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APPROXIMATION OF FIXED POINTS OF ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS IN BANACH SPACES

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Abstract. Let T be an asymptotically pseudocontractive self-mapping of a nonempty closed convex subset D of a reflexive Banach space X with a Gâteaux differentiable norm. We deal with the problem of strong convergence of almost fixed points $x_n = \mu_n T^n x_n + (1 - \mu_n)u$ to fixed point of T. Next, this result is applied to deal with the strong convergence of explicit iteration process $z_{n+1} = v_{n+1}(\alpha_n T^n z_n + (1-\alpha_n)z_n) + (1 - v_{n+1})u$ to fixed point of T

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1. Introduction

Let D be a nonempty closed convex subset of a real Banach space X and let $T : D \to D$ be a mapping. Given an $x_0 \in D$ and a t $\in (0, 1)$, then, for a nonexpansive mapping T, we can define contraction $G_t : D \to D$ by $G_t x = tTx+(1-t)x_0, x \in D$. By Banach contraction principle, G_t has a unique fixed point x_t in D, i.e., we have

$$x_t = tTx_t + (1 - t)x_0.$$

The strong convergence of path $\{x_t\}$ as $t \to 1$ for a nonexpansive mapping T on a bounded D was proved in Hilbert space independently by Browder [2] and Halpern

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[7] in 1967 and in a uniformly smooth Banach space by Reich [10]. Later, it has been studied in various papers (see [12], [14], [15], [23], [28]).

The asymptotically nonexpansive mappings were introduced by Goebel and Kirk [4] and further studied by various authors (see [1], [6], [7], [12], [17], [19], [21], [22], [24], [25], [27]).

Recently, Schu [20] has considered the strong convergence of almost fixed points $x_n = \mu_n T^n x_n$ of an asymptotically nonexpansive mapping T in a smooth and reflexive Banach space having a weakly sequentially continuous duality mapping. Unfortunately, Schu's results do not apply to L^p spaces if $p \neq 2$, since none of these spaces possess weakly sequentially duality mapping.

The object of this paper is to deal with the problem of strong convergence of the sequence of almost fixed points defined by the equation

$$x_n = \mu_n T^n x_n + (1-\mu_n)u \tag{1}$$

for an asymptotically pseudocontractive mapping T in a reflexive Banach space with the Gâteaux differentiable norm. In particular, Corollary 1 improves and extends the results of [12], [14], [16], [20] and [23] to the larger class of asymptotically pseudocontractive mappings. Further, we deal with the problem of strong convergence of the explicit iteration process

$$z_{n+1} = v_{n+1}(\alpha_n T^n x_n + (1-\alpha_n)x_n) + (1-v_{n+1})u$$

by applying Corollary 1.

It is well known that the Mann iteration process ([13]) is not guaranteed to converge to a fixed point of a Lipschitz pseudocontractive defined even on a compact convex subset of a Hilbert space (see [10]). In [11], Ishikawa introduced a new iteration process, which converges to a fixed point of a Lipschitz pseudocontractive mapping defined on a compact convex subset of a Hilbert space. Schu [22] first studied the convergence of the modified Ishikawa iterative sequence for completely continuous asymptotically pseudocontractive mappings in Hilbert spaces. Schu's result has been extended to asymptotically pseudocontractive type mappings defined on compact convex subsets of a Hilbert space (see [4], [15]). In application point of view, compactness is a very strong condition. One of important features of our approach is that it allows relaxation of compactness.

2. Preliminaries

Let X be a real Banach space and D a subset of X. An operator $T : D \to D$ is said to be *asymptotically pseudocontractive* ([24]) if and only if, for each $n \in N$ and $u, v \in D$, there exist $j \in J(u-v)$ and a constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\langle T^n u - T^n v, j\rangle \leq k_n \|u-v\|^2,$$

where $J : X \to 2^{X^*}$ is the *normalized duality mapping* defined by

$$J(u) = \{j \in X^* : \langle u, j\rangle = \|u\|^2, \|j\| = \|u\|\}.$$

The class of asymptotically pseudocontractive mappings is essentially wider than the class of asymptotically nonexpansive mappings ($T : D \to D$ for which there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n u - T^n v\| \leq k_n \|u-v\|$$

for all $u, v \in D$ and $n \in N$). In fact, if T is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$, then for each $u, v \in D$, $j \in J(u-v)$ and $n \in N$, we have

$$\langle T^n u - T^n v, j\rangle \leq \|T^n u - T^n v\|\|u-v\| \leq k_n\|u-v\|^2.$$

The *normal structure coefficient* $N(X)$ of X is defined ([2]) by

$$N(X) = \left\{ \frac{\mathrm{diam} D}{r_D D} : D \text{ is a nonempty bounded convex subset of } X \right.$$
$$\left. \text{with diam } D > 0 \right\},$$

where $r_D(D) = \inf_{x\in D}\{\sup_{y\in D}\|x-y\|\}$ is the Chebyshev radius of D relative to itself and $\mathrm{diam} D = \sup_{x,y\in D}\|x-y\|$ is the diameter of D. The space X is said to have the *uniformly normal structure* if $N(X) > 1$.

Recall that a nonempty subset D of a Banach space X is said to satisfy the *property* (P) ([12]) if the following holds:

(P) $$x \in D \Rightarrow \omega_\omega(x) \subset D,$$

where ω_ω (x) is *weak ω-limit set* of T at x, i.e.,

$$\{y \in C : y = weak - \lim_j T^{n_j}x \text{ for some } n_j \to \infty\}.$$

The following result can be found in [12].

Lemma 1. *Let D be a nonempty bounded subset of a Banach space X with uniformly normal structure and $T : D \to D$ be a uniformly L-Lipschitzian mapping with $L < N(X)^{1/2}$. Suppose that there exists a nonempty bounded closed convex subset C of D with property (P). Then T has a fixed point in C.*

A *Banach limit LIM* is a bounded linear functional on ℓ^∞ such that

$$\liminf_{n\to\infty} t_n \leq LIM t_n \leq \limsup_{n\to\infty} t_n$$

and

$$LIM t_n = LIM t_{n+1}$$

for all bounded sequence $\{t_n\}$ in ℓ^∞. Let $\{x_n\}$ be a bounded sequence of X. Then we can define the real-valued continuous convex function f on X by $f(z) = LIM\|x_n - z\|^2$ for all $z \in X$.

The following Lemma was give in [8].

Lemma 2 [8]**.** *Let X be a Banach space with the uniformly Gâteaux differentiable norm and $u \in X$. Then*

$$f(u) = \inf_{z\in X} f(z)$$

if and only if

$$LIM\langle z, J(x_n - u)\rangle = 0$$

for all $z \in X$, where $J : X \to X^*$ is the normalized duality mapping and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing.

3. The Main Results

In this section, we establish strong convergence of sequence $\{x_n\}$ defined by the equation (1) in a reflexive Banach space with uniformly Gâteaux differentiable norm.

Suppose now that D is a nonempty closed and convex subset of a Banach space X and $T : D \to D$ is an asymptotically pseudocontractive mapping (we may always assume $k_n \geq 1$ for all $n \geq 1$). Suppose also that $\{\lambda_n\}$ is a sequence of real number in (0,1) such that $\lim_{n\to\infty} \lambda_n = 1$.

Now, for $u \in D$ and a positive integer $n \in N$, consider a mapping T_n on D defined by

$$T_n x = \Big(1 - \frac{\lambda_n}{k_n}\Big)u + \frac{\lambda_n}{k_n} T^n x, \quad x \in D.$$

In the sequel, we use the notations $F(T)$ for the set of fixed points of T and μ_n for $\frac{\lambda_n}{k_n}$.

Lemma 3. *For each $n \geq 1$, T_n has exactly one fixed point x_n in D such that*

$$x_n = \mu_n T^n x_n + (1 - \mu_n)u.$$

Proof. Since T_n is a strictly pseudocontractive mapping on D, it follows from Corollary 1 of [5] that T_n possesses exactly one fixed point x_n in D.

Lemma 4. *If the set*

$$G(u, Tu) = \{x \in D : \langle T^n u - u, j\rangle > 0 \text{ for all } j \in j(x - u),\ n \geq 1\}$$

is bounded, then the sequence $\{x_n\}$ is bounded.

Proof. Since T is asymptotically pseudocontractive, for $j \in J(x_n - u)$, we have

$$\langle \mu_n(T^n x_n - u) + \mu_n(u - T^n u), j\rangle \leq \lambda_n \|x_n - u\|^2,$$

which implies

$$\langle T^n u - u, j\rangle \geq \frac{1 - \lambda_n}{\mu_n} \|x_n - u\|^2$$

since $\mu_n(T^n x_n - u) = x_n - u$. If $x \neq 0$, we have

$$\langle T^n u - u, j\rangle > 0$$

and it follows that $x_n \in G(u, Tu)$ for all $n \geq 1$ and hence $\{x_n\}$ is bounded.

Before presenting our main result, we need the following:

Definition 1. Let D be a nonempty closed subset of a Banach spaces X, $T : D \to D$ be a nonlinear mapping and $M = \{x \in D : f(x) = \min_{z\in D} f(z)\}$. Then T is said to satisfy the *property* (S) if the following holds:

$$\text{(S)} \qquad \begin{array}{l} \text{For any bounded sequence } \{x_n\} \text{ in } D, \\ \displaystyle\lim_{n\to\infty} \|x_n - Tx_n\| = 0 \text{ implies } M \cap F(T) \neq \emptyset. \end{array}$$

Theorem 1. *Let D be a nonempty closed and convex subset of a reflexive Banach space X with a uniformly Gâteaux differentiable norm, $T : D \to D$ be a continuous asymptotically pseudocontractive mapping with a sequence $\{k_n\}$ and $\{\lambda_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$. Suppose that for $u \in D$, the set $G(u, Tu)$ is bounded and the mapping T satisfies the property (S). Then we have the following:*

(a) *For each $n \geq 1$, there is exactly one $x_n \in D$ such that*

$$x_n = \mu_n T^n x_n + (1 - \mu_n)u.$$

(b) *If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then it follows that there exists the sunny nonexpansive retraction P from D onto $F(T)$ such that $\{x_n\}$ converges strongly to Px.*

Proof. The part (a) follows from Lemma 3. So, it remains to prove part (b). From Lemma 4, $\{x_n\}$ is bounded and so we can define a function $f : D \to R^+$ by

$$f(z) = LIM\|x_n - z\|^2$$

for all $z \in D$. Since f is continuous and convex, $f(z) \to \infty$ as $\|z\| \to \infty$ and X is reflexive, f attains it infimum over D. Let $z_0 \in D$ such that $f(z_0) = \min_{z\in D} f(z)$ and let $M = \{x \in D : f(x) = \min_{z\in D} f(z)\}$. Then M is nonempty because $z_0 \in M$. Since $\{x_n\}$ is bounded by Lemma 4 and T satisfied the property (S), it follows that $M \cap F(T) \neq \emptyset$. Suppose that $v \in M \cap F(T)$. Then, by Lemma 2, we have

$$LIM\langle x - v, J(x_n - v)\rangle \leq 0$$

for all $x \in D$. In particular, we have

$$LIM\langle u - v, J(x_n - v)\rangle \leq 0. \tag{2}$$

On the other hand, from the equation (1), we have

$$x_n - T^n x_n = (1 - \mu_n)(u - T^n x_n) = \frac{1 - \mu_n}{\mu_n}(u - x_n). \tag{3}$$

Now, for any $v \in F(T)$, we have

$$\begin{aligned}\langle x_n - T^n x_n, J(x_n - v)\rangle &= \langle x_n - v + T_n v - T^n x_n, J(x_n - v)\rangle \\ &\geq -(k_n - 1)\|x_n - v\|^2 \\ &\geq -(k_n - 1)K^2\end{aligned}$$

for some $K > 0$ and it follows from (3) that

$$\langle x_n - u, J(x_n - v)\rangle \leq \frac{\lambda_n(k_n - 1)}{k_n - \lambda_n} K^2.$$

Hence we have

$$LIM\langle x_n - u, J(x_n - v)\rangle \leq 0. \tag{4}$$

Combining (2) and (4), then we have

$$LIM\langle x_n - v, J(x_n - v)\rangle = LIM\|x_n - v\|^2 \leq 0.$$

Therefore, there is a subsequence $\{x_{n_i}\}$ which converges strongly to v. To complete the proof, suppose there is another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to y (say). Since $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$ and T is continuous, then y is a fixed point of T. It then follows from (4) that

$$\langle v - u, J(v - y)\rangle \leq 0$$

and

$$\langle y - u, J(y - v)\rangle \leq 0.$$

Adding these two inequalities yields

$$\langle v - y, J(v - y)\rangle = \|v - y\|^2 \leq 0$$

and thus $v = y$. This prove the strong convergence of $\{x_n\}$ to $v \in F(T)$. Now we can define a mapping P from D onto $F(T)$ by $\lim_{n\to\infty} x_n = Pu$. From (4), we have

$$\langle u - Pu, J(v - Pu)\rangle \leq 0$$

for all $u \in D$ and $v \in F(T)$. Therefore, P is the sunny nonexpansive retraction. This completes the proof.

Remark 1. The assumption of λ_n such that $\lambda_n \in (\frac{1}{2}, 1)$ with $k_n \leq \frac{2\lambda_n^2}{2\lambda_n - 1}$ implies $\lim_{n\to\infty} \frac{\lambda_n(k_n-1)}{(k_n-\lambda_n)} = 0$ (see Lemma 1.4 of [16]).

Next, we substitute the property (S) mentioned in Theorem 1 by assuming that T is uniformly L-Lipschitzian in Banach space with the uniformly normal structure and D does have the property (P) (see [12]).

Corollary 1. *Let X be a Banach space with the uniformly Gâteaux differentiable norm, $N(X)$ be the normal structure coefficient of X such that $N(X) > 1$, D be nonempty closed convex subset of X. $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\}$ and $L < N(X)^{1/2}$ and $\{\lambda_n\}$ be a sequence of real numbers in $(0, 1)$ such that $\lim_{n\to\infty}\lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$. Suppose that every closed convex bounded subset of D satisfies the property (P). Then we have*

(a) *For each $n \geq 1$, there is exactly one $x_n \in D$ such that*

$$x_n = \mu_n T^n x_n + (1 - \mu_n)u.$$

(b) *If $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then it follows that there exists the sunny nonexpansive retraction P from D onto $F(T)$ such that the sequence $\{x_n\}$ converges strongly to Px.*

Remark 2. (1) Theorem 1 and Corollary 1 can be applied to all uniformly convex and uniformly smooth Banach spaces and, in particular, all L^P spaces, $1 < p < \infty$.

(2) As was mentioned in the introduction, Theorem 1 extends and improves the corresponding results of [12], [14], [16], [20] and [23] to much larger class of asymptotically pseudocontractive mappings and to more general Banach spaces X considered here.

If we choose $\{\lambda_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$ (such a sequence $\{\lambda_n\}$ always exists. For example, taking $\lambda_n = \min\{1-\sqrt{k_n-1}, 1-\frac{1}{n}\}$), then the following result is a direct consequence of Corollary 1:

Corollary 2. *Let D be nonempty closed convex and bounded subset of a uniformly smooth Banach space X, $T : D \to D$ be an asymptotically nonexpansive mapping with Lipschitzian constant k_n and $\{\lambda_n\}$ be a sequence of real numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$. Then we have the following:*

(a) *For $u \in D$ each $n \geq 1$, there is exactly one $x_n \in D$ such that*

$$x_n = \mu_n T^n x_n + (1-\mu_n)u.$$

(b) *If $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists the sunny nonexpansive retraction P from D onto $F(T)$ such that $\{x_n\}$ converges strongly to Px.*

We immediately obtain from Corollary 2 the following result (Theorem 1 of Lim and Xu [8]) with additional information that almost fixed points converges to y, where y is fixed point of T nearest point to u.

Corollary 3. *Let D be a nonempty closed convex and bounded subset of a uniformly smooth Banach space and $T : D \to D$ be an asymptotically nonexpansive mapping. Let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\lim_{n\to\infty} \frac{k_n-1}{k_n-\lambda_n} = 0$. Suppose that, for any $x \in D$, $\{x_n\}$ is a sequence in the defined by (1). Suppose in addition that the following condition:*

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

holds. Then there exists the sunny nonexpansive retraction P from D onto $F(T)$ such that $\{x_n\}$ converges strongly to Px.

4. Applications

Halpern [9] has introduced the explicit iteration process $\{z_{n+1}\}$ defined by $z_{n+1} = \lambda_{n+1} T z_n$ for approximation of a fixed point for a nonexpansive self-mapping T defined on the unit ball of a Hilbert space. Later, this iteration process has been studied extensively by various authors and has been successfully employed to approximate fixed points of various class of nonlinear mappings (see [15], [20], [23]).

In this section, we establish some strong convergence theorems for the results of the explicit iteration process $\{z_{n+1}\}$ defined by

$$z_{n+1} = v_{n+1}(\alpha_n T^n z_n + (1-\alpha_n) z_n) + (1-\mu_{n+1})u$$

by applying the results concerning the implicit iteration process $\{x_n\}$ defined by

$$x_n = \mu_n T^n x_n + (1 - \mu_n)u$$

of the last section.

First, we shall introduce a definition, which is partly due to Halpern [7].

Let $\{a_n\}$ and $\{v_n\}$ be sequence of real numbers in $(0, \infty)$ and (0,1), respectively. Then $(\{a_n\}, \{v_n\})$ is said to have *property* (A) ([17]):
(a) $\{a_n\}$ is decreasing,
(b) $\{v_n\}$ is strictly increasing,
(c) there is a sequence $\{\beta_n\}$ of natural number such that
(c-1) $\{\beta_n\}$ is strictly increasing,
(c-2) $\lim_{n\to\infty}\beta_n(1 - v_n) = \infty$,
(c-3) $\lim_{n\to\infty}\frac{1-v_{n+\beta_n}}{1-v_n} = 1$,
(c-4) $\lim_{n\to\infty}\frac{a_n-a_{n+\beta_n}}{1-v_n} = 0$.

The following lemma was proved in [23]:

Lemma 5 [23]. *Let D be a nonempty bounded and convex subset of a normed space X, $0 \in D$, $\{S_n\}$ be a sequence self-mappings on D, $\{L_n\}$ be a sequence of real numbers in $[1, \infty]$ such that $\|S_n x - S_n y\| \leq L_n\|x - y\|$ for all $x, y \in D$ and $n \geq 1, \{\lambda_n\} \subset (0, 1)$, $\{a_n\} \subset (0, \infty)$ be such that $(\{a_n\}, \{v_n\})$ has property* (A) *and $\{\frac{1-v_n}{1-\lambda_n}\}$ is bounded, where $v_n = \lambda_n/L_n$, and $\{x_n\}$ be a sequence in D such that $x_n = v_n S_n(x_n)$ for all $n \geq 1$ and $\lim_{n\to\infty} x_n = v$. Suppose that there exists a constant $d > 0$ such that*

$$\|S_m(x) - S_n(x)\| \leq d|a_m - a_n|$$

for all $m, n \geq 1$ and $x \in D$. Suppose also that, for an arbitrary points $z_0 \in D, \{z_n\}$ is a sequence in D such that $z_{n+1} = v_{n+1}S_n(z_n)$ for all $n \geq 1$. Then $\lim_{n\to\infty} z_n = v$.

Xu [26] has proved that, if X is q-uniformly smooth ($q > 1$), then there exists a constant $c > 0$ such that

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c\|y\|^q \tag{5}$$

for all $x, y \in X$, where the mapping $J_q : X \to 2^{X^*}$ is a generalized duality mapping defined by

$$J_q(x) = \{j \in X^* : \langle x, j\rangle = \|x\|^q, \|j\| = \|x\|^{q-1}\}.$$

Typical examples of such space are the Lesbesgue L_p, the sequence ℓ_p and the Sobolev W_p^m spaces for $1 < p < \infty$. In fact, these spaces are p-uniformly smooth if $1 < p \leq 2$ and 2-uniformly smooth for $p \geq 2$.

Before, presenting our results, we need the following:

Lemma 6. *Let $q > 1$ be a real number, D be a nonempty closed subset of a q-uniformly smooth Banach space X, $T : D \to D$ be a uniformly L-Lipschitzian and asymptotically pseudocontractive mapping with a sequence $\{k_n\}$ and $\{\lambda_n\}$ and $\{\alpha_n\}$*

be two sequences of real numbers in $(0,1)$. *Suppose that* $\{G_n\}$ *is a self-mapping on* D *defined by* $G_n x = \alpha_n T^n x + (1-\alpha_n)x$ *for all* $x \in D$. *Then we have the following:*

(a) $\|G_n x - G_n y\| \le L_n \|x-y\|$ *for all* $x, y \in D$ *and* $n \ge 1$, *where*

$$L_n = [1 + q\alpha_n(k_n - 1) + c\alpha_n^q(1+L)^q]^{\frac{1}{q}}.$$

(b) *For* $u \in D$ *and each* $n \ge 1$, *there is exactly one* $x_n \in D$ *such that*

$$x_n = v_n G_n(x_n) + (1-v_n)u,$$

where $v_n = \lambda_n / L_n$.

(c) *If* $u = 0$, *then it follows that* $x_n = \frac{v_n \alpha_n}{1 - v_n(1-\alpha_n)} T^n x_n$ *for all* $n \ge 1$.

Proof. To prove part (a), set $F_n = I - T^n$, where I denotes the identity operator. Then, for each $n \ge 1$, $G_n = I - \alpha_n F_n$ and $\|F_n x - F_n y\| \le (1+L)\|x-y\|$ for all $x, y \in D$. Since

$$\langle F_n x - F_n y, J_q(x-y) \rangle \ge -(k_n - 1)\|x-y\|^2$$

for all $x, y \in D$ and $n \ge 1$, using (5), we obtain

$$\begin{aligned}
&\|G_n x - G_n y\|^q \\
&= \|x - y - \alpha_n(F_n x - F_n y)\|^q \\
&\le \|x-y\|^q - q\alpha_n \langle F_n x - F_n y, J_q(x-y) \rangle + c\alpha_n^q(1+L)^q \|x-y\|^q \\
&\le [1 + q\alpha_n(k_n-1) + c\alpha_n^q(1+L)^q]\|x-y\|^q.
\end{aligned}$$

To prove part (b), for $u \in D$ and $n \ge 1$, define a mapping $T_n : D \to D$ by

$$T_n x = v_n G_n x + (1-v_n)u, \quad x \in D.$$

Since $v_n \in (0,1), T_n$ is a contraction mapping on D. Thus, by the Banach contraction principle, T_n has exactly one $x_n \in D$ such that $x_n = v_n G_n x_n + (1-v_n)u$. This completes the proof.

The following lemma can be shown by simple calculation:

Lemma 7. *Let* D *be a nonempty closed convex subset of a Banach space* X, $T : D \to D$ *be an asymptotically nonexpansive mapping with a sequence* $\{k_n\}$ *and* $\{\lambda_n\}$ *and* $\{\alpha_n\}$ *be two sequences of real numbers in* $(0,1)$. *Suppose that* $\{G_n\}$ *is a sequence of self-mappings on* D *defined by* $G_n x = \alpha_n T^n x + (1-\alpha_n)x$ *for any* $x \in D$. *Then we have the following:*

(a) $\|G_n x - G_n y\| \le k_n \|x-y\|$ *for all* $x, y \in D$ *and* $n \ge 1$.

(b) *For* $u \in D$ *and each* $n \ge 1$, *there is exactly one* $x_n \in D$ *such that*

$$x_n = \mu_n G_n x_n + (1-\mu_n)u,$$

where $\mu_n = \lambda_n / k_n$.

We now prove the main result of this section.

Theorem 2. *Let $q > 1$ be a real number, D be a nonempty closed convex and bounded subset of a q-uniformly smooth Banach space X, $T : D \to D$ be a uniformly L-Lipschitzian asymptotically pseudocontractive mapping with a sequence $\{k_n\}$ and $L < N(X)^{\frac{1}{2}}$ and $\{\lambda_n\}$ and $\{\alpha_n\}$ be two sequences of real numbers in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$, $\lim_{n\to\infty} \frac{L_n-1}{L_n-\lambda_n} = 0$ and $\lim_{n\to\infty} \frac{1-v_n}{\alpha_n} = 0$, where $L_n = [1 + q\alpha_n(k_n - 1) + c\alpha_n^q(1 + L)^q]^{\frac{1}{q}}$ and $v_n = \lambda_n/L_n$. Suppose that $(\{\alpha_n\}, \{v_n\})$ has property (A), $\{\frac{1-v_n}{1-\lambda_n}\}$ is bounded and $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$ for any bounded sequence $\{y_n\}$ in D with $\lim_{n\to\infty} \|y_n - T^n y_n\| = 0$. Suppose also that, for any $u, z_0 \in D$, $\{z_n\}$ is a sequence in D defined by*

$$z_{n+1} = v_{n+1}(\alpha_n T^n z_n + (1 - \alpha_n)z_n) + (1 - v_{n+1})u.$$

Then there exists the sunny nonexpansive retraction P from D onto $F(T)$ such that $\{z_n\}$ converges strongly to Pu.

Proof. Without loss of generality, we may assume that $u = 0$. For $n \geq 1$, set $\eta_n = \frac{v_n\alpha_n}{(1-v_n)+v_n\alpha_n}$. Then $\{\eta_n\} \subset (0,1)$ and $\eta_n = (1 + \frac{1}{v_n}(\frac{1-v_n}{\alpha_n}))^{-1}$ for all $n \geq 1$. Since $\lim_{n\to\infty} v_n = 1$ and $\lim_{n\to\infty} \frac{1-v_n}{\alpha_n} = 0$, it follows that $\lim_{n\to\infty} \eta_n = 1$ and hence, by Lemma 6 and Corollary 1, the sequence $\{x_n\}$ defined by $x_n = \eta_n T^n x_n$ converges strongly to Pu. Let $\{G_n\}$ be a sequence of self mappings on D defined by

$$G_n(x) = \alpha_n T^n x + (1 - \alpha_n)x, \quad x \in D.$$

By Lemma 6, for each $n \geq 1$, there is exactly one $x_n \in D$ such that $x_n = v_n G_n(x_n)$ and hence $x_n = \eta_n T^n x_n$. By Corollary 1, we have that $\{x_n\}$ converges strongly to some fixed point of T. Since $z_n = \eta_n G_n(z_n)$ for all $n \geq 1$ and $\|G_m(x) - G_n(x)\| \leq |\alpha_m - \alpha_n|$ diam D for all $m, n \geq 1$ and $x \in D$. It follows from Lemma 5 that $\{z_n\}$ converges strongly to Pu. This completes the proof.

Remark 3. (1) Theorem 2 extends Theorem 2.4 of Schu [23] to the wider class of asymptotically pseudocontractive mappings and from a Hilbert spaces to the more general Banach space X considered here.

(2) Another iteration procedure for uniformly L-Lipschitzian asymptotically pseudocontractive mapping T in a Hilbert space may be found in the work of Schu [22] with the condition that the given mapping T is completely continuous.

Corollary 3. *Let D be a nonempty closed convex and bounded subset of a uniformly smooth Banach space X, $T : D \to D$ be a uniformly asymptotically regular and asymptotically nonexpansive mapping with a sequence $\{k_n\}$ and $\{\alpha_n\}$ be sequence of real numbers in $(0,1)$ with $\lim_{n\to\infty} \lambda_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \frac{(k_n-1)}{(k_n-\lambda_n)} = 0$ and $\lim_{n\to\infty} \frac{1-\mu_n}{a_n} = 0$. Suppose also that, for any $u, z_0 \in D$, $\{z_n\}$ is a sequence in D defined by*

$$z_{n+1} = \mu_{n+1}(\alpha_n T^n z_n + (1 - \alpha_n)z_n) + (1 - \mu_n)u, \quad n \geq 1.$$

Then $\{z_n\}$ converges strongly to some fixed point of T.

Remark 4. Schu [19], [21] and Tan and Xu [24] have studied the weak convergence for the sequence $\{z_n\}$ defined by (the modified Mann iteration process) $z_{n+1} = \alpha_n T^n z_n + (1 - \alpha_n)z_n$ to fixed point of asymptotically nonexpansive mapping T in a uniformly convex Banach space with the Fréchet differentiable norm or with a weakly sequentially duality mapping.

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