in X such that $Fz = \{z\}$.

Proof.. First we will establish the existence of a fixed point. Put $p = max\{(a+2b)^{\frac{1}{2}}, (a+c)^{\frac{1}{2}}\}$, take any x_o in X. Since we may assume that $D(x_0, Fx_0)$ is positive, we can choose $x_1 \in Fx_0$ which satisfies $\phi(d(x_0,x_1)) \geq p(D(x_0,Fx_0))\phi(H(x_0,Fx_0))$, we may assume that $p(d(x_0,x_1))$ is positive. Assuming now that $D(x_1,Fx_1)$ is positive, we choose $x_2 \in Fx_1$ such that $\phi(d(x_1,x_2)) \geq p(d(x_0,x_1))\phi(H(x_1,Fx_1))$ and $\phi(d(x_1,x_2)) \geq p(D(x_1,Fx_1))\phi(d(x_1,Fx_1))$, since $d(x_0,x_1) \geq D(x_0,Fx_0)$ and p is deceasing then we have also
$\phi(d(x_0,x_1)) \geq p(d(x_0,x_1))\phi(H(x_0,Fx_0))$. Now

$$\begin{aligned}\phi(d(x_1,x_2)) &\leq \phi(\delta(Fx_0,Fx_1)) \\ &\leq a(d(x_0,x_1))\phi(d(x_0,x_1)) + b(d(x_0,x_1))[\phi(H(x_0,Fx_0)) + \phi(H(x_1,Fx_1))] \\ &+ c(d(x_0,x_1))\ min\{\phi(D(Fx_0,x_1)), \phi(D(x_0,Fx_1))\} \\ &\leq ap^{-1}\phi(d(x_0,x_1)) + bp^{-1}[\phi(d(x_0,x_1)) + \phi(d(x_1,x_2))],\end{aligned}$$

which implies

$$\phi(d(x_1,x_2)) \leq q(d(x_0,x_1))\phi(d(x_0,x_1))$$

where

$$q : (0,\infty) \to [0,1)$$

is defined by

$$q = \frac{a+b}{p-b}.$$

Note that $r \geq t$ implies $q(r) \leq p(t) < 1$. By induction, assumunig that $D(x_i, Fx_i)$ and $p(d(x_{i-1}, x_i))$ are positive, we obtain a sequence $\{x_i\}$ which satisfies $x_i \in Fx_{i-1}$, $\phi(d(x_{i-1}, x_i)) \geq p(d(x_{i-1}, x_i))\phi(H(x_{i-1}, Fx_{i-1}))$,

$$\begin{aligned}\phi(d(x_i, x_{i+1})) &\geq p(d(x_{i-1}, x_i))\phi(H(x_i, Fx_i)),\\ \phi(d(x_i, x_{i+1})) &\leq q(d(x_{i-1}, x_i))\phi(d(x_{i-1}, x_i))\\ &\leq p(d(x_{i-1}, x_i))\phi(d(x_{i-1}, x_i))\\ &< \phi(d(x_{i-1}, x_i)).\end{aligned}$$

It is not difficult to verify that $lim_i\ d(x_i, x_{i+1}) = 0$. If $\{x_i\}$ is not Cauchy, there exists $\varepsilon > 0$ and two sequences of natural numbers $\{m(i)\}, \{n(i)\}$,

$m(i) > n(i) > i$ such that $d(x_{m(i)}, x_{n(i)}) > \varepsilon$ while $d(x_{m(i)-1}, x_{n(i)}) \leq \varepsilon$. It is not difficult to verify that

$$d(x_{m(i)}, x_{n(i)}) \to \varepsilon \text{ as } i \to \infty \text{ and } d(x_{m(i)+1}, x_{n(i)+1}) \to \varepsilon \text{ as } i \to \infty.$$

For i sufficiently large $d(x_{m(i)}, x_{m(i)+1}) < \varepsilon$ and $d(x_{n(i)}, x_{n(i)+1}) < \varepsilon$. For these i we have

$$\begin{aligned}\phi(d(x_{m(i)+1}, x_{n(i)+1})) &\leq \phi(\delta(Fx_{m(i)}, Fx_{n(i)}))\\ &\leq a(d(x_{m(i)}, x_{n(i)}))\phi(d(x_{m(i)}, x_{n(i)}))\\ &+ b(d(x_{m(i)}, x_{n(i)}))[\phi(H(x_{m(i)}, Fx_{m(i)})) + \phi(H(x_{n(i)}, Fx_{n(i)}))]\\ &+ c(d(x_{m(i)}, x_{n(i)}))\ min\{\phi(D(x_{m(i)}, Fx_{n(i)})), \phi(D(x_{n(i)}, Fx_{m(i)})\}\\ &\leq a(d(x_{m(i)}, x_{n(i)}))\phi(d(x_{m(i)}, x_{n(i)}))\\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{n(i)}, x_{n(i)+1}))\phi(d(x_{n(i)}, x_{n(i)+1}))\\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{m(i)}, x_{m(i)+1}))\phi(d(x_m(i), x_{m(i)+1}))\\ &+ c(d(x_{n(i)}, x_{m(i)})\phi(d(x_{m(i)}, x_{n(i)+1}))\\ &\leq a(d(x_{m(i)}, x_{n(i)}))\phi(d(x_{m(i)}, x_{n(i)+1}) + d(x_{n(i)+1}, x_{n(i)})\\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{n(i)}, x_{n(i)+1}))\phi(d(x_{n(i)}, x_{n(i)+1}))\\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{m(i)}, x_{m(i)+1}))\phi(d(x_{m(i)}, x_{m(i)+1}))\\ &+ c(d(x_{n(i)}, x_{m(i)})\phi(d(x_{m(i)}, x_{n(i)+1} + d(x_{n(i)+1}, x_{n(i)}))\\ &\leq [a(\varepsilon) + c(\varepsilon)]\phi(d(x_{m(i)}, x_{n(i)}) + d(x_{n(i)}, x_{n(i)+1}))\\ &+ \phi(d(x_{m(i)}, x_{m(i)+1})) + \phi(d(x_{n(i)}, x_{n(i)+1}))\quad (*)\end{aligned}$$

Letting $i \to \infty$ in $(*)$, we have: $\phi(\varepsilon) \leq [a(\varepsilon) + c(\varepsilon)]\phi(\varepsilon) < \phi(\varepsilon)$. This is contradiction. Hence $\{x_i\}$ is cauchy sequence in a complete metric space X, then there existe a point $x \in X$ such that $x_n \to x$ as $i \to \infty$. This x is a fixed point of F because

$$\begin{aligned}\phi(H(x_{i+1}, Fx)) = \phi(\delta(x_{i+1}, Fx)) &\leq \phi(\delta(Fx_i, Fx))\\ &\leq a(d(x_i, x))\phi(d(x_i, x))\\ &+ b(d(x_i, x))[\phi(H(x, Fx)) + \phi(H(x_i, Fx_i))]\\ &+ c(d(x_i, x))\ min\{\phi(D(x_i, Fx)), \phi(D(x, Fx_i))\}\\ &\leq a(d(x_i, x))\phi(d(x_i, x))\\ &+ b(d(x_i, x))p^{-1}(d(x_i, x_{i+1})\phi(d(x_i, x_{i+1}))\\ &+ b(d(x_i, x))\phi(H(x, Fx)) + c(d(x_i, x))\phi(d(x, x_{i+1})\quad (**)\end{aligned}$$

Using $b < \frac{1}{2}$, $p^{-1}(d(x_i, x_{i+1})) < p^{-1}(d(x_0, x_1))$ and letting $i \to \infty$ in $(**)$, we have:

$$\phi(\delta(x, Fx)) \leq \frac{1}{2}\phi(H(x, Fx)).$$

That is $\phi(H(x, Fx)) = 0$ and therefore $H(x, Fx) = 0$ i.e, $Fx = x$. $Fx = \{x\}$. We claim that x is unique fixed point of F. For this, we suppose that y $(x \neq y)$ is another fixed point of F such that $Fy = \{y\}$. Then

$$\begin{aligned}\phi(d(y, x)) &\leq \phi(\delta(Fy, Fx)) \\ &\leq a\phi(d(x, y)) + b[\phi(H(x, Fx)) + \phi(H(y, Fy))] \\ &+ c\ min\{\phi(D(x, Fy)), \phi(D(y, Fx))\} \\ &\leq [a + c]\phi(d(x, y)) < \phi(d(x, y)),\end{aligned}$$

a contradiction. This completes the proof of the theorem.

We may establish a common fixed point theorem for a pair of mappings F and G which stisfying the contractive condition corresponding to $(C.1)$, i.e., for all $x, y \in X$

$$(C.2)\quad \begin{aligned}\phi(\delta(Fx, Gy)) &\leq a\phi(d(x, y)) + b[\phi(H(x, Fx)) + \phi(H(y, Gy))] \\ &+ c\ min\{\phi(D(y, Fx)), \phi(D(x, Gy))\},\end{aligned}$$

2. 3 A Common Fixed Point Theorem.

Theorem 2.3. *Let (X, d) be a metric space. Let F and G be two mappings of X into $BN(X)$ and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$. Furthermore, let a, b, c be three nonnegative constants such that $a + 2b < 1$ and $a + c < 1$. Suppose that F and G satisfy $(C.2)$. Then F and G have a unique common fixed point. This fixed point satisfies $Fx = Gx = \{x\}$.*

Proof. Put $p = max\{(a+2b)^{\frac{1}{2}},\ c^{\frac{1}{2}}\}$. we may assume that is positive. We define by using the Axiom of choice the two single-valued functions $f, g : X \to X$ by letting $f(x)$ be a point $w_1 \in Fx$ and $g(x)$ be a point $w_2 \in Gx$ such that $\phi(d(x, w_1)) \geq p\phi(H(x, Fx))$ and $\phi(d(x, w_2)) \geq p\phi(H(x, Gx))$. Then for every $x, y \in X$ we have:

$$\begin{aligned}\phi(d(f(x), g(y))) \leq \phi(\delta(Fx, Gy)) &\leq a\phi(d(x, y)) + b[\phi(H(x, Fx)) + \phi(H(y, Gy))] \\ &+ c\ min\{\phi(D(y, Fx)), \phi(D(x, Gy))\} \\ &\leq a\phi(d(x, y)) + p^{-1}b[\phi(d(x, fx)) + \phi(d(y, gy))] \\ &+ c\ min\{\phi(d(y, fx)), \phi(d(x, gy))\}.\end{aligned}$$

Since $a+2p^{-1}b \leq p^{-1}(a+2b) \leq p < 1$, from [7, Theorem 2.1] we conclude that f and g has a common fixed point. That is, there exists a point x such that $0 = d(x, fx) = \phi(d(x, fx)) \geq p\phi(H(x, Fx))$ and $0 = d(x, gx) = \phi(d(x, gx)) \geq p\phi(H(x, Gx))$ which implies $\phi(H(x, Fx)) = 0$ and $\phi(H(x, Gx)) = 0$, then $H(x, Fx) = \delta(x, Fx) = 0$ and $H(x, Gx) = \delta(x, Gx) = 0$ i.e. $Fx = Gx = \{x\}$. Hence F and G have a common fixed point $x \in X$. We claim that x is unique common fixed point of F and G. For this, we suppose that y $(x \neq y$ $)$ is another fixed point of F and G. Since $y \in Fy$ and $y \in Gy$, from $(C.2)$ we have

$$\begin{aligned}\max\{\phi(H(y, Fy)), \phi(H(y, Gy))\} &\leq \phi(\delta(Fy, Gy)) \\ &\leq b[\phi(H(y, Fy)) + \phi(H(y, Gy))] \\ &\leq 2b \max\{\phi(H(y, Fy)), \phi(H(y, Gy))\}\end{aligned}$$

which implies $\delta(Fy, Gy) = 0$, that is $Fy = Gy = \{y\}$. Then

$$\begin{aligned}\phi(d(y,x)) &= \phi(\delta(Fy, Gx)) \\ &\leq a\phi(d(x,y)) + b[\phi(H(x,Gx)) + \phi(H(y,Fy))] \\ &+ c\min\{\phi(D(x,Fy)), \phi(D(y,Gx))\} \\ &\leq [a+c]\phi(d(x,y)) < \phi(d(x,y)),\end{aligned}$$

a contradiction. This completes the proof of the theorem.

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NON-AUTONOMOUS INHOMOGENEOUS BOUNDARY CAUCHY PROBLEMS AND RETARDED EQUATIONS

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Abstract. In this paper we prove the existence and the uniqueness of classical solution of non-autonomous inhomogeneous boundary Cauchy problems, this solution is given by a variation of constants formula. Then, we apply this result to show the existence of solution of a retarded equation.

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Key Words and Phrases. Boundary Cauchy problem, Evolution families, Classical solution, Wellposedness, Variation of constants formula, Retarded equation

1 Introduction

Consider the following Cauchy problem with boundary conditions

$$(IBCP)\begin{cases} \dfrac{d}{dt}u(t) = A(t)u(t), & 0 \le s \le t \le T, \\ L(t)u(t) = \phi(t)u(t) + f(t), & 0 \le s \le t \le T, \\ u(s) = u_0. \end{cases}$$

This type of problems presents an abstract formulation of several natural equations such as retarded differential equations, retarded (difference) equations, dynamical population equations and neutral differential equations.

In the autonomous case $(A(t) = A, L(t) = L, \phi(t) = \phi)$ the Cauchy problem $(IBCP)$ was studied by Greiner [2,3]. He used a perturbation of domain of generator of semigroups, and showed the existence of classical solutions of $(IBCP)$ via variation of constants formula. In the homogeneous case $(f = 0)$, Kellermann [6] and Nguyen Lan [8] have showed the existence of an evolution family $(U(t,s))_{t\ge s\ge 0}$ as the classical solution of the problem $(IBCP)$.

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The aim of this paper is to show well-posedness in the general case ($f \neq 0$) and apply this result to get a solution of a retarded equation. In Section 2 we prove the existence and uniqueness of the classical solution of $(IBCP)$. For that purpose, we transform $(IBCP)$ into an ordinary Cauchy problem and prove the equivalence of the two problems. Moreover, the solution of $(IBCP)$ is explicitly given by a variation of constants formula similar to the one given in [3] in the autonomous case. We note that the operator matrices method was also used in [4, 8, 9] for the investigation of inhomogeneous Cauchy problems without boundary conditions.

Section 3 is devoted to an application to the retarded equation

$$(RE)\begin{cases} v(t) = K(t)v_t + f(t), & t \geq s \geq 0, \\ v_s = \varphi. \end{cases}$$

We introduce now the following basic definitions which will be used in the sequel. A family of linear (unbounded) operators $(A(t))_{0\leq t\leq T}$ on a Banach space X is called a stable family if there are constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $]\omega, \infty[\subset \rho(A(t))$ for all $0 \leq t \leq T$ and

$$\left\| \prod_{i=1}^{k} R(\lambda, A(t_i)) \right\| \leq M(\lambda - \omega)^{-k} \text{ for } \lambda > \omega$$

for any finite sequence $0 \leq t_1 \leq ... \leq t_k \leq T$.
A family of bounded linear operators $(U(t,s))_{0\leq s\leq t}$ on X is said an evolution family if
(1) $U(t,t) = I_d$ and $U(t,r)U(r,s) = U(t,s)$ for all $0 \leq s \leq r \leq t$,
(2) the mapping $\left\{(t,s) \in \mathbb{R}_+^2 : t \geq s\right\} \ni (t,s) \longmapsto U(t,s)$ is strongly continuous.
For evolution families and their applications to non-autonomous Cauchy problems we refer to [1,5,10].

2 Well-posedness of Cauchy problem with boundary coditions

Let D, X and Y be Banach spaces, D densely and continuously embedded in X, consider families of operators $A(t) \in L(D,X)$, $L(t) \in L(D,Y)$ and $\phi(t) \in L(X,Y), 0 \leq t \leq T$. In this section we will use the operator matrices method in order to prove the existence of classical solution for the non-autonomous Cauchy problem with inhomogeneous boundary conditions

$$(IBCP)\begin{cases} \dfrac{d}{dt}u(t) = A(t)u(t), & 0 \leq s \leq t \leq T, \\ L(t)u(t) = \phi(t)u(t) + f(t), & 0 \leq s \leq t \leq T, \\ u(s) = u_0, \end{cases}$$

it means that we will transform this Cauchy problem into an ordinary homogeneous one.

In all this section we consider the following hypotheses :
(H_1) $t \longmapsto A(t)x$ is continuously differentiable for all $x \in D$;
(H_2) the family $(A^0(t))_{0\leq t\leq T}$, $A^0(t) := A(t)|_{kerL(t)}$, is stable, with (M_0, ω_0) constants of stability;
(H_3) the operator $L(t)$ is surjective for every $t \in [0,T]$ and $t \longmapsto L(t)x$ is continuously differentiable for all $x \in D$;

(H_4) $t \longmapsto \phi(t)x$ is continuously differentiable for all $x \in X$;
(H_5) there exist constants $\gamma > 0$ and $\omega \in \mathbb{R}$ such that

$$||L(t)x||_Y \geq \gamma^{-1}(\lambda - \omega)||x||_X \text{ for } x \in ker(\lambda - A(t)), \lambda > \omega \text{ and } t \in [0, T].$$

Definition 2.1. *A function* $u : [s, T] \longrightarrow X$ *is called a classical solution of* $(IBCP)$ *if it is continuously differentiable,* $u(t) \in D$, $\forall t \in [s, T]$ *and* u *satisfies* $(IBCP)$*. If* $(IBCP)$ *has a classical solution, we say that it is well-posed.*

We recall the following results which will be used in the sequel.

Lemma 2.1. [6,7] *For* $t \in [0, T]$ *and* $\lambda \in \rho(A^0(t))$ *we have the following properties*
i) $D = D(A^0(t)) \oplus ker(\lambda - A(t))$.
ii) $L(t)|_{ker(\lambda - A(t))}$ *is an isomorphism from* $ker(\lambda - A(t))$ *onto* Y.
iii) $t \longmapsto L_{\lambda,t} := \left(L(t)|_{ker(\lambda - A(t))}\right)^{-1}$ *is strongly continuously differentiable.*

As consequences of this lemma we have $L(t)L_{\lambda,t} = I_{d_Y}$, $L_{\lambda,t}L(t)$ and $(I - L_{\lambda,t}L(t))$ are the projections in D onto $ker(\lambda - A(t))$ and $D(A^0(t))$ respectively.
In order to get the homogenization of $(IBCP)$, we introduce the Banach space $E := X \times C^1([0, T], Y) \times Y$, where $C^1([0, T], Y)$ is the space of continuously differentiable functions from $[0, T]$ into Y equipped with the norm $\|g\| := \|g\|_\infty + \|g'\|_\infty$, for $g \in C^1([0, T], Y)$.
Let $\mathcal{A}^\phi(t)$ be a matrix operator defined on E by

$$\mathcal{A}^\phi(t) := \begin{pmatrix} A(t) & 0 & 0 \\ 0 & 0 & 0 \\ L(t) - \phi(t) & -\delta_t & 0 \end{pmatrix},\ D(\mathcal{A}^\phi(t)) := D \times C^1([0, T], Y) \times \{0\},\ t \in [0, T],$$

here $\delta_t : C([0, T], Y) \longrightarrow Y$ is such that $\delta_t(g) = g(t)$.
To the family $\mathcal{A}^\phi(\cdot)$ we associate the homogeneous Cauchy problem

$$(NCP) \begin{cases} \dfrac{d}{dt}\mathcal{U}(t) = \mathcal{A}^\phi(t)\mathcal{U}(t), \quad 0 \leq s \leq t \leq T, \\ \mathcal{U}(s) = \left(\begin{smallmatrix} u_0 \\ f \\ 0 \end{smallmatrix}\right). \end{cases}$$

In the following proposition we give an equivalence between solutions of $(IBCP)$ and those of (NCP).

Proposition 2.1. *Let* $\left(\begin{smallmatrix} u_0 \\ f \end{smallmatrix}\right) \in D \times C^1([0, T], Y)$.
(i) If the function $t \longmapsto \mathcal{U}(t) := \left(\begin{smallmatrix} u_1(t) \\ u_2(t) \\ 0 \end{smallmatrix}\right)$ *is a classical solution of* (NCP) *with initial value* $\left(\begin{smallmatrix} u_0 \\ f \\ 0 \end{smallmatrix}\right)$. *Then* $t \longmapsto u_1(t)$ *is a classical solution of* $(IBCP)$ *with initial value* u_0.
(ii) Let u *be a classical solution of* $(IBCP)$ *with initial value* u_0*. Then, the function* $t \longmapsto \mathcal{U}(t) = \left(\begin{smallmatrix} u(t) \\ f \\ 0 \end{smallmatrix}\right)$ *is a classical solution of* (NCP) *with initial value* $\left(\begin{smallmatrix} u_0 \\ f \\ 0 \end{smallmatrix}\right)$.

Proof.. *i)* Since $\mathcal{U}$ is a classical solution, then, from Definition 2.1, u_1 is continuously differentiable and $u_1(t) \in D$, for $t \in [s, T]$. Moreover we have

$$\begin{aligned} \mathcal{U}'(t) &= \begin{pmatrix} u_1'(t) \\ u_2'(t) \\ 0 \end{pmatrix} \\ &= \mathcal{A}^{\phi}(t)\mathcal{U}(t) \\ &= \begin{pmatrix} A(t)u_1(t) \\ 0 \\ L(t)u_1(t) - \phi(t)u_1(t) - \delta_t u_2(t) \end{pmatrix}. \end{aligned} \tag{2.1}$$

Therefore

$$u_1'(t) = A(t)u_1(t) \text{ and } u_2'(t) = 0.$$

This implies that $u_2(t) = u_2(s) = f, \forall t \in [s, T]$, hence the equation (2.1) yields to

$$L(t)u_1(t) = \phi(t)u_1(t) + f(t), \; 0 \le s \le t \le T.$$

The initial value condition is obvious.
The assertion (ii) is obvious. $\square$

Now we return to the study of the Cauchy problem (NCP). For that aim, we recall the following result.

Theorem 2.1. *(*[11]*, Theorem 1.3) Let $(A(t))_{0 \le t \le T}$ be a stable family of linear operators on a Banach space X such that*
i) the domain $D := (D(A(t)), \| \cdot \|_D)$ is a Banach space independent of t,
ii) the mapping $t \longmapsto A(t)x$ is continuously differentiable in X for every $x \in D$. Then there is an evolution family $(U(t, s))_{0 \le s \le t \le T}$ on $\overline{D}$. Moreover $U(t, s)$ has the following properties :
(1) *$U(t, s)D(s) \subset D(t)$ for all $0 \le s \le t \le T$, where $D(r)$ is defined by*

$$D(r) := \left\{ x \in D : A(r)x \in \overline{D} \right\},$$

(2) *the mapping $t \longmapsto U(t, s)x$ is continuously differentiable in X on $[s, T]$ and*

$$\frac{d}{dt}U(t, s)x = A(t)U(t, s)x \text{ for all } x \in D(s) \text{ and } t \in [s, T].$$

In order to apply Theorem 2.1, we need the following lemma.

Lemma 2.2. *The family of operators $(\mathcal{A}^{\phi}(t))_{0 \le t \le T}$ is stable.*

Proof.. For $t \in [0, T]$, we write $\mathcal{A}^{\phi}(t)$ as

$$\mathcal{A}^{\phi}(t) = \mathcal{A}(t) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\phi(t) & -\delta_t & 0 \end{pmatrix},$$

where $\mathcal{A}(t) = \begin{pmatrix} A(t) & 0 & 0 \\ 0 & 0 & 0 \\ L(t) & 0 & 0 \end{pmatrix}$, with domain $D(\mathcal{A}(t)) = D(\mathcal{A}^{\phi}(t))$.

Since $\mathcal{A}^{\phi}(t)$ is a perturbation of $\mathcal{A}(t)$ by a linear bounded operator on E, hence, in view of a perturbation result ([10], Thm. 5.2.3) it is sufficient to show the stability of $(\mathcal{A}(t))_{0\leq t\leq T}$.

Let $\lambda > \omega_0$ and $\begin{pmatrix} x \\ f \\ y \end{pmatrix}$, we have

$$(\lambda - \mathcal{A}(t)) \begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ f \\ y \end{pmatrix} = \begin{pmatrix} (\lambda - A(t))R(\lambda, A^0(t))x - (\lambda - A(t))L_{\lambda,t}y \\ f \\ -L(t)R(\lambda, A^0(t))x + L(t)L_{\lambda,t}y \end{pmatrix}$$

Since $R(\lambda, A^0(t))x \in D(A^0(t)) = ker(L(t))$, $L_{\lambda,t}y \in ker(\lambda - A(t))$ and $L(t)L_{\lambda,t} = I_{d_Y}$, we obtain

$$(\lambda - \mathcal{A}(t)) \begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} = I_{d_E}. \qquad (2.2)$$

On the other hand, for $\begin{pmatrix} x \\ f \\ 0 \end{pmatrix} \in D(\mathcal{A}(t))$, we have

$$\begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} (\lambda - \mathcal{A}(t)) \begin{pmatrix} x \\ f \\ 0 \end{pmatrix} = \begin{pmatrix} R(\lambda, A^0(t))(\lambda - A(t))x + L_{\lambda,t}L(t)x \\ f \\ 0 \end{pmatrix}.$$

¿From Lemma 2.1, let $x_1 \in D(A^0(t))$ and $x_2 \in ker(\lambda - A(t))$ such that $x = x_1 + x_2$. Then

$$\begin{aligned} R(\lambda, A^0(t))(\lambda - A(t))x + L_{\lambda,t}L(t)x &= R(\lambda, A^0(t))(\lambda - A(t))(x_1 + x_2) \\ &\quad + L_{\lambda,t}L(t)(x_1 + x_2) \\ &= R(\lambda, A^0(t))(\lambda - A(t))x_1 + L_{\lambda,t}L(t)x_2 \\ &= x_1 + x_2 \\ &= x. \end{aligned}$$

As a consequence, we get

$$\begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} (\lambda - \mathcal{A}(t)) = I_{D(\mathcal{A}(t))}.$$

¿From (2.2) and (2.3) , we obtain that the resolvent of $\mathcal{A}(t)$ is given by

$$R(\lambda, \mathcal{A}(t)) = \begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, by a direct computation one can obtain, for a finite sequence $0 \leq t_1 \leq ... \leq t_k \leq T$,

$$\prod_{i=1}^{k} R(\lambda, \mathcal{A}(t_i)) = \begin{pmatrix} \prod_{i=1}^{k} R(\lambda, A^0(t_i)) & 0 & -\prod_{i=2}^{k} R(\lambda, A^0(t_i))L_{\lambda,t_1} \\ 0 & \frac{1}{\lambda^k} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

¿From the hypothesis (H_5), we conclude that $||L_{\lambda,t}|| \leq \gamma(\lambda-\omega)^{-1}$ for all $t \in [0,T]$ and $\lambda > \omega$. Define $\omega_1 = \max(\omega_0, \omega)$. Therefore, by using (H_2), we obtain for $\begin{pmatrix} x \\ f \\ y \end{pmatrix} \in E$

$$\begin{aligned}\left\|\prod_{i=1}^{k} R(\lambda, \mathcal{A}(t_i)) \begin{pmatrix} x \\ f \\ y \end{pmatrix}\right\| &= \left\|\prod_{i=1}^{k} R(\lambda, A^0(t_i))x - \prod_{i=2}^{k} R(\lambda, A^0(t_i)) L_{\lambda,t_1} y + \frac{1}{\lambda^k} f\right\| \\ &\leq M(\lambda-\omega_1)^{-k}||x|| + M(\lambda-\omega_1)^{-(k-1)}\gamma(\lambda-\omega_1)^{-1}||y|| \\ &\quad + (\lambda-\omega_1)^{-k}||f|| \\ &\leq M^{'}(\lambda-\omega_1)^{-k}\left\|\begin{pmatrix} x \\ f \\ y \end{pmatrix}\right\|,\end{aligned}$$

where $M^{'} := \max(M, M\gamma)$. Thus the lemma is proved. □

Now we are ready to state the main result.

Theorem 2.2. *Let f be a continuously differentiable function on $[0,T]$ onto Y. Then, for all initial value $u_0 \in D$, such that $L(s)u_0 = \phi(s)u_0 + f(s)$, the Cauchy problem $(IBCP)$ has a unique classical solution u. Moreover, u is given by the variation of constants formula*

$$\begin{aligned}u(t) &= U(t,s)(I - L_{\lambda,s}L(s))u_0 + L_{\lambda,t}f(t,u(t)) \\ &\quad + \int_s^t U(t,r)\left[\lambda L_{\lambda,r}f(r,u(r)) - (L_{\lambda,r}f(r,u(r)))^{'}\right] dr, \qquad (2.4)\end{aligned}$$

where $(U(t,s))_{t\geq s\geq 0}$ is the evolution family generated by $A^0(t)$ and $f(t,u(t)) := \phi(t)u(t) + f(t)$.

Proof. First, for the existence of $U(t,s)$ we refer to [7]. Since $(\mathcal{A}^{\phi}(t))_{0\leq t\leq T}$ is stable and from assumptions (H_1), (H_3) and (H_4), $(\mathcal{A}^{\phi}(t))_{0\leq t\leq T}$ satisfies all conditions of Theorem 2.1, then there exists an evolution family $\mathcal{U}^{\phi}(t,s)$ such that, for all initial value $\begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix} \in D(\mathcal{A}^{\phi}(s))$, the function $\begin{pmatrix} u_1(t) \\ u_2(t) \\ 0 \end{pmatrix} := \mathcal{U}^{\phi}(t,s)\begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix}$ is a classical solution of (NCP). Therefore, from (i) of Proposition 2.1, u_1 is a classical solution of $(IBCP)$. The uniqueness of u_1 comes from the uniqueness of the solution of (NCP) and Proposition 2.1.
Let u be a classical solution of $(IBCP)$, at first, we assume that $\phi(t) = 0$, then

$$\begin{aligned}u_2(t) &:= L_{\lambda,t}L(t)u(t) \\ &= L_{\lambda,t}f(t),\end{aligned}$$

and $u_1(t) := (I - L_{\lambda,t}L(t))u(t)$ are differentiable on t and we have

$$\begin{aligned}u_1'(t) &= u'(t) - u_2'(t) \\ &= A(t)(u_1(t) + u_2(t)) - (L_{\lambda,t}f(t))^{'} \\ &= A^0(t)u_1(t) + \lambda L_{\lambda,t}f(t) - (L_{\lambda,t}f(t))'.\end{aligned}$$

If we define $\tilde{f}(t) := \lambda L_{\lambda,t} f(t) - (L_{\lambda,t} f(t))'$, we obtain

$$u_1(t) = U(t,s)u_1(s) + \int_s^t U(t,r)\tilde{f}(r)\,dr.$$

Replacing $u_1(s)$ by $(I - L_{\lambda,s}L(s))u_0$, we obtain

$$u_1(t) = U(t,s)(I - L_{\lambda,s}L(s))u_0 + \int_s^t U(t,r)\tilde{f}(r)\,dr,$$

consequently,

$$u(t) = U(t,s)(I - L_{\lambda,s}L(s))u_0 + L_{\lambda,t}f(t) + \int_s^t U(t,r)\left[\lambda L_{\lambda,r}f(r) - (L_{\lambda,r}f(r))'\right] dr. \quad (2.5)$$

Now in the case $\Phi(t) \neq 0$, since $f(\cdot, u(\cdot))$ is continuously differentiable, similar arguments are used to obtain the formula (2.5) for $f(\cdot) := f(\cdot, u(\cdot))$ which is exactly (2.4). $\square$

3 Retarded equation

On the Banach space $C_E^1 := C^1([-r,0],E)$, where $r > 0$ and E is a Banach space, we consider the retarded equation

$$(RE)\begin{cases} v(t) = K(t)v_t + f(t), & 0 \le s \le t \le T, \\ v_s = \varphi \in C_E^1. \end{cases}$$

Where $v_t(\tau) := v(t+\tau)$, for $\tau \in [-r,0]$, and $f : [0,T] \longrightarrow E$.

Definition 3.1. *A function $v : [s-r,T] \longrightarrow E$ is said a solution of (RE), if it is continuously differentiable, $K(t)v_t$ is well defined, $\forall t \in [0,T]$ and v satisfies (RE).*

In this section we are interested in the study of the retarded equation (RE), we will apply the abstract result obtained in the previous section in order to get a solution of (RE). As a first step, we show that this problem can be written as a boundary Cauchy one. More precisely, we show in the following theorem that solutions of (RE) are equivalent to those of the boundary Cauchy problem

$$(IBCP)'\begin{cases} \dfrac{d}{dt}u(t) = A(t)u(t), \quad 0 \le s \le t \le T, & (3.1) \\ L(t)u(t) = \phi(t)u(t) + f(t), & (3.2) \\ u(s) = \varphi. & (3.3) \end{cases}$$

Where $A(t)$ is defined by

$$\begin{cases} A(t)u := u' \\ D := D(A(t)) = C^1([-r,0],E), \end{cases}$$

$L(t) : D \longrightarrow E : L(t)\varphi = \varphi(0)$ and $\phi(t) : C([-r,0],E) \longrightarrow E : \phi(t)\varphi = K(t)\varphi$.
Note that the spaces D, X and Y in section 2, are given here by $D := C^1([-r,0],E)$, $X := C([-r,0],E)$ and $Y := E$.
We have the following theorem

Theorem. *i) If u is a classical solution of $(IBCP)'$, then the function v defined by*

$$v(t) := \begin{cases} u(t)(0), & s \leq t \leq T, \\ \varphi(t-s), & -r+s \leq t \leq s, \end{cases}$$

is a solution of (RE).
ii) If v is a solution of (RE), then $t \longmapsto u(t) := v_t$ is a classical solution of $(IBCP)'$.

Proof. *i)* Let u be a classical solution of $(IBCP)'$, then from Definition 2.1, v is continuously differentiable. On the other hand, (3.1) and (3.3) implies that u verifies the translation property

$$u(t)(\tau) = \begin{cases} u(t+\tau)(0), & s \leq t+\tau \leq T \\ \varphi(t+\tau-s), & -r+s \leq t+\tau \leq s, \end{cases}$$

which implies that $v_t(\cdot) = u(t)(\cdot)$. Therefore, from (3.2), we obtain

$$\begin{aligned} v(t) &= u(t)(0) \\ &= L(t)u(t)(\cdot) + f(t) \\ &= K(t)u(t)(\cdot) + f(t) \\ &= K(t)v_t(\cdot) + f(t). \end{aligned}$$

Hence v satisfies (RE).
ii) Now, let v be a solution of (RE). From Definition 3.1, $u(t) \in C^1([-r,0],E) = D(A(t))$, for $s \leq t \leq T$. Moreover,

$$\begin{aligned} L(t)u(t) &= u(t)(0) \\ &= v(t) \\ &= K(t)v(t) + f(t) \\ &= \phi(t)u(t) + f(t). \end{aligned}$$

The equation (3.1) is obvious. $\square$

This theorem allows us to concentrate our self on the problem $(IBCP)'$. So, it remains to show that the hypotheses $(H_1) - (H_5)$ are satisfied.
The hypotheses (H_1), (H_3) are obvious and (H_4) can be obtained from the assumptions on the operator $K(t)$.
For (H_2), let $\varphi \in D(A^0(t)) = \left\{\varphi \in C^1([-r,0],E)\,;\, \varphi(0) = 0\right\}$ and $f \in C([-r,0],E)$ such that $(\lambda - A^0(t))\varphi = f$, for $\lambda > 0$. Then

$$\varphi(\tau) = e^{\lambda\tau}\varphi(0) + \int_\tau^0 e^{\lambda(\tau-\sigma)} f(\sigma)\, d\sigma, \quad \tau \in [-r,0].$$

Since $\varphi(0) = 0$, we get

$$(R(\lambda, A^0(t))f)(\tau) = \int_\tau^0 e^{\lambda(\tau-\sigma)} f(\sigma)\, d\sigma.$$

By induction , we can show that

$$\left(\prod_{i=1}^{k} R(\lambda, A^0(t_i))f\right)(\tau) = \frac{1}{(k-1)!}\int_{\tau}^{0} (\sigma-\tau)^{k-1} e^{\lambda(\tau-\sigma)} f(\sigma)\, d\sigma,$$

for a finite sequence $0 \leq t_1 \leq ... \leq t_k \leq T$. Hence

$$\begin{aligned}\left\|\left(\prod_{i=1}^{k} R(\lambda, A^0(t_i))f\right)(\tau)\right\| &\leq \frac{1}{(k-1)!}\int_{\tau}^{0} (\sigma-\tau)^{k-1} e^{\lambda(\tau-\sigma)}\, d\sigma \|f\| \\ &= e^{\lambda\tau}\sum_{i=k}^{\infty} \frac{\lambda^{i-k}(-\tau)^i}{i!}\|f\| \\ &= \frac{e^{\lambda\tau}}{\lambda^k}\sum_{i=k}^{\infty} \frac{(-\lambda\tau)^i}{i!}\|f\| \\ &\leq \frac{1}{\lambda^k}\|f\|, \quad \text{for } \tau \in [-r, 0].\end{aligned}$$

Therefore

$$\left\|\prod_{i=1}^{k} R(\lambda, A^0(t_i))f\right\| \leq \frac{1}{\lambda^k}\|f\|, \quad \lambda > 0.$$

This proves the stability of $A^0(t))_{t\in[0,T]}$.
Now, if $\varphi \in ker(\lambda - A(t))$, then $\varphi(\tau) = e^{\lambda\tau}\varphi(0)$, for $\tau \in [-r, 0]$. Hence

$$\begin{aligned}\|L(t)\varphi\| &= \|\varphi(0)\| \\ &= \|e^{-\lambda\tau}\varphi(\tau)\|,\end{aligned}$$

since $\lim_{\lambda\to+\infty} \frac{e^{-\lambda\cdot}}{\lambda} = +\infty$, in C_E, we can take $c > 0$ such that $\frac{e^{-\lambda\cdot}}{\lambda} \geq c$, therefore

$$\|L(t)\varphi\| \geq c\lambda\|\varphi\|,\ \forall t \in [0, T].$$

So (H_5) holds. As a conclusion, we get the following corollary

Corollary 3.1. *Let f be a continuously differentiable function from $[0, T]$ onto E, then for all $\varphi \in C_E^1$ such that, $\varphi(0) = K(s)\varphi + f(s)$, the retarded equation (RE) has a solution v, moreover, v satisfies*

$$v_t = T(t-s)(\varphi - e^{\lambda\cdot}\varphi(0)) + e^{\lambda\cdot} f(t, v_t) + \int_s^t T(t-r)e^{\lambda\cdot}\left[\lambda f(r, v_r) - (f(r, v_r))'\right] dr,$$

where $(T(t))_{t\geq 0}$ is the c_0-semigroup generated by $A^0(t)$.

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SMOOTHERS AND THEIR APPLICATIONS IN AUTONOMOUS SYSTEM THEORY

J. E. Palomar Tarancón

Abstract. In this paper the author introduces the concept of smoother. Roughly speaking, a smoother is a pair (s, K) consisting of a continuous map s sending each point p of its domain into a closed neighborhood V_p of p, and an operator K that transforms any function f into another Kf being smoother than f. This property allows us to remove the effect of a perturbation P from the solutions of an autonomous system the vector field of which is modified by P.

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0. Introduction

The main aim of this paper consists of introducing the concept of smoother together with an application in differential equation theory. Roughly speaking a smoother is an operator transforming an arbitrary function f_1 into a similar one f_2 being smoother than f_1. In general, smoothers perform integral transforms in function spaces. To get a first approximation to the smoother concept consider the following facts. Let $y = f(x)$ be any integrable function defined in $\mathbb{R}$ and $\sigma : \mathbb{R} \to \mathcal{C}$ a map such that $\mathcal{C}$ stands for the collection of all closed subsets of $\mathbb{R}$ the interior of each of which is non-void. For every $x \in \mathbb{R}$, let λ_x be any non-negative real number, and let $\sigma(x) = [x - \lambda_x, x + \lambda_x]$. With these assumptions, consider the linear transform $\mathfrak{K}$ defined as follows. If $\lambda_x \neq 0$ is a finite number, then

$$\mathfrak{K}f(x) = \frac{1}{2\lambda_x}\int_{-\lambda_x}^{\lambda_x} f(x+\tau)\, d\tau \tag{0.0.1}$$

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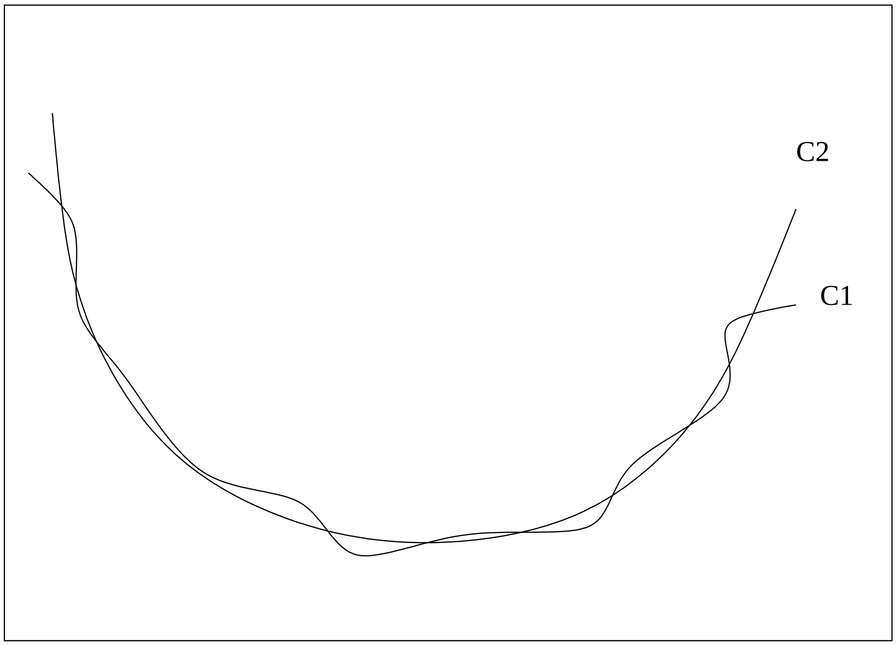

Figure 1.

Conversely, for λ_x infinite

(0.0.2) $$\mathfrak{K}f(x) = \lim_{k\to\infty} \frac{1}{2k} \int_{-k}^{k} f(x+\tau)\, d\tau$$

Finally, if $\lambda_x = 0$, then

(0.0.3) $$\mathfrak{K}f(x) = \lim_{k\to 0} \frac{1}{2k} \int_{-k}^{k} f(x+\tau)\, d\tau = f(x)$$

Of course, assuming that such a limit exists. Thus, the integral transform $\mathfrak{K}$ sends the value of $f(x)$ at x into the average of all values of $f(x)$ in a closed neighborhood $[x-\lambda_x, x+\lambda_x]$ of x. Obviously, in general, the transform $\mathfrak{K}f(x)$ is smoother than $f(x)$. To see this fact consider the following cases. If for every x, $\sigma(x) = \mathbb{R}$, then $\forall x \in \mathbb{R}: \ \lambda_x = \infty$ and $\mathfrak{K}f(x)$ is a constant function, that is the smoothest one that can be built. If for every $x \in \mathbb{R}$, $\sigma(x) = \{x\}$, then $\forall x \in : \lambda_x = 0$, therefore $\mathfrak{K}f(x) = f(x)$, and consequently both functions have the same smoothness degree. Thus, between both extreme cases one can build several degrees of smoothness. In the former example, what we term smoother is nothing but the pair $(\sigma, \mathfrak{K})$.

Perhaps the most natural smoother application consists of removing, from a given function, the noise arising from some perturbation. For instance consider the curves $C1$ and $C2$ in Figure 1. Suppose that the differences between $C1$ and $C2$ are consequence of some perturbation working over $C2$. If both curves are the plots of two functions $f_1(x)$ and $f_2(x)$ respectively, in general, one can build a smoother $(\sigma, \mathfrak{K})$ such that $\mathfrak{K} f_1(x) = f_2(x)$. Now, consider a vector field $\boldsymbol{X}$ and the result $\boldsymbol{Y}$ of a perturbation P working over $\boldsymbol{X}$, and assume $(\sigma, \mathfrak{K})$ to satisfy the relation $\mathfrak{K}\boldsymbol{Y} = \boldsymbol{X}$. If $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are the general solutions for the ordinary differential equations $\dfrac{d}{dt}\,\boldsymbol{x}(t) = \boldsymbol{X}\big(\boldsymbol{x}(t)\big)$ and $\dfrac{d}{dt}\,\boldsymbol{y}(t) = \boldsymbol{Y}\big(\boldsymbol{y}(t)\big)$ respectively, then we shall say the smoother $(\sigma, \mathfrak{K})$ to be compatible with the vector field $\boldsymbol{Y}$, provided that the following relation holds: $\mathfrak{K}\boldsymbol{y}(t) = \boldsymbol{x}(t)$. Thus, one can obtain the corresponding perturbation-free function from solutions of $\dfrac{d}{dt}\,\boldsymbol{y}(t) = \boldsymbol{Y}\big(\boldsymbol{y}(t)\big)$ using the smooth vector field $\boldsymbol{X} = \mathfrak{K}\boldsymbol{Y}$ instead. The main aim of this paper consists of investigating a compatibility criterion.

1. Smoothers

Let $\boldsymbol{Top}$ stand for the category of all topological spaces, and let $\mathfrak{N} : \boldsymbol{Top} \to \boldsymbol{Top}$ be the endofunctor carrying each object $(E, \mathcal{T})$ in $\boldsymbol{Top}$ into the topological space $\mathfrak{N}(E, \mathcal{T}) = (\wp(E) \setminus \{\{\emptyset\}\}, \mathcal{T}^*)$ the underlying set of which $\wp(E) \setminus \{\{\emptyset\}\}$ consists of all nonempty subsets of E. Let $\mathcal{T}^*$ be the topology a subbase S of which is defined as follows. Denote by $\mathcal{C}$ the collection of all $\mathcal{T}$-closed subsets of E and for every pair $(A, B) \in \mathcal{C} \times \mathcal{T}$, let $K_{A,B} = \{C \in \wp(E) | A \subset C \subseteq B\}$. With these assumptions, define the subbase S as follows.

$$S = \{K_{A,B} | (A, B) \in \mathcal{C} \times \mathcal{T}\}$$

Obviously, if $A \supset B$, then $K_{A,B} = \emptyset$. Likewise, if $A = \emptyset$ and $B = E$, then $K_{A,B} = \wp(E) \setminus \{\{\emptyset\}\}$.

Let the arrow-map of $\mathfrak{N}$ be the law sending each continuous map

$$f : (E_1, \mathcal{T}_1) \to (E_2, \mathcal{T}_2)$$

into the map $\mathfrak{N}f$ carrying each subset $A \subseteq E_1$ into $f[A] \subseteq E_2$. It is not difficult to see $\mathfrak{N}f$ to be a continuous map with respect to the associated topology $\mathcal{T}^*$.

Definition 1.0.1. Let $\boldsymbol{cAlg}(\mathfrak{N})$ denote the concrete category of $\mathfrak{N}$-co-algebras. Thus, every object in $\boldsymbol{cAlg}(\mathfrak{N})$ is a pair $\big((E, \mathcal{T}), \sigma_{(E,\mathcal{T})}\big)$, consisting of a topological space $(E, \mathcal{T})$ together with a continuous map $\sigma_{(E,\mathcal{T})} : (E, \mathcal{T}) \to \mathfrak{N}(E, \mathcal{T})$.

Recall that a continuous mapping $f : (E_1, \mathcal{T}_1) \to (E_2, \mathcal{T}_2)$ is a morphism in $\boldsymbol{cAlg}(\mathfrak{N})$ from $\big((E_1, \mathcal{T}_1), \sigma_{(E_1,\mathcal{T}_1)}\big)$ into $\big((E_2, \mathcal{T}_1), \sigma_{(E_2,\mathcal{T}_2)}\big)$, provided that the following diagram commutes.

(1.0.4)
$$\begin{tikzcd}
(E_1, \mathcal{T}_1) \arrow[r, "\sigma_{(E_1,\mathcal{T}_1)}"] \arrow[d, "f"'] & \mathfrak{N}(E_1, \mathcal{T}_1) \arrow[d, "\mathfrak{N}f"] \\
(E_2, \mathcal{T}_2) \arrow[r, "\sigma_{(E_2,\mathcal{T}_2)}"'] & \mathfrak{N}(E_2, \mathcal{T}_2)
\end{tikzcd}$$

Now, let $\boldsymbol{Tvec}$ be the topological vector space category, and let $\boldsymbol{TopVec}$ denote the category the objects of which are products of the form $\mathfrak{N}(E,\mathcal{T})\times\big(\mathcal{C}^0(E,V),T^*\big)$, where (V,T) is a topological vector space and T^* the pointwise topology for the set $\mathcal{C}^0(E,V)$ of all continuous maps from $(E,\mathcal{T})$ into $|(V,T)|$; where the functor $|\ |:\boldsymbol{Tvec}\to\boldsymbol{Top}$ forgets the vector space structure and preserves the topological one. In addition, a $\boldsymbol{TopVec}$-morphism with domain $\mathfrak{N}(E_1,\mathcal{T}_1)\times\big(\mathcal{C}^0(E_1,V),T_1^*\big)$ and codomain $\mathfrak{N}(E_2,\mathcal{T}_2)\times\big(\mathcal{C}^0(E_2,V),T_2^*\big)$ is of the form $\mathfrak{N}f\times g$ where f lies in $\hom_{\boldsymbol{Top}}(E_1,E_2)$ and g is a continuous mapping with domain $\mathcal{C}^0(E_1,V)$ and codomain $\mathcal{C}^0(E_2,V)$.

Given any topological space $(E,\mathcal{T})$, let

$$\mathfrak{P}_{(E,\mathcal{T})}:\boldsymbol{Tvec}\to\boldsymbol{TopVec}$$

denote the functor carrying each $\boldsymbol{Tvec}$-object (V,T) into the product

$$\mathfrak{N}(E,\mathcal{T})\times\big(\mathcal{C}^0(E,V),T^*\big)$$

and sending every $\boldsymbol{Tvec}$-morphism $f:(V_1,T_1)\to(V_2,T_2)$ into $Id\times f_*$; where $f_*=\hom_{\boldsymbol{Top}}((E,\mathcal{T}),|f|)$ stands for the morphism carrying each $g\in\mathcal{C}^0(E,V_1)$ into $f\circ g\in\mathcal{C}^0(E,V_2)$, and as usual $\hom_{\boldsymbol{Top}}((E,\mathcal{T}),|-|)$ denotes the covariant hom-functor.

Finally, let $\boldsymbol{Alg}(\mathfrak{P}_{(E,\mathcal{T})})$denote the category of $\mathfrak{P}_{(E,\mathcal{T})}$-algebras, that is, each object is a pair of the form $\Big((V,T),\mathfrak{K}_{(V,T)}\Big)$ where

$$\mathfrak{K}_{(V,T)}:\mathfrak{P}_{(E,\mathcal{T})}(V,T)=\mathfrak{N}(E,\mathcal{T})\times\big(\mathcal{C}^0(E,V),T^*\big)\to|(V,T)|$$

is a continuous map. In addition, a given continuous linear mapping

$$f:(V_1,T_1)\to(V_2,T_2)$$

is an $\boldsymbol{Alg}(\mathfrak{P}_{(E,\mathcal{T})})$-morphism whenever the following quadrangle commutes.

(1.0.5)
$$\begin{tikzcd}
\mathfrak{N}(E,\mathcal{T})\times\big(\mathcal{C}^0(E,V_1),T_1^*\big) \arrow[r, "\mathfrak{K}_{(V_1,T_1)}"] \arrow[d, "Id\times\hom_{\boldsymbol{Top}}((E,\mathcal{T}),|f|)"'] & |(V_1,T_1)| \arrow[d, "|f|"] \\
\mathfrak{N}(E,\mathcal{T})\times\big(\mathcal{C}^0(E,V_2),T_2^*\big) \arrow[r, "\mathfrak{K}_{(V_2,T_2)}"'] & |(V_2,T_2)|
\end{tikzcd}$$

Definition 1.0.2. A smoother will be any pair

$$\Big(\big((E,\mathcal{T}),\sigma_{(E,\mathcal{T})}\big),\big((V,T),\mathfrak{K}_{(V,T)}\big)\Big)$$

such that $\big((E,\mathcal{T}),\sigma_{(E,\mathcal{T})}\big)$ is a co-algebra lying in $\boldsymbol{cAlg}(\mathfrak{N})$ and $\big((V,T),\mathfrak{K}_{(V,T)}\big)$ is an algebra in $\boldsymbol{Alg}(\mathfrak{P}_{(E,\mathcal{T})})$ satisfying the following conditions.

a) For every $p\in E$: $p\in\sigma_{(E,\mathcal{T})}(p)$.

b) For every $(p,f)\in E\times\mathcal{C}^0(E,V)$:

$$\mathfrak{K}_{(V,T)}\big(\sigma_{(E,\mathcal{T})}(p),f\big)\in\mathfrak{C}\left(\mathfrak{N}f\big(\sigma_{(E,\mathcal{T})}(p)\big)\right)$$

where, for any subset $A \subseteq E$, $\mathfrak{C}(A)$ denotes the convex cover of A.

c) $\mathfrak{K}_{(V,T)}$ is linear with respect to its second argument, that is to say, for every couple of scalars (α, β) and each pair of maps (f, g) the following holds.

$$\begin{aligned}\mathfrak{K}_{(V,T)}\big(\sigma_{(E,\mathcal{T})}(p), \alpha f + \beta g\big) = \\ \alpha \mathfrak{K}_{(V,T)}\big(\sigma_{(E,\mathcal{T})}(p), f\big) + \beta \mathfrak{K}_{(V,T)}\big(\sigma_{(E,\mathcal{T})}(p), g\big)\end{aligned} \tag{1.0.6}$$

1.0.1. Transformation associated to a smoother. Given a homeomorphism $\varphi : (E, \mathcal{T}) \to |(V, T)|$, a smoother

$$\mathfrak{S} = \Big(\big((E, \mathcal{T}), \sigma_{(E,\mathcal{T})}\big), \big((V, T), \mathfrak{K}_{(V,T)}\big)\Big)$$

induces a transformation $\mathfrak{S}_\varphi$ carrying each point $p \in E$ into

$$\varphi^{-1}\left(\mathfrak{K}_{(V,T)}\big(\sigma_{(E,\mathcal{T})}(p), \varphi\big)\right)$$

which will be said to be induced by $\mathfrak{S}$. Likewise, for every one-parameter continuous map family $h : E \times I \subseteq E \times \mathbb{R} \to E$ one can define the induced transformation by

$$\mathfrak{S}_{h(p,t)}(p) = \varphi^{-1}\left(\mathfrak{K}_{(V,T)}\left(\sigma_{(E,\mathcal{T})}(h(p,t), \varphi\right)\right) \tag{1.0.7}$$

1.0.2. Ordering. Smoothers form a category ***Smtr*** the morphism-class of which consists of every $\boldsymbol{cAlg}(\mathfrak{N})$-morphism $f : (E_1, \mathcal{T}_1) \to (E_2, \mathcal{T}_2)$ such that the following quadrangle commutes

$$\begin{tikzcd} \big(\mathfrak{N}(E_1, \mathcal{T}_1) \times \mathcal{C}^0(E_1, V)\big) \arrow[r, "\mathfrak{K}_{(V,T)}"] \arrow[d, "\mathfrak{N}f \times \hom_{\boldsymbol{Top}}(f, |(V,T)|)"'] & |(V,T)| \arrow[d, "|Id|"] \\ \big(\mathfrak{N}(E_2, \mathcal{T}_2) \times \mathcal{C}^0(E_2, V)\big) \arrow[r, "\mathfrak{K}_{(V,T)}"'] & |(V,T)| \end{tikzcd} \tag{1.0.8}$$

where $\hom_{\boldsymbol{Top}}(-, |(V,T)|) : \boldsymbol{Top} \to \boldsymbol{Top}^{op}$ stands for the contravariant hom-functor.

Regarding ***Smtr*** as a concrete category over ***Set*** via the forgetful functor such that

$$\Big(\big((E, \mathcal{T}), \sigma_{(E,\mathcal{T})}\big), \big((V, T), \mathfrak{K}_{(V,T)}\big)\Big) \mapsto E$$

with the obvious arrow-map, in each fibre one can define an ordering $\preceq$ as follows. For any smoother $\Big(\big((E, \mathcal{T}), \sigma_{(E,\mathcal{T})}\big), \big((V, T), \mathfrak{K}_{(V,T)}\big)\Big)$, let

$$\Omega\Big(\big((E, \mathcal{T}), \sigma_{(E,\mathcal{T})}\big), \big((V, T), \mathfrak{K}_{(V,T)}\big)\Big)$$

denote the intersection of all topologies for E containing the set family

$$K = \big\{\sigma_{(E,\mathcal{T})}(p) \mid p \in E\big\}$$

then

$$\begin{aligned}\Big(\big((E_1, \mathcal{T}_1), \sigma_{(E_1,\mathcal{T}_1)}\big), \big((V, T), \mathfrak{K}_{(V,T)}\big)\Big) \\ \preceq \Big(\big((E_2, \mathcal{T}_2), \sigma_{(E_2,\mathcal{T}_2)}\big), \big((V, T), \mathfrak{K}_{(V,T)}\big)\Big)\end{aligned} \tag{1.0.9}$$

if and only if the topology $\left(\Omega\left((E_1,\mathcal{T}_1),\sigma_{(E_1,\mathcal{T}_1)}\right),\left((V,T),\mathfrak{K}_{(V,T)}\right)\right)$ is finer than the topology $\Omega\left(\left((E_2,\mathcal{T}_2),\sigma_{(E_2,\mathcal{T}_2)}\right),\left((V,T),\mathfrak{K}_{(V,T)}\right)\right)$. For a maximal element, the topology $\Omega\left(\left((E,\mathcal{T}),\sigma_{(E,\mathcal{T})}\right),\left((V,T),\mathfrak{K}_{(V,T)}\right)\right)$ must be indiscrete. In this case, for every p in E, $\sigma_{(E,\mathcal{T})}(p)=E$, and consequently, for every $p,q\in E$,

$$\mathfrak{K}_{E,\mathcal{T}}(\sigma_{(E,\mathcal{T})}(p),\varphi)=\mathfrak{K}_{(E,\mathcal{T})}(E,\varphi)=\mathfrak{K}(\sigma_{(E,\mathcal{T})}(q),\varphi)$$

therefore $\mathfrak{S}_\varphi h(p,t)=\varphi^{-1}\left(\mathfrak{K}_{(V,T)}\left(\sigma_{(E,\mathcal{T})}(h(p,t),\varphi\right)\right)$ transforms $h(p,t)$ into a constant map, which is the smoothest function one can build. Conversely, a minimal element corresponds to the discrete topology. In this case, by virtue of both conditions **a**) and **b**), the transformation (1.0.7) is the identity, so then $h(p,t)$ remains unaltered. Between both extremes one can build several degrees of smoothness.

1.1 Smoothers in smooth manifolds.. Let $(M,\mathcal{A}_n)$ be a smooth manifold, $\varphi:U\subseteq M\to\mathbb{R}^n$ a chart and $\mathcal{T}$ the induced topology for U. Henceforth, the pair $(U,\mathcal{T})$ will be assumed to be a Hausdorff space. In the most natural way, one can build a smoother $\mathfrak{S}=\left(\left((U,\mathcal{T}),\sigma_{(U,\mathcal{T})}\right),\left((\mathbb{R}^n,T),\mathfrak{K}_{(\mathbb{R}^n,T)}\right)\right)$ over $(U,\mathcal{T})$ the associated set of continuous maps $\mathcal{C}^0(U,\mathbb{R}^n)$ contains each smooth one like the diffeomorphism associated to each chart.

For those smooth manifolds such that, for each $p\in M_n$, each tangent space T_p is isomorphic to $\mathbb{R}^n$, that is to say, there is an isomorphism $\lambda_p:T_p\to\mathbb{R}^n$, one can associate a map $\omega_{\boldsymbol{X}}:U\to\mathbb{R}^n$ to every smooth vector field $\boldsymbol{X}$ letting

$$\forall p\in U:\quad \omega_{\boldsymbol{X}}(p)=\lambda_p\big(\boldsymbol{X}_p)\big) \tag{1.1.1}$$

Accordingly, the image of the vector field $\boldsymbol{X}$ under $\mathfrak{S}$ is

$$\lambda_p\big(\boldsymbol{Y}\big)=\mathfrak{K}_{(\mathbb{R}^n,T)}\big(\sigma_{(U,\mathcal{T})}(p),\omega_{\boldsymbol{X}}\big) \tag{1.1.2}$$

therefore

$$\boldsymbol{Y}=\lambda_p^{-1}\left(\mathfrak{K}_{(\mathbb{R}^n,T)}\big(\sigma_{(U,\mathcal{T})}(p),\omega_{\boldsymbol{X}}\big)\right) \tag{1.1.3}$$

From the former equations it follows immediately that if $h_t:U\to U$ is the one-parameter group associated to $\boldsymbol{X}$, then

$$\omega_{\boldsymbol{X}}(p)=\lambda_p\left(\boldsymbol{X}\right)\Big|_p=\lim_{t\to 0}\frac{\varphi\circ h_t(p)-\varphi(p)}{t}\in\mathbb{R}^n \tag{1.1.4}$$

accordingly

$$\begin{aligned}&\mathfrak{K}_{(\mathbb{R}^n,T)}\big(\sigma_{(U,\mathcal{T})}(p),\omega_{\boldsymbol{X}}\big)=\\&\qquad\lim_{t\to 0}\frac{\mathfrak{K}_{(\mathbb{R}^n,T)}\big(\sigma_{(U,\mathcal{T})}(p),\varphi\circ h_t\big)-\mathfrak{K}_{(\mathbb{R}^n,T)}\big(\sigma_{(U,\mathcal{T})}(p),\varphi\big)}{t}\end{aligned} \tag{1.1.5}$$

Definition 1.1.1. Let $\boldsymbol{X}$ be a smooth vector field the coordinates of which are $(X^1, \dots X^n)$, and consider the differential equation

$$(1.1.6)\qquad \begin{cases} \dfrac{d}{dt}\, x^1(\gamma(p,t)) = X^1\big(x^1(\gamma(p,t), x^2(\gamma(p,t)\dots\big) \\ \dfrac{d}{dt}\, x^2(\gamma(p,t)) = X^2\big(x^1(\gamma(p,t), x^2(\gamma(p,t)\dots\big) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{d}{dt}\, x^n(\gamma(p,t)) = X^n\big(x^1(\gamma(p,t), x^2(\gamma(p,t)\dots\big) \end{cases}$$

where the differentiable curve $\gamma : I \subseteq \mathbb{R} \to U$ is assumed to be solution of the former equation for the initial value $\gamma(p,t_0) = p$. Say, a smoother $\mathfrak{S} = \big(((U,\mathcal{T}), \sigma_{(U,\mathcal{T})}), ((\mathbb{R}^n, T), \mathfrak{K}_{(\mathbb{R}^n,T)})\big)$ to be compatible with $\boldsymbol{X}$ provided that the curve $\boldsymbol{y}\big(\rho(p,t)\big) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(\gamma(p,t)\right), \varphi\right)$ is solution of the equation

$$(1.1.7)\qquad \begin{cases} \dfrac{d}{dt}\, y^1(\rho(p,t)) = Y^1\big(y^1(\rho(p,t), y^2(\rho(p,t)\dots\big) \\ \dfrac{d}{dt}\, y^2(\rho(p,t)) = Y^2\big(y^1(\rho(p,t), y^2(\rho(p,t)\dots\big) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{d}{dt}\, y^n(\rho(p,t)) = Y^n\big(y^1(\rho(p,t), y^2(\rho(p,t)\dots\big) \end{cases}$$

where $q = \mathfrak{S}_\varphi(y(p,t_0)) = \mathfrak{S}_\varphi(p)$, and

$$(1.1.8)\qquad (Y^1, Y^2, \dots Y^n) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \omega_{\boldsymbol{X}}\right)$$

Obviously, if p is a fixed point for $\mathfrak{S}_\varphi$, then $\boldsymbol{y}(q,t)$ and $\boldsymbol{x}(p,t)$ are solutions of (1.1.6) and (1.1.7), respectively, for the same initial value $p = x(p,t_0) = y(p,t_0)$.

Remark. If $p = q$, that is to say, if p is a fixed-point for $\mathfrak{S}_\varphi$, then from Definition 1.0.2 the relations

$$(1.1.9)\qquad \forall p \in U: \quad \boldsymbol{x}(p) \in \mathfrak{C}\left(\mathfrak{N}\varphi\left(\sigma_{(U,\mathcal{T})}(p)\right)\right)$$

and

$$(1.1.10)\qquad \forall p \in U: \quad \boldsymbol{y}(p) \in \mathfrak{C}\left(\mathfrak{N}\varphi\left(\sigma_{(U,\mathcal{T})}(p)\right)\right)$$

are true, therefore

$$(1.1.11)\qquad \|\boldsymbol{x}(p) - \boldsymbol{y}(p)\| \le \max_{q_1,q_2 \in \mathfrak{C}\left(\sigma_{(U,\mathcal{T})}(p)\right)} \|\varphi(q_1) - \varphi(q_2)\|$$

From the former relation one can build some proximity criteria. If the maximum distance among points in any set $\sigma_{(U,\mathcal{T})}(p)$ is bounded, that is to say, if there is $\delta > 0$ such that

$$\forall p \in U: \quad \max_{q_1,q_2 \in \mathfrak{C}\left(\sigma_{(U,\mathcal{T})}(p)\right)} \|\varphi(q_1) - \varphi(q_2)\| < \delta$$

then

$$\forall t > 0: \quad \|\boldsymbol{x}(p,t) - \boldsymbol{y}(p,t)\| < \delta$$

Proposition 1.1.3. *Let $\big((U,\mathcal{T}),\sigma_{(U,\mathcal{T})}\big)$ be a co-algebra in $\boldsymbol{cAlg}(\mathfrak{N})$ and for every point p of U let $\mu_p : \sigma_{(U,\mathcal{T})} \to [0,\infty)$ be a measure for $\sigma_{(U,\mathcal{T})}(p)$ such that the set $\sigma_{(U,\mathcal{T})}(p)$ is μ_p-measurable. If for every $p \in U$ the following condition hold,*

a) $p \in \sigma_{(U,\mathcal{T})}(p)$.

b) $\sigma_{(U,\mathcal{T})}(p)$ is a closed subset of U.

c) $\mu_p\big(\sigma_{(U,\mathcal{T})}(p)\big) = 1$

then the pair $\Big(\big((U,\mathcal{T}),\sigma_{(U,\mathcal{T})}\big)\,,\big((V,T),\mathfrak{K}_{(V,T)}\big)\Big)$ is a smoother, where

$$\mathfrak{K}_{(V,T)}\Big(\sigma_{(U,\mathcal{T})}(p),\boldsymbol{f}\Big) = \underset{\sigma_{(U,\mathcal{T})}(p)}{\int\cdots\int} \boldsymbol{f}\, d\mu_p \tag{1.1.12}$$

Proof. Obviously, $\mathfrak{K}_{(V,T)}$ is linear with respect to its second coordinate, and by definition, it satisfies condition **a**) in Definition 1.0.2, therefore it remains to be proved $\mathfrak{K}_{(V,T)}$ to satisfy condition **b**) too.

It is a well-known fact that each coordinate f^j of any measurable function $\boldsymbol{f}$ is the limit of a sequence $\big\{f^j_n \mid n \in \mathbb{N}\big\}$ of step-functions each of which of the form

$$f^j_n = \sum_{i=1}^{m} c^j_{i,n}\chi_{E_{i,n}} \tag{1.1.13}$$

such that each of the $E_{i,n}$ is μ_p-measurable and for every $i \in \mathbb{N}$, $c^j_{i,n} = f^j(\alpha_i)$ for some $\alpha_i \in E_{i,n}$, besides, $\forall n \in \mathbb{N} : E_{i,n} \cap E_{j,n} = \emptyset\ (i \neq j)$ and $\cup_{i=1}^m E_{i,n} = \sigma_{(U,\mathcal{T})}(p)$. In addition

$$\underset{\sigma_{(U,\mathcal{T})}(p)}{\int\cdots\int} f^j\, d\mu_p = \lim_{n\to\infty}\sum_{i=1}^{m} c^j_{i,n}\mu_p(\chi_{E_{i,n}}) \tag{1.1.14}$$

Now, from statement **c**) it follows that

$$\forall n \in \mathbb{N} : \qquad \sum_{i=1}^{m}\mu_p(\chi_{E_{i,n}}) = \mu_p(\sigma_{(U,\mathcal{T})}(p)) = 1 \tag{1.1.15}$$

therefore

$$\forall n \in \mathbb{N} : \qquad \sum_{i=1}^{m}\boldsymbol{c}_{i,n}\mu_p(\chi_{E_{i,n}}) \in \mathfrak{C}\left(\mathfrak{N}\boldsymbol{f}(\sigma_{(U,\mathcal{T})}(p))\right) \tag{1.1.16}$$

where $\boldsymbol{c}_{i,n} = (c^1_{i,n}, c^2_{i,n},\cdots)$. Finally, since $\sigma_{(U,\mathcal{T})}(p)$ is assumed to be closed, the proposition follows. $\square$

2. A compatibility criterion

Although smoothers can be useful in several areas, the aim of this paper is its application in differential equations in which only those smoothers being compatible with the associated vectors are useful. To build a compatibility criterion the following result is a powerful tool.

Theorem 2.0.4. *Let $(M_n, \mathcal{A}_n)$ be a smooth manifold and (U, φ) a chart. Let $h_t : U \to U$ stand for the one-parameter group associated to a smooth vector field $\boldsymbol{X}$ and*

$$\mathfrak{S} = \left(\left((U, \mathcal{T}), \sigma_{(U,\mathcal{T})}\right), \left((\mathbb{R}^n, T), \mathfrak{K}_{(\mathbb{R}^n,T)}\right)\right)$$

a smoother. If the following relation holds

$$\exists \delta > 0, \forall t < \delta : \quad \mathfrak{K}_{(\mathbb{R}^n,T)}\Big(\sigma_{(U,\mathcal{T})}\big(h_t(p)\big), \varphi\Big) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \varphi \circ h_t\right) \tag{2.0.17}$$

then $\mathfrak{S}$ is compatible with $\boldsymbol{X}$.

Proof. First, from

$$\boldsymbol{y}(q,t) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_t(p)\right), \varphi\right)$$

we obtain that

$$\varphi(q) = \boldsymbol{y}(q, t_0) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \varphi\right) = \varphi\left(\mathfrak{S}_\varphi(p)\right)$$

Now, it is not difficult to see that

$$\frac{d}{dt}\boldsymbol{y}\Big|_q = \lim_{t\to 0} \frac{\mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_t(p)\right), \varphi\right) - \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_0(p)\right), \varphi\right)}{t} \tag{2.0.18}$$

and using (2.0.17) the former equation becomes

$$\begin{aligned}\frac{d}{dt}\boldsymbol{y}\Big|_q &= \\ &\lim_{t\to 0} \frac{\mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \varphi \circ h_t\right)\big) - \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \varphi\right)}{t} = \\ &\lim_{t\to 0} \frac{\mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \varphi \circ h_t - \varphi\right)}{t} = \\ &\lim_{t\to 0} \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \frac{\varphi \circ h_t - \varphi}{t}\right)\end{aligned} \tag{2.0.19}$$

and by continuity

$$\lim_{t\to 0} \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \frac{\varphi \circ h_t - \varphi}{t}\right) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \lim_{t\to 0}\frac{\varphi \circ h_t - \varphi}{t}\right) \tag{2.0.20}$$

therefore, taking into account (1.1.1) and (1.1.4),

$$\frac{d}{dt}\boldsymbol{y}\Big|_q = \lim_{t\to 0} \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \omega_{\boldsymbol{X}}\right) \tag{2.0.21}$$

accordingly, if $\varphi\big(h_t(p)\big) = (x^1(p,t), x^2(p,t)\dots)$ is solution of (1.1.6) for the initial value p, then

$$(y^1(q,t), y^2(q,t)\dots) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_t(p)\right), \varphi\right)$$

is solution of the equation (1.1.7) for the initial value $q = \mathfrak{S}_\varphi(p)$, being

$$(Y^1, Y^2\dots) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \omega_{\boldsymbol{X}}\right)$$

□

Corollary 2.0.5. *With the same conditions as in the preceding theorem, if p is a fixed point for $\mathfrak{S}_\varphi$ and $\boldsymbol{x}(p,t) = \big(x^1(p,t), x^2(p,t)\dots\big)$ is solution of the initial value problem*

$$\begin{cases} \dfrac{d}{dt}\,\boldsymbol{x}(p,t) = \boldsymbol{X}\big(\boldsymbol{x}(p,t)\big) \\ \boldsymbol{x}(p,t_0) = \varphi(p) \end{cases} \tag{2.0.22}$$

then $\boldsymbol{y}(p,t) = \big(y^1(p,t), y^2(p,t)\dots\big) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_t(p)\right),\varphi\right)$ is solution for the initial value problem

$$\begin{cases} \dfrac{d}{dt}\,\boldsymbol{y}(p,t) = \boldsymbol{Y}\big(\boldsymbol{y}(p,t)\big) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right),\omega_{\boldsymbol{X}}\right) \\ \boldsymbol{y}(p,t_0) = \varphi(p) \end{cases} \tag{2.0.23}$$

Remark 2.0.6. The smoother defined in (2.0.39) satisfies the conditions of the former corollary, because each point (x,y) of $\mathbb{R}^2$ is a fixed-point. However, the smoother

$$\mathfrak{S} = \Big(\big((U,\mathcal{T}),\sigma_{(U,\mathcal{T})}\big),\big((\mathbb{R}^n,T),\mathfrak{K}_{(\mathbb{R}^n,T)}\big)\Big)$$

such that the law $\sigma_{(U,\mathcal{T})}$ sends each point $(x_0,y_0)\in\mathbb{R}^2$ into the subset

$$A_{(x_0,y_0)} = \left\{(x,y)\in\mathbb{R}^2 \mid x_0\le x\le x_0+1-\frac{y}{y_0},\quad 0\le y\le e^{x_0}\right\}$$

the associated transform of which is

$$\begin{aligned} \mathfrak{K}_{(\mathbb{R}^2,T)} : f(x,y) \mapsto \frac{2}{3y}\iint_{\sigma_{(U,\mathcal{T})}(x,y)} f(x+u,y+v)\,dudv = \\ \frac{2}{3y}\int_{-y}^{0}\int_{0}^{1-\frac{v}{y}} f(x+u,y+v)\,dudv \end{aligned} \tag{2.0.24}$$

is compatible with the vector $\binom{1}{y}$ and sends the point (x,y) into $(x+\frac{7}{9},\frac{4}{9}y)$, this is to say, $\mathfrak{S}_\varphi(x,y) = (x+\frac{7}{9},\frac{4}{9}y)$, Thus, there is no fixed point for $\mathfrak{S}_\varphi$. Of course, this smoother transforms the solution $(t+x_0, y_0e^t)$ of the equation

$$\begin{cases} \dfrac{d}{dt}\,x(t) = 1 \\ \dfrac{d}{dt}\,y(t) = y(t) \end{cases} \tag{2.0.25}$$

for the initial value (x_0,y_0) at $t=0$, into the solution $(t+x_0+\frac{7}{9},\frac{4}{9}y_0e^t)$ of the same equation for the initial value $(x_0+\frac{7}{9},\frac{4}{9}y_0)$.

Definition 2.0.7. Given a smoother

$$\mathfrak{S} = \Big(\big((U,\mathcal{T}),\sigma_{(U,\mathcal{T})}\big),\big((\mathbb{R}^n,T),\mathfrak{K}_{(\mathbb{R}^n,T)}\big)\Big)$$

defined over a chart (U,φ) of a smooth manifold $(M,\mathcal{A}_n)$, and a smooth vector field $\boldsymbol{X}$, define the derivative $\nabla_{\boldsymbol{X}}\mathfrak{S}$ by the following expression.

$$\begin{aligned} &\nabla_{\boldsymbol{X}}\mathfrak{S}|_p = \\ &\lim_{t\to 0}\frac{\mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_t(p)\right),\varphi\right) - \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right),\varphi\circ h_t\right)}{t} \end{aligned} \tag{2.0.26}$$

Corollary 2.0.8. *If $\nabla_{\boldsymbol{X}}\mathfrak{S}=0$, then $\mathfrak{S}$ is compatible with $\boldsymbol{X}$.*

Proof. Obviously, taking into account Definition 2.0.7, from $\nabla_{\boldsymbol{X}}\mathfrak{S}=0$, the statement (2.0.17) follows. □

Definition 2.0.9. Given a smooth vector field $\boldsymbol{X}$, the associated one-parameter group of which is $h_t : U \to U$, say a measure-field $\{\mu_p \mid p \in U\}$ to be invariant with respect to $\boldsymbol{X}$, provided that for every $p \in U$ and each measurable subset E of $\sigma_{(U,\mathcal{T})}(p)$ the following relation holds

$$\forall t \in \mathbb{R} : \qquad \mu_{h_t(p)}\left(\mathfrak{N}h_t(E)\right) = \mu_p(E) \tag{2.0.27}$$

accordingly the measure $\mu_p(E)$ remains unaltered under the one-parameter group $h_t : U \to U$ associated to $\boldsymbol{X}$.

Remark 2.0.10. In [4] it is shown that, for a wide class of vector fields, each differentiable-map $\psi : U \to \mathbb{C}$ satisfying the equation

$$\boldsymbol{X}\check{}\psi(p) = \left(\sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}\,\psi(x^1, x^2 \dots)\right)\check{} = 0 \tag{2.0.28}$$

satisfies also the equation

$$\frac{d}{dt}\,\psi(h_t(p)) = 0 \tag{2.0.29}$$

accordingly, $\psi(h_t(p))$ does not depend upon the parameter t; where for every continuous function f, $f\check{}$ denotes the maximal extension by continuity. Thus, an invariance criterion can consist of proving the existence of a differentiable map $\psi_E : U \to \mathbb{R}$, for each measurable subset $E \subseteq U$, such that

$$\begin{cases} \psi_E(p) = \mu_p(E) \\ \boldsymbol{X}\check{}\psi_E(p) = 0 \end{cases} \tag{2.0.30}$$

Theorem 2.0.11. *If a measure field $\{\mu_p \mid p \in U\}$ is invariant with respect to $\boldsymbol{X}$ and, for each $t \in \mathbb{R}$, the member of corresponding one-parameter group $h_t : U \to U$ is a $\boldsymbol{cAlg}(\mathfrak{N})$-morphism, then the smoother*

$$\mathfrak{S} = \left(\left((U,\mathcal{T}), \sigma_{(U,\mathcal{T})}\right), \left((\mathbb{R}^n, T), \mathfrak{K}_{(\mathbb{R}^n,T)}\right)\right)$$

such that

$$\mathfrak{K}_{(V,T)}\left(\sigma_{(U,\mathcal{T})}(p), \varphi\right) = \underset{\sigma_{(U,\mathcal{T})}(p)}{\int\cdots\int} \varphi\, d\mu_p \tag{2.0.31}$$

is compatible with $\boldsymbol{X}$.

Proof. First, because, for each $t \in \mathbb{R}$, the map $h_t : U \to U$ is assumed to be a $\boldsymbol{cAlg}(\mathfrak{N})$-morphism, then by virtue of (1.0.4) we have that

$$\begin{aligned} &\mathfrak{K}_{(V,T)}\left(\sigma_{(U,\mathcal{T})}(h_t(p)), \varphi\right) = \\ &\qquad \mathfrak{K}_{(V,T)}\left(\mathfrak{N}h_t\big(\sigma_{(U,\mathcal{T})}(p)\big), \varphi\right) = \underset{\mathfrak{N}h_t\left(\sigma_{(U,\mathcal{T})}(p)\right)}{\int\cdots\int} \varphi\, d\mu_p \end{aligned} \tag{2.0.32}$$

and because

$$\mathfrak{N}h_t\left(\sigma_{(U,\mathcal{T})}(p)\right) = \left\{h_t(q) \mid q \in \sigma_{(U,\mathcal{T})}(p)\right\} \tag{2.0.33}$$

then

$$\underset{\mathfrak{N}h_t\left(\sigma_{(U,\mathcal{T})}(p)\right)}{\int \cdots \int} \varphi \, d\mu_p = \underset{\sigma_{(U,\mathcal{T})}(p)}{\int \cdots \int} \varphi \circ h_t \, d\mu_{h_t(p)} \tag{2.0.34}$$

therefore, since the invariance of $\{\mu_p \mid p \in U\}$ with respect to $\boldsymbol{X}$ is assumed, then for every subset $E \subseteq \sigma_{(U,\mathcal{T})}(p)$ the following relation holds

$$\mu_{h_t(p)}\big(\mathfrak{N}h_t(E)\big) = \mu_p(E)$$

therefore from (2.0.34) it follows that

$$\underset{\sigma_{(U,\mathcal{T})}(p)}{\int \cdots \int} \varphi \circ h_t \, d\mu_{h_t(p)} = \underset{\sigma_{(U,\mathcal{T})}(p)}{\int \cdots \int} \varphi \circ h_t \, d\mu_p \tag{2.0.35}$$

consequently,

$$\begin{aligned} &\mathfrak{K}_{(V,T)}\left(\sigma_{(U,\mathcal{T})}(h_t(p)), \varphi\right) = \\ &\qquad \underset{\sigma_{(U,\mathcal{T})}(p)}{\int \cdots \int} \varphi \circ h_t \, d\mu_p = \mathfrak{K}_{(V,T)}\left(\sigma_{(U,\mathcal{T})}(p), \varphi \circ h_t\right) \end{aligned} \tag{2.0.36}$$

accordingly, the smoother $\mathfrak{S}$ satisfies the conditions of the preceding theorem. $\square$

Example 2.0.12. Consider the initial value problem

$$\begin{cases} \dfrac{d}{dt}\, x(t) = 1 \\ \dfrac{d}{dt}\, y(t) = 0.1 \cos\big(x(t)\big) \\ \big(x(0), y(0)\big) = (x_0, y_0) \end{cases} \tag{2.0.37}$$

the solution of which is

$$\begin{cases} x(t) = x_0 + t \\ y(t) = y_0 + 0.1\big(\sin(t + x_0) - \sin(x_0)\big) \end{cases} \tag{2.0.38}$$

where we are assuming the function $0.1\cos\big(x(t)\big)$, in the second equation of (2.0.37), to be the consequence of a perturbation working over the vector field $\binom{1}{0}$. The map σ, sending each $(x, y) \in \mathbb{R}^2$ into the closed set $[x-\pi, x+\pi] \times [y-1, y+1]$, together with the operator $\mathfrak{K}$ defined as follows

(2.0.39)

$$\begin{aligned} \mathfrak{K} : \big(f_1(x,y), f_2(x,y)\big) \mapsto \Big(&\frac{1}{4\pi}\int_{-1}^{1}\int_{-\pi}^{\pi} f_1(x+u, y+v)\,dudv, \\ &\qquad \frac{1}{4\pi}\int_{-1}^{1}\int_{-\pi}^{\pi} f_2(x+u, y+v)\,dudv\Big) \end{aligned}$$

form a smoother such that $\mathfrak{K}$ transforms the vector $\left(\begin{smallmatrix}1\\ 0.1\cos(x)\end{smallmatrix}\right)$ of the equation (2.0.37) into the perturbation-free vector $\binom{1}{0}$, therefore it transforms also (2.0.37) into the following initial value problem,

$$\begin{cases} \dfrac{d}{dt}x(t) = 1 \\ \dfrac{d}{dt}y(t) = 0 \\ \big(x(0), y(0)\big) = (x_0, y_0) \end{cases} \tag{2.0.40}$$

Now, it is not difficult to see $\mathfrak{K}$ to be compatible with the vector field of the equation (2.0.37), therefore $\mathfrak{K}$ transforms also the solution (2.0.38) of (2.0.37) into the solution of (2.0.40), as one can see in the following equality

$$\begin{cases} \dfrac{1}{4\pi}\displaystyle\int_{-1}^{1}\int_{-\pi}^{\pi}(x_0 + u + t)\colon dudv = x_0 + t \\ \dfrac{1}{4\pi}\displaystyle\int_{-1}^{1}\int_{-\pi}^{\pi}\big(y_0 + v + 0.1\big(\sin(x_0 + u + t) - \\ \qquad\qquad\qquad \sin(x_0 + u)\big)\big)dudv = y_0 \end{cases} \tag{2.0.41}$$

and

$$\begin{cases} x(t) = x_0 + t \\ y(t) = y_0 \end{cases} \tag{2.0.42}$$

is nothing but the general solution of (2.0.40). Thus, $\mathfrak{K}$ sends (2.0.37) into (2.0.40) and also sends the general solution of (2.0.37) into the perturbation-free solution (2.0.42) of (2.0.40).

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APPROXIMATION OF FIXED POINTS OF ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS IN BANACH SPACES

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Abstract. Let T be an asymptotically pseudocontractive self-mapping of a nonempty closed convex subset D of a reflexive Banach space X with a Gâteaux differentiable norm. We deal with the problem of strong convergence of almost fixed points $x_n = \mu_n T^n x_n + (1-\mu_n)u$ to fixed point of T. Next, this result is applied to deal with the strong convergence of explicit iteration process $z_{n+1} = v_{n+1}(\alpha_n T^n z_n + (1-\alpha_n)z_n) + (1-v_{n+1})u$ to fixed point of T

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1. Introduction

Let D be a nonempty closed convex subset of a real Banach space X and let $T : D \to D$ be a mapping. Given an $x_0 \in D$ and a t $\in (0,1)$, then, for a nonexpansive mapping T, we can define contraction $G_t : D \to D$ by $G_t x = tTx + (1-t)x_0, x \in D$. By Banach contraction principle, G_t has a unique fixed point x_t in D, i.e., we have

$$x_t = tTx_t + (1-t)x_0.$$

The strong convergence of path $\{x_t\}$ as $t \to 1$ for a nonexpansive mapping T on a bounded D was proved in Hilbert space independently by Browder [2] and Halpern

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[7] in 1967 and in a uniformly smooth Banach space by Reich [10]. Later, it has been studied in various papers (see [12], [14], [15], [23], [28]).

The asymptotically nonexpansive mappings were introduced by Goebel and Kirk [4] and further studied by various authors (see [1], [6], [7], [12], [17], [19], [21], [22], [24], [25], [27]).

Recently, Schu [20] has considered the strong convergence of almost fixed points $x_n = \mu_n T^n x_n$ of an asymptotically nonexpansive mapping T in a smooth and reflexive Banach space having a weakly sequentially continuous duality mapping. Unfortunately, Schu's results do not apply to L^p spaces if $p \neq 2$, since none of these spaces possess weakly sequentially duality mapping.

The object of this paper is to deal with the problem of strong convergence of the sequence of almost fixed points defined by the equation

$$x_n = \mu_n T^n x_n + (1 - \mu_n)u \tag{1}$$

for an asymptotically pseudocontractive mapping T in a reflexive Banach space with the Gâteaux differentiable norm. In particular, Corollary 1 improves and extends the results of [12], [14], [16], [20] and [23] to the larger class of asymptotically pseudocontractive mappings. Further, we deal with the problem of strong convergence of the explicit iteration process

$$z_{n+1} = v_{n+1}(\alpha_n T^n x_n + (1 - \alpha_n)x_n) + (1 - v_{n+1})u$$

by applying Corollary 1.

It is well known that the Mann iteration process ([13]) is not guaranteed to converge to a fixed point of a Lipschitz pseudocontractive defined even on a compact convex subset of a Hilbert space (see [10]). In [11], Ishikawa introduced a new iteration process, which converges to a fixed point of a Lipschitz pseudocontractive mapping defined on a compact convex subset of a Hilbert space. Schu [22] first studied the convergence of the modified Ishikawa iterative sequence for completely continuous asymptotically pseudocontractive mappings in Hilbert spaces. Schu's result has been extended to asymptotically pseudocontractive type mappings defined on compact convex subsets of a Hilbert space (see [4], [15]). In application point of view, compactness is a very strong condition. One of important features of our approach is that it allows relaxation of compactness.

2. Preliminaries

Let X be a real Banach space and D a subset of X. An operator $T : D \to D$ is said to be *asymptotically pseudocontractive* ([24]) if and only if, for each $n \in N$ and $u, v \in D$, there exist $j \in J(u - v)$ and a constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n u - T^n v, j\rangle \leq k_n \|u - v\|^2,$$

where $J : X \to 2^{X^*}$ is the *normalized duality mapping* defined by

$$J(u) = \{j \in X^* : \langle u, j\rangle = \|u\|^2, \|j\| = \|u\|\}.$$

The class of asymptotically pseudocontractive mappings is essentially wider than the class of asymptotically nonexpansive mappings ($T : D \to D$ for which there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n u - T^n v\| \leq k_n \|u - v\|$$

for all $u, v \in D$ and $n \in N$). In fact, if T is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$, then for each $u, v \in D$, $j \in J(u-v)$ and $n \in N$, we have

$$\langle T^n u - T^n v, j\rangle \leq \|T^n u - T^n v\|\|u - v\| \leq k_n \|u - v\|^2.$$

The *normal structure coefficient* $N(X)$ of X is defined ([2]) by

$$N(X) = \left\{ \frac{\text{diam} D}{r_D D} : D \text{ is a nonempty bounded convex subset of } X \right.$$
$$\left. \text{with diam } D > 0 \right\},$$

where $r_D(D) = \inf_{x \in D}\{\sup_{y \in D} \|x - y\|\}$ is the Chebyshev radius of D relative to itself and $\text{diam} D = \sup_{x,y \in D} \|x - y\|$ is the diameter of D. The space X is said to have the *uniformly normal structure* if $N(X) > 1$.

Recall that a nonempty subset D of a Banach space X is said to satisfy the *property* (P) ([12]) if the following holds:

$$\text{(P)} \qquad x \in D \Rightarrow \omega_\omega(x) \subset D,$$

where ω_ω (x) is *weak ω-limit set* of T at x, i.e.,

$$\{y \in C : y = weak - \lim_j T^{n_j} x \text{ for some } n_j \to \infty\}.$$

The following result can be found in [12].

Lemma 1. *Let D be a nonempty bounded subset of a Banach space X with uniformly normal structure and $T : D \to D$ be a uniformly L-Lipschitzian mapping with $L < N(X)^{1/2}$. Suppose that there exists a nonempty bounded closed convex subset C of D with property (P). Then T has a fixed point in C.*

A *Banach limit LIM* is a bounded linear functional on ℓ^∞ such that

$$\liminf_{n\to\infty} t_n \leq LIM t_n \leq \limsup_{n\to\infty} t_n$$

and

$$LIM t_n = LIM t_{n+1}$$

for all bounded sequence $\{t_n\}$ in ℓ^∞. Let $\{x_n\}$ be a bounded sequence of X. Then we can define the real-valued continuous convex function f on X by $f(z) = LIM\|x_n - z\|^2$ for all $z \in X$.

The following Lemma was give in [8].

Lemma 2 [8]**.** *Let X be a Banach space with the uniformly Gâteaux differentiable norm and $u \in X$. Then*

$$f(u) = \inf_{z \in X} f(z)$$

if and only if

$$LIM\langle z, J(x_n - u)\rangle = 0$$

for all $z \in X$, where $J : X \to X^*$ is the normalized duality mapping and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing.

3. The Main Results

In this section, we establish strong convergence of sequence $\{x_n\}$ defined by the equation (1) in a reflexive Banach space with uniformly Gâteaux differentiable norm.

Suppose now that D is a nonempty closed and convex subset of a Banach space X and $T : D \to D$ is an asymptotically pseudocontractive mapping (we may always assume $k_n \geq 1$ for all $n \geq 1$). Suppose also that $\{\lambda_n\}$ is a sequence of real number in (0,1) such that $\lim_{n\to\infty} \lambda_n = 1$.

Now, for $u \in D$ and a positive integer $n \in N$, consider a mapping T_n on D defined by

$$T_n x = \Big(1 - \frac{\lambda_n}{k_n}\Big)u + \frac{\lambda_n}{k_n} T^n x, \quad x \in D.$$

In the sequel, we use the notations $F(T)$ for the set of fixed points of T and μ_n for $\frac{\lambda_n}{k_n}$.

Lemma 3. *For each $n \geq 1$, T_n has exactly one fixed point x_n in D such that*

$$x_n = \mu_n T^n x_n + (1 - \mu_n)u.$$

Proof. Since T_n is a strictly pseudocontractive mapping on D, it follows from Corollary 1 of [5] that T_n possesses exactly one fixed point x_n in D.

Lemma 4. *If the set*

$$G(u, Tu) = \{x \in D : \langle T^n u - u, j\rangle > 0 \text{ for all } j \in j(x - u),\ n \geq 1\}$$

is bounded, then the sequence $\{x_n\}$ is bounded.

Proof. Since T is asymptotically pseudocontractive, for $j \in J(x_n - u)$, we have

$$\langle \mu_n(T^n x_n - u) + \mu_n(u - T^n u), j\rangle \leq \lambda_n \|x_n - u\|^2,$$

which implies

$$\langle T^n u - u, j\rangle \geq \frac{1 - \lambda_n}{\mu_n} \|x_n - u\|^2$$

since $\mu_n(T^n x_n - u) = x_n - u$. If $x \neq 0$, we have

$$\langle T^n u - u, j\rangle > 0$$

and it follows that $x_n \in G(u, Tu)$ for all $n \geq 1$ and hence $\{x_n\}$ is bounded.

Before presenting our main result, we need the following:

Definition 1. Let D be a nonempty closed subset of a Banach spaces X, $T : D \to D$ be a nonlinear mapping and $M = \{x \in D : f(x) = \min_{z\in D} f(z)\}$. Then T is said to satisfy the *property* (S) if the following holds:

(S) For any bounded sequence $\{x_n\}$ in D,
$\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0$ implies $M \cap F(T) \neq \emptyset$.

Theorem 1. *Let D be a nonempty closed and convex subset of a reflexive Banach space X with a uniformly Gâteaux differentiable norm, $T : D \to D$ be a continuous asymptotically pseudocontractive mapping with a sequence $\{k_n\}$ and $\{\lambda_n\}$ be a sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$. Suppose that for $u \in D$, the set $G(u, Tu)$ is bounded and the mapping T satisfies the property (S). Then we have the following:*

(a) *For each $n \geq 1$, there is exactly one $x_n \in D$ such that*

$$x_n = \mu_n T^n x_n + (1-\mu_n)u.$$

(b) *If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then it follows that there exists the sunny nonexpansive retraction P from D onto $F(T)$ such that $\{x_n\}$ converges strongly to Px.*

Proof. The part (a) follows from Lemma 3. So, it remains to prove part (b). From Lemma 4, $\{x_n\}$ is bounded and so we can define a function $f : D \to R^+$ by

$$f(z) = LIM\|x_n - z\|^2$$

for all $z \in D$. Since f is continuous and convex, $f(z) \to \infty$ as $\|z\| \to \infty$ and X is reflexive, f attains it infimum over D. Let $z_0 \in D$ such that $f(z_0) = \min_{z\in D} f(z)$ and let $M = \{x \in D : f(x) = \min_{z\in D} f(z)\}$. Then M is nonempty because $z_0 \in M$. Since $\{x_n\}$ is bounded by Lemma 4 and T satisfied the property (S), it follows that $M \cap F(T) \neq \emptyset$. Suppose that $v \in M \cap F(T)$. Then, by Lemma 2, we have

$$LIM\langle x - v, J(x_n - v)\rangle \leq 0$$

for all $x \in D$. In particular, we have

$$LIM\langle u - v, J(x_n - v)\rangle \leq 0. \tag{2}$$

On the other hand, from the equation (1), we have

$$x_n - T^n x_n = (1-\mu_n)(u - T^n x_n) = \frac{1-\mu_n}{\mu_n}(u - x_n). \tag{3}$$

Now, for any $v \in F(T)$, we have

$$\begin{aligned}\langle x_n - T^n x_n, J(x_n - v)\rangle &= \langle x_n - v + T_n v - T^n x_n, J(x_n - v)\rangle \\ &\geq -(k_n - 1)\|x_n - v\|^2 \\ &\geq -(k_n - 1)K^2\end{aligned}$$

for some $K > 0$ and it follows from (3) that

$$\langle x_n - u, J(x_n - v)\rangle \leq \frac{\lambda_n(k_n - 1)}{k_n - \lambda_n}K^2.$$

Hence we have

$$LIM\langle x_n - u, J(x_n - v)\rangle \leq 0. \tag{4}$$

Combining (2) and (4), then we have

$$LIM\langle x_n - v, J(x_n - v)\rangle = LIM\|x_n - v\|^2 \le 0.$$

Therefore, there is a subsequence $\{x_{n_i}\}$ which converges strongly to v. To complete the proof, suppose there is another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to y (say). Since $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and T is continuous, then y is a fixed point of T. It then follows from (4) that

$$\langle v - u, J(v - y)\rangle \le 0$$

and

$$\langle y - u, J(y - v)\rangle \le 0.$$

Adding these two inequalities yields

$$\langle v - y, J(v - y)\rangle = \|v - y\|^2 \le 0$$

and thus $v = y$. This prove the strong convergence of $\{x_n\}$ to $v \in F(T)$. Now we can define a mapping P from D onto $F(T)$ by $\lim_{n\to\infty} x_n = Pu$. From (4), we have

$$\langle u - Pu, J(v - Pu)\rangle \le 0$$

for all $u \in D$ and $v \in F(T)$. Therefore, P is the sunny nonexpansive retraction. This completes the proof.

Remark 1. The assumption of λ_n such that $\lambda_n \in (\frac{1}{2}, 1)$ with $k_n \le \frac{2\lambda_n^2}{2\lambda_n - 1}$ implies $\lim_{n\to\infty} \frac{\lambda_n(k_n-1)}{(k_n-\lambda_n)} = 0$ (see Lemma 1.4 of [16]).

Next, we substitute the property (S) mentioned in Theorem 1 by assuming that T is uniformly L-Lipschitzian in Banach space with the uniformly normal structure and D does have the property (P) (see [12]).

Corollary 1. *Let X be a Banach space with the uniformly Gâteaux differentiable norm, N(X) be the normal structure coefficient of X such that $N(X) > 1$, D be nonempty closed convex subset of X. $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\}$ and $L < N(X)^{1/2}$ and $\{\lambda_n\}$ be a sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty}\lambda_n = 1$ and $\lim_{n\to\infty}\frac{k_n-1}{k_n-\lambda_n} = 0$. Suppose that every closed convex bounded subset of D satisfies the property (P). Then we have*

(a) *For each $n \ge 1$, there is exactly one $x_n \in D$ such that*

$$x_n = \mu_n T^n x_n + (1 - \mu_n)u.$$

(b) *If $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then it follows that there exists the sunny nonexpansive retraction P from D onto F(T) such that the sequence $\{x_n\}$ converges strongly to Px.*

Remark 2. (1) Theorem 1 and Corollary 1 can be applied to all uniformly convex and uniformly smooth Banach spaces and, in particular, all L^P spaces, $1 < p < \infty$.

(2) As was mentioned in the introduction, Theorem 1 extends and improves the corresponding results of [12], [14], [16], [20] and [23] to much larger class of asymptotically pseudocontractive mappings and to more general Banach spaces X considered here.

If we choose $\{\lambda_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$ (such a sequence $\{\lambda_n\}$ always exists. For example, taking $\lambda_n = \min\{1-\sqrt{k_n-1}, 1-\frac{1}{n}\}$), then the following result is a direct consequence of Corollary 1:

Corollary 2. *Let D be nonempty closed convex and bounded subset of a uniformly smooth Banach space X, $T : D \to D$ be an asymptotically nonexpansive mapping with Lipschitzian constant k_n and $\{\lambda_n\}$ be a sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$. Then we have the following:*

(a) *For $u \in D$ each $n \geq 1$, there is exactly one $x_n \in D$ such that*

$$x_n = \mu_n T^n x_n + (1-\mu_n)u.$$

(b) *If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists the sunny nonexpansive retraction P from D onto $F(T)$ such that $\{x_n\}$ converges strongly to Px.*

We immediately obtain from Corollary 2 the following result (Theorem 1 of Lim and Xu [8]) with additional information that almost fixed points converges to y, where y is fixed point of T nearest point to u.

Corollary 3. *Let D be a nonempty closed convex and bounded subset of a uniformly smooth Banach space and $T : D \to D$ be an asymptotically nonexpansive mapping. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$. Suppose that, for any $x \in D$, $\{x_n\}$ is a sequence in the defined by (1). Suppose in addition that the following condition:*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

holds. Then there exists the sunny nonexpansive retraction P from D onto $F(T)$ such that $\{x_n\}$ converges strongly to Px.

4. Applications

Halpern [9] has introduced the explicit iteration process $\{z_{n+1}\}$ defined by $z_{n+1} = \lambda_{n+1} T z_n$ for approximation of a fixed point for a nonexpansive self-mapping T defined on the unit ball of a Hilbert space. Later, this iteration process has been studied extensively by various authors and has been successfully employed to approximate fixed points of various class of nonlinear mappings (see [15], [20], [23]).

In this section, we establish some strong convergence theorems for the results of the explicit iteration process $\{z_{n+1}\}$ defined by

$$z_{n+1} = v_{n+1}(\alpha_n T^n z_n + (1-\alpha_n)z_n) + (1-\mu_{n+1})u$$

by applying the results concerning the implicit iteration process $\{x_n\}$ defined by

$$x_n = \mu_n T^n x_n + (1 - \mu_n)u$$

of the last section.

First, we shall introduce a definition, which is partly due to Halpern [7].

Let $\{a_n\}$ and $\{v_n\}$ be sequence of real numbers in $(0, \infty)$ and (0,1), respectively. Then $(\{a_n\}, \{v_n\})$ is said to have *property* (A) ([17]):
(a) $\{a_n\}$ is decreasing,
(b) $\{v_n\}$ is strictly increasing,
(c) there is a sequence $\{\beta_n\}$ of natural number such that
(c-1) $\{\beta_n\}$ is strictly increasing,
(c-2) $\lim_{n\to\infty} \beta_n(1 - v_n) = \infty$,
(c-3) $\lim_{n\to\infty} \frac{1-v_{n+\beta_n}}{1-v_n} = 1$,
(c-4) $\lim_{n\to\infty} \frac{a_n - a_{n+\beta_n}}{1-v_n} = 0$.

The following lemma was proved in [23]:

Lemma 5 [23]. *Let D be a nonempty bounded and convex subset of a normed space X, $0 \in D$, $\{S_n\}$ be a sequence self-mappings on D, $\{L_n\}$ be a sequence of real numbers in $[1, \infty]$ such that $\|S_n x - S_n y\| \leq L_n \|x - y\|$ for all $x, y \in D$ and $n \geq 1, \{\lambda_n\} \subset (0, 1)$, $\{a_n\} \subset (0, \infty)$ be such that $(\{a_n\}, \{v_n\})$ has property* (A) *and $\{\frac{1-v_n}{1-\lambda_n}\}$ is bounded, where $v_n = \lambda_n / L_n$, and $\{x_n\}$ be a sequence in D such that $x_n = v_n S_n(x_n)$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = v$. Suppose that there exists a constant $d > 0$ such that*

$$\|S_m(x) - S_n(x)\| \leq d|a_m - a_n|$$

for all $m, n \geq 1$ and $x \in D$. Suppose also that, for an arbitrary points $z_0 \in D, \{z_n\}$ is a sequence in D such that $z_{n+1} = v_{n+1} S_n(z_n)$ for all $n \geq 1$. Then $\lim_{n\to\infty} z_n = v$.

Xu [26] has proved that, if X is q-uniformly smooth ($q > 1$), then there exists a constant $c > 0$ such that

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c\|y\|^q \tag{5}$$

for all $x, y \in X$, where the mapping $J_q : X \to 2^{X^*}$ is a generalized duality mapping defined by

$$J_q(x) = \{j \in X^* : \langle x, j\rangle = \|x\|^q, \|j\| = \|x\|^{q-1}\}.$$

Typical examples of such space are the Lesbesgue L_p, the sequence ℓ_p and the Sobolev W_p^m spaces for $1 < p < \infty$. In fact, these spaces are p-uniformly smooth if $1 < p \leq 2$ and 2-uniformly smooth for $p \geq 2$.

Before, presenting our results, we need the following:

Lemma 6. *Let $q > 1$ be a real number, D be a nonempty closed subset of a q-uniformly smooth Banach space X, $T : D \to D$ be a uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with a sequence $\{k_n\}$ and $\{\lambda_n\}$ and $\{\alpha_n\}$*

be two sequences of real numbers in $(0,1)$. *Suppose that* $\{G_n\}$ *is a self-mapping on* D *defined by* $G_n x = \alpha_n T^n x + (1-\alpha_n)x$ *for all* $x \in D$. *Then we have the following:*

(a) $\|G_n x - G_n y\| \leq L_n \|x-y\|$ *for all* $x, y \in D$ *and* $n \geq 1$, *where*

$$L_n = [1 + q\alpha_n(k_n - 1) + c\alpha_n^q (1+L)^q]^{\frac{1}{q}}.$$

(b) *For* $u \in D$ *and each* $n \geq 1$, *there is exactly one* $x_n \in D$ *such that*

$$x_n = v_n G_n(x_n) + (1-v_n)u,$$

where $v_n = \lambda_n / L_n$.

(c) *If* $u = 0$, *then it follows that* $x_n = \frac{v_n \alpha_n}{1 - v_n(1-\alpha_n)} T^n x_n$ *for all* $n \geq 1$.

Proof. To prove part (a), set $F_n = I - T^n$, where I denotes the identity operator. Then, for each $n \geq 1$, $G_n = I - \alpha_n F_n$ and $\|F_n x - F_n y\| \leq (1+L)\|x-y\|$ for all $x, y \in D$. Since

$$\langle F_n x - F_n y, J_q(x-y)\rangle \geq -(k_n - 1)\|x-y\|^2$$

for all $x, y \in D$ and $n \geq 1$, using (5), we obtain

$$\begin{aligned}
&\|G_n x - G_n y\|^q \\
&= \|x - y - \alpha_n(F_n x - F_n y)\|^q \\
&\leq \|x-y\|^q - q\alpha_n \langle F_n x - F_n y, J_q(x-y)\rangle + c\alpha_n^q (1+L)^q \|x-y\|^q \\
&\leq [1 + q\alpha_n(k_n-1) + c\alpha_n^q(1+L)^q]\|x-y\|^q.
\end{aligned}$$

To prove part (b), for $u \in D$ and $n \geq 1$, define a mapping $T_n : D \to D$ by

$$T_n x = v_n G_n x + (1-v_n)u, \quad x \in D.$$

Since $v_n \in (0,1), T_n$ is a contraction mapping on D. Thus, by the Banach contraction principle, T_n has exactly one $x_n \in D$ such that $x_n = v_n G_n x_n + (1-v_n)u$. This completes the proof.

The following lemma can be shown by simple calculation:

Lemma 7. *Let* D *be a nonempty closed convex subset of a Banach space* X, $T : D \to D$ *be an asymptotically nonexpansive mapping with a sequence* $\{k_n\}$ *and* $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *be two sequences of real numbers in* $(0,1)$. *Suppose that* $\{G_n\}$ *is a sequence of self-mappings on* D *defined by* $G_n x = \alpha_n T^n x + (1-\alpha_n)x$ *for any* $x \in D$. *Then we have the following:*

(a) $\|G_n x - G_n y\| \leq k_n \|x-y\|$ *for all* $x, y \in D$ *and* $n \geq 1$.

(b) *For* $u \in D$ *and each* $n \geq 1$, *there is exactly one* $x_n \in D$ *such that*

$$x_n = \mu_n G_n x_n + (1-\mu_n)u,$$

where $\mu_n = \lambda_n / k_n$.

We now prove the main result of this section.

Theorem 2. *Let $q > 1$ be a real number, D be a nonempty closed convex and bounded subset of a q-uniformly smooth Banach space X, $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\}$ and $L < N(X)^{\frac{1}{2}}$ and $\{\lambda_n\}$ and $\{\alpha_n\}$ be two sequences of real numbers in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 1$, $\lim_{n\to\infty} \frac{L_n-1}{L_n-\lambda_n} = 0$ and $\lim_{n\to\infty} \frac{1-v_n}{\alpha_n} = 0$, where $L_n = [1 + q\alpha_n(k_n - 1) + c\alpha_n^q(1 + L)^q]^{\frac{1}{q}}$ and $v_n = \lambda_n/L_n$. Suppose that $(\{\alpha_n\}, \{v_n\})$ has property (A), $\{\frac{1-v_n}{1-\lambda_n}\}$ is bounded and $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$ for any bounded sequence $\{y_n\}$ in D with $\lim_{n\to\infty} \|y_n - T^n y_n\| = 0$. Suppose also that, for any $u, z_0 \in D$, $\{z_n\}$ is a sequence in D defined by*

$$z_{n+1} = v_{n+1}(\alpha_n T^n z_n + (1 - \alpha_n)z_n) + (1 - v_{n+1})u.$$

Then there exists the sunny nonexpansive retraction P from D onto $F(T)$ such that $\{z_n\}$ converges strongly to Pu.

Proof. Without loss of generality, we may assume that $u = 0$. For $n \geq 1$, set $\eta_n = \frac{v_n\alpha_n}{(1-v_n)+v_n\alpha_n}$. Then $\{\eta_n\} \subset (0, 1)$ and $\eta_n = (1 + \frac{1}{v_n}(\frac{1-v_n}{\alpha_n}))^{-1}$ for all $n \geq 1$. Since $\lim_{n\to\infty} v_n = 1$ and $\lim_{n\to\infty} \frac{1-v_n}{\alpha_n} = 0$, it follows that $\lim_{n\to\infty} \eta_n = 1$ and hence, by Lemma 6 and Corollary 1, the sequence $\{x_n\}$ defined by $x_n = \eta_n T^n x_n$ converges strongly to Pu. Let $\{G_n\}$ be a sequence of self mappings on D defined by

$$G_n(x) = \alpha_n T^n x + (1 - \alpha_n)x, \quad x \in D.$$

By Lemma 6, for each $n \geq 1$, there is exactly one $x_n \in D$ such that $x_n = v_n G_n(x_n)$ and hence $x_n = \eta_n T^n x_n$. By Corollary 1, we have that $\{x_n\}$ converges strongly to some fixed point of T. Since $z_n = \eta_n G_n(z_n)$ for all $n \geq 1$ and $\|G_m(x) - G_n(x)\| \leq |\alpha_m - \alpha_n|$ diam D for all $m, n \geq 1$ and $x \in D$. It follows from Lemma 5 that $\{z_n\}$ converges strongly to Pu. This completes the proof.

Remark 3. (1) Theorem 2 extends Theorem 2.4 of Schu [23] to the wider class of asymptotically pseudocontractive mappings and from a Hilbert spaces to the more general Banach space X considered here.

(2) Another iteration procedure for uniformly L-Lipschitzian asymptotically pseudocontractive mapping T in a Hilbert space may be found in the work of Schu [22] with the condition that the given mapping T is completely continuous.

Corollary 3. *Let D be a nonempty closed convex and bounded subset of a uniformly smooth Banach space X, $T : D \to D$ be a uniformly asymptotically regular and asymptotically nonexpansive mapping with a sequence $\{k_n\}$ and $\{\alpha_n\}$ be sequence of real numbers in $(0, 1)$ with $\lim_{n\to\infty} \lambda_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{(k_n-1)}{(k_n-\lambda_n)} = 0$ and $\lim_{n\to\infty} \frac{1-\mu_n}{a_n} = 0$. Suppose also that, for any $u, z_0 \in D$, $\{z_n\}$ is a sequence in D defined by*

$$z_{n+1} = \mu_{n+1}(\alpha_n T^n z_n + (1 - \alpha_n)z_n) + (1 - \mu_n)u, \quad n \geq 1.$$

Then $\{z_n\}$ converges strongly to some fixed point of T.

Remark 4. Schu [19], [21] and Tan and Xu [24] have studied the weak convergence for the sequence $\{z_n\}$ defined by (the modified Mann iteration process) $z_{n+1} = \alpha_n T^n z_n + (1 - \alpha_n)z_n$ to fixed point of asymptotically nonexpansive mapping T in a uniformly convex Banach space with the Fréchet differentiable norm or with a weakly sequentially duality mapping.

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RADIAL MINIMIZER OF A P-GINZBURG-LANDAU TYPE FUNCTIONAL WITH NORMAL IMPURITY INCLUSION

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Abstract. The author proves the $W^{1,p}$ and $C^{1,\alpha}$ convergence of the radial minimizers u_ε of an Ginzburg-Landau type functional as $\varepsilon \to 0$. The zeros of the radial minimizer are located and the convergent rate of the module of the minimizer is estimated.

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§1. Introduction

Let $n \geq 2, B_r = \{x \in R^n; |x| < r\}, g(x) = x$ on ∂B_1. Recall the Ginzburg-Landau type functional

$$E_\varepsilon(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_{B_1\setminus B_\Gamma} (1-|u|^2)^2 + \frac{1}{4\varepsilon^2}\int_{B_\Gamma} |u|^2,$$

on the class functions $H_g^1(B_1, R^n)$. The functional $E_\varepsilon(u)$ is related to the Ginzburg-Landau model of superconductivity with normal impurity inclusion such as superconducting normal junctions (cf. [5]) if $n = 2$. $B_1 \setminus \overline{B_\Gamma}$ and B_Γ represent the domains occupied by superconducting materials and normal conducting materials, respectively. The minimizer u_ε is the order parameter. Zeros of u_ε are known as Ginzburg-Landau vortices which are of significance in the theory of superconductivity(cf. [1]). The paper [7] studied the asymptotic behaviors of the minimizer of $E_\varepsilon(u, B_1)$ on the function class $H_g^1(B_1, R^2)$ and discussed the vortex-pinning effect. For the simplified Ginzburg-Landau functional, many papers stated the asymptotic behavior of the minimizer u_ε as $\varepsilon \to 0$. When $n = 2$, the asymptotics of u_ε were well-studied by [1]. In the case of higher dimension, for the radial minimizer u_ε of $E_\varepsilon(u, B_1)$, some results on the convergence had been shown in [14] as $\varepsilon \to 0$. There

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were many works for the radial minimizer in [12]. Other related works can be seen in [2] [3] [8] and [17] etc.

Assume $p > n$. Consider the minimizers of the p-Ginzburg-Landau type functional

$$E_\varepsilon(u, B_1) = \frac{1}{p}\int_{B_1} |\nabla u|^p + \frac{1}{4\varepsilon^p}\int_{B_1\setminus B_\Gamma}(1-|u|^2)^2 + \frac{1}{4\varepsilon^p}\int_{B_\Gamma}|u|^4,$$

on the class functions

$$W = \{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B_1, R^n); f(1) = 1, r = |x|\}.$$

By the direct method in the calculus of variations we can see that the minimizer u_ε exists and it will be called *radial minimizer*. In this paper, we suppose that $\Gamma \in (0, \varepsilon]$. The conclusion of the case of $\Gamma = O(\varepsilon)$ as $\varepsilon \to 0$ is still true by the same argument. we will discuss he location of the zeros of the radial minimizer. Based on the result, we shall establish the uniqueness of the radial minimizer. The asymptotic behavior of the radial minimizer be concerned with as $\varepsilon \to 0$. The estimates of the rate of the convergence for the module of minimizer will be presented.

We will prove the following theorems.

Theorem 1.1. *Assume $u_\varepsilon$is a radial minimizer of $E_\varepsilon(u, B_1)$. Then for any given $\eta \in (0, 1/2)$ there exists a constant $h = h(\eta) > 0$ such that*

$$Z_\varepsilon = \{x \in B_1; |u_\varepsilon(x)| < 1 - 2\eta\} \subset B(0, h\varepsilon) \cup B_\Gamma.$$

Moreover, the zeros of the radial minimizer are contained in $B_{h\varepsilon}$ as $\Gamma \in (0, h\varepsilon]$. When $\Gamma \in (h\varepsilon, \varepsilon]$, the zeros are contained in $B_\Gamma \setminus B(0, h\varepsilon)$.

Theorem 1.2. *For any given $\varepsilon \in (0, 1)$, the radial minimizers of $E_\varepsilon(u, B_1)$ are unique on W.*

Theorem 1.3. *Assume u_ε is the radial minimizer of $E_\varepsilon(u, B_1)$. Then as $\varepsilon \to 0$,*

$$u_\varepsilon \to \frac{x}{|x|}, \quad in \quad W^{1,p}_{loc}(\overline{B_1} \setminus \{0\}, R^n); \tag{1.1}$$

$$u_\varepsilon \to \frac{x}{|x|}, \quad in \quad C^{1,\beta}_{loc}(B_1 \setminus \{0\}, R^n), \tag{1.2}$$

for some $\beta \in (0, 1)$.

Theorem 1.4. *Let $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ be the radial minimizer of $E_\varepsilon(u, B_1)$. Then for any $T > 0$, there exist $C, \varepsilon_0 > 0$ such that as $\varepsilon \in (0, \varepsilon_0)$,*

$$\int_T^1 r^{n-1}[(f'_\varepsilon)^p + \frac{1}{\varepsilon^p}(1 - f_\varepsilon^2)^2]dr \le C\varepsilon^p.$$

$$\sup_{r\in[T,1]}(1 - f_\varepsilon(r)) \le C\varepsilon^{p-\frac{n}{2}}.$$

Some basic properties of minimizers are given in §2. The main purpose of §3 is to prove Theorem 1.1. In §4 and §5 we present the proof of (1.1). The proof of Theorem 1.2 is given in §6. §7 gives the proof of (1.2). Theorem 1.4 is derived in §8.

§2. Preliminaries

In polar coordinates, for $u(x) = f(r)\frac{x}{|x|}$ we have

$$|\nabla u| = (f_r^2 + (n-1)r^{-2}f^2)^{1/2}, \quad \int_{B_1} |u|^p = |S^{n-1}| \int_0^1 r^{n-1}|f|^p \, dr,$$

$$\int_{B_1} |\nabla u|^p = |S^{n-1}| \int_0^1 r^{n-1}(f_r^2 + (n-1)r^{-2}f^2)^{p/2} \, dr.$$

It is easily seen that $f(r)\frac{x}{|x|} \in W^{1,p}(B_1, R^n)$ implies $f(r)r^{\frac{n-1}{p}-1}, f_r(r)r^{\frac{n-1}{p}} \in L^p(0,1)$. Conversely, if $f(r) \in W^{1,p}_{loc}(0,1], f(r)r^{\frac{n-1}{p}-1}, f_r(r)r^{\frac{n-1}{p}} \in L^p(0,1)$, then $f(r)\frac{x}{|x|} \in W^{1,p}(B_1, R^n)$. Thus if we denote

$$V = \{f \in W^{1,p}_{loc}(0,1]; r^{\frac{n-1}{p}} f_r, r^{(n-1-p)/p} f \in L^p(0,1), f(r) \geq 0, f(1) = 1\},$$

then $V = \{f(r); u(x) = f(r)\frac{x}{|x|} \in W\}$.

Substituting $u(x) = f(r)\frac{x}{|x|} \in W$ into $E_\varepsilon(u, B_1)$, we obtain

$$E_\varepsilon(u, B_1) = |S^{n-1}| E_\varepsilon(f)$$

where

$$E_\varepsilon(f) = \frac{1}{p}\int_0^1 (f_r^2 + (n-1)r^{-2}f^2)^{p/2} r^{n-1} dr + \frac{1}{4\varepsilon^p}\int_\Gamma^1 (1-f^2)^2] r^{n-1} \, dr + \frac{1}{4\varepsilon^p}\int_0^\Gamma f^4 r^{n-1} \, dr.$$

This implies that $u = f(r)\frac{x}{|x|} \in W$ is the minimizer of $E_\varepsilon(u, B_1)$ if and only if $f(r) \in V$ is the minimizer of $E_\varepsilon(f)$.

Proposition 2.1. *The set V defined above is a subset of $\{f \in C[0,1]; f(0) = 0\}$.*

Proof. Let $f \in V$ and $h(r) = f(r^{\frac{p-1}{p-n}})$.Then

$$\int_0^1 |h'(r)|^p \, dr = (\frac{p-1}{p-n})^p \int_0^1 |f'(r^{\frac{p-1}{p-n}})|^p r^{\frac{p(n-1)}{p-n}} \, dr = (\frac{p-1}{p-n})^{p-1} \int_0^1 s^{n-1}|f'(s)|^p \, ds < \infty$$

by noting $f_s(s)s^{(n-1)/p} \in L^p(0,1)$. Using interpolation inequality and Young inequality, we have that for some $y > 1$,

$$\|h\|_{W^{1,y}((0,1),R)} < \infty,$$

which implies that $h(r) \in C[0,1]$ and hence $f(r) \in C[0,1]$.

Suppose $f(0) > 0$, then $f(r) \geq s > 0$ for $r \in [0,t)$ with $t > 0$ small enough since $f \in C[0,1]$. We have

$$\int_0^1 r^{n-1-p} f^p \, dr \geq s^p \int_0^t r^{n-1-p} \, dr = \infty,$$

which contradicts $r^{(n-1)/p-1} f \in L^p(0,1)$. Therefore $f(0) = 0$ and the proof is complete.

It is not difficult to prove the following

Proposition 2.2. *The functional $E_\varepsilon(u, B_1)$ achieves its minimum on W by a function $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$.*

Proposition 2.3. *The minimizer u_ε satisfies the equality*

$$\int_{B_1} |\nabla u|^{p-2}\nabla u\nabla\phi dx - \frac{1}{\varepsilon^p}\int_{B_1\setminus B_\Gamma} u\phi(1-|u|^2)dx + \frac{1}{\varepsilon^p}\int_{B_\Gamma} u\phi|u|^2 dx = 0, \tag{2.1}$$

$$\forall\phi = f(r)\frac{x}{|x|} \in C_0^\infty(B_1, R^n). \quad u|_{\partial B_1} = x. \tag{2.2}$$

Proof. Denote u_ε by u. For any $t \in [0,1)$and $\phi = f(r)\frac{x}{|x|} \in C_0^\infty(B_1, R^n)$, we have $u + t\phi \in W$ as long as t is small sufficiently. Since u is a minimizer we obtain

$$\frac{dE_\varepsilon(u+t\phi, B_1)}{dt}|_{t=0} = 0,$$

namely,

$$\begin{aligned}0 &= \frac{d}{dt}|_{t=0}\int_{B_1}\frac{1}{p}|\nabla(u+t\phi)|^p + \frac{1}{4\varepsilon^p}\int_{B_1\setminus B_\Gamma}(1-|u+t\phi|^2)^2dx \\ &\quad + \frac{1}{4\varepsilon^p}\int_{B_\Gamma}|u+t\phi|^4dx \\ &= \int_{B_1}|\nabla u|^{p-2}\nabla u\nabla\phi dx - \frac{1}{\varepsilon^p}\int_{B_1\setminus B_\Gamma}u\phi(1-|u|^2)dx + \frac{1}{\varepsilon^p}\int_{B_\Gamma}u\phi|u|^2dx.\end{aligned}$$

By a limit process we see that the test function ϕ can be any member of $\{\phi = f(r)\frac{x}{|x|} \in W^{1,p}(B_1, R^n); \phi|_{\partial B_1} = 0\}$.

Similarly, we also derive

The minimizer $f_\varepsilon(r)$ of the functional $E_\varepsilon(f)$ satisfies

$$\begin{aligned}&\int_0^1 r^{n-1}(f_r^2 + (n-1)r^{-2}f^2)^{(p-2)/2}(f_r\phi_r + (n-1)r^{-2}f\phi)\,dr \\ &= \frac{1}{\varepsilon^p}\int_\Gamma^1 r^{n-1}(1-f^2)f\phi\,dr - \frac{1}{\varepsilon^p}\int_0^\Gamma r^{n-1}f^3\phi\,dr, \quad \forall\phi \in C_0^\infty(0,1).\end{aligned} \tag{2.3}$$

By a limit process we see that the test function ϕ in (2.3) can be any member of

$$X = \{\phi(r) \in W_{loc}^{1,p}(0,1]; \phi(0) = \phi(1) = 0, \phi(r) \geq 0, r^{\frac{n-1}{p}}\phi', r^{\frac{n-p-1}{p}}\phi \in L^p(0,1)\}$$

Proposition 2.4. *Let f_ε satisfies (2.3) and $f(1) = 1$. Then $f_\varepsilon \leq 1$ on [0,1].*

Proof. Denote $f = f_\varepsilon$ in (2.3) and set $\phi = f(f^2-1)_+$. Then

$$\begin{aligned}&\int_0^1 r^{n-1}(f_r^2 + (n-1)r^{-2}f^2)^{(p-2)/2}[f_r^2(f^2-1)_+ + ff_r[(f^2-1)_+]_r \\ &+ (n-1)r^{n-3}f^2(f^2-1)_+]\,dr + \frac{1}{\varepsilon^p}\int_\Gamma^1 r^{n-1}f^2(f^2-1)_+^2\,dr \\ &+ \frac{1}{\varepsilon^p}\int_0^\Gamma r^{n-1}f^4(f^2-1)_+\,dr = 0\end{aligned}$$

from which it follows that

$$\frac{1}{\varepsilon^p}\int_\Gamma^1 r^{n-1}f^2(f^2-1)_+^2\,dr + \frac{1}{\varepsilon^p}\int_0^\Gamma r^{n-1}f^4(f^2-1)_+\,dr = 0$$

Thus $f = 0$ or $(f^2-1)_+ = 0$ on $[0,1]$ and hence $f = f_\varepsilon \leq 1$ on [0,1].

Proposition 2.5. *Assume u_ε is a weak radial solution of (2.1)(2.2). Then there exist positive constants C_1, ρ which are both independent of ε such that*

$$\|\nabla u_\varepsilon(x)\|_{L^\infty(B(x,\rho\varepsilon/8))} \le C_1\varepsilon^{-1}, \quad if \quad x \in B(0, 1-\rho\varepsilon), \tag{2.4}$$

$$|u_\varepsilon(x)| \ge \frac{10}{11}, \quad if \quad x \in \overline{B_1} \setminus B(0, 1-2\rho\varepsilon). \tag{2.5}$$

Proof. Let $y = x\varepsilon^{-1}$ in (2.1) and denote $v(y) = u(x)$, $B_\varepsilon = B(0, \varepsilon^{-1})$. Then

$$\int_{B_\varepsilon} |\nabla v|^{p-2}\nabla v\nabla\phi dy = \int_{B_\varepsilon\setminus B(0,\Gamma\varepsilon^{-1})} v(1-|v|^2)\phi dy - \int_{B(0,\Gamma\varepsilon^{-1})} v\phi|v|^2 dy \tag{2.6}$$

$\forall\phi \in W_0^{1,p}(B_\varepsilon, R^n)$. This implies that $v(y)$ is a weak solution of (2.6). By using the standard discuss of the Holder continuity of weak solution of (2.6) on the boundary (for example see Theorem 1.1 and Line 19-21 of Page 104 in [4]) we can see that for any $y_0 \in \partial B_\varepsilon$ and $y \in B(y_0, \rho_0)$ (where $\rho_0 > 0$ is a constant independent of ε), there exist positive constants $C = C(\rho_0)$ and $\alpha \in (0,1)$ which are both independent of ε such that

$$|v(y) - v(y_0)| \le C(\rho_0)|y - y_0|^\alpha.$$

Choose $\rho > 0$ sufficiently small such that

$$y \in B(y_0, 2\rho) \subset B(y_0, \rho_0), \quad and \quad C(\rho_0)|y-y_0|^\alpha \le \frac{1}{11}, \tag{2.7}$$

then

$$|v(y)| \ge |v(y_0)| - C(\rho_0)|y-y_0|^\alpha = 1 - C(\rho_0)|y-y_0|^\alpha \ge \frac{10}{11}.$$

Let $x = y\varepsilon$. Thus

$$|u_\varepsilon(x)| \ge \frac{10}{11}, \quad if \quad x \in B(x_0, 2\rho\varepsilon)$$

where $x_0 \in \partial B_1$. This implies (2.5).

Taking $\phi = v\zeta^p, \zeta \in C_0^\infty(B_\varepsilon, R)$ in (2.6), we obtain

$$\int_{B_\varepsilon} |\nabla v|^p\zeta^p dy \le p\int_{B_\varepsilon} |\nabla v|^{p-1}\zeta^{p-1}|\nabla\zeta||v|dy + \int_{B_\varepsilon\setminus B(0,\Gamma\varepsilon^{-1})} |v|^2(1-|v|^2)\zeta^p dy$$
$$+ \int_{B(0,\Gamma\varepsilon^{-1})} |v^4|\zeta^p dy.$$

For the ρ in (2.7), setting $y \in B(0, \varepsilon^{-1} - \rho), B(y, \rho/2) \subset B_\varepsilon$, and

$$\zeta = 1 \ in \ B(y, \rho/4), \zeta = 0 \ in \ B_\varepsilon \setminus B(y, \rho/2), |\nabla\zeta| \le C(\rho),$$

we have

$$\int_{B(y,\rho/2)} |\nabla v|^p\zeta^p \le C(\rho)\int_{B(y,\rho/2)} |\nabla v|^{p-1}\zeta^{p-1} + C(\rho).$$

Using Holder inequality we can derive $\int_{B(y,\rho/4)} |\nabla v|^p \le C(\rho)$. Combining this with the Tolksdroff' theorem in [19] (Page 244 Line 19-23) yields

$$\|\nabla v\|^p_{L^\infty(B(y,\rho/8))} \le C(\rho)\int_{B(y,\rho/4)} (1+|\nabla v|)^p \le C(\rho)$$

which implies

$$\|\nabla u\|_{L^\infty(B(x,\varepsilon\rho/8))} \le C(\rho)\varepsilon^{-1}.$$

Proposition 2.6. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B_1)$. Then*

$$E_\varepsilon(u_\varepsilon, B_1) \le C\varepsilon^{n-p} + C, \tag{2.8}$$

with a constant C independent of $\varepsilon \in (0, 1)$.

Proof. Denote

$$I(\varepsilon, R) = Min\{\int_{B(0,R)} [\frac{1}{p}|\nabla u|^p + \frac{1}{\varepsilon^p}(1 - |u|^2)^2]; u \in W_R\},$$

where $W_R = \{u(x) = f(r)\frac{x}{|x|} \in W^{1,p}(B(0, R), R^n); r = |x|, f(R) = 1\}$. Then

$$\begin{aligned} &I(\varepsilon, 1) = E_\varepsilon(u_\varepsilon, B_1) \\ &= \frac{1}{p}\int_{B_1} |\nabla u_\varepsilon|^p dx + \frac{1}{4\varepsilon^p}\int_{B_1\setminus B_\Gamma} (1 - |u_\varepsilon|^2)^2 dx + \frac{1}{4\varepsilon^p}\int_{B_\Gamma} |u_\varepsilon|^4 dx \\ &= \varepsilon^{n-p}[\frac{1}{p}\int_{B(0,\varepsilon^{-1})} |\nabla u_\varepsilon|^p dy + \frac{1}{4}\int_{B(0,\varepsilon^{-1})\setminus B(0,\Gamma\varepsilon^{-1})} (1 - |u_\varepsilon|^2)^2 dy \\ &\quad + \frac{1}{4}\int_{B(0,\Gamma\varepsilon^{-1})} |u_\varepsilon|^4 dy] = \varepsilon^{n-p} I(1, \varepsilon^{-1}). \end{aligned} \tag{2.9}$$

Let u_1 be a solution of $I(1, 1)$ and define

$$u_2 = u_1, \quad if \quad 0 < |x| < 1; \quad u_2 = \frac{x}{|x|}, \quad if \quad 1 \le |x| \le \varepsilon^{-1}.$$

Thus $u_2 \in W_{\varepsilon^{-1}}$ and,

$$\begin{aligned} &I(1, \varepsilon^{-1}) \\ &\le \frac{1}{p}\int_{B(0,\varepsilon^{-1})} |\nabla u_2|^p + \frac{1}{4}\int_{B(0,\varepsilon^{-1})\setminus B(0,\Gamma\varepsilon^{-1})} (1 - |u_2|^2)^2 + \frac{1}{4}\int_{B(0,\Gamma\varepsilon^{-1})} |u_\varepsilon|^4 \\ &= \frac{1}{p}\int_{B_1} |\nabla u_1|^p + \frac{1}{4}\int_{B_1} (1 - |u_1|^2)^2 + \frac{1}{4}\int_{B_1} |u_1|^4 dx + \frac{1}{p}\int_{B(0,\varepsilon^{-1})\setminus B_1} |\nabla \frac{x}{|x|}|^p \\ &= I(1, 1) + \frac{(n - 1)^{p/2}|S^{n-1}|}{p}\int_1^{\varepsilon^{-1}} r^{n-p-1} dr \\ &= I(1, 1) + \frac{(n - 1)^{p/2}|S^{n-1}|}{p(p - n)}(1 - \varepsilon^{p-n}) \le C. \end{aligned}$$

Substituting this into (2.9) yields (2.8).

§3. Proof of Theorem 1.1

Proposition 3.1. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B_1)$. Then for some constant C independent of $\varepsilon \in (0, 1]$*

$$\frac{1}{\varepsilon^n}\int_{B_1\setminus B_\Gamma} (1 - |u_\varepsilon|^2)^2 + \frac{1}{\varepsilon^n}\int_{B_\Gamma} |u_\varepsilon|^4 \le C. \tag{3.1}$$

Proof. (3.1) can be derived by multiplying (2.8) by ε^{p-n}.

Proposition 3.2. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B_1)$. Then for any $\eta \in (0, 1/2)$, there exist positive constants λ, μ independent of $\varepsilon \in (0, 1)$ such that if*

$$\frac{1}{\varepsilon^n}\int_{A_{\Gamma,1-\rho\varepsilon}\cap B^{2l\varepsilon}} (1-|u_\varepsilon|^2)^2 \le \mu, \tag{3.2}$$

where $A_{\Gamma,1-\rho\varepsilon} = B(0, 1-\rho\varepsilon) \setminus B_\Gamma$, $B^{2l\varepsilon}$ is some ball of radius $2l\varepsilon$ with $l \ge \lambda$, then

$$|u_\varepsilon(x)| \ge 1-\eta, \quad \forall x \in A_{\Gamma,1-\rho\varepsilon} \cap B^{l\varepsilon}. \tag{3.3}$$

Proof. First we observe that there exists a constant $C_2 > 0$ which is independent of ε such that for any $x \in B_1$ and $0 < \rho \le 1$, $|B_1 \cap B(x,r)| \ge |A_{\Gamma,1-\rho\varepsilon} \cap B(x,r)| \ge C_2 r^n$. To prove the proposition, we choose

$$\lambda = \frac{\eta}{2C_1}, \quad \mu = \frac{C_2}{C_1^n}(\frac{\eta}{2})^{n+2}, \tag{3.4}$$

where C_1 is the constant in (2.4). Suppose that there is a point $x_0 \in A_{\Gamma,1-\rho\varepsilon} \cap B^{l\varepsilon}$ such that $|u_\varepsilon(x_0)| < 1-\eta$. Then applying (2.4) we have

$$\begin{aligned} |u_\varepsilon(x) - u_\varepsilon(x_0)| &\le C_1\varepsilon^{-1}|x-x_0| \le C_1\varepsilon^{-1}(\lambda\varepsilon) \\ &= C_1\lambda = \frac{\eta}{2}, \quad \forall x \in B(x_0, \lambda\varepsilon), \end{aligned}$$

hence $(1-|u_\varepsilon(x)|^2)^2 > \frac{\eta^2}{4}, \quad \forall x \in B(x_0, \lambda\varepsilon)$. Thus

(3.5)

$$\begin{aligned} \int_{B(x_0,\lambda\varepsilon)\cap A_{\Gamma,1-\rho\varepsilon}} (1-|u_\varepsilon|^2)^2 &> \frac{\eta^2}{4}|A_{\Gamma,1-\rho\varepsilon} \cap B(x_0,\lambda\varepsilon)| \\ &\ge C_2\frac{\eta^2}{4}(\lambda\varepsilon)^n = C_2\frac{\eta^2}{4}(\frac{\eta}{2C_1})^n\varepsilon^n = \mu\varepsilon^n. \end{aligned}$$

Since $x_0 \in B^{l\varepsilon} \cap B_1$, and $(B(x_0,\lambda\varepsilon) \cap A_{\Gamma,1-\rho\varepsilon}) \subset (B^{2l\varepsilon} \cap A_{\Gamma,1-\rho\varepsilon})$, (3.5) implies

$$\int_{B^{2l\varepsilon}\cap A_{\Gamma,1-\rho\varepsilon}} (1-|u_\varepsilon|^2)^2 > \mu\varepsilon^n,$$

which contradicts (3.2) and thus (3.3) is proved.

Let u_ε be a radial minimizer of $E_\varepsilon(u, B_1)$. Given $\eta \in (0, 1/2)$. Let λ, μ be constants in Proposition 3.2 corresponding to η. If

$$\frac{1}{\varepsilon^n}\int_{B(x^\varepsilon,2\lambda\varepsilon)\cap A_{\Gamma,1-\rho\varepsilon}} (1-|u_\varepsilon|^2)^2 \le \mu, \tag{3.6}$$

then $B(x^\varepsilon, \lambda\varepsilon)$ is called $\eta-$ good ball, or simply good ball. Otherwise it is called $\eta-$ bad ball or simply bad ball.

Now suppose that $\{B(x_i^\varepsilon, \lambda\varepsilon), i \in I\}$ is a family of balls satisfying

$$(i): x_i^\varepsilon \in A_{\Gamma,1-\rho\varepsilon}, i \in I; \quad (ii): A_{\Gamma,1-\rho\varepsilon} \subset \cup_{i\in I}B(x_i^\varepsilon, \lambda\varepsilon);$$

$$(iii): B(x_i^\varepsilon, \lambda\varepsilon/4) \cap B(x_j^\varepsilon, \lambda\varepsilon/4) = \emptyset, i \ne j. \tag{3.7}$$

Denote $J_\varepsilon = \{i \in I; B(x_i^\varepsilon, \lambda\varepsilon)\ \textit{is a bad ball}\}$.

Proposition 3.3. *There exists a positive integer N such that the number of bad balls*

$$Card\ J_\varepsilon \le N.$$

Proof. Since (3.7) implies that every point in B_1 can be covered by finite, say m (independent of ε) balls, from (3.1)(3.6) and the definition of bad balls,we have

$$\begin{aligned}
\mu\varepsilon^n Card J_\varepsilon &\le \sum_{i\in J_\varepsilon}\int_{B(x_i^\varepsilon,2\lambda\varepsilon)\cap A_{\Gamma,1-\rho\varepsilon}}(1-|u_\varepsilon|^2)^2\\
&\le m\int_{\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,2\lambda\varepsilon)\cap A_{\Gamma,1-\rho\varepsilon}}(1-|u_\varepsilon|^2)^2\\
&\le m\int_{B_1\setminus B_\Gamma}(1-|u_\varepsilon|^2)^2\le mC\varepsilon^n
\end{aligned}$$

and hence $Card\ J_\varepsilon \le \frac{mC}{\mu}\le N$.

Proposition 3.3 is an important result since the number of bad balls $Card J_\varepsilon$ is always finite as ε turns sufficiently small.

Similar to the argument of Theorem IV.1 in [1], we have

Proposition 3.4. *There exist a subset $J\subset J_\varepsilon$ and a constant $h\ge\lambda$ such that $\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,\lambda\varepsilon)\subset\cup_{i\in J}B(x_j^\varepsilon,h\varepsilon)$ and*

$$|x_i^\varepsilon-x_j^\varepsilon|>8h\varepsilon,\quad i,j\in J,\quad i\ne j. \tag{3.8}$$

Proof. If there are two points x_1,x_2 such that (3.8) is not true with $h=\lambda$, we take $h_1=9\lambda$ and $J_1=J_\varepsilon\setminus\{1\}$. In this case, if (3.8) holds we are done. Otherwise we continue to choose a pair points x_3,x_4 which does not satisfy (3.8) and take $h_2=9h_1$ and $J_2=J_\varepsilon\setminus\{1,3\}$. After at most N steps we may choose $\lambda\le h\le\lambda 9^N$ and conclude this proposition.

Applying Proposition 3.4, we may modify the family of bad balls such that the new one, denoted by $\{B(x_i^\varepsilon,h\varepsilon);i\in J\}$, satisfies

$$\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,\lambda\varepsilon)\subset\cup_{i\in J}B(x_i^\varepsilon,h\varepsilon),\quad Card\ J\le Card\ J_\varepsilon,$$

$$|x_i^\varepsilon-x_j^\varepsilon|>8h\varepsilon, i,j\in J, i\ne j.$$

The last condition implies that every two balls in the new family are not intersected.

Now we prove our main result of this section.

Theorem 3.5. *Let u_ε be a radial minimizer of $E_\varepsilon(u,B_1)$. Then for any $\eta\in(0,1/2)$, there exists a constant $h=h(\eta)$ independent of $\varepsilon\in(0,1)$ such that $Z_\varepsilon=\{x\in B_1;|u_\varepsilon(x)|<1-\eta\}\subset B(0,h\varepsilon)\cup B_\Gamma$. In particular the zeros of u_ε are contained in $B(0,h\varepsilon)\cup B_\Gamma$.*

Proof. Suppose there exists a point $x_0\in Z_\varepsilon$ such that $x_0\overline{\in}B(0,h\varepsilon)$. Then all points on the circle $S_0=\{x\in B_1;\ |x|=|x_0|\}$ satisfy $|u_\varepsilon(x)|<1-\eta$ and hence by virtue of Proposition 3.2 and (2.5), all points on S_0 are contained in bad balls. However, since $|x_0|\ge h\varepsilon, S_0$ can not be covered by a single bad ball. S_0 can be covered by at least two bad balls. However this is impossible. Theorem is proved.

Complete the proof of Theorem 1.1.. Using Theorem 3.5 and (2.5), we can see that $|u_\varepsilon(x)| \geq min(\frac{10}{11}, 1-2\eta), \quad x\overline{\in} B(0, h(\eta)\varepsilon) \cup B_\Gamma$. When $\Gamma \in (0, h\varepsilon]$, this means

$$|u_\varepsilon(x)| \geq min(\frac{10}{11}, 1-2\eta), \quad x\overline{\in} B(0, h(\eta)\varepsilon). \tag{3.9}$$

When $\Gamma \in (h\varepsilon, \varepsilon]$, from Theorem 3.5 we know that $|u_\varepsilon| \geq 1-\eta$ on $B_1 \setminus \overline{B_\Gamma}$. Moreover, similar to the proof of Proposition 3.2, we may still obtain: for any given $\eta \in (0, 1/2)$, there are $\lambda = \frac{\eta}{2C_1}, \quad \mu_2 = C_2\lambda^n(\frac{\eta}{2})^{n+2}$, such that if for $l > \lambda$,

$$\frac{1}{\varepsilon^n}\int_{B_\Gamma \cap B^{2l\varepsilon}} |u_\varepsilon|^4 \leq \mu_2 \tag{3.10}$$

holds, then $|u_\varepsilon(x)| \leq \eta, \quad \forall x \in B_\Gamma \cap B^{l\varepsilon}$. We will take (3.10) as the ruler which distinguishes the good and the bad balls. The ball $B(x^\varepsilon, \lambda\varepsilon)$ satisfying

$$\frac{1}{\varepsilon^2}\int_{B_\Gamma \cap B(x^\varepsilon, 2\lambda\varepsilon)} |u_\varepsilon|^4 \leq \mu_2$$

is named the bad ball in B_Γ. Otherwise, the ball $B(x^\varepsilon, \lambda\varepsilon)$ is named the good ball in B_Γ. Similar to the proof of Proposition 3.3, from proposition 3.1 we may also conclude that the number of the good balls is finite. Moreover, by the same way to the proof of Theorem 3.5, we obtain that

$$\{x \in B_\Gamma; |u_\varepsilon(x)| > \eta\} \subset B_{h\varepsilon} \quad and \quad |u_\varepsilon(x)| \leq \eta \quad as \quad x \in B_\Gamma \setminus B_{h\varepsilon}. \tag{3.11}$$

§4. Uniform estimate

Let $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u, B_1)$, namely f_ε be a minimizer of $E_\varepsilon(f)$ in V. From Proposition 2.6, we have

$$E_\varepsilon(f_\varepsilon) \leq C\varepsilon^{n-p}. \tag{4.1}$$

for some constant C independent of $\varepsilon \in (0, 1)$.

In this section we further prove that for any given $R \in (0, 1)$, there exists a constant $C(R)$ such that

$$E_\varepsilon(f_\varepsilon; R) \leq C(R) \tag{4.2}$$

for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ sufficiently small, where

$$E_\varepsilon(f; R) = \frac{1}{p}\int_R^1 (f_r^2 + (n-1)r^{-2}f^2)^{p/2} r^{n-1}\, dr + \frac{1}{4\varepsilon^p}\int_R^1 (1-f^2)^2 r^{n-1}\, dr.$$

Proposition 4.1. *Given $T \in (0, 1)$. There exist constants $T_j \in [\frac{(j-1)T}{N+1}, \frac{jT}{N+1}]$, $(N = [p])$ and C_j, such that*

$$E_\varepsilon(f_\varepsilon; T_j) \leq C_j \varepsilon^{j-p} \tag{4.3}$$

for $j = n, n+1, ..., N$, where $\varepsilon \in (0, \varepsilon_0)$ with ε_0 sufficiently small.

Proof. For $j = n$, the inequality (4.3) can be obtained by (4.1) easily. Suppose that (4.3) holds for all $j \leq m$. Then we have, in particular,

$$E_\varepsilon(f_\varepsilon; T_m) \leq C_m \varepsilon^{m-p}. \tag{4.4}$$

If $m = N$ then we have done. Suppose $m < N$, we want to prove (4.3) for $j = m+1$.

From (4.4) and integral mean value theorem, we can see that there exists $T_{m+1} \in [\frac{mT}{N+1}, \frac{(m+1)T}{N+1}]$ such that

$$\frac{1}{\varepsilon^p}(1 - f_\varepsilon^2)^2|_{r=T_{m+1}} \leq CE_\varepsilon(u_\varepsilon, \partial B(0, T_{m+1})) \leq C_m \varepsilon^{m-p}. \tag{4.5}$$

Consider the minimizer ρ_1 of the functional

$$E(\rho, T_{m+1}) = \frac{1}{p}\int_{T_{m+1}}^1 (\rho_r^2 + 1)^{p/2} dr + \frac{1}{2\varepsilon^p}\int_{T_{m+1}}^1 (1 - \rho)^2 dr$$

It is easy to prove that the minimizer ρ_ε of $E(\rho, T_{m+1})$ on $W^{1,p}_{f_\varepsilon}((T_{m+1}, 1), R^+)$ exists and satisfies

$$-\varepsilon^p(v^{(p-2)/2}\rho_r)_r = 1 - \rho, \quad in\ (T_{m+1}, 1), \tag{4.6}$$

$$\rho|_{r=T_{m+1}} = f_\varepsilon,\ \ \rho|_{r=1} = f_\varepsilon(1) = 1 \tag{4.7}$$

where $v = \rho_r^2 + 1$. Since $f_\varepsilon \leq 1$, it follows from the maximum principle

$$\rho_\varepsilon \leq 1. \tag{4.8}$$

Applying (4.1) we see easily that

$$E(\rho_\varepsilon; T_{m+1}) \leq E(f_\varepsilon; T_{m+1}) \leq CE_\varepsilon(f_\varepsilon; T_{m+1}) \leq C\varepsilon^{m-p}. \tag{4.9}$$

Now choosing a smooth function $0 \leq \zeta(r) \leq 1$ in (0,1] such that $\zeta = 1$ on $(0, T_{m+1}), \zeta = 0$ near $r = 1$ and $|\zeta_r| \leq C(T_{m+1})$, multiplying (4.6) by $\zeta\rho_r(\rho = \rho_\varepsilon)$ and integrating over $(T_{m+1}, 1)$ we obtain

(4.10)

$$v^{(p-2)/2}\rho_r^2|_{r=T_{m+1}} + \int_{T_{m+1}}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})\, dr = \frac{1}{\varepsilon^p}\int_{T_{m+1}}^1 (1 - \rho)\zeta\rho_r\, dr.$$

Using (4.9) we have

(4.11)

$$\begin{aligned}
&|\int_{T_{m+1}}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r + \zeta\rho_{rr})\, dr| \\
&\leq \int_{T_{m+1}}^1 v^{(p-2)/2}|\zeta_r|\rho_r^2\, dr + \frac{1}{p}|\int_{T_{m+1}}^1 (v^{p/2}\zeta)_r\, dr - \int_{T_{m+1}}^1 v^{p/2}\zeta_r\, dr| \\
&\leq C\int_{T_{m+1}}^1 v^{p/2} + \frac{1}{p}v^{p/2}|_{r=T_{m+1}} + \frac{C}{p}\int_{T_{m+1}}^1 v^{p/2} dr \\
&\leq C\varepsilon^{m-p} + \frac{1}{p}v^{p/2}|_{r=T_{m+1}}
\end{aligned}$$

and using (4.5)(4.7)(4.9) we have

(4.12)
$$|\frac{1}{\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)\zeta\rho_r\,dr| = \frac{1}{2\varepsilon^p}|\int_{T_{m+1}}^1 ((1-\rho)^2\zeta)_r\,dr - \int_{T_{m+1}}^1 (1-\rho)^2\zeta_r\,dr|$$
$$\leq \frac{1}{2\varepsilon^p}(1-\rho)^2|_{r=T_{m+1}} + \frac{C}{2\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho)^2\,dr| \leq C\varepsilon^{m-p}.$$

Combining (4.10) with (4.11)(4.12) yields
$$v^{(p-2)/2}\rho_r^2|_{r=T_{m+1}} \leq C\varepsilon^{m-p} + \frac{1}{p}v^{p/2}|_{r=T_{m+1}}.$$

Hence for any $\delta \in (0,1)$,
$$v^{p/2}|_{r=T_{m+1}} = v^{(p-2)/2}(\rho_r^2+1)|_{r=T_{m+1}} = v^{(p-2)/2}\rho_r^2|_{r=T_{m+1}} + v^{(p-2)/2}|_{r=T_{m+1}}$$
$$\leq C\varepsilon^{m-p} + \frac{1}{p}v^{p/2}|_{r=T_{m+1}} + v^{(p-2)/2}|_{r=T_{m+1}}$$
$$= C\varepsilon^{m-p} + (\frac{1}{p}+\delta)v^{p/2}|_{r=T_{m+1}} + C(\delta)$$

from which it follows by choosing $\delta > 0$ small enough that

$$v^{p/2}|_{r=T_{m+1}} \leq C\varepsilon^{m-p}. \tag{4.13}$$

Now we multiply both sides of (4.6) by $\rho - 1$ and integrate. Then
$$-\varepsilon^p\int_{T_{m+1}}^1 [v^{(p-2)/2}\rho_r(\rho-1)]_r\,dr + \varepsilon^p\int_{T_{m+1}}^1 v^{(p-2)/2}\rho_r^2\,dr + \int_{T_{m+1}}^1 (\rho-1)^2\,dr = 0.$$

From this, using(4.5)(4.7)(4.13), we obtain
$$E(\rho_\varepsilon; T_{m+1}) \leq C|\int_{T_{m+1}}^1 [v^{(p-2)/2}\rho_r(\rho-1)]_r\,dr| \tag{4.14}$$
$$= Cv^{(p-2)/2}|\rho_r||\rho-1|_{r=T_{m+1}} \leq Cv^{(p-1)/2}|\rho-1|_{r=T_{m+1}}$$
$$\leq (C\varepsilon^{m-p})^{(p-1)/p}(C\varepsilon^m)^{1/2} \leq C\varepsilon^{m-p+1}.$$

Define $w_\varepsilon = f_\varepsilon,\ for\ r \in (0, T_{m+1});\quad w_\varepsilon = \rho_\varepsilon,\ for\ r \in [T_{m+1}, 1]$. Since that f_ε is a minimizer of $E_\varepsilon(f)$, we have $E_\varepsilon(f_\varepsilon) \leq E_\varepsilon(w_\varepsilon)$. Thus, it follows that
$$E_\varepsilon(f_\varepsilon; T_{m+1}) \leq \frac{1}{n}\int_{T_{m+1}}^1 (\rho_r^2 + (n-1)r^{-2}\rho^2)^{p/2}r^{n-1}\,dr + \frac{1}{4\varepsilon^p}\int_{T_{m+1}}^1 (1-\rho^2)^2r^{n-1}\,dr$$

by virtue of $\Gamma \leq \varepsilon < T_{m+1}$ since ε is sufficiently small. Noticing that
$$\int_{T_{m+1}}^1 (\rho_r^2 + (n-1)r^{-2}\rho^2)^{p/2}r^{n-1}dr - \int_{T_{m+1}}^1 ((n-1)r^2\rho^2)^{p/2}r^{n-1}dr$$
$$= \frac{p}{2}\int_{T_{m+1}}^1\int_0^1 [\rho_r^2 + (n-1)r^{-2}\rho^2)s + (n-1)r^{-2}\rho^2(1-s)]^{(p-2)/2}ds\rho_r^2r^{n-1}dr$$
$$\leq C\int_{T_{m+1}}^1 (\rho_r^2 + (n-1)r^{-2}\rho^2)^{(p-2)/2}\rho_r^2r^{n-1}dr$$
$$+ C\int_{T_{m+1}}^1 ((n-1)r^{-2}\rho^2)^{(p-2)/2}\rho_r^2r^{n-1}dr$$
$$\leq C\int_{T_{m+1}}^1 (\rho_r^p + \rho_r^2)dr$$

and using (4.8) we obtain

$$
\begin{aligned}
&E_\varepsilon(f_\varepsilon; T_{m+1}) \\
&\le \frac{1}{p}\int_{T_{m+1}}^{1} ((n-1)r^{-2}\rho^2)^{p/2} r^{n-1}\, dr + C\int_{T_{m+1}}^{1} (\rho_r^p + \rho_r^2) dr \\
&+ \frac{C}{4\varepsilon^p}\int_{T_{m+1}}^{1} (1-\rho^2)^2 dr \\
&\le \frac{1}{p}\int_{T_{m+1}}^{1} ((n-1)r^{-2})^{p/2} r^{n-1}\, dr + CE(\rho_\varepsilon; T_{m+1}).
\end{aligned}
$$

Combining this with (4.14) yields (4.3) for $j = m+1$. It is just (4.3) for $j = m+1$.

Proposition 4.2. *Given $T \in (0,1)$. There exist constants $T_{N+1} \in [\frac{NT}{N+1}, T]$ and C_{N+1} such that*

$$
\begin{aligned}
E_\varepsilon(u_\varepsilon; T_{N+1}) \le (n-1)^{p/2}\frac{|S^{n-1}|}{p}\int_{T_{N+1}}^{1} r^{n-p-1} dr \\
+ C_{N+1}\varepsilon^{N+1-p}, \quad N = [p].
\end{aligned}
$$

Proof. From (4.3) we can see $E_\varepsilon(u_\varepsilon; T_N) \le C\varepsilon^{N-p}$. Hence by using integral mean value theorem we know that there exists $T_{N+1} \in [\frac{NT}{N+1}, T]$ such that

$$
\frac{1}{p}\int_{\partial B(0,T_{N+1})} |\nabla u_\varepsilon|^p dx + \frac{1}{4\varepsilon^p}\int_{\partial B(0,T_{N+1})} (1-|u_\varepsilon|^2)^2 dx \le C\varepsilon^{N-p}. \tag{4.15}
$$

Denote ρ_2 is a minimizer of the functional

$$
E(\rho, T_{N+1}) = \frac{1}{p}\int_{T_{N+1}}^{1} (\rho_r^2 + 1)^{p/2} dr + \frac{1}{2\varepsilon^p}\int_{T_{N+1}}^{1} (1-\rho)^2 dr
$$

on $W^{1,p}_{f_\varepsilon}((T_{N+1}, 1), R^+ \cup \{0\})$. It is not difficult to prove by maximum principle that

$$
\rho_2 \le 1. \tag{4.16}
$$

By the same way of the derivation of (4.14), from (4.3) and (4.15) it can be concluded that

$$
E(\rho_2, T_{N+1}) \le C(T_{N+1})\varepsilon^{N+1-p}. \tag{4.17}
$$

Noticing that u_ε is a minimizer and $\rho_2 \frac{x}{|x|} \in W_2$, we also have

$$
\begin{aligned}
&E_\varepsilon(f_\varepsilon; T_{N+1}) \le E_\varepsilon(\rho_2; T_{N+1}) \\
&\le \frac{1}{p}\int_{T_{N+1}}^{1} [\rho_{2r}^2 + \rho_2^2(n-1)r^{-2}]^{p/2} r^{n-1} dr + \frac{1}{2\varepsilon^p}\int_{T_{N+1}}^{1} (1-\rho_2)^2 dr.
\end{aligned} \tag{4.18}
$$

On the other hand,

$$
\begin{aligned}
&\int_{T_{N+1}}^1 [\rho_r^2+(n-1)r^{-2}\rho^2]^{p/2}r^{n-1}dr-\int_{T_{N+1}}^1 [(n-1)r^{-2}\rho^2]^{p/2}r^{n-1}dr\\
=&\frac{p}{2}\int_{T_{N+1}}^1\int_0^1 [\rho_r^2+(n-1)r^{-2}\rho^2]^{(p-2)/2}s+(n-1)r^{-2}\rho^2(1-s)ds\rho_r^2r^{n-1}dr\\
\le& C\int_{T_{N+1}}^1 [\rho_r^2+(n-1)r^{-2}\rho^2]^{(p-2)/2}\rho_r^2r^{n-1}dr\\
&+C\int_{T_{N+1}}^1[(n-1)r^{-2}\rho^2]^{(p-2)/2}\rho_r^2r^{n-1}dr\le C\int_{T_{N+1}}^1[\rho_r^p+\rho_r^2]dr.
\end{aligned}
$$

Substituting this into (4.18), we have

$$
\begin{aligned}
&E_\varepsilon(f_\varepsilon;T_{N+1})\\
\le&\frac{1}{p}\int_{T_{N+1}}^1(n-1)^{p/2}\rho_2^pr^{n-p-1}dr+C\int_{T_{N+1}}(\rho_{2r}^p+\rho_{2r}^2)dr\\
&+\frac{1}{2\varepsilon^p}\int_{T_{N+1}}^1(1-\rho_2)^2dr\\
\le&\frac{1}{p}\int_{T_{N+1}}^1(n-1)^{p/2}\rho_2^pr^{n-p-1}dr+C\varepsilon^{N+1-p}\\
\le&\frac{1}{p}(n-1)^{p/2}\int_{T_{N+1}}^1 r^{n-p-1}dr+C\varepsilon^{N+1-p},
\end{aligned}
$$

by using (4.16) and (4.17). This is the conclusion of Proposition.

§5. $W^{1,p}$ CONVERGENCE

Based on the Proposition 4.2, we may obtain better convergence for radial minimizers.

Theorem 5.1. *Let $u_\varepsilon=f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u,B_1)$. Then*

$$\lim_{\varepsilon\to0}u_\varepsilon=\frac{x}{|x|},\quad \text{in } W^{1,p}(K,R^n) \tag{5.1}$$

for any compact subset $K\subset\overline{B_1}\setminus\{0\}$.

Proof. Without loss of generality, we may assume $K=\overline{B_1}\setminus B(0,T_{N+1})$. From Proposition 4.2, we have

$$E_\varepsilon(u_\varepsilon,K)=|S^{n-1}|E_\varepsilon(f_\varepsilon;T_{N+1})\le C \tag{5.2}$$

where C is independent of ε. This and $|u_\varepsilon|\le1$ imply the existence of a subsequence u_{ε_k} of u_ε and a function $u_*\in W^{1,p}(K,R^n)$, such that

$$\lim_{\varepsilon_k\to0}u_{\varepsilon_k}=u_*,\quad \text{weakly in } W^{1,p}(K,R^n), \tag{5.3}$$

$$\lim_{\varepsilon_k \to 0} |u_{\varepsilon_k}| = 1, \quad \textit{in } C^\alpha(K,R),\ \alpha \in (0, 1-n/p). \tag{5.4}$$

(5.4) implies $u_* = \frac{x}{|x|}$. Noticing that any subsequence of u_ε has a convergence subsequence and the limit is always $\frac{x}{|x|}$, we can assert

$$\lim_{\varepsilon \to 0} u_\varepsilon = \frac{x}{|x|}, \quad \textit{weakly in } W^{1,p}(K, R^n). \tag{5.5}$$

From this and the weakly lower semicontinuity of $\int_K |\nabla u|^p$, using Proposition 4.2, we know that

$$\begin{aligned}\int_K |\nabla \frac{x}{|x|}|^p &\le \underline{\lim}_{\varepsilon_k \to 0} \int_K |\nabla u_\varepsilon|^p \le \overline{\lim}_{\varepsilon_k \to 0} \int_K |\nabla u_\varepsilon|^p \\ &\le C\varepsilon^{[p]+1-p} + |S^{n-1}| \int_{T_{N+1}}^1 ((n-1)r^{-2})^{p/2} r^{n-1}\, dr\end{aligned}$$

and hence

$$\lim_{\varepsilon \to 0} \int_K |\nabla u_\varepsilon|^p = \int_K |\nabla \frac{x}{|x|}|^p$$

since

$$\int_K |\nabla \frac{x}{|x|}|^p = |S^{n-1}| \int_{T_{N+1}}^1 ((n-1)r^{-2})^{p/2} r^{n-1}\, dr.$$

Combining this with (5.4)(5.5) completes the proof of (5.1).

From (3.5) we also see that the zeroes of the radial minimizer $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ are in B_R for given $R > 0$ if ε is small enough.

§6 Uniqueness and regularized property

Theorem 6.1. *For any given $\varepsilon \in (0,1)$, the radial minimizers of $E_\varepsilon(u, B_1)$ are unique on W.*

Proof. Fix $\varepsilon \in (0,1)$. Suppose $u_1(x) = f_1(r)\frac{x}{|x|}$ and $u_2(x) = f_2(r)\frac{x}{|x|}$ are both radial minimizers of $E_\varepsilon(u, B_1)$ on W, then they are both weak radial solutions of (2.1) (2.2). Thus

$$\begin{aligned}&\int_{B_1} (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla\phi dx \\ =& \frac{1}{\varepsilon^p}\int_{B_1\setminus B_\Gamma} [(u_1 - u_2) - (u_1|u_1|^2 - u_2|u_2|^2)]\phi dx \\ &- \frac{1}{\varepsilon^p}\int_{B_\Gamma} (u_1|u_1|^2 - u_2|u_2|^2)\phi dx.\end{aligned}$$

Set $\phi = u_1 - u_2 = (f_1 - f_2)\frac{x}{|x|}$. Take η sufficiently small such that $h < 1$.

Case 1. When $\Gamma \leq h\varepsilon$, we have

(6.1)
$$\begin{aligned}
&\int_{B_1} (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1 - u_2)dx \\
&= \frac{1}{\varepsilon^p}\int_{B_1}(f_1 - f_2)^2 dx - \frac{1}{\varepsilon^p}\int_{B_1}(f_1 - f_2)^2(f_1^2 + f_2^2 + f_1 f_2)dx \\
&= \frac{1}{\varepsilon^p}\int_{B_1\setminus B(0,h\varepsilon)}(f_1 - f_2)^2[1 - (f_1^2 + f_2^2 + f_1 f_2)]dx \\
&+ \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)}(f_1 - f_2)^2 dx - \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)}(f_1 - f_2)^2(f_1^2 + f_2^2 + f_1 f_2)dx.
\end{aligned}$$

Letting $\eta < \frac{1}{2} - \frac{1}{2\sqrt{2}}$ in (3.9), we have $f_1, f_2 \geq 1/\sqrt{2}$ *on* $B_1 \setminus B(0, h\varepsilon)$ for any given $\varepsilon \in (0,1)$. Hence

$$\int_{B_1}(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1 - u_2)dx \leq \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)}(f_1 - f_2)^2 dx.$$

Applying (2.11) of [19], we can see that there exists a positive constant γ independent of ε and h such that

$$\gamma\int_{B_1}|\nabla(u_1 - u_2)|^2 dx \leq \frac{1}{\varepsilon^p}\int_{B(0,h\varepsilon)}(f_1 - f_2)^2 dx, \tag{6.2}$$

which implies

$$\int_{B_1}|\nabla(f_1 - f_2)|^2 dx \leq \frac{1}{\gamma\varepsilon^p}\int_{B(0,h\varepsilon)}(f_1 - f_2)^2 dx. \tag{6.3}$$

Denote $G = B(0, h\varepsilon)$. Applying Theorem 2.1 in Ch II of [16], we have $\|f\|_{\frac{2n}{n-2}} \leq \beta\|\nabla f\|_2$ as $n > 2$, where $\beta = \frac{2(n-1)}{n-2}$. Taking $f = f_1 - f_2$ and applying (6.3), we obtain $f(|x|) = 0$ as $x \in \partial B_1$ and

$$[\int_{B_1}|f|^{\frac{2n}{n-2}}dx]^{\frac{n-2}{n}} \leq \beta^2\int_{B_1}|\nabla f|^2 dx \leq \beta^2\gamma^{-1}\int_G |f|^2 dx\varepsilon^{-p}.$$

Using Holder inequality, we derive

$$\int_G |f|^2 dx \leq |G|^{1-\frac{n-2}{n}}[\int_G |f|^{\frac{2n}{n-2}}dx]^{\frac{n-2}{n}} \leq |B_1|^{1-\frac{n-2}{n}}h^2\varepsilon^{2-p}\frac{\beta^2}{\gamma}\int_G |f|^2 dx.$$

Hence for any given $\varepsilon \in (0,1)$,

$$\int_G |f|^2 dx \leq C(\beta, |B_1|, \gamma, \varepsilon)h^2\int_G |f|^2 dx. \tag{6.4}$$

Denote $F(\eta) = \int_{B(0,h(\eta)\varepsilon)}|f|^2 dx$, then $F(\eta) \geq 0$ and (6.4) implies that

$$F(\eta)(1 - C(\beta, |B_1|, \gamma, \varepsilon)h^2) \leq 0. \tag{6.5}$$

On the other hand, since $C(\beta,|B_1|,\gamma,\varepsilon)$ is independent of η, we may take $0<\eta<\frac{1}{2}-\frac{1}{2\sqrt{2}}$ so small that $h=h(\eta)\leq\lambda 9^N=9^N\frac{\eta}{2C_1}$ (which is implied by (3.4)) satisfies $1<C(\beta,|B_1|,\gamma,\varepsilon)h^2$ for the fixed $\varepsilon\in(0,1)$, which and (6.5) imply that $F(\eta)=0$. Namely $f=0$ a.e. on G, or $f_1=f_2$, a.e. on $B(0,h\varepsilon)$. Substituting this into (6.2), we know that $u_1-u_2=C$ a.e. on B_1. Noticing the continuity of u_1,u_2 which is implied by Proposition 2.1, and $u_1=u_2=x$ on ∂B_1, we can see at last that

$$u_1=u_2,\quad on\quad \overline{B_1}.$$

When $n=2$, using

$$\|f\|_6\leq\beta\|\nabla f\|_{3/2} \tag{6.6}$$

which implied by Theorem 2.1 in Ch II of [16], and by the same argument above we can also derive $u_1=u_2$ on $\overline{B_1}$.

Case 2. When $h\varepsilon<\Gamma\leq\varepsilon$. Similar to (6.1), by taking $\eta<\frac{1}{2}-\frac{1}{\sqrt{2}}$ and using (3.11) we get

(6.7)
$$\begin{aligned}
&\int_{B_1}|\nabla(f_1-f_2)|^p dx\leq\int_{B_1}(|\nabla u_1|^{p-2}\nabla u_1-|\nabla u_2|^{p-2}\nabla u_2)\nabla(u_1-u_2)dx\\
&\leq\frac{1}{\varepsilon^p}\int_{B_1\setminus B_\Gamma}(f_1-f_2)^2[1-(f_1^2+f_2^2+f_1f_2)]dx\\
&\ +C(\varepsilon)\eta^2\int_{B_\Gamma\setminus B_{h\varepsilon}}(f_1-f_2)^2dx+C(\varepsilon)\int_{B(0,h\varepsilon)}(f_1-f_2)^2dx\\
&\leq C(\varepsilon)\eta^2\int_{B_\Gamma\setminus B_{h\varepsilon}}(f_1-f_2)^2dx+C(\varepsilon)\int_{B(0,h\varepsilon)}(f_1-f_2)^2dx.
\end{aligned}$$

Substituting

$$\begin{aligned}
&\eta^2C(\varepsilon)\int_{B_\Gamma\setminus B_{h\varepsilon}}(f_1-f_2)^2dx\leq C\eta^2\int_{B_1}(f_1-f_2)^2dx\\
&\leq C\eta^2(\int_{B_1}(f_1-f_2)^6dx)^{1/3}\leq C\eta^2\int_{B_1}|\nabla(f_1-f_2)|^2dx
\end{aligned}$$

(which implied by (6.6)) into (6.7) and choosing η sufficiently small, we have

$$\int_{B_1}|\nabla(f_1-f_2)|^2dx\leq C\int_{B_{h\varepsilon}}(f_1-f_2)^2dx,$$

this is (6.3). The other part of the proof is as same as the Case 1. The theorem is proved.

In the following, we will prove that the radial minimizer u_ε can be obtained as the limit of a subsequence $u_\varepsilon^{\tau_k}$of the radial minimizer u_ε^τ of the regularized functionals

$$E_\varepsilon^\tau(u,B_1)=\frac{1}{p}\int_{B_1}(|\nabla u|^2+\tau)^{p/2}+\frac{1}{4\varepsilon^p}\int_{B_1\setminus\Gamma}(1-|u|^2)^2+\frac{1}{4\varepsilon^p}\int_\Gamma|u|^4,\quad(\tau>0)$$

on W as $\tau_k\to 0$, namely

Theorem 6.2. *Assume that u_ε^τ be the radial minimizer of $E_\varepsilon^\tau(u, B_1)$ in W. Then there exist a subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ and $\tilde{u}_\varepsilon \in W$ such that*

$$\lim_{\tau_k \to 0} u_\varepsilon^{\tau_k} = \tilde{u}_\varepsilon, \quad in \quad W^{1,p}(B_1, R^n). \tag{6.8}$$

Here $\tilde{u}_\varepsilon$ is just the radial minimizer of $E_\varepsilon(u, B_1)$ in W.

It is not difficult to proof that the minimizer u_ε^τ is a classical solution of the equation

$$-div(v^{(p-2)/2}\nabla u) = \frac{1}{\varepsilon^p} u(1-|u|^2), \quad on \quad B_1 \setminus B_\Gamma; \tag{6.9}$$

$$-div(v^{(p-2)/2}\nabla u) = \frac{1}{\varepsilon^p} u|u|^2, \quad on \quad B_\Gamma$$

and also satisfies the maximum principle: $|u_\varepsilon^\tau| \le 1$ on B_1, where $v = |\nabla u|^2 + \tau$. By virtue of the uniqueness of the radial minimizer, we know $\tilde{u}_\varepsilon = u_\varepsilon$. Thus the radial minimizer u_ε can be regularized by the radial minimizer u_ε^τ of $E_\varepsilon^\tau(u, B_1)$.

Proof of Theorem 6.2.. First, from (2.8) we have

$$E_\varepsilon^\tau(u_\varepsilon^\tau, B_1) \le E_\varepsilon^\tau(u_\varepsilon, B_1) \le CE_\varepsilon(u_\varepsilon, B_1) \le C\varepsilon^{2-p} \tag{6.10}$$

as $\tau \in (0,1)$, where C does not depend on ε and τ. This and $|u_\varepsilon^\tau| \le 1$ imply that $\|u_\varepsilon^\tau\|_{W^{1,p}(B_1)} \le C(\varepsilon)$. Applying the embedding theorem we see that there exist a subsequence $u_\varepsilon^{\tau_k}$ of u_ε^τ and $\tilde{u}_\varepsilon \in W^{1,p}(B_1, R^n)$ such that

$$u_\varepsilon^{\tau_k} \to \tilde{u}_\varepsilon, \quad weakly \quad in \quad W^{1,p}(B_1, R^n), \tag{6.11}$$

$$u_\varepsilon^{\tau_k} \longrightarrow \tilde{u}_\varepsilon, \quad in\ C(\overline{B_1}, R^n),, \tag{6.12}$$

as $\tau_k \to 0$. Since (6.11) and the weakly low semicontinuity of the functional $\int_{B_1} |\nabla u|^p$, we obtain

$$\int_{B_1} |\nabla \tilde{u}_\varepsilon|^p \le \underline{\lim}_{\tau_k \to 0} \int_{B_1} |\nabla u_\varepsilon^{\tau_k}|^p. \tag{6.13}$$

From (6.12) it follows $\tilde{u}_\varepsilon \in W$. This means $E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, B_1) \le E_\varepsilon^{\tau_k}(\tilde{u}_\varepsilon, B_1)$, i.e.,

$$\overline{\lim}_{\tau_k \to 0} E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, B_1) \le \lim_{\tau_k \to 0} E_\varepsilon^{\tau_k}(\tilde{u}_\varepsilon, B_1). \tag{6.14}$$

We can also deduce

$$\int_{B_1 \setminus \Gamma} (1 - |u_\varepsilon^{\tau_k}|^2)^2 + \int_\Gamma |u_\varepsilon^{\tau_k}|^4 \to \int_{B_1 \setminus \Gamma} (1 - |\tilde{u}_\varepsilon|^2)^2 + \int_\Gamma |\tilde{u}_\varepsilon|^4$$

from (6.12) as $\tau_k \to 0$. This and (6.14) show

$$\overline{\lim}_{\tau_k \to 0} \int_{B_1} (|\nabla u_\varepsilon^{\tau_k}|^2 + \tau_k)^{p/2} \le \lim_{\tau_k \to 0} \int_{B_1} (|\nabla \tilde{u}_\varepsilon|^2 + \tau_k)^{p/2} = \int_{B_1} |\nabla \tilde{u}_\varepsilon|^p.$$

Combining this with (6.13) we obtain $\int_{B_1} |\nabla u_\varepsilon^{\tau_k}|^p \to \int_{B_1} |\nabla \tilde{u}_\varepsilon|^p$ as $\tau_k \to 0$, which together with (6.11) implies $\nabla u_\varepsilon^{\tau_k} \to \nabla \tilde{u}_\varepsilon,\ in\ L^p(B_1, R^n)$. Noticing (6.12) we have the conclusion $u_\varepsilon^{\tau_k} \to \tilde{u}_\varepsilon,\ in\ W^{1,p}(B_1, R^n)$ as $\tau_k \to 0$. This is (6.8).

On the other hand, we know

$$E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, B_1) \le E_\varepsilon^{\tau_k}(u, B_1) \tag{6.15}$$

for all $u \in W$. Noticing the conclusion $\lim_{\tau_k \to 0} E_\varepsilon^{\tau_k}(u_\varepsilon^{\tau_k}, B_1) = E_\varepsilon(\tilde{u}_\varepsilon, B_1)$ which had been proved just now we can say $E_\varepsilon(\tilde{u}_\varepsilon, B_1) \le E_\varepsilon(u, B_1)$ when $\tau_k \to 0$ in (6.15), which implies $\tilde{u}_\varepsilon$ be a minimizer of $E_\varepsilon(u, B_1)$.

§7. Proofs of (1.2)

Proposition 7.1. *Assume* $u_\varepsilon^\tau = u = f(r)\frac{x}{|x|}$. *Then there exists* $C > 0$ *which is independent of* ε, τ *such that*

$$\|f\|_{C^{1,\alpha}(K,R)} \le C, \quad \forall \alpha \le 1/2,$$

where $K \subset (0,1)$ *is an arbitrary closed interval.*

Proof. From (6.9) it follows that f solves

$$\begin{aligned} &-(A^{(p-2)/2}f_r)_r - (n-1)r^{-1}A^{(p-2)/2}f_r + r^{-2}A^{(p-2)/2}f \\ &= \frac{1}{\varepsilon^p} f(1-f^2), \quad on \quad (\Gamma, 1) \end{aligned} \tag{7.1}$$

where $A = f_r^2 + (n-1)r^{-2}f^2 + \tau$. Take $R > 0$ sufficiently small such that $K \subset\subset (2R, 1-2R)$. Let $\zeta \in C_0^\infty([0,1],[0,1])$ be a function satisfying $\zeta = 0$ on $[0,R] \cup [1-R,1]$, $\zeta = 1$ on $[2R, 1-2R]$ and $|\nabla\zeta| \le C(R)$ on $(0,1)$. Differentiating (7.1), multiplying with $f_r\zeta^2$ and integrating, we have

$$\begin{aligned} &-\int_0^1 (A^{(p-2)/2}f_r)_{rr}(f_r\zeta^2)dr - (n-1)\int_0^1 (r^{-1}A^{(p-2)/2}f_r)_r(f_r\zeta^2)dr \\ &+ \int_0^1 (r^{-2}A^{(p-2)/2}f)_r(f_r\zeta^2)dr = \frac{1}{\varepsilon^p}\int_0^1 [f(1-f^2)]_r(f_r\zeta^2)dr. \end{aligned}$$

Integrating by parts yields

$$\begin{aligned} &\int_0^1 (A^{(p-2)/2}f_r)_r(f_r\zeta^2)_r dr + \int_0^1 A^{(p-2)/2}(f_r\zeta^2)_r[(n-1)r^{-1}f_r \\ &- r^{-2}f]dr \le \frac{1}{\varepsilon^p}\int_0^1 (1-f^2)f_r^2\zeta^2 dr. \end{aligned}$$

Denote $I = \int_R^{1-R} \zeta^2(A^{(p-2)/2}f_{rr}^2 + (p-2)A^{(p-4)/2}f_r^2 f_{rr}^2)dr$, then for any $\delta \in (0,1)$, there holds

$$I \le \delta I + C(\delta)\int_R^{1-R} A^{p/2}\zeta_r^2 dr + \frac{1}{\varepsilon^p}\int_R^{1-R} f_r^2(1-f^2)\zeta^2 dr \tag{7.2}$$

by using Young inequality. From (7.1) we can see that

$$\frac{1}{\varepsilon^p}(1-f^2) = f^{-1}[-(A^{(p-2)/2}f_r)_r - (n-1)r^{-1}A^{(p-2)/2}f_r + r^{-2}A^{(p-2)/2}f].$$

Applying Young inequality again we obtain that for any $\delta \in (0,1)$,

$$\frac{1}{\varepsilon^p}\int_0^1 (1-f^2)f_r^2\zeta^2 dr \le \delta I + C(\delta)\int_R^{1-R} A^{(p+2)/2}\zeta^2 dr.$$

Substituting this into (7.2) and choosing δ sufficiently small, we have

$$I \le C\int_R^{1-R} A^{p/2}\zeta_r^2 dr + C\int_R^{1-R} A^{(p+2)/2}\zeta^2 dr. \tag{7.3}$$

To estimate the second term of the right hand side of (7.3), we take $\phi = \zeta^{2/q} f_r^{(p+2)/q}$ in the interpolation inequality (Ch II, Theorem 2.1 in [16])

$$\|\phi\|_{L^q} \le C\|\phi_r\|_{L^1}^{1-1/q}\|\phi\|_{L^1}^{1/q}, \quad q \in (1+\frac{2}{p}, 2).$$

We derive by applying Young inequality that for any $\delta \in (0,1)$,

$$\begin{aligned}(7.4) \qquad \int_R^{1-R} f_r^{p+2}\zeta^2 dr \le{}& C(\int_R^{1-R} \zeta^{2/q}|f_r|^{(p+2)/q}dr)\\ &\cdot(\int_R^{1-R}\zeta^{2/q-1}|\zeta_r||f_r|^{(p+2)/q} + \zeta^{2/q}|f_r|^{(p+2)/q-1}|f_{rr}|dr)^{q-1}\\ \le{}& C(\int_R^{1-R}\zeta^{2/q}|f_r|^{(p+2)/q}dr)(\int_R^{1-R}\zeta^{2/q-1}|\zeta_r||f_r|^{(p+2)/q}\\ &+\delta I + C(\delta)\int_R^{1-R} A^{\frac{p+2}{q}-\frac{p}{2}}\zeta^{4/q-2}dr)^{q-1}.\end{aligned}$$

We may claim

$$(7.5) \qquad \int_R^{1-R} A^{p/2}dr \le C,$$

by the same argument of the proof of Proposition 4.2, where C is independent of ε and τ. In fact, from (6.10) we may also derive (4.17). Noting u_ε^τ is a radial minimizer of $E_\varepsilon^\tau(u, B_1)$, replacing (4.18) we obtain

$$\begin{aligned}&E_\varepsilon^\tau(f_\varepsilon\frac{x}{|x|}; B_1 \setminus B(0,T_{N+1})) \le CE(\rho_2; T_{N+1})\\ \le{}& \frac{C}{p}(n-1)^{p/2}\int_{T_{N+1}}^1 r^{n-p-1}dr + C\varepsilon^{N+1-p}.\end{aligned}$$

This means that (7.5) holds.

Noting $q \in (1+\frac{2}{p}, 2)$, we may using Holder inequality to the right hand side of (7.4). Thus, by virtue of (7.5),

$$\int_R^{1-R} f_r^{p+2}\zeta^2 dr \le \delta I + C(\delta).$$

Substituting this into (7.3) we obtain

$$\int_R^{1-R} A^{(p-2)/2}f_{rr}^2\zeta^2 dr \le C,$$

which, together with (7.5), implies that $\|A^{p/4}\zeta\|_{H^1(R,1-R)} \le C$. Noticing $\zeta = 1$ on K, we have $\|A^{p/4}\|_{H^1(K)} \le C$. Using embedding theorem we can see that for any $\alpha \le 1/2$, there holds $\|A^{p/4}\|_{C^\alpha(K)} \le C$. It is not difficult to prove our proposition.

Theorem 7.2. *Let $u_\varepsilon = f_\varepsilon(r)\frac{x}{|x|}$ be a radial minimizer of $E_\varepsilon(u, B_1)$. Then for any compact subset $K \subset B_1 \setminus \{0\}$, we have*

$$\lim_{\varepsilon\to 0} u_\varepsilon = \frac{x}{|x|}, \quad in\ C^{1,\beta}(K, R^n), \quad \beta \in (0,1)$$

Proof. For every compact subset $K \subset B_1 \setminus \{0\}$, applying Proposition 7.1 yields that for some $\beta \in (0, 1/2]$ one has

$$\|u_\varepsilon^\tau\|_{C^{1,\beta}(K)} \le C = C(K), \tag{7.6}$$

where the constant does not depend on ε, τ.

Applying (7.6) and the embedding theorem we know that for any ε and some $\beta_1 < \beta$, there exist $w_\varepsilon \in C^{1,\beta_1}(K, R^n)$ and a subsequence of τ_k of τ such that as $k \to \infty$,

$$u_\varepsilon^{\tau_k} \to w_\varepsilon, \quad in \quad C^{1,\beta_1}(K, R^n). \tag{7.7}$$

Combining this with (6.8) we know that $w_\varepsilon = u_\varepsilon$.

Applying (7.6) and the embedding theorem again we can see that for some $\beta_2 < \beta$, there exist $w \in C^{1,\beta_2}(K, R^n)$ and a subsequence of τ_k which can be denoted by τ_m such that as $m \to \infty$,

$$u_{\varepsilon_m}^{\tau_m} \to w, \quad in \quad C^{1,\beta_2}(K, R^n). \tag{7.8}$$

Denote $\gamma = min(\beta_1, \beta_2)$. Then as $m \to \infty$, we have

$$\begin{aligned} \|u_{\varepsilon_m} - w\|_{C^{1,\beta}(K,R^n)} &\le \|u_{\varepsilon_m} - u_{\varepsilon_m}^{\tau_m}\|_{C^{1,\beta}(K,R^n)} \\ &\quad + \|u_{\varepsilon_m}^{\tau_m} - w\|_{C^{1,\beta}(K,R^n)} \le o(1) \end{aligned} \tag{7.9}$$

by applying (7.7) and (7.8). Noting (1.1) we know that $w = \frac{x}{|x|}$.

Noting the limit $\frac{x}{|x|}$ is unique, we can see that the convergence (7.9) holds not only for some subsquence but for all u_ε. Applying the uniqueness theorem (Theorem 6.1) of the radial minimizers, we know that the regularizable radial minimizer just is the radial minimizer. Theorem is proved.

§8. Proof of Theorem 1.4

First (3.1) shows one rate that the minimizer f_ε converge to 1 as $\varepsilon \to 0$. Moreover, proposition 4.2 implies that for any $T > 0$,

$$\frac{1}{4\varepsilon^p}\int_T^1 (1 - f_\varepsilon^2)^2 r^{n-1} dr \le C. \tag{8.1}$$

In the following we shall give other better estimates of the rate of the convergence for the radial minimizer f_ε than (8.1).

Theorem 8.1. *Let $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ be the radial minimizer of $E_\varepsilon(u, B_1)$. For any $T > 0$, there exists a constant $C > 0$ which is independent of ε such that as ε sufficiently small,*

$$\int_T^1 |f_\varepsilon'|^p r^{n-1} dr + \frac{1}{\varepsilon^p}\int_T^1 (1-f_\varepsilon^2)^2 r^{n-1} dr \leq C\varepsilon^{[p]+1-p}. \tag{8.2}$$

Here $[p]$ is the integer number part of p. Moreover, as $\varepsilon \to 0$,

$$\frac{1}{p}\int_{B_1\setminus B_T} |\nabla u_\varepsilon|^p + \frac{1}{4\varepsilon^p}\int_{B_1\setminus B_T}(1-|u_\varepsilon|^2)^2 \to \frac{1}{p}\int_{B_1\setminus B_T(0)} |\nabla \frac{x}{|x|}|^p. \tag{8.3}$$

Proof. By proposition 4.2 we have

$$E_\varepsilon(f_\varepsilon; B_T) \leq \frac{1}{p}\int_T^1 (n-1)^{p/2} r^{n-p-1} dr + C\varepsilon^{2([p]+1-p)/p}, \tag{8.4}$$

thus,

$$\int_T^1 (1-f_\varepsilon)^2 dr \leq C(T)\varepsilon^p, \tag{8.5}$$

for any $T > 0$. On the other hand, Jensen's inequality implies

$$\begin{aligned} E_\varepsilon(f_\varepsilon; B_T) \geq & \frac{1}{p}\int_T^1 |f_\varepsilon'|^p r^{n-1} dr \\ & + \frac{1}{p}\int_T^1 ((n-1)\frac{f_\varepsilon^2}{r^2})^{p/2} r^{n-1} dr + \frac{1}{4\varepsilon^p}\int_T^1 (1-f_\varepsilon^2)^2 r^{n-1} dr. \end{aligned}$$

Combining this with (8.4) we have

$$\begin{aligned} \frac{1}{p}\int_T^1 ((n-1)\frac{f_\varepsilon^2}{r^2})^{p/2} r^{n-1} dr &\leq E_\varepsilon(f_\varepsilon; B_T) \\ &\leq C\varepsilon^{2([p]+1-p)/p} + \frac{1}{p}\int_T^1 (n-1)^{p/2} r^{n-p-1} dr. \end{aligned} \tag{8.6}$$

Applying (8.5) and Hölder's inequality we obtain

$$\begin{aligned} & \int_T^1 ((n-1)r^{-2})^{p/2} r^{n-1} dr - \int_T^1 ((n-1)r^{-2} f_\varepsilon^2)^{p/2} r^{n-1} dr \\ = & \int_T^1 (n-1)^{p/2} r^{n-p-1}(1-f_\varepsilon^p) dr \leq C(T)\int_T^1 (1-f_\varepsilon) dr \\ \leq & C(\int_T^1 (1-f_\varepsilon)^2 dr)^{1/2} \leq C\varepsilon^{p/2}. \end{aligned}$$

Substituting this into (8.6) we obtain

$$\begin{aligned} -C\varepsilon^{p/2} \leq & E_\varepsilon(f_\varepsilon; B_T) \\ & - \frac{1}{p}\int_T^1 ((n-1)r^{-2})^{p/2} r^{n-1} dr \leq C\varepsilon^{[p]+1-p}. \end{aligned} \tag{8.7}$$

Noticing

$$\frac{1}{p}\int_{B_1\setminus B_T(0)} |\nabla \frac{x}{|x|}|^p = \frac{|S^{n-1}|}{p}\int_T^1 ((n-1)r^{-2})^{p/2} r^{n-1} dr,$$

from (8.7) we can see that both (8.2) and (8.3) hold.

Theorem 8.2. *Let* $u_\varepsilon(x) = f_\varepsilon(r)\frac{x}{|x|}$ *be the radial minimizer of* $E_\varepsilon(u, B_1)$ *on* W*. Then there exist* $C, \varepsilon_0 > 0$ *such that as* $\varepsilon \in (0, \varepsilon_0)$*,*

$$\int_T^1 r^{n-1}[(f'_\varepsilon)^p + \frac{1}{\varepsilon^p}(1 - f_\varepsilon^2)^2]dr \le C\varepsilon^p. \tag{8.8}$$

$$\sup_{r\in[T,1]}(1 - f_\varepsilon(r)) \le C\varepsilon^{p-\frac{n}{2}}. \tag{8.9}$$

(8.8) gives the estimate of the rate of f_ε's convergence to 1 in $W^{1,p}[T,1]$ sense, and that in $C^0[T,1]$ sense is showed by (8.9).

Proof. It follows from Jensen's inequality that

$$\begin{aligned} E_\varepsilon(f_\varepsilon; T) &= \frac{1}{p}\int_T^1 [(f'_\varepsilon)^2 + \frac{(n-1)}{r^2}f_\varepsilon^2]^{p/2} r^{n-1}dr \\ &+ \frac{1}{4\varepsilon^p}\int_T^1 (1 - f_\varepsilon^2)^2 r^{n-1}dr \\ &\ge \frac{1}{p}\int_T^1 (f'_\varepsilon)^p r^{n-1}dr + \frac{1}{4\varepsilon^p}\int_T^1 (1 - f_\varepsilon^2)^2 r^{n-1}dr \\ &+ \frac{1}{p}\int_T^1 \frac{[n-1]^{p/2}}{r^p} f_\varepsilon^p r^{n-1}dr. \end{aligned}$$

Combining this with Proposition 4.2 yields

$$\begin{aligned} &\frac{1}{p}\int_T^1 (f'_\varepsilon)^p r^{n-1}dr + \frac{1}{4\varepsilon^p}\int_T^1 (1 - f_\varepsilon^2)^2 r^{n-1}dr \\ \le &\frac{1}{p}\int_T^1 \frac{[n-1]^{p/2}}{r^p}(1 - f_\varepsilon^p) r^{n-1}dr + C\varepsilon^{[p]+1-p}. \end{aligned}$$

Noticing (8.1), we obtain

$$\begin{aligned} &\frac{1}{p}\int_T^1 (f'_\varepsilon)^p r^{n-1}dr + \frac{1}{4\varepsilon^p}\int_T^1 (1 - f_\varepsilon^2)^2 r^{n-1}dr \\ \le &C\int_T^1 \frac{[n-1]^{p/2}}{r^p}(1 - f_\varepsilon) r^{n-1}dr + C\varepsilon^{[p]+1-p} \\ \le &C\varepsilon^{p/2} + C\varepsilon^{[p]+1-p} \le C\varepsilon^{[p]+1-p}. \end{aligned} \tag{8.10}$$

Using Proposition 4.2 and (8.10), as well as the integral mean value theorem we can see that there exists

$$T_1 \in [T, T(1 + 1/2)] \subset [R/2, R]$$

such that

$$[(f_\varepsilon)_r^2 + (n-1)r^{-2}f_\varepsilon^2]_{r=T_1} \le C_1, \tag{8.11}$$

$$[\frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2]_{r=T_1} \le C_1\varepsilon^{[p]+1-p}. \tag{8.12}$$

Consider the functional

$$E(\rho,T_1)=\frac{1}{p}\int_{T_1}^1(\rho_r^2+1)^{p/2}dr+\frac{1}{2\varepsilon^p}\int_{T_1}^1(1-\rho)^2dr.$$

It is easy to prove that the minimizer ρ_3 of $E(\rho,T_1)$ in $W^{1,p}_{f_\varepsilon}((T_1,1),R^+\cup\{0\})$ exists.

By the same way to proof of (4.14), using (8.11) and (8.12) we have

$$E(\rho_3,T_1)\le v^{\frac{p-2}{2}}\rho_{3r}(1-\rho_3)|_{r=T_1}\le C_1(1-\rho_3(T_1))\le C\varepsilon^{F[1]},$$

where $F[j]=\frac{[p]+1-p}{2^j}+\frac{(2^j-1)p}{2^j}, j=1,2,\cdots$. Hence, similar to the proof of Proposition 4.2, we obtain

$$E_\varepsilon(f_\varepsilon;T_1)\le C\varepsilon^{F[1]}+\frac{1}{p}\int_{T_1}^1\frac{[n-1]^{p/2}}{r^{p-1}}dr.$$

Furthermore, similar to the derivation of (8.10), using (8.1) we may get

$$\int_{T_1}^1(f_\varepsilon')^pr^{n-1}dr+\frac{1}{\varepsilon^p}\int_{T_1}^1(1-f_\varepsilon^2)^2r^{n-1}dr\le C\varepsilon^{F[1]}+C\varepsilon^{p/2}\le C_2\varepsilon^{F[1]}.$$

Set $T_m=R(1-\frac{1}{2^m})$. Proceeding in the way above (whose idea is improving the exponents of ε from $F[k]$ to $F[k+1]$ step by step), we can see that there exists some $m\in N$ satisfying $F[m-1]\le\frac{p}{2}\le F[m]$ such that

$$\begin{aligned}&\int_{T_m}^1(f_\varepsilon')^prdr+\frac{1}{\varepsilon^p}\int_{T_m}^1(1-f_\varepsilon^2)^2r^{n-1}dr\\ &\le C\varepsilon^{\frac{[p]+1-p}{2^m}+\frac{(2^m-1)p}{2^m}}+C\varepsilon^{p/2}\le C\varepsilon^{p/2}.\end{aligned}\tag{8.13}$$

Similar to the derivation of (8.11) and (8.12), it is known that there exists $T_{m+1}\in[T_m,3T_m/2]$ such that

$$[(f_\varepsilon)_r^2+(n-1)r^{-2}f_\varepsilon^2]_{r=T_{m+1}}\le C, \tag{8.14}$$

$$[\frac{1}{\varepsilon^p}(1-f_\varepsilon^2)^2]_{r=T_{m+1}}\le C\varepsilon^{p/2}. \tag{8.15}$$

The minimizer ρ_4 of the functional

$$E(\rho,T_{m+1})=\frac{1}{p}\int_{T_{m+1}}^1(\rho_r^2+1)^{p/2}dr+\frac{1}{2\varepsilon^p}\int_{T_{m+1}}^1(1-\rho)^2dr$$

in $W^{1,p}_{f_\varepsilon}((T_1,1),R^+)$ exists. By the same way to proof of (4.14), using (8.15) and (8.14) we have

$$E(\rho_4,T_{m+1})\le v^{\frac{p-2}{2}}\rho_{4r}(1-\rho_3)|_{r=T_{m+1}}\le C(1-\rho_4(T_{m+1}))\le C\varepsilon^{G[1]},$$

where $G[j]=\frac{p/2}{2^j}+\frac{(2^j-1)p}{2^j}, j=m+1,m+2,\cdots$. By the argument of proof of Proposition 4.2, we obtain

$$E_\varepsilon(f_\varepsilon;T_{m+1})\le C\varepsilon^{G[1]}+\frac{1}{p}\int_{T_{m+1}}^1\frac{[n-1]^{p/2}}{r^{p-1}}dr.$$

Furthermore, similar to the derivation of (8.10), using (8.13) we may get

$$\int_{T_{m+1}}^1(f'_\varepsilon)^p r^{n-1}dr+\frac{1}{\varepsilon^p}\int_{T_{m+1}}^1(1-f_\varepsilon^2)^2r^{n-1}dr\le C\varepsilon^{G[1]}.$$

Proceeding in the way above (whose idea is improving the exponents of ε from $G[k]$ to $G[k+1]$ step by step), we can see that for any $k\in N$,

$$\int_{T_{m+k}}^1(f'_\varepsilon)^p r^{n-1}dr+\frac{1}{\varepsilon^p}\int_{T_{m+k}}^1(1-f_\varepsilon^2)^2r^{n-1}dr\le C\varepsilon^{\frac{p/2}{2^k}+\frac{(2^k-1)p}{2^k}}.$$

Letting $k\to\infty$, we derive

$$\int_R^1(f'_\varepsilon)^p r^{n-1}dr+\frac{1}{\varepsilon^p}\int_R^1(1-f_\varepsilon^2)^2r^{n-1}dr\le C\varepsilon^p.$$

This is (8.8).

From (8.8) we can see that

$$\int_T^1(1-f_\varepsilon^2)^2r^{n-1}dr\le C\varepsilon^{2p}. \tag{8.16}$$

On the other hand, from (5.2) and $|u_\varepsilon|\le 1$ it follows that $\|f_\varepsilon\|_{W^{1,p}((T,1),R)}\le C$. Applying the embedding theorem we know that for any $r_0\in[T,1]$,

$$|f_\varepsilon(r)-f_\varepsilon(r_0)|\le C|r-r_0|^{1-1/p},\ \forall r\in(r_0-\varepsilon,r_0+\varepsilon).$$

Thus

$$(1-f_\varepsilon(r))^2\ge(1-f_\varepsilon(r_0))^2-\varepsilon^{1-1/p}\ge\frac{1}{2}(1-f_\varepsilon(r_0))^2.$$

Substituting this into (8.16) we obtain

$$C\varepsilon^{2p}\ge\int_T^1(1-f_\varepsilon^2)^2r^{n-1}dr\ge\int_{r_0-\varepsilon}^{r_0+\varepsilon}(1-f_\varepsilon^2)^2r^{n-1}dr\ge\frac{1}{2}(1-f_\varepsilon(r_0))^2\varepsilon^n$$

which implies $1-f_\varepsilon(r_0)\le C\varepsilon^{p-\frac{n}{2}}$. Noting r_0 is an arbitrary point in $[T,1]$, we have

$$\sup_{r\in[T,1]}(1-f_\varepsilon(r))\le C\varepsilon^{p-\frac{n}{2}}.$$

Thus (8.9) is derived and the proof of Theorem is complete.

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ON THE LARGE PROPER SUBLATTICES OF FINITE LATTICES

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Abstract. In this present note, We study and prove some properties of the large proper sublattices of finite lattices. It is shown that every finite lattice L with $|L| > 4$ contains a proper sublattice S with $|S| \geq [2(|L|-2)]^{1/3} + 2 > (2|L|)^{1/3}$.

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1 Introduction

In [2], Tom Whaley proved the following classic result about sublattices of lattices.

Theorem 1.1. *If L is a lattice with $k = |L|$ infinite and regular, then either*
(1) there is a proper principal ideal of L of size k,or
(2) there is a proper principal filter of L of size k, or
(3) M_k, the modular lattice of height 2 and size k, is a 0, 1-sublattice of L.

Corollary 1.2. *If L is infinite and regular, then L has a proper sublattice of cardinality $|L|$.*

In [4], Ralph Freese, Jennifer Hyndman, and J. B. Nation proved the following classic result about sublattices of finite ordered set and finite lattices.

Theorem 1.3. *Let P be a finite ordered set with $|P| = n$. Let $\gamma = \lceil n^{1/3} \rceil$. Then either*
(1) there is a principal ideal I of P with $|I| \geq \gamma$, or
(2) there is a principal filter F of P with $|F| \geq \gamma$, or
(3) P contains a super-antichain A with $|A| \geq \gamma$.

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Theorem 1.4. *Let L be a finite lattice with $|L| = n+2$. Then one of the following must hold.*

(1) There exists $x < 1$ with $|(x]| \geq n^{1/3}$;

(2) There exists $y > 0$ with $|[y)| \geq n^{1/3}$;

(3) M_β is a 0, 1-sublattice of L, where $\beta = \lceil n^{1/3} \rceil$.

The above result gives some lower bound of the large proper sublattices of finite ordered set and finite lattices, but they are not so good. In this present note, We give a new lower bound of the large proper sublattices of finite lattices. Our first main result is that:

Theorem 1.5. *Let L be a finite lattice with $|L| = n > 4$. Then there exists proper sublattice $S \subset L$ with $|S| \geq [2(n-2)]^{1/3} + 2 > (2|L|)^{1/3}$.*

2 Definitions and Lemmas

Let $(P, \leq)$ be a poset and $H \subset P$, $a \in P$. The a is an *upper bound* of H if and only if $h \leq a$ for all $h \in H$. An upper bound a of H is the *least upper bound* of H if and only if , for any upper bound b of H, we have $a \leq b$. We shall write $a = \sup H$. The concepts of *lower bound* and *greatest lower bound* are similarly defined; the latter is denoted by $\inf H$. Set

$M(P) = \{(a, b) \in P \times P| \ \sup\{a, b\}\text{and}\inf\{a, b\}\text{exist in } P\}$

$$(x] = \{a \in P | a \leq x\}; \quad [x) = \{a \in P | x \leq a\}; \quad [x]_P = (x] \cup [x).$$

$$N_x = \bigcup_{a \geq x} (a] \cup \bigcup_{b \leq x} [b).$$

where $x \in P$.

Definition 2.1. *A poset $(L, \leq)$ is a lattice if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in L$.*

Theorem 2.2. . *A poset $(P, \leq)$ is a lattice if and only if $M(P) = P$*

Definition 2.3. *If $(A, \leq)$ is a poset, $a, b \in A$, then a and b are comparable if $a \leq b$ or $a \geq b$. Otherwise, a and b are incomparable, in notation $a \| b$. A chain is, therefore, a poset in which there are no incomparable elements. An unorderedposet is one in which $a \| b$ for all $a \neq b$. $(A, \leq)$ is a convex poset if $C(P) = \{a \in P | a \neq \inf P, a \neq \sup P$ and $a \not\| x$ for all $x \in P\} = \varnothing$.*

Definition 2.4. *Let $(A, \leq)$ be a poset and let B be a non-void subset of A. Then there is a natural partial order $\leq_B$ on B induced by $\leq$: for a, $b \in B.a \leq_B b$ if and only if $a \leq b$, we call $(B, \leq_B)$, (or simply, $(B, \leq)$) a subposet of $(A, \leq)$*

Definition 2.5. *Let $(A, \leq)$ be a poset and let B be a subposet of A. If $M(B) = M(A) \cap (B \times B)$ and $sup_B\{a, b\} = sup_A\{a, b\}$, $\inf_B\{a, b\} = \inf_A\{a, b\}$ for all $(a, b) \in M(B)$, then we call $(B, \leq)$ a semi $-$ sublattice of $(A, \leq)$*

Definition 2.6. *A chain C in a poset P is a nonvoid subset which, as a subposet, is a chain. An antichain C in a poset P is a nonvoid subset which, as a subposet, is unordered.*

Definition 2.7. *The length, $l(C)$ of a finite chain is $|C|-1$. A poset P is said to be of length n (in formula $l(P)=n$) where n is a natural number, if and only if there is a chain in P of length n and all chain in P are of length $\leq n$. The width of poset P is m, where m is a natural number, if and only if there is an antichain in P of m elements and all antichain in P have $\leq m$ elements.*

We say that $A \subset P$ is a super-antichain if no pair of distinct elements of A has a common upper bound or a common lower bound. Let $S \subset P$

Lemma 2.8. *. If P is a convex poset , let $x \in P$, then $(x]$, $[x)$ and $[x]_P = (x] \cup [x)$ are proper semi-sublattices of P.*

Lemma 2.9. *If P is a convex poset with $|P| > 1$, let $\eta = \max_{x \in P} |[x]_P|$, then $|N_a| - 1 \leq \frac{1}{2}(\eta-1)^2$ for all $a \in P$.*

Proof. Since η be the largest size of a proper semi-sublattice $[x]_P$ of P, so that $|[x]_P| \leq \eta$ for all $x \in P$. For $a \in P$, if $s = |\{y \in (a] | x \leq y \Rightarrow x = y \text{ for all } x \in P\}|$ and $t = |\{y \in [a) | y \leq x \Rightarrow y = x \text{ for all } x \in P\}|$, then

$$|N_a| \leq t(\eta - s - 1) + s(\eta - t - 1) + 1 = (\eta - 1)(s + t) - 2ts + 1$$

where $0 \leq s + t \leq \eta - 1$. A little calculus shows that this is at most $\frac{1}{2}(\eta-1)(\eta-1)+1$. Then $|N_a| - 1 \leq \frac{1}{2}(\eta-1)^2$ □

Lemma 2.10. *If P is a finite convex poset with $k = (2|P|)^{1/3}$, then either*

(1) there is a proper semi-sublattice $[a]_P$ of P of size $|[a]_P| \geq k$, or

(2) P contains a super-antichain of size k.

Proof. . Suppose that (1) fail. We will construct a super-antichain by transfinite induction. Let $|P| = n$. For every $a \in P$ set

$$N_a = \bigcup_{x \geq a} (x] \cup \bigcup_{y \leq a} [y).$$

We form a super-antichain A as follows. Choose $a_1 \in P$ arbitrarily. Given $a_1, \cdots, a_m$, choose $a_{m+1} \in P - \bigcup_{1 \leq i \leq m} N_{a_i}$ as long as this last set is nonempty. Thus we obtain a sequence $a_1, \cdots, a_r$ where $r \geq \lceil n/(\frac{1}{2}(\eta-1)^2+1) \rceil \geq n/(\frac{1}{2}\eta^2)$ such that $\{a_1, \cdots, a_r\}$ is a super-antichain. Since $r(\frac{1}{2}\eta^2) \geq n$, either $\eta \geq (2n)^{1/3}$ or $r \geq (2n)^{1/3}$, that is. either $\eta \geq (2|P|)^{1/3}$ or $r \geq (2|P|)^{1/3}$. □

Definition 2.11. *Let $(L, \vee, \wedge)$ is a finite lattice, $a \in L$, it is join-irreducible if $a = b \vee c$ implies that $a = b$ or $a = c$; it is meet-irreducible if $a = b \wedge c$ implies that $a = b$ or $a = c$. An element which is both join- and meet- irreducible is called doubly irreducible, let $Irr(L)$ denote the set of all doubly irreducible elements of L.*

3 Main Theorems

Theorem 3.1. *Let L be a finite lattice with $|L| = n > 4$ and $P = L \setminus \{0, 1\}$, $C(P) = \varnothing$. Then one of the following must hold.*

(1) There exists proper sublattice $S = [a]_P \cup \{0\}$ (or $S = [a]_P \cup \{1\}$) $\subset L$ with $|S| \geq (2(n-2))^{1/3} + 1$;

(2) M_k is a 0, 1-sublattice of L, where $k = \lceil (2(n-2))^{1/3} + 2 \rceil$.

Proof. Let $P = L \setminus \{0,1\}$. Note that this makes $|L| = n$ and let $\eta = \max_{x \in P} |[x]_P|$. Then by lemma 2.10 we have :

either there is a semi-sublattice $[a]_P$ of P of size $|[a]_P| \geq (2(n-2))^{1/3}$,

or P contains a super-antichain of size $(2(n-2))^{1/3}$.

Observe that $[a] \cup \{0\}$and $[a] \cup \{1\}$ are proper sublattice of lattice L(for all $a \in P$). Thus either there is a proper sublattice $S = [a]_P \cup \{0\}$(or $S = [a]_P \cup \{1\}$) of L of size $|S| \geq \sqrt{[}3]2(n-2)+1$, or L contains a M_k of size $(2(n-2))^{1/3}+2$. □

Corollary 3.2. *Let L be a finite lattice with $|L| = n > 4$ and $C(L \setminus \{0,1\}) = \varnothing$. Then there exists proper sublattice $S \subset L$ with $|S| \geq (2(n-2))^{1/3}+2$;*

Theorem 3.3. *Let L be a finite lattice with $|L| = n > 4$ and $C(L \setminus \{0,1\}) \neq \varnothing$. Then there exists proper sublattice $S \subset L$ with $|S| \geq \frac{1}{2}(n+3)$.*

Proof. Let $a \in C(L \setminus \{0,1\}) \neq \varnothing$. Then either $a \leq x$ or $a \geq x$ for all $x \in L$. Thus we have

$$L = [a]_L = (a] \cup [a).$$

Therefore

(1) If $\min\{|(a]|, |[a)|\} = 2$, then we have: $\max\{|(a]|, |[a)|\} = n-1 \geq \frac{1}{2}(n+3)$;

(2) If $\min\{|(a]|, |[a)|\} > 2$, then we have: $\max\{|(a]|, |[a)|\} \geq \frac{1}{2}(n-3)+3 = \frac{1}{2}(n+3)$.

The proof is complete. □

Proof. [Proof of Theorem 1.5] This proof is obvious from Lemma 2.10 and Theorem 3.3. □

4 The large proper sublattices of finite modular lattices

In the construction of the super-antichain in Lemma 2.10 we started with an arbitrary element a_1. We record this stronger fact in the next theorem.

Theorem 4.1. *Let L be a finite lattice with $|L| = n > 4$ and $C(L \setminus \{0,1\}) = \varnothing$, let $P = L \setminus \{0,1\}$ and $\eta = \max_{x \in P} |[x]_P \cup \{0,1\}|$. Then every element of L is contained in a 0, 1-sublattice M_k of L with $k(\frac{1}{2}(\eta-2)^2) \geq n-2$. In particular, if $n-2 > \frac{1}{2}(\eta-2)^2$, then L is complemented.*

Proof. This proof is obvious from Lemma 2.8 and Theorem 3.3. □

A simple application is:

Theorem 4.2. *Let L be a finite modular lattice with $|L| = n > 4$. Then L has a proper sublattice S with $|S| \geq \sqrt{2n}$.*

Proof. Let $P = L \setminus \{0,1\}$.

Case 1. When $C(P) = C(L \setminus \{0,1\}) \neq \varnothing$. the proof is trivial(by Theorem 3.3).

Case 2. When $C(P) = C(L \setminus \{0,1\}) = \varnothing$. If $n-2 \leq \frac{1}{2}(\eta-2)^2$, then L has a sublattice $S = [a]_P \cup \{0,1\}$ with $|S| = \eta \geq \sqrt{2(|L|-2)}+2 \geq \sqrt{2n}$ (by Theorem 4.1). So we may assume that $n-2 > \frac{1}{2}(\eta-2)^2$, whence by Theorem 4.1, L is complemented. The result is true for $L \cong M_k$, so we may assume that L has height greater than 2. There is a element $b \in L \setminus \{0,1\}$ with $|[b]_L| = \eta > 3$ (L has height greater than 2), then there exists a element $b' \in [b]_L \setminus \{0,b,1\}$. And we have $N_b \neq P$ by $n-2 > \frac{1}{2}(\eta-2)^2$. Hence there exists a element $c \in P \setminus N_a$. Since $c \notin N_a$, sublattice $\{0, b, b', c'1\}$ of L is a pentagon, contrary to our assumption. □

Definition 4.3. *For a finite lattice L , let*

$$\lambda(L) = \max_{S \in Sub(L), S \neq L} |S|$$

Theorem 4.4. *Let L be a finite lattice with $|L| = n$. Then $\lambda(L) = n - 1$ if and only if $Irr(L) \setminus \{0, 1\} \neq \varnothing$.*

Proof. . The proof is trivial. □

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A CLASS OF RUSCHEWEYH - TYPE HARMONIC UNIVALENT FUNCTIONS WITH VARYING ARGUMENTS

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Abstract. A comprehensive class of complex-valued harmonic univalent functions with varying arguments defined by Ruscheweyh derivatives is introduced. Necessary and sufficient coefficient bounds are given for functions in this class to be starlike. Distortion bounds and extreme points are also obtained.

A.M.S. (MOS) Subject Classification Codes. 30C45, 30C50

Key Words and Phrases. Harmonic, Univalent, Starlike

1. Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain G if both u and v are real and harmonic in G. In any simply connected domain $D \subset G$ we can write $f = h + \overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]).

Denote by H the family of functions $f = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disc $U = \{z : |z| < 1\}$ for which $f(0) = h(0) = 0 = f_z(0) - 1$. Thus for $f = h + \overline{g}$ in H we may express the analytic functions for h and g as

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \ \ g(z) = b_1 z + \sum_{m=2}^{\infty} b_m z^m \ \ (0 \le b_1 < 1). \tag{1}$$

Note that the family H of orientation preserving,normalized harmonic univalent functions reduces to S the class of normalized analytic univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero that is $g \equiv 0$.

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For $f = h + \overline{g}$ given by (1) and $n > -1$, we define the Ruscheweyh derivative of the harmonic function $f = h + \overline{g}$ in H by

$$D^n f(z) = D^n h(z) + \overline{D^n g(z)} \tag{2}$$

where D the Ruscheweyh derivative (see[5]) of a power series $\phi(z) = z + \sum_{m=2}^{\infty} \phi_m z^m$ is given by

$$D^n \phi(z) = \frac{z}{(1-z)^{n+1}} * \phi(z) = z + \sum_{m=2}^{\infty} C(n,m)\phi_m z^m$$

where

$$C(n,m) = \frac{(n+1)_{m-1}}{(m-1)!} = \frac{(n+1)(n+2)\dots(n+m-1)}{(m-1)!}.$$

The operator $*$ stands for the hadamard product or convolution product of two power series

$$\phi(z) = \sum_{m=1}^{\infty} \phi_m z^m \ \ and \ \ \psi(z) = \sum_{m=1}^{\infty} \psi_m z^m$$

defined by

$$(\phi * \psi)(z) = \phi(z) * \psi(z) = \sum_{m=1}^{\infty} \phi_m \psi_m z^m.$$

For fixed values of $n (n > -1)$, let $R_H(n, \alpha)$ denote the family of harmonic functions $f = h + \overline{g}$ of the form (1) such that

$$\frac{\partial}{\partial \theta}(\arg D^n f(z)) \geq \alpha, \ \ 0 \leq \alpha < 1, \ \ |z| = r < 1. \tag{3}$$

We also let $V_{\overline{H}}(n, \alpha) = R_H(n, \alpha) \cap V_H$, where V_H[3], the class of harmonic functions $f = h + \overline{g}$ for which h and g are of the form (1) and their exists ϕ so that , $\mod 2\pi$,

$$\beta_m + (m-1)\phi \equiv \pi, \ \ \delta_m + (m-1)\phi \equiv 0, \ \ m \geq 2, \tag{4}$$

where $\beta_m = \arg(a_m)$ and $\delta_m = \arg(b_m)$.

Note that $R_H(0, \alpha) = SH(\alpha)$ [4], is the class of orientation preserving harmonic univalent functions f which are starlike of order α in U, that is $\frac{\partial}{\partial \theta}(arg f(re^{i\theta})) > \alpha$ where $z = re^{i\theta}$ in U. In [1] , it is proved that the coefficient condition

$$\sum_{m=2}^{\infty} m\left(|a_m| + |b_m|\right) \leq 1 - b_1$$

is sufficient for functions $f = h + \overline{g}$ and of the form (1) to be in $SH(0)$. Recently Jahangiri and Silverman [3] gave the sufficient and necessary conditions for functions $f = h + \overline{g}$ of the form (1) to be in $V_H(\alpha)$ where $0 \leq \alpha < 1$. Further note that if $n = 0$ and the co-analytic part of $f = h + \overline{g}$ is zero, then the class $V_{\overline{H}}(n, \alpha)$ reduces to the class studied in [6].

In this paper, we will give the sufficient condition for $f = h + \overline{g}$ given by (1) to be in the class $R_H(n, \alpha)$, and it is shown that these coefficient condition is also necessary for functions in the class $V_{\overline{H}}(n, \alpha)$. Finally we obtain distortion theorems and characterize the extreme points for functions in $V_{\overline{H}}(n, \alpha)$.

2.Coefficient Bounds

In our first theorem we obtain a sufficient coefficient bound for harmonic functions in $R_H(n, \alpha)$

Theorem 1. Let $f = h + \overline{g}$ given by (1). If

$$\sum_{m=2}^{\infty} \left(\frac{m-\alpha}{1-\alpha}|a_m| + \frac{m+\alpha}{1-\alpha}|b_m| \right) C(n,m) \leq 1 - \frac{1+\alpha}{1-\alpha} b_1 \tag{5}$$

where $a_1 = 1$ and $0 \leq \alpha \leq 1$, then $f \in R_H(n, \alpha)$.

Proof. To prove $f \in R_H(n, \alpha)$, by definition of $R_H(n, \alpha)$ we only need to show that if (5) holds then the required condition (3) satisfied. For (3) we can write

$$\frac{\partial}{\partial \theta}(\arg D^n f(z)) = Re \left\{ \frac{z(D^n h(z))' - \overline{z(D^n g(z))'}}{D^n h(z) - D^n g(z)} \right\} = Re\ \frac{A(z)}{B(z)}.$$

Using the fact that $Re\ w \geq \alpha$ if and only if $|1 - \alpha + w| \geq |1 + \alpha - w|$, it suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0. \tag{6}$$

Substituting for $A(z)$ and $B(z)$ in (6), which yields

$$\begin{aligned}
&|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \\
\geq\ & (2-\alpha)|z| - \sum_{m=2}^{\infty} [mC(n,m) + (1-\alpha)C(n,m)]|a_m||z|^m \\
&- \sum_{m=1}^{\infty} [mC(n,m) - (1-\alpha)C(n,m)]|\overline{b}_m|\ |z|^m - \alpha|z| \\
&- \sum_{m=2}^{\infty} [mC(n,m) - (1+\alpha)C(n,m)]|a_m||z|^m \\
&- \sum_{m=1}^{\infty} [mC(n,m) + (1+\alpha)C(n,m)]|\overline{b}_m|\ |z|^m \\
\geq\ & 2(1-\alpha)|z| \left\{ 1 - \sum_{m=2}^{\infty} \tfrac{m-\alpha}{1-\alpha}|a_m||z|^{m-1}C(n,m) - \sum_{m=1}^{\infty} \tfrac{m+\alpha}{1-\alpha}|b_m||z|^{m-1}C(n,m) \right\} \\
\geq\ & 2(1-\alpha)|z| \left\{ 1 - \frac{1+\alpha}{1-\alpha}b_1 - \left(\sum_{m=2}^{\infty} \frac{m-\alpha}{1-\alpha}C(n,m)|a_m| + \sum_{m=2}^{\infty} \frac{m+\alpha}{1-\alpha}C(n,m)|b_m| \right) \right\}.
\end{aligned} \tag{7}$$

The last expression is non negative by (5), and so $f \in R_H(n, \alpha)$.

Now we obtain the necessary and sufficient conditions for function $f = h + \overline{g}$ be given with (4).

Theorem 2. Let $f = h + \overline{g}$ be given by (1). Then $f \in V_{\overline{H}}(n, \alpha)$ if and only if

$$\sum_{m=2}^{\infty} \left\{ \frac{m-\alpha}{1-\alpha}|a_m| + \frac{m+\alpha}{1-\alpha}|b_m| \right\} C(n,m) \leq 1 - \frac{1+\alpha}{1-\alpha} b_1 \tag{8}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$.

Proof. Since $V_{\overline{H}}(n, \alpha) \subset R_H(n, \alpha)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f \in V_{\overline{H}}(n, \alpha)$, we notice that the condition $\frac{\partial}{\partial \theta}(\arg D^n f(z)) \geq \alpha$ is equivalent to

$$\frac{\partial}{\partial \theta}(\arg D^n f(z)) - \alpha = Re \left\{ \frac{z(D^n h(z))' - \overline{z(D^n g(z))'}}{D^n h(z) - D^n g(z)} - \alpha \right\} \geq 0.$$

That is

$$Re \left[\frac{(1-\alpha)z + \left(\sum\limits_{m=2}^{\infty} (m-\alpha) C(n,m)|a_m| z^m - \sum\limits_{m=1}^{\infty} (m+\alpha) C(n,m)|b_m| \overline{z}^m \right)}{z + \sum\limits_{m=2}^{\infty} C(n,m)|a_m| z^m + \sum\limits_{m=1}^{\infty} C(n,m)|b_m| \overline{z}^m} \right] \geq 0. \tag{9}$$

The above condition must hold for all values of z in U. Upon choosing ϕ according to (4) we must have

$$\frac{(1-\alpha) - (1+\alpha)b_1 - \left(\sum\limits_{m=2}^{\infty} (m-\alpha) C(n,m)|a_m| r^{m-1} + \sum\limits_{m=2}^{\infty} (m+\alpha) C(n,m)|b_m| r^{m-1} \right)}{1 + |b_1| + \left(\sum\limits_{m=2}^{\infty} C(n,m)|a_m| + \sum\limits_{m=2}^{\infty} C(n,m)|b_m| \right) r^{m-1}} \geq 0. \tag{10}$$

If the condition (8) does not hold then the numerator in (10) is negative for r sufficiently close to 1. Hence there exist a $z_0 = r_0$ in (0,1) for which quotient of (10) is negative. This contradicts the fact $f \in V_{\overline{H}}(n, \alpha)$ and so proof is complete.

Corollary 1. A necessary and sufficient condition for $f = h + \overline{g}$ satisfying (8) to be starlike is that $\arg(a_m) = \pi - 2(m-1)\pi/k$, and $\arg(b_m) = 2\pi - 2(m-1)\pi/k$, $(k = 1, 2, 3, \dots)$.

Our next theorem on distortion bounds for functions in $V_{\overline{H}}(n, \alpha)$ which yields a covering result for the family $V_{\overline{H}}(n, \alpha)$.

Theorem 3. If $f \in V_{\overline{H}}(n, \alpha)$ then

$$|f(z)| \leq (1 + |b_1|)r + \frac{1}{C(n,2)} \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1| \right) r^2, \ |z| = r < 1$$

and

$$|f(z)| \geq (1 + |b_1|)r - \frac{1}{C(n,2)} \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2+\alpha}|b_1| \right) r^2, \ |z| = r < 1. \tag{11}$$

Proof. We will only prove the right hand inequality in (11). The argument for the left hand inequality is similar. Let $f \in V_{\overline{H}}(n,\alpha)$ taking the absolute value of f, we obtain

$$\begin{aligned} |f(z)| &\leq ((1+|b_1|)|r| + \sum_{m=2}^{\infty}(|a_m|+|b_m|)|r|^m \\ &\leq \ (1+b_1)r + r^2\sum_{m=2}^{\infty}(|a_m|+|b_m| \end{aligned}$$

That is

$$\begin{aligned} |f(z)| &\leq (1+|b_1|)r + \frac{1-\alpha}{C(n,2)(2-\alpha)}\left(\sum_{m=2}^{\infty}\frac{(2-\alpha)C(n,2)}{1-\alpha}|a_m| + \frac{(2-\alpha)C(n,2)}{1-\alpha}|b_m|\right)r^2 \\ &\leq \ (1+|b_1|)r + \frac{1-\alpha}{C(n,2)(2-\alpha)}\left[1-\frac{1+\alpha}{1-\alpha}|b_1|\right]r^2 \\ &\leq \ (1+|b_1|)r + \frac{1}{C(n,2)}\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1|\right)r^2. \end{aligned}$$

Corollary 2. Let f of the form (1) be so that $f \in V_{\overline{H}}(n,\alpha)$. Then

$$\left\{w : |w| < \frac{2C(n,2)-1-[C(n,2)-1]\alpha}{(2-\alpha)C(n,2)} - \frac{2C(n,2)-1-[C(n,2)-1]\alpha}{(2+\alpha)C(n,2)}b_1\right\} \subset f(U). \tag{12}$$

We use the coefficient bounds to examine the extreme points for $V_{\overline{H}}(n,\alpha)$ and determine extreme points of $V_{\overline{H}}(n,\alpha)$.

Theorem 4. Set $\lambda_m = \frac{(1-\alpha)}{(m-\alpha)C(n,m)}$ and $\mu_m = \frac{1+\alpha}{(m+\alpha)C(n,m)}$. For b_1 fixed, the extreme points for $V_{\overline{H}}(n,\alpha)$ are

$$\{z + \lambda_m x z^m + \overline{b_1 z}\} \cup \{z + \overline{b_1 z + \mu_m x z^m}\} \tag{13}$$

where $m \geq 2$ and $|x| = 1 - |b_1|$.

Proof. Any function f in $V_{\overline{H}}(n,\alpha)$ may expressed as

$$f(z) = z + \sum_{m=2}^{\infty}|a_m|e^{i\beta_m}z^m + \overline{b_1 z} + \overline{\sum_{m=2}^{\infty}|b_m|e^{i\delta_m}z^m},$$

where the coefficients satisfy the inequality (5). Set

$$h_1(z) = z,\ g_1(z) = b_1 z,\ h_m(z) = z+\lambda_m e^{i\beta_m}z^m,\ g_m(z) = b_1 z+\mu_m e^{i\delta_m}z^m\ for\ m = 2,3,\ldots.$$

Writing $X_m = \frac{|a_m|}{\lambda_m}$, $Y_m = \frac{|b_m|}{\mu_m}$, $m = 2,3,\ldots$ and $X_1 = 1-\sum_{m=2}^{\infty}X_m$; $Y_1 = 1-\sum_{m=2}^{\infty}Y_m$ we have,

$$f(z) = \sum_{m=1}^{\infty}(X_m h_m(z) + Y_m g_m(z)).$$

In particular, setting

$$f_1(z) = z+\overline{b_1 z} \textit{ and } f_m(z) = z+\lambda_m x z^m+\overline{b_1 z}+\overline{\mu_m y z^m},\ (m \geq 2, |x|+|y| = 1-|b_1|)$$

we see that extreme points of $V_{\overline{H}}(n,\alpha)$ are contained in $\{f_m(z)\}$.

To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$f_1(z) = \frac{1}{2}\{f_1(z) + \lambda_2(1-|b_1|)z^2\} + \frac{1}{2}\{f_1(z) - \lambda_2(1-|b_1|)z^2\},$$

a convex linear combination of functions in $V_{\overline{H}}(n,\alpha)$.

To see that is not an extreme point if both $|x| \neq 0$ and $|y| \neq 0$, we will show that it can then also be expressed as a convex linear combinations of functions in $V_{\overline{H}}(n,\alpha)$. Without loss of generality, assume $|x| \geq |y|$. Choose $\epsilon > 0$ small enough so that $\epsilon < \frac{|x|}{|y|}$. Set $A = 1+\epsilon$ and $B = 1 - |\frac{\epsilon x}{y}|$. We then see that both

$$t_1(z) = z + \lambda_m A x z^m + \overline{b_1 z + \mu_m y B z^m}$$

and

$$t_2(z) = z + \lambda_m(2-A)xz^m + \overline{b_1 z + \mu_m y(2-B)z^m},$$

are in $V_{\overline{H}}(n,\alpha)$ and note that

$$f_n(z) = \frac{1}{2}\{t_1(z) + t_2(z)\}.$$

The extremal coefficient bounds shows that functions of the form (13) are the extreme points for $V_{\overline{H}}(n,\alpha)$, and so the proof is complete.

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MULTIPLE RADIAL SYMMETRIC SOLUTIONS FOR NONLINEAR BOUNDARY VALUE PROBLEMS OF p-LAPLACIAN

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Abstract. We discuss the existence of multiple radial symmetric solutions for nonlinear boundary value problems of p-Laplacian, based on Leggett-Williams's fixed point theorem.

A.M.S. (MOS) Subject Classification Codes. 35J40, 35J65, 35J67.

Key Words and Phrases. Multiple radial symmetric solutions, p-Laplacian equation, Leggett-Williams's fixed point theorem.

1. Introduction.

In this paper, we consider the existence of multiple radial symmetric solutions of the p-Laplacian equation

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = g(x)f(x,u), \quad x \in \Omega, \tag{1.1}$$

subject to the nonlinear boundary value condition

$$B\left(\frac{\partial u}{\partial \nu}\right) + u = 0, \quad x \in \partial\Omega \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is the unit ball centered at the origin, ν denotes the unit outward normal to the boundary $\partial\Omega$, $g(x)$, $f(x,s)$ and $B(s)$ are all the given functions. In order to discuss the radially symmetric solutions, we assume that $g(x)$ and $f(x,s)$ are radially symmetric, namely, $g(x) = g(|x|)$, $f(x,s) = f(|x|,s)$. Let $w(t) \equiv u(|x|)$ with $t = |x|$ be a radially symmetric solution. Then a direct calculation shows that

$$(t^{n-1}\varphi(w'(t)))' + t^{n-1}g(t)f(t,w(t)) = 0, \qquad 0 < t < 1, \tag{1.3}$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where $\varphi(s) = |s|^{p-2}s$ and $p > 1$, with the boundary value condition

$$w'(0) = 0, \tag{1.4}$$

$$w(1) + B(w'(1)) = 0. \tag{1.5}$$

Such a problem arises in many different areas of applied mathematics and the fields of mechanics, physics and has been studied extensively, see [1]–[6]. In particular, the Leggett-Williams fixed point theorem has been used to discuss the multiplicity of solutions. For example, He, Ge and Peng [1] considered the following ordinary differential equation

$$(\varphi(y'))' + g(t)f(t, y) = 0, \qquad 0 < t < 1,$$

which corresponds to the special case $n = 1$ of the equation (1.3), with the boundary value conditions

$$y(0) - B_0(y'(0)) = 0,$$
$$y(1) - B_1(y'(1)) = 0.$$

They used the Leggett-Williams fixed point theorem and proved the existence of multi-nonnegative solutions.

In this paper, we extent the results in [1] with $n \geq 1$. We want to use Leggett-Williams's fixed-point theorem to search for solutions of the problem (1.3)–(1.5) too.

This paper is organized as follows. Section 2 collects the preliminaries and statements of results. The proofs of theorems will be given subsequently in Section 3.

2. Preliminaries and Main Results

As a preliminary, we first assume that the given functions satisfy the following conditions Preliminaries and Main Results

(A1) $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ is a continuous function.

(A2) $g : (0, 1) \to [0, +\infty)$ is continuous and is allowed to be singular at the end points of $(0, 1)$, $g(t) \not\equiv 0$ on any subinterval of $(0, 1)$. In addition,

$$0 < \int_0^1 g(r)dr < +\infty.$$

(A3) $B(s)$ is a continuous, nondecreasing, odd function, defined on $(-\infty, +\infty)$. And there exists a constant $m > 0$, such that

$$0 \leq B(s) \leq ms, \qquad s \geq 0.$$

In order to prove the existence of the multi-radially symmetric solutions of the problem (1.3)–(1.5), we need some lemmas.

First, we introduce some denotations. Let $E = (E, \|\cdot\|)$ be a Banach space, $P \subset E$ is a cone. By a nonnegative continuous concave functional α on P, we mean a mapping $\alpha : P \to [0, +\infty)$ that is α is continuous and

$$\alpha(tw_1 + (1-t)w_2) \geq t\alpha(w_1) + (1-t)\alpha(w_2),$$

for all $w_1, w_2 \in P$, and all $t \in [0,1]$. Let $0 < a < b$, $r > 0$ be constants. Denote

$$P_r = \{w \in P | \|w\| < r\},$$

and

$$P(\alpha, a, b) = \{w \in P | a \leq \alpha(w), \|w\| \leq b\}.$$

We need the following two useful lemmas.

Lemma 2.1 (Leggett-Williams's fixed point theorem)*Let $T : \overline{P}_c \to \overline{P}_c$ be completely continuous and α be a nonnegative continuous concave functional on P such that $\alpha(w) \leq \|w\|$, for all $w \in \overline{P}_c$. Suppose there exist $0 < a < b < d \leq c$ such that*

(B1) $\{w \in P(\alpha, b, d) | \alpha(w) > b\} \neq \emptyset$ and $\alpha(Tw) > b$, for $w \in P(\alpha, b, d)$,

(B2) $\|Tw\| < a$, for $\|w\| \leq a$, and

(B3) $\alpha(Tw) > b$, for $w \in P(\alpha, b, c)$ with $\|Tw\| > d$.

Then, T has at least three fixed points w_1, w_2 and w_3 satisfying

$$\|w_1\| < a, \quad b < \alpha(w_2), \quad \textit{and} \quad \|w_3\| > a, \quad \alpha(w_3) < b.$$

Lemma 2.2 *Let $w \in P$ and $\delta \in (0, 1/2)$. Then*

(C1) If $0 < \sigma < 1$,

$$w(t) \geq \begin{cases} \dfrac{\|w\|t}{\sigma}, 0 \leq t \leq \sigma, \\ \dfrac{\|w\|(1-t)}{(1-\sigma)}, \sigma \leq t \leq 1. \end{cases}$$

(C2) $w(t) \geq \delta\|w\|$, for all $t \in [\delta, 1-\delta]$.

(C3) $w(t) \geq \|w\|t$, $0 \leq t \leq 1$, if $\sigma = 1$.

(C4) $w(t) \geq \|w\|(1-t)$, $0 \leq t \leq 1$, if $\sigma = 0$.

Here $\sigma \in [0,1]$, such that

$$w(\sigma) = \|w\| \equiv \sup_{t \in [0,1]} |w(t)|.$$

We want to use the fixed-point theorem in Lemma 2.1 to search for solutions of the problem (1.3)–(1.5). By (A2), there exists a constant $\delta \in (0, 1/2)$, so that

$$L(x) \equiv \psi\Big(\int_\delta^x g(t)dt\Big) + \psi\Big(\int_x^{1-\delta} g(t)dt\Big), \quad \delta \leq x \leq 1-\delta,$$

is a positive and continuous function in $[\delta, 1-\delta]$, where $\psi(s) \equiv |s|^{\frac{1}{(p-1)}} \operatorname{sgn} s$ is the inverse function of $\varphi(s) = |s|^{p-2}s$. For convenience, we set

$$L \equiv \min_{\delta \leq x \leq 1-\delta} L(x),$$

and

$$\lambda = (m+1)\psi\Big(\int_0^1 g(r)dr\Big).$$

And in this paper, we set the Banach space $E = C[0,1]$ with the norm defined by

$$\|w\| = \sup_{t \in [0,1]} |w(t)|, \quad w \in E.$$

The cone $P \subset E$ is specified as,

$$P = \{w \in E | w \text{ is a nonnegative concave function in } [0,1]\}.$$

Furthermore, we define the nonnegative and continuous concave function α satisfying

$$\alpha(w) = \frac{w(\delta) + w(1-\delta)}{2}, \quad w \in P.$$

Obviously,

$$\alpha(w) \leq \|w\|, \quad \text{for all } w \in P.$$

Under all the assumptions (A1)–(A3), we can get the main result as follows

Theorem 2.1 *Let a, b, d, δ be given constants with $0 < a < \delta b < b < b/\delta \leq d$, and let the following conditions on f and φ are fulfilled:*

(D1) For all $(t, w) \in [0,1] \times [0,a]$, $f(t,w) < \varphi\left(\dfrac{a}{\lambda}\right)$;

(D2) Either

i) $\limsup\limits_{w \to +\infty} \frac{f(t,w)}{w^{p-1}} < \varphi\left(\frac{1}{\lambda}\right)$, uniformly all $t \in [0,1]$, or

ii) $f(t,w) \leq \varphi\left(\frac{\eta}{\lambda}\right)$, for all $(t,w) \in [0,1] \times [0,\eta]$ with some $\eta \geq d$, $\lambda > 0$;

(D3) $f(t,w) > \varphi\left(\dfrac{2b}{\delta L}\right)$, for $(t,w) \in [\delta, 1-\delta] \times [\delta b, d]$ with some $L > 0$.

Then, the problem (1.3)–(1.5) have at least three radially symmetric solutions w_1, w_2 and w_3, such that

$$\|w_1\| < a, \quad \alpha(w_2) > b, \quad \textit{and} \quad \|w_3\| > a, \quad \alpha(w_3) < b.$$

3. Proofs of the Main Results

We are now in a position to prove our main results.

Proof of Theorem 2.1. Define $T : P \to E$, $w \mapsto W$, where W is determined by

$$\begin{aligned} W(t) =&(Tw)(t) \\ \triangleq& B \circ \psi\Big(\int_0^1 r^{n-1} g(r) f(r, w(r)) dr\Big) \\ &+ \int_t^1 \psi\Big(s^{-(n-1)} \int_0^s r^{n-1} g(r) f(r, w(r)) dr\Big) ds, \qquad t \in [0,1], \end{aligned}$$

for each $w \in P$.

First we prove each fixed point of W in P is a solution of (1.3)– (1.5). By the definition of W, we have

$$W'(t) = (Tw)'(t) = -\psi\Big(t^{-(n-1)} \int_0^t r^{n-1} g(r) f(r, w(r)) dr\Big).$$

Noticing that

$$\begin{aligned} &\Big| -\psi\Big(t^{-(n-1)} \int_0^t r^{n-1} g(r) f(r, w(r)) dr\Big)\Big| \\ =&\Big| -\psi\Big(\int_0^t \left(\frac{r}{t}\right)^{n-1} g(r) f(r, w(r)) dr\Big)\Big| \\ \leq&\Big| -\psi\Big(\int_0^t g(r) f(r, w(r)) dr\Big)\Big|, \end{aligned}$$

and by the integrability of g and f, we have

$$\lim_{t\to 0^+} W'(t) = \lim_{t\to 0^+} \psi\Big(\int_0^t g(r)f(r,w(r))dr\Big) = 0. \tag{3.1}$$

Considering

$$W'(0) = \lim_{t\to 0} \frac{W(t)-W(0)}{t},$$

and

$$\begin{aligned}
&W(t)-W(0)\\
=&\int_t^1 \psi\Big(s^{-(n-1)}\int_0^s r^{n-1}g(r)f(r,w(r))dr\Big)ds\\
&\qquad -\int_0^1 \psi\Big(s^{-(n-1)}\int_0^s r^{n-1}g(r)f(r,w(r))dr\Big)ds\\
=&-\int_0^t \psi\Big(s^{-(n-1)}\int_0^s r^{n-1}g(r)f(r,w(r))dr\Big)ds,
\end{aligned}$$

and by using L$'$Hospital's rule, we get

$$\begin{aligned}
W'(0) =&\lim_{t\to 0}\frac{W(t)-W(0)}{t}\\
=&\lim_{t\to 0}\Big(W(t)-W(0)\Big)'\\
=&-\lim_{t\to 0}\psi\Big(t^{-(n-1)}\int_0^t r^{n-1}g(r)f(r,w(r))dr\Big)ds\\
=&0.
\end{aligned}$$

Recalling (3.1), we know that $W'(t)$ is right-continuous at the point $t=0$, and $W'(0)=0$, namely, the fixed point of W satisfies (1.4). By the assumption (A1) and (A2), we also have

$$W'(t) = (Tw)'(t) \le 0.$$

Then $\|Tw\| = (Tw)(0)$. On the other hand, since

$$W(1) = B\psi\left(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\right),$$

and

$$B(W'(1)) = -B\psi\left(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\right),$$

we see that

$$W(1) + B(w'(1)) = 0,$$

namely, the fixed point of W also satisfies (1.5).

Next we show that the conditions in Lemma 2.1 are satisfied. We first prove that condition (D2) implies the existence of a number c where $c>d$ such that

$$W : \overline{P}_c \to \overline{P}_c.$$

If ii) of (D2) holds, by the condition (A3), we see that

$$\begin{aligned}
\|Tw\| =&(Tw)(0)\\
=&B\circ\psi\Big(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\Big)\\
&\qquad+\int_0^1\psi\Big(s^{-(n-1)}\int_0^s r^{n-1}g(r)f(r,w(r))dr\Big)ds\\
\le& m\psi\Big(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\Big)\\
&\qquad+\int_0^1\psi\Big(\int_0^s\left(\frac{r}{s}\right)^{n-1}g(r)f(r,w(r))dr\Big)ds\\
\le&(m+1)\psi\Big(\int_0^1 g(r)f(r,w(r))dr\Big)\\
\le&(m+1)\psi\Big(\int_0^1 g(r)\varphi\left(\frac{\eta}{\lambda}\right)dr\Big)\\
=&(m+1)\psi\Big(\int_0^1 g(r)dr\Big)\psi\Big(\varphi\left(\frac{\eta}{\lambda}\right)\Big)\\
=&\frac{\eta}{\lambda}(m+1)\psi\Big(\int_0^1 g(r)dr\Big)\\
=&\eta, \qquad\qquad \text{for } w\in\overline{P}_\eta.
\end{aligned}$$

Then, if we select $c=\eta$, there must be $W:\overline{P}_c\to\overline{P}_c$.

If i) of (D2) is satisfied, then there must exist $D>0$ and $\epsilon<\varphi(1/\lambda)$, so that

$$\frac{f(t,w)}{w^{p-1}}<\epsilon,\quad \text{for } (t,w)\in[0,1]\times[D,+\infty). \tag{3.2}$$

Let $M=\max\{f(t,w)|\ 0\le t\le 1, 0\le w\le D\}$. By (3.2), we obtain

$$f(t,w)\le M+\epsilon w^{p-1},\quad \text{for } (t,w)\in[0,1]\times[0,+\infty). \tag{3.3}$$

Selecting a proper real number c, so that

$$\varphi(c)>\max\Big\{\varphi(d), M\Big(\varphi\left(\frac{1}{\lambda}\right)-\epsilon\Big)^{-1}\Big\}. \tag{3.4}$$

Utilizing (3.2), (3.3) and (3.4), we have

$$\begin{aligned}
\|Tw\| =&(Tw)(0)\\
=&B\circ\psi\Big(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\Big)\\
&+\int_0^1\psi\Big(s^{-(n-1)}\int_0^s r^{n-1}g(r)f(r,w(r))dr\Big)ds\\
\le& m\psi\Big(\int_0^1 r^{n-1}g(r)f(r,w(r))dr\Big)\\
&+\int_0^1\psi\Big(\int_0^s\Big(\frac{r}{s}\Big)^{n-1}g(r)f(r,w(r))dr\Big)ds\\
\le&(m+1)\psi\Big(\int_0^1 g(r)f(r,w(r))dr\Big)\\
\le&(m+1)\psi\Big(\int_0^1 g(r)(M+\epsilon w^{p-1})dr\Big)\\
=&(m+1)\psi\left(\int_0^1 g(r)\left(M\left(\varphi\left(\frac{1}{\lambda}\right)-\epsilon\right)^{-1}\left(\varphi\left(\frac{1}{\lambda}\right)-\epsilon\right)+\epsilon w^{p-1}\right)dr\right)\\
\le&(m+1)\psi\left(\int_0^1 g(r)\left(\varphi(c)\left(\varphi\left(\frac{1}{\lambda}\right)-\epsilon\right)+\epsilon c^{p-1}\right)dr\right)\\
=&(m+1)\psi\Big(\int_0^1 g(r)dr\Big)\frac{c}{\lambda}=c, \qquad\qquad \text{for } w\in\overline{P}_c.
\end{aligned}$$

So we obtain $\|W\|\le c$, that is $W:\overline{P}_c\to\overline{P}_c$.

Then we want to verify that W satisfies the condition (B2) in Lemma 2.1. If $\|w\|\le a$, then by the condition (D1), we know

$$f(t,w)<\varphi\left(\frac{a}{\lambda}\right),\qquad \text{for } 0\le t\le 1,\quad 0\le w\le a.$$

We use the methods similarly to the above, and can get $\|W\|=\|Tw\|<a$, that is, W satisfies (B2).

To fulfill condition (B1) of Lemma 2.1, we note that $w(t)\equiv(b+d)/2>b$, $0\le t\le 1$, is the member of $P(\alpha,b,d)$ and $\alpha(w)=\alpha((b+d)/2)>b$, hence $\{w\in P(\alpha,b,d)|\ \alpha(w)>b\}\ne\emptyset$. Now assume $w\in P(\alpha,b,d)$. Then

$$\alpha(w)=\frac{w(\delta)+w(1-\delta)}{2}\ge b,\quad \text{and } b\le\|w\|\le d.$$

Utilizing the condition (C2) in Lemma 2.2, we know that for all s, which satisfying $\delta\le s\le 1-\delta$, there has

$$\delta b\le\delta\|w\|\le w(s)\le d.$$

And meanwhile, we can select a proper ε, so that

$$\left(\frac{\varepsilon}{s}\right)^{n-1}>\left(\frac{\varepsilon}{1-\delta}\right)^{n-1}>\frac{1}{2}.$$

Combining the condition (D3), we can see

$$\begin{aligned}
\alpha(Tw) &= \frac{(Tw)(\delta)+(Tw)(1-\delta)}{2} \\
&\geq (Tw)(1-\delta) \\
&\geq \int_{1-\delta}^{1} \psi\left(\int_0^s \left(\frac{r}{s}\right)^{n-1} g(r)f(r,w(r))dr\right) ds \\
&\geq \int_{1-\delta}^{1} \psi\left(\int_\varepsilon^s \left(\frac{r}{s}\right)^{n-1} g(r)f(r,w(r))dr\right) ds \\
&\geq \int_{1-\delta}^{1} \psi\left(\int_\varepsilon^s \left(\frac{\varepsilon}{s}\right)^{n-1} g(r)f(r,w(r))dr\right) ds \\
&\geq \int_{1-\delta}^{1} \psi\left(\left(\frac{\varepsilon}{s}\right)^{n-1} \int_\delta^{1-\delta} g(r)f(r,w(r))dr\right) ds \\
&> \int_{1-\delta}^{1} \psi\left(\frac{1}{2}\int_\delta^{1-\delta} g(r)\varphi\left(\frac{2b}{\delta L}\right) dr\right) ds \\
&= \frac{1}{2}\delta\psi\left(\int_\delta^{1-\delta} g(r)dr\right)\frac{2b}{\delta L} \\
&\geq b.
\end{aligned}$$

That is (B1) is well verified.

Finally, we prove (B3) of Lemma 2.1 is also satisfied. For $w \in P(\alpha, b, c)$, we have $\|Tw\| > d$. By using the condition (C2) in Lemma 2.2, we get

$$\alpha(Tw) = \frac{(Tw)(\delta)+(Tw)(1-\delta)}{2} \geq \delta\|Tw\| > \delta d > b.$$

Then, the condition (B3) in Leggett-Williams's fixed point theorem is well verified.

Using the above results and applying Leggett-Williams's fixed point theorem, we can see that the operator W has at least three fixed points, that is the problem (1.3)–(1.5) have at least three radially symmetric solutions w_1, w_2 and w_3, which satisfying

$$\|w_1\| < a, \quad \alpha(w_2) > b, \quad \text{and} \quad \|w_3\| > a, \quad \alpha(w_3) < b.$$

The proof is complete.

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CLASSES OF RUSCHEWEYH-TYPE ANALYTIC UNIVALENT FUNCTIONS

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Abstract. A class of univalent functions is defined by making use of the Ruscheweyh derivatives. This class provides an interesting transition from starlike functions to convex functions. In special cases it has close inter-relations with uniformly starlike and uniformly convex functions. We study the effects of certain integral transforms and convolutions on the functions in this class.

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1. Introduction

Let A denote the family of functions f that are analytic in the open unit disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Consider the subclass T consisting of functions f in A, which are univalent in U and are of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, where $a_n \geq 0$. Such functions were first studied by Silverman [6]. For $\alpha \geq 0,\ 0 \leq$ ¡1 and fixed $\lambda > -1$, we let $D(\alpha, \beta, \lambda)$ denote the set of all functions in T for which

$$Re\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}\right) > \alpha \left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right| + \beta.$$

Here, the operator $D^\lambda f(z)$ is the Ruscheweyh derivative of f, (see [4]), given by

$$D^\lambda f(z) = \frac{z}{(1-z)^{1+\lambda}} * f(z) = z - \sum_{n=2}^{\infty} a_n B_n(\lambda) z^n,$$

where $*$ stands for the convolution or Hadamard product of two power series and

$$B_n(\lambda) = \frac{(\lambda+1)(\lambda+2) - (\lambda+n-1)}{(n-1)!}.$$

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The family $D(\alpha,\beta,\lambda)$, which has been studied in [5], is of special interest for it contains many well-known as well as new classes of analytic univalent functions. In particular, for $\alpha=0$ and $0\le\lambda\le 1$ it provides a transition from starlike functions to convex functions. More specifically, $D(0,\beta,0)$ is the family of functions starlike of order β and $D(0,\beta,1)$ is the family of functions convex of order β. For $D(\alpha,0,0)$, we obtain the class of uniformly α-starlike functions introduced by Kanas and Wisniowski [2], which can be generalized to $D(\alpha,\beta,0)$, the class of uniformly α-starlike functions of order β. Generally speaking, $D(\alpha,\beta,\lambda)$ consists of functions $F(z)=D^\lambda f(z)$ which are uniformly α-starlike functions of order β in$U$. In Section 2 we study the effects of certain integral operators on the class $D(\alpha,\beta,\lambda)$. Section 3 deals with the convolution properties of the class $D(\alpha,\beta,\lambda)$ in connection with Gaussian hypergeometric functions.

2. Integral transform of the class $D(\alpha,\beta,\lambda)$.

For $f\in A$ we define the integral transform

$$V_\mu(f)(z)=\int_0^1\mu(t)\frac{f(tz)}{t}dt,$$

where μ is a real-valued, non-negative weight function normalized so that $\int_0^1\mu(t)dt=1$. Some special cases of $\mu(t)$ are particularly interesting such as $\mu(t)=(1+c)t^c, c>-1$, for which V_μ is known as the Bernardi operator, and

$$\mu(t)=\frac{(c+1)^\delta}{\mu(\delta)}\,t^c\left(\log\frac{1}{t}\right)^{\delta-1},\ c>-1,\delta\ge 0$$

which gives the Komatu operator. For more details see [3].

First we show that the class $D(\alpha,\beta,\lambda)$ is closed under $V_\mu(f)$.

Theorem 1. Let $f\in D(\alpha,\beta,\lambda)$. Then $V_\mu(f)\in D(\alpha,\beta,\lambda)$.

Proof. By definition, we have

$$\begin{aligned}V_\mu(f)(z)&=\frac{(c+1)^\delta}{\mu(\delta)}\int_0^1(-1)^{\delta-1}t^c(\log t)^{\delta-1}\left(z-\sum_{n=2}^\infty a_nz^nt^{n-1}\right)dt\\&=\frac{(-1)^{\delta-1}(c+1)^\delta}{\mu(\delta)}\lim_{\gamma\to 0^+}\left[\int_\gamma^1 t^c(\log t)^{\delta-1}\left(z-\sum_{n=2}^\infty a_nz^nt^{n-1}\right)dt\right].\end{aligned}$$

A simple calculation gives

$$V_\mu(f)(z)=z-\sum_{n=2}^\infty\left(\frac{c+1}{c+n}\right)^\delta a_nz^n.$$

We need to prove that

$$\sum_{n=2}^\infty\frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta}\left(\frac{c+1}{c+n}\right)^\delta a_nB_n(\lambda)<1. \tag{2.1}$$

On the other hand (see [5], Theorem 1), $f\in D(\alpha,\beta,\lambda)$ if and only if

$$\sum_{n=2}^\infty\frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta}a_nB_n(\lambda)<1.$$

Hence $\frac{c+1}{c+n} < 1$. Therefore (2.1) holds and the proof is complete.

The above theorem yields the following two special cases.

Corollary 1. If f is starlike of order β then $V_\mu(f)$ is also starlike of order β.

Corollary 2. If f is convex of order β then $V_\mu(f)$ is also convex of order β.

Next we provide a starlikeness condition for functions in $D(\alpha,\beta,\lambda)$ under $V_\mu(f)$.

Theorem 2. Let $f \in D(\alpha,\beta,\lambda)$. Then $V_\mu(f)$ is starlike of order $0 \le \gamma < 1$ in $|z| < R_1$ where

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^\delta \frac{(1-\gamma)[n(1+\alpha)-(\alpha+\beta)]}{(n-\gamma)(1-\beta)} \; B_n(\lambda)\right]^{\frac{1}{n-1}}.$$

Proof. It is sufficient to prove

$$\left|\frac{z(V_\mu(f)(z))'}{V_\mu(f)(z)} - 1\right| < 1-\gamma. \tag{2.2}$$

For the left hand side of (2.2) we have

$$\left|\frac{z(V_\mu(f)(z))'}{V_\mu(f)(z)} - 1\right| = \left|\frac{\sum_{n=2}^\infty (1-n)\left(\frac{c+1}{c+n}\right)^\delta a_n z^{n-1}}{1-\sum_{n=2}^\infty \left(\frac{c+1}{c+n}\right)^\delta a_n z^{n-1}}\right|$$
$$\le \frac{\sum_{n=2}^\infty (n-1)\left(\frac{c+1}{c+n}\right)^\delta a_n |z|^{n-1}}{1-\sum_{n=2}^\infty \left(\frac{c+1}{c+n}\right)^\delta a_n |z|^{n-1}}.$$

This last expression is less than $1-\gamma$ since.

$$|z|^{n-1} < \left(\frac{c+n}{c+1}\right) \frac{(1-\gamma)[n(1+\alpha)-(\alpha+\beta)]}{(n-\gamma)(1-\beta)} B_n(\lambda).$$

Therefore, the proof is complete.

Using the fact that f is convex if and only if zf' is starlike, we obtain the following

Theorem 3. Let $f \in D(\alpha,\beta,\lambda)$. Then $V_\mu(f)$ is convex of order $0 \le \gamma < 1$ in $|z| < R_2$ where

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^\delta \frac{(1-\gamma)[n(1+\alpha)-(\alpha+\beta)]}{n(n-\gamma)(1-\beta)} B_n(\lambda)\right]^{\frac{1}{n-1}}.$$

3. A Convolution Operator on $D(\alpha,\beta,\lambda)$.

Denote by $F_1(a,b,c;z)$ the usual Gaussian hypergeometric functions defined by

$$F_1(a,b,c;z) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n n!} z^n, \;\; |z| < 1, \tag{3.1}$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \;\; c > b > 0 \; and \; c > a+b.$$

It is well known (see [1]) that under the condition $c > b > 0$ and $c > a+b$ we have

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

For every $f \in T$ we define the convolution operator $H_{a,b,c}(f)(z)$ as

$$H_{a,b,c}(f)(z) ={}_2 F_1(a,b,c;z) * f(z),$$

where ${}_2F_1(a,b,c;z)$ is the Gaussian hypergeometric function defined in (3.1). For determining the resultant of $H_{a,b,c}(f)(z)$ if we set

$$k = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}$$

then we have

$H_{a,b,c}(f)(z) = \left(z + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^{n+1}\right) * \left(z - \sum_{n=1}^{\infty} a_{n+1} z^{n+1}\right)$

$z - \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} a_{n+1} z^{n+1} = z - \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=1}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(n+1)} a_{n+1} z^{n+1}$

$= z - k \sum_{n=1}^{\infty} \left[a_{n+1} z^{n+1} \sum_{n=0}^{\infty} \left[\frac{(c-a)_m (1-a)_m}{(c-a-b+1)_m m!} \int_0^1 t^{b+n-1}(1-t)^{c-a-b+m} dt\right]\right]$

$= z + k \int_0^1 \frac{t^{b-1}}{t} (1-t)^{c-a-b} {}_2F_1(c-a, 1-a, c-a-b+1; 1-t) \left(-\sum_{n=1}^{\infty} a_{n+1} t^{n+1} z^{n+1}\right) dt$

$= z + k \int_0^1 t^{b-1}(1-t)^{c-a-b} {}_2F_1(c-a, 1-a, c-a-b+1; 1-t) \frac{f(tz)-tz}{t} dt$

$= z + k \int_0^1 t^{b-1}(1-t)^{c-a-b} {}_2F_1(c-a, 1-a, c-a-b+1; 1-t) \frac{f(tz)}{t} dt$

$-zk \int_0^1 t^{b-1}(1-t)^{c-a-b} {}_2F_1(c-a, 1-a, c-a-b+1; 1-t) dt.$

If we set

$$\mu(t) = kt^{b-1}(1-t)^{c-a-b} {}_2F_1(c-a, 1-a, c-a-b+1; 1-t) \tag{3.2}$$

then it is easy to see that $\int_0^1 \mu(t)dt = 1$. Consequently

$$H_{a,b,c}(f)(z) = \int_0^1 \mu(t) \frac{f(tz)}{t} dt,$$

where $\mu(t)$ is as in (3.2).

This paves the way to state and prove our next theorem.

Theorem 4. Let $f \in D(\alpha, \beta, \lambda)$. Then $H_{a,b,c}(f) \in D(\alpha, \beta, \lambda_1)$ where

$$\lambda_1 \le \inf_n \left[\frac{(c)_{n-1}(n-1)!}{(a)_{n-1}(b)_{n-1}} \ B_n(\lambda) - 1\right].$$

Proof. Since

$$H_{a,b,c}(f)(z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^n$$

we need to show that

$$\sum_{n=2}^{\infty} \frac{[n(1+\alpha)-(\alpha+\beta)](a)_{n-1}(b)_{n-1}}{(1-\beta)(c)_{n-1}(n-1)!} \ a_n B_n(\lambda_1) < 1. \tag{3.3}$$

The inequality (3.3) holds if

$$\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}\ B_n(\lambda_1) < B_n(\lambda).$$

Therefore

$$\lambda_1 < \frac{(c)_{n-1}(n-1)!}{(a)_{n-1}(b)_{n-1}}\ B_n(\lambda) - 1$$

which completes the proof.

The starlikeness of the functions in $D(\alpha,\beta,\lambda)$ under $H_{a,b,c}$ is investigated in the following theorem.

Theorem 5. Let $f \in D(\alpha,\beta,\lambda)$. Then $H_{a,b,c}(f) \in S^*(\gamma)$ for $|z| < R$ and

$$R = \inf_n \left[\frac{[n(1+\alpha)-(\alpha+\beta)](1-\gamma)(c)_{n-1}(n-1)!}{(n-\gamma)(1-\beta)(a)_{n-1}(b)_{n-1}}\ B_n(\lambda)\right]^{\frac{1}{n-1}}.$$

Proof. We need to show that

$$\left|\frac{zH'_{a,b,c}(f)(z)}{H_{a,b,c}(f)(z)} - 1\right| < 1-\gamma. \tag{3.4}$$

For the left hand side of (3.4) we have

$$\left|\frac{zH'_{a,b,c}(f)(z)}{H_{a,b,c}(f)(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty} \frac{(n-1)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n|z|^{n-1}}.$$

This last expression is less than $1-\gamma$ if

$$\sum_{n=2}^{\infty} \frac{(n-\gamma)(a)_{n-1}(b)_{n-1}}{(1-\gamma)(c)_{n-1}(n-1)!}\ a_n|z|^{n-1} < 1. \tag{3.5}$$

Using the fact, see [1], that $f \in D(\alpha,\beta,\lambda)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta} a_n B_n(\lambda) < 1,$$

we can say (3.5) is true if

$$\frac{(n-\gamma)(a)_{n-1}(b)_{n-1}}{(1-\gamma)(c)_{n-1}(n-1)!}\ |z|^{n-1} < \frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta}\ B_n(\lambda).$$

Or equivalently

$$|z|^{n-1} < \frac{[n(1+\alpha)-(\alpha+\beta)](1-\gamma)(c)_{n-1}(n-1)!}{(n-\gamma)(1-\beta)(a)_{n-1}(b)_{n-1}}\ B_n(\lambda)$$

which yields the starlikeness of the family $H_{a,b,c}(f)$.

For $\alpha = \lambda = 0$ and $\gamma = \beta$ we obtain the following

Corollary 3. Let $f \in S^*(\beta)$. Then $H_{a,b,c}(f) \in S^*(\beta)$ in $|z| < R_1$ for

$$R_1 = \inf_n \left[\frac{(c)_{n-1}(n-1)!}{(a)_{n-1}(b)_{n-1}}\right]^{\frac{1}{n-1}}.$$

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