Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 2, December 2003, pp. 36–48.
Submitted: February, 2003. Published: December 31 2003.

SMOOTHERS AND THEIR APPLICATIONS IN AUTONOMOUS SYSTEM THEORY

J. E. Palomar Tarancón

Abstract. In this paper the author introduces the concept of smoother. Roughly speaking, a smoother is a pair (s, K) consisting of a continuous map s sending each point p of its domain into a closed neighborhood V_p of p, and an operator K that transforms any function f into another Kf being smoother than f. This property allows us to remove the effect of a perturbation P from the solutions of an autonomous system the vector field of which is modified by P.

A.M.S. (MOS) Subject Classification Codes. 44A05, 34A99, 18B99

Key Words and Phrases. Smoother, co-algebra, integral transform, perturbation

0. Introduction

The main aim of this paper consists of introducing the concept of smoother together with an application in differential equation theory. Roughly speaking a smoother is an operator transforming an arbitrary function f_1 into a similar one f_2 being smoother than f_1. In general, smoothers perform integral transforms in function spaces. To get a first approximation to the smoother concept consider the following facts. Let $y = f(x)$ be any integrable function defined in $\mathbb{R}$ and $\sigma : \mathbb{R} \to \mathcal{C}$ a map such that $\mathcal{C}$ stands for the collection of all closed subsets of $\mathbb{R}$ the interior of each of which is non-void. For every $x \in \mathbb{R}$, let λ_x be any non-negative real number, and let $\sigma(x) = [x - \lambda_x, x + \lambda_x]$. With these assumptions, consider the linear transform $\mathfrak{K}$ defined as follows. If $\lambda_x \neq 0$ is a finite number, then

$$\mathfrak{K}f(x) = \frac{1}{2\lambda_x} \int_{-\lambda_x}^{\lambda_x} f(x + \tau)\, d\tau \tag{0.0.1}$$

Department of Mathematics, Institut Jaume I C/. Europa, 2-3E, 12530, Burriana(Castellón) Spain
E-mail Address: jepalomar@ono.com

Figure 1.

Conversely, for λ_x infinite

$$\mathfrak{K}f(x) = \lim_{k\to\infty} \frac{1}{2k} \int_{-k}^{k} f(x+\tau)\, d\tau \tag{0.0.2}$$

Finally, if $\lambda_x = 0$, then

$$\mathfrak{K}f(x) = \lim_{k\to 0} \frac{1}{2k} \int_{-k}^{k} f(x+\tau)\, d\tau = f(x) \tag{0.0.3}$$

Of course, assuming that such a limit exists. Thus, the integral transform $\mathfrak{K}$ sends the value of $f(x)$ at x into the average of all values of $f(x)$ in a closed neighborhood $[x-\lambda_x, x+\lambda_x]$ of x. Obviously, in general, the transform $\mathfrak{K}f(x)$ is smoother than $f(x)$. To see this fact consider the following cases. If for every x, $\sigma(x) = \mathbb{R}$, then $\forall x \in \mathbb{R} :\ \lambda_x = \infty$ and $\mathfrak{K}f(x)$ is a constant function, that is the smoothest one that can be built. If for every $x \in \mathbb{R}$, $\sigma(x) = \{x\}$, then $\forall x \in : \lambda_x = 0$, therefore $\mathfrak{K}f(x) = f(x)$, and consequently both functions have the same smoothness degree. Thus, between both extreme cases one can build several degrees of smoothness. In the former example, what we term smoother is nothing but the pair $(\sigma, \mathfrak{K})$.

Perhaps the most natural smoother application consists of removing, from a given function, the noise arising from some perturbation. For instance consider the curves $C1$ and $C2$ in Figure 1. Suppose that the differences between $C1$ and $C2$ are consequence of some perturbation working over $C2$. If both curves are the plots of two functions $f_1(x)$ and $f_2(x)$ respectively, in general, one can build a smoother $(\sigma, \mathfrak{K})$ such that $\mathfrak{K} f_1(x) = f_2(x)$. Now, consider a vector field $\boldsymbol{X}$ and the result $\boldsymbol{Y}$ of a perturbation P working over $\boldsymbol{X}$, and assume $(\sigma, \mathfrak{K})$ to satisfy the relation $\mathfrak{K}\boldsymbol{Y} = \boldsymbol{X}$. If $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are the general solutions for the ordinary differential equations $\dfrac{d}{dt}\,\boldsymbol{x}(t) = \boldsymbol{X}\big(\boldsymbol{x}(t)\big)$ and $\dfrac{d}{dt}\,\boldsymbol{y}(t) = \boldsymbol{Y}\big(\boldsymbol{y}(t)\big)$ respectively, then we shall say the smoother $(\sigma, \mathfrak{K})$ to be compatible with the vector field $\boldsymbol{Y}$, provided that the following relation holds: $\mathfrak{K}\boldsymbol{y}(t) = \boldsymbol{x}(t)$. Thus, one can obtain the corresponding perturbation-free function from solutions of $\dfrac{d}{dt}\,\boldsymbol{y}(t) = \boldsymbol{Y}\big(\boldsymbol{y}(t)\big)$ using the smooth vector field $\boldsymbol{X} = \mathfrak{K}\boldsymbol{Y}$ instead. The main aim of this paper consists of investigating a compatibility criterion.

1. Smoothers

Let $\boldsymbol{Top}$ stand for the category of all topological spaces, and let $\mathfrak{N} : \boldsymbol{Top} \to \boldsymbol{Top}$ be the endofunctor carrying each object $(E, \mathcal{T})$ in $\boldsymbol{Top}$ into the topological space $\mathfrak{N}(E, \mathcal{T}) = (\wp(E) \setminus \{\{\emptyset\}\}, \mathcal{T}^*)$ the underlying set of which $\wp(E) \setminus \{\{\emptyset\}\}$ consists of all nonempty subsets of E. Let $\mathcal{T}^*$ be the topology a subbase S of which is defined as follows. Denote by $\mathcal{C}$ the collection of all $\mathcal{T}$-closed subsets of E and for every pair $(A, B) \in \mathcal{C} \times \mathcal{T}$, let $K_{A,B} = \{C \in \wp(E) | A \subset C \subseteq B\}$. With these assumptions, define the subbase S as follows.

$$S = \{K_{A,B} | (A, B) \in \mathcal{C} \times \mathcal{T}\}$$

Obviously, if $A \supset B$, then $K_{A,B} = \emptyset$. Likewise, if $A = \emptyset$ and $B = E$, then $K_{A,B} = \wp(E) \setminus \{\{\emptyset\}\}$.

Let the arrow-map of $\mathfrak{N}$ be the law sending each continuous map

$$f : (E_1, \mathcal{T}_1) \to (E_2, \mathcal{T}_2)$$

into the map $\mathfrak{N}f$ carrying each subset $A \subseteq E_1$ into $f[A] \subseteq E_2$. It is not difficult to see $\mathfrak{N}f$ to be a continuous map with respect to the associated topology $\mathcal{T}^*$.

Definition 1.0.1. Let $\boldsymbol{cAlg}(\mathfrak{N})$ denote the concrete category of $\mathfrak{N}$-co-algebras. Thus, every object in $\boldsymbol{cAlg}(\mathfrak{N})$ is a pair $\big((E, \mathcal{T}), \sigma_{(E,\mathcal{T})}\big)$, consisting of a topological space $(E, \mathcal{T})$ together with a continuous map $\sigma_{(E,\mathcal{T})} : (E, \mathcal{T}) \to \mathfrak{N}(E, \mathcal{T})$.

Recall that a continuous mapping $f : (E_1, \mathcal{T}_1) \to (E_2, \mathcal{T}_2)$ is a morphism in $\boldsymbol{cAlg}(\mathfrak{N})$ from $\big((E_1, \mathcal{T}_1), \sigma_{(E_1,\mathcal{T}_1)}\big)$ into $\big((E_2, \mathcal{T}_1), \sigma_{(E_2,\mathcal{T}_2)}\big)$, provided that the following diagram commutes.

(1.0.4)
$$\begin{array}{ccc} (E_1, \mathcal{T}_1) & \xrightarrow{\sigma_{(E_1,\mathcal{T}_1)}} & \mathfrak{N}(E_1, \mathcal{T}_1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{N}f} \\ (E_2, \mathcal{T}_2) & \xrightarrow[\sigma_{(E_2,\mathcal{T}_2)}]{} & \mathfrak{N}(E_2, \mathcal{T}_2) \end{array}$$

Now, let $\boldsymbol{Tvec}$ be the topological vector space category, and let $\boldsymbol{TopVec}$ denote the category the objects of which are products of the form $\mathfrak{N}(E,\mathcal{T})\times\left(\mathcal{C}^0(E,V),T^*\right)$, where (V,T) is a topological vector space and T^* the pointwise topology for the set $\mathcal{C}^0(E,V)$ of all continuous maps from $(E,\mathcal{T})$ into $|(V,T)|$; where the functor $|\ |:\boldsymbol{Tvec}\to\boldsymbol{Top}$ forgets the vector space structure and preserves the topological one. In addition, a $\boldsymbol{TopVec}$-morphism with domain $\mathfrak{N}(E_1,\mathcal{T}_1)\times\left(\mathcal{C}^0(E_1,V),T_1^*\right)$ and codomain $\mathfrak{N}(E_2,\mathcal{T}_2)\times\left(\mathcal{C}^0(E_2,V),T_2^*\right)$ is of the form $\mathfrak{N}f\times g$ where f lies in $\hom_{\boldsymbol{Top}}(E_1,E_2)$ and g is a continuous mapping with domain $\mathcal{C}^0(E_1,V)$ and codomain $\mathcal{C}^0(E_2,V)$.

Given any topological space $(E,\mathcal{T})$, let

$$\mathfrak{P}_{(E,\mathcal{T})}:\boldsymbol{Tvec}\to\boldsymbol{TopVec}$$

denote the functor carrying each $\boldsymbol{Tvec}$-object (V,T) into the product

$$\mathfrak{N}(E,\mathcal{T})\times\left(\mathcal{C}^0(E,V),T^*\right)$$

and sending every $\boldsymbol{Tvec}$-morphism $f:(V_1,T_1)\to(V_2,T_2)$ into $Id\times f_*$; where $f_*=\hom_{\boldsymbol{Top}}((E,\mathcal{T}),|f|)$ stands for the morphism carrying each $g\in\mathcal{C}^0(E,V_1)$ into $f\circ g\in\mathcal{C}^0(E,V_2)$, and as usual $\hom_{\boldsymbol{Top}}((E,\mathcal{T}),|-|)$ denotes the covariant hom-functor.

Finally, let $\boldsymbol{Alg}(\mathfrak{P}_{(E,\mathcal{T})})$denote the category of $\mathfrak{P}_{(E,\mathcal{T})}$-algebras, that is, each object is a pair of the form $\Big((V,T),\mathfrak{K}_{(V,T)}\Big)$ where

$$\mathfrak{K}_{(V,T)}:\mathfrak{P}_{(E,\mathcal{T})}(V,T)=\mathfrak{N}(E,\mathcal{T})\times\left(\mathcal{C}^0(E,V),T^*\right)\to|(V,T)|$$

is a continuous map. In addition, a given continuous linear mapping

$$f:(V_1,T_1)\to(V_2,T_2)$$

is an $\boldsymbol{Alg}(\mathfrak{P}_{(E,\mathcal{T})})$-morphism whenever the following quadrangle commutes.

(1.0.5)
$$\begin{tikzcd}
\mathfrak{N}(E,\mathcal{T})\times\left(\mathcal{C}^0(E,V_1),T_1^*\right) \arrow[r, "\mathfrak{K}_{(V_1,T_1)}"] \arrow[d, "Id\times\hom_{\boldsymbol{Top}}((E,\mathcal{T}),|f|)"'] & |(V_1,T_1)| \arrow[d, "|f|"] \\
\mathfrak{N}(E,\mathcal{T})\times\left(\mathcal{C}^0(E,V_2),T_2^*\right) \arrow[r, "\mathfrak{K}_{(V_2,T_2)}"'] & |(V_2,T_2)|
\end{tikzcd}$$

Definition 1.0.2. A smoother will be any pair

$$\left(\left((E,\mathcal{T}),\sigma_{(E,\mathcal{T})}\right),\left((V,T),\mathfrak{K}_{(V,T)}\right)\right)$$

such that $\left((E,\mathcal{T}),\sigma_{(E,\mathcal{T})}\right)$ is a co-algebra lying in $\boldsymbol{cAlg}(\mathfrak{N})$ and $\left((V,T),\mathfrak{K}_{(V,T)}\right)$ is an algebra in $\boldsymbol{Alg}(\mathfrak{P}_{(E,\mathcal{T})})$ satisfying the following conditions.

a) For every $p\in E$: $p\in\sigma_{(E,\mathcal{T})}(p)$.

b) For every $(p,f)\in E\times\mathcal{C}^0(E,V)$:

$$\mathfrak{K}_{(V,T)}\left(\sigma_{(E,\mathcal{T})}(p),f\right)\in\mathfrak{C}\left(\mathfrak{N}f\left(\sigma_{(E,\mathcal{T})}(p)\right)\right)$$

where, for any subset $A \subseteq E$, $\mathfrak{C}(A)$ denotes the convex cover of A.

c) $\mathfrak{K}_{(V,T)}$ is linear with respect to its second argument, that is to say, for every couple of scalars (α, β) and each pair of maps (f, g) the following holds.

$$\begin{aligned} \mathfrak{K}_{(V,T)}\big(\sigma_{(E,\mathcal{T})}(p), \alpha f + \beta g\big) = \\ \alpha \mathfrak{K}_{(V,T)}\big(\sigma_{(E,\mathcal{T})}(p), f\big) + \beta \mathfrak{K}_{(V,T)}\big(\sigma_{(E,\mathcal{T})}(p), g\big) \end{aligned} \tag{1.0.6}$$

1.0.1. Transformation associated to a smoother. Given a homeomorphism $\varphi : (E, \mathcal{T}) \to |(V,T)|$, a smoother

$$\mathfrak{S} = \big(\big((E,\mathcal{T}), \sigma_{(E,\mathcal{T})}\big), \big((V,T), \mathfrak{K}_{(V,T)}\big)\big)$$

induces a transformation $\mathfrak{S}_\varphi$ carrying each point $p \in E$ into

$$\varphi^{-1}\left(\mathfrak{K}_{(V,T)}\big(\sigma_{(E,\mathcal{T})}(p), \varphi\big)\right.$$

which will be said to be induced by $\mathfrak{S}$. Likewise, for every one-parameter continuous map family $h : E \times I \subseteq E \times \mathbb{R} \to E$ one can define the induced transformation by

$$\mathfrak{S}_{h(p,t)}(p) = \varphi^{-1}\left(\mathfrak{K}_{(V,T)}\left(\sigma_{(E,\mathcal{T})}(h(p,t), \varphi)\right)\right) \tag{1.0.7}$$

1.0.2. Ordering. Smoothers form a category $\boldsymbol{Smtr}$ the morphism-class of which consists of every $\boldsymbol{cAlg}(\mathfrak{N})$-morphism $f : (E_1, \mathcal{T}_1) \to (E_2, \mathcal{T}_2)$ such that the following quadrangle commutes

$$\begin{array}{ccc} \big(\mathfrak{N}(E_1,\mathcal{T}_1) \times \mathcal{C}^0(E_1,V)\big) & \xrightarrow{\mathfrak{K}_{(V,T)}} & |(V,T)| \\ {\scriptstyle \mathfrak{N}f \times \hom_{\boldsymbol{Top}}(f, |(V,T)|)} \downarrow & & \downarrow {\scriptstyle |Id|} \\ \big(\mathfrak{N}(E_2,\mathcal{T}_2) \times \mathcal{C}^0(E_2,V)\big) & \xrightarrow[\mathfrak{K}_{(V,T)}]{} & |(V,T)| \end{array} \tag{1.0.8}$$

where $\hom_{\boldsymbol{Top}}(-, |(V,T)|) : \boldsymbol{Top} \to \boldsymbol{Top}^{op}$ stands for the contravariant hom-functor.

Regarding $\boldsymbol{Smtr}$ as a concrete category over $\boldsymbol{Set}$ via the forgetful functor such that

$$\big(\big((E,\mathcal{T}), \sigma_{(E,\mathcal{T})}\big), \big((V,T), \mathfrak{K}_{(V,T)}\big)\big) \mapsto E$$

with the obvious arrow-map, in each fibre one can define an ordering $\preceq$ as follows. For any smoother $\big(\big((E,\mathcal{T}), \sigma_{(E,\mathcal{T})}\big), \big((V,T), \mathfrak{K}_{(V,T)}\big)\big)$, let

$$\Omega\left(\big((E,\mathcal{T}), \sigma_{(E,\mathcal{T})}\big), \big((V,T), \mathfrak{K}_{(V,T)}\big)\right)$$

denote the intersection of all topologies for E containing the set family

$$K = \left\{\sigma_{(E,\mathcal{T})}(p) \mid p \in E\right\}$$

then

$$\begin{aligned} \Big(\big((E_1,\mathcal{T}_1), \sigma_{(E_1,\mathcal{T}_1)}\big), \big((V,T), \mathfrak{K}_{(V,T)}\big)\Big) \\ \preceq \Big(\big((E_2,\mathcal{T}_2), \sigma_{(E_2,\mathcal{T}_2)}\big), \big((V,T), \mathfrak{K}_{(V,T)}\big)\Big) \end{aligned} \tag{1.0.9}$$

if and only if the topology $\left(\Omega\left((E_1,\mathcal{T}_1),\sigma_{(E_1,\mathcal{T}_1)}\right),\left((V,T),\mathfrak{K}_{(V,T)}\right)\right)$ is finer than the topology $\Omega\left(\left((E_2,\mathcal{T}_2),\sigma_{(E_2,\mathcal{T}_2)}\right),\left((V,T),\mathfrak{K}_{(V,T)}\right)\right)$. For a maximal element, the topology $\Omega\left(\left((E,\mathcal{T}),\sigma_{(E,\mathcal{T})}\right),\left((V,T),\mathfrak{K}_{(V,T)}\right)\right)$ must be indiscrete. In this case, for every p in E, $\sigma_{(E,\mathcal{T})}(p)=E$, and consequently, for every $p,q\in E$,

$$\mathfrak{K}_{E,\mathcal{T}}(\sigma_{(E,\mathcal{T})}(p),\varphi)=\mathfrak{K}_{(E,\mathcal{T})}(E,\varphi)=\mathfrak{K}(\sigma_{(E,\mathcal{T})}(q),\varphi)$$

therefore $\mathfrak{S}_\varphi h(p,t)=\varphi^{-1}\left(\mathfrak{K}_{(V,T)}\left(\sigma_{(E,\mathcal{T})}(h(p,t),\varphi\right)\right)$ transforms $h(p,t)$ into a constant map, which is the smoothest function one can build. Conversely, a minimal element corresponds to the discrete topology. In this case, by virtue of both conditions **a**) and **b**), the transformation (1.0.7) is the identity, so then $h(p,t)$ remains unaltered. Between both extremes one can build several degrees of smoothness.

1.1 Smoothers in smooth manifolds.. Let $(M,\mathcal{A}_n)$ be a smooth manifold, $\varphi:U\subseteq M\to\mathbb{R}^n$ a chart and $\mathcal{T}$ the induced topology for U. Henceforth, the pair $(U,\mathcal{T})$ will be assumed to be a Hausdorff space. In the most natural way, one can build a smoother $\mathfrak{S}=\left(\left((U,\mathcal{T}),\sigma_{(U,\mathcal{T})}\right),\left((\mathbb{R}^n,T),\mathfrak{K}_{(\mathbb{R}^n,T)}\right)\right)$ over $(U,\mathcal{T})$ the associated set of continuous maps $\mathcal{C}^0(U,\mathbb{R}^n)$ contains each smooth one like the diffeomorphism associated to each chart.

For those smooth manifolds such that, for each $p\in M_n$, each tangent space T_p is isomorphic to $\mathbb{R}^n$, that is to say, there is an isomorphism $\lambda_p:T_p\to\mathbb{R}^n$, one can associate a map $\omega_{\boldsymbol{X}}:U\to\mathbb{R}^n$ to every smooth vector field $\boldsymbol{X}$ letting

$$\forall p\in U:\quad \omega_{\boldsymbol{X}}(p)=\lambda_p\big(\boldsymbol{X}_p)\big) \tag{1.1.1}$$

Accordingly, the image of the vector field $\boldsymbol{X}$ under $\mathfrak{S}$ is

$$\lambda_p\big(\boldsymbol{Y}\big)=\mathfrak{K}_{(\mathbb{R}^n,T)}\big(\sigma_{(U,\mathcal{T})}(p),\omega_{\boldsymbol{X}}\big) \tag{1.1.2}$$

therefore

$$\boldsymbol{Y}=\lambda_p^{-1}\left(\mathfrak{K}_{(\mathbb{R}^n,T)}\big(\sigma_{(U,\mathcal{T})}(p),\omega_{\boldsymbol{X}}\big)\right) \tag{1.1.3}$$

From the former equations it follows immediately that if $h_t:U\to U$ is the one-parameter group associated to $\boldsymbol{X}$, then

$$\omega_{\boldsymbol{X}}(p)=\lambda_p\left(\boldsymbol{X}\right)\Big|_p=\lim_{t\to 0}\frac{\varphi\circ h_t(p)-\varphi(p)}{t}\in\mathbb{R}^n \tag{1.1.4}$$

accordingly

$$\begin{aligned}&\mathfrak{K}_{(\mathbb{R}^n,T)}\big(\sigma_{(U,\mathcal{T})}(p),\omega_{\boldsymbol{X}}\big)=\\&\qquad\lim_{t\to 0}\frac{\mathfrak{K}_{(\mathbb{R}^n,T)}\big(\sigma_{(U,\mathcal{T})}(p),\varphi\circ h_t\big)-\mathfrak{K}_{(\mathbb{R}^n,T)}\big(\sigma_{(U,\mathcal{T})}(p),\varphi\big)}{t}\end{aligned} \tag{1.1.5}$$

Definition 1.1.1. Let $\boldsymbol{X}$ be a smooth vector field the coordinates of which are $(X^1, \dots X^n)$, and consider the differential equation

$$\left\{\begin{array}{l}\frac{d}{dt}x^1(\gamma(p,t)) = X^1\big(x^1(\gamma(p,t), x^2(\gamma(p,t)\dots\big)\\ \frac{d}{dt}x^2(\gamma(p,t)) = X^2\big(x^1(\gamma(p,t), x^2(\gamma(p,t)\dots\big)\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \frac{d}{dt}x^n(\gamma(p,t)) = X^n\big(x^1(\gamma(p,t), x^2(\gamma(p,t)\dots\big)\end{array}\right. \tag{1.1.6}$$

where the differentiable curve $\gamma : I \subseteq \mathbb{R} \to U$ is assumed to be solution of the former equation for the initial value $\gamma(p, t_0) = p$. Say, a smoother $\mathfrak{S} = \big(\big((U,\mathcal{T}), \sigma_{(U,\mathcal{T})}\big), \big((\mathbb{R}^n, T), \mathfrak{K}_{(\mathbb{R}^n,T)}\big)\big)$ to be compatible with $\boldsymbol{X}$ provided that the curve $\boldsymbol{y}\big(\rho(p,t)\big) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(\gamma(p,t)\right), \varphi\right)$ is solution of the equation

$$\left\{\begin{array}{l}\frac{d}{dt}y^1(\rho(p,t)) = Y^1\big(y^1(\rho(p,t), y^2(\rho(p,t)\dots\big)\\ \frac{d}{dt}y^2(\rho(p,t)) = Y^2\big(y^1(\rho(p,t), y^2(\rho(p,t)\dots\big)\\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \frac{d}{dt}y^n(\rho(p,t)) = Y^n\big(y^1(\rho(p,t), y^2(\rho(p,t)\dots\big)\end{array}\right. \tag{1.1.7}$$

where $q = \mathfrak{S}_\varphi(y(p, t_0)) = \mathfrak{S}_\varphi(p)$, and

$$(Y^1, Y^2, \dots Y^n) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \omega_{\boldsymbol{X}}\right) \tag{1.1.8}$$

Obviously, if p is a fixed point for $\mathfrak{S}_\varphi$, then $\boldsymbol{y}(q,t)$ and $\boldsymbol{x}(p,t)$ are solutions of (1.1.6) and (1.1.7), respectively, for the same initial value $p = x(p,t_0) = y(p,t_0)$.

Remark. If $p = q$, that is to say, if p is a fixed-point for $\mathfrak{S}_\varphi$, then from Definition 1.0.2 the relations

$$\forall p \in U: \quad \boldsymbol{x}(p) \in \mathfrak{C}\left(\mathfrak{N}\varphi\left(\sigma_{(U,\mathcal{T})}(p)\right)\right) \tag{1.1.9}$$

and

$$\forall p \in U: \quad \boldsymbol{y}(p) \in \mathfrak{C}\left(\mathfrak{N}\varphi\left(\sigma_{(U,\mathcal{T})}(p)\right)\right) \tag{1.1.10}$$

are true, therefore

$$\|\boldsymbol{x}(p) - \boldsymbol{y}(p)\| \leq \max_{q_1,q_2 \in \mathfrak{C}\left(\sigma_{(U,\mathcal{T})}(p)\right)} \|\varphi(q_1) - \varphi(q_2)\| \tag{1.1.11}$$

From the former relation one can build some proximity criteria. If the maximum distance among points in any set $\sigma_{(U,\mathcal{T})}(p)$ is bounded, that is to say, if there is $\delta > 0$ such that

$$\forall p \in U: \quad \max_{q_1,q_2 \in \mathfrak{C}\left(\sigma_{(U,\mathcal{T})}(p)\right)} \|\varphi(q_1) - \varphi(q_2)\| < \delta$$

then

$$\forall t > 0: \quad \|\boldsymbol{x}(p,t) - \boldsymbol{y}(p,t)\| < \delta$$

Proposition 1.1.3. *Let* $\big((U,\mathcal{T}),\sigma_{(U,\mathcal{T})}\big)$ *be a co-algebra in* $\boldsymbol{cAlg}(\mathfrak{N})$ *and for every point* p *of* U *let* $\mu_p:\sigma_{(U,\mathcal{T})}\to[0,\infty)$ *be a measure for* $\sigma_{(U,\mathcal{T})}(p)$ *such that the set* $\sigma_{(U,\mathcal{T})}(p)$ *is* μ_p*-measurable. If for every* $p\in U$ *the following condition hold,*

a) $p\in\sigma_{(U,\mathcal{T})}(p)$.

b) $\sigma_{(U,\mathcal{T})}(p)$ *is a closed subset of* U.

c) $\mu_p\big(\sigma_{(U,\mathcal{T})}(p)\big)=1$

then the pair $\Big(\big((U,\mathcal{T}),\sigma_{(U,\mathcal{T})}\big)\,,\big((V,T),\mathfrak{K}_{(V,T)}\big)\Big)$ *is a smoother, where*

$$\mathfrak{K}_{(V,T)}\Big(\sigma_{(U,\mathcal{T})}(p),\boldsymbol{f}\Big)=\int\!\cdots\!\int_{\sigma_{(U,\mathcal{T})}(p)}\boldsymbol{f}\,d\mu_p \tag{1.1.12}$$

Proof. Obviously, $\mathfrak{K}_{(V,T)}$ is linear with respect to its second coordinate, and by definition, it satisfies condition **a**) in Definition 1.0.2, therefore it remains to be proved $\mathfrak{K}_{(V,T)}$ to satisfy condition **b**) too.

It is a well-known fact that each coordinate f^j of any measurable function $\boldsymbol{f}$ is the limit of a sequence $\big\{f_n^j\mid n\in\mathbb{N}\big\}$ of step-functions each of which of the form

$$f_n^j=\sum_{i=1}^{m}c_{i,n}^j\chi_{E_{i,n}} \tag{1.1.13}$$

such that each of the $E_{i,n}$ is μ_p-measurable and for every $i\in\mathbb{N}$, $c_{i,n}^j=f^j(\alpha_i)$ for some $\alpha_i\in E_{i,n}$, besides, $\forall n\in\mathbb{N}:E_{i,n}\cap E_{j,n}=\emptyset\ (i\neq j)$ and $\cup_{i=1}^m E_{i,n}=\sigma_{(U,\mathcal{T})}(p)$. In addition

$$\int\!\cdots\!\int_{\sigma_{(U,\mathcal{T})}(p)}f^j\,d\mu_p=\lim_{n\to\infty}\sum_{i=1}^{m}c_{i,n}^j\mu_p(\chi_{E_{i,n}}) \tag{1.1.14}$$

Now, from statement **c**) it follows that

$$\forall n\in\mathbb{N}:\qquad \sum_{i=1}^{m}\mu_p(\chi_{E_{i,n}})=\mu_p(\sigma_{(U,\mathcal{T})}(p))=1 \tag{1.1.15}$$

therefore

$$\forall n\in\mathbb{N}:\qquad \sum_{i=1}^{m}\boldsymbol{c}_{i,n}\mu_p(\chi_{E_{i,n}})\in\mathfrak{C}\left(\mathfrak{N}\boldsymbol{f}(\sigma_{(U,\mathcal{T})}(p))\right) \tag{1.1.16}$$

where $\boldsymbol{c}_{i,n}=(c_{i,n}^1,c_{i,n}^2,\cdots)$. Finally, since $\sigma_{(U,\mathcal{T})}(p)$ is assumed to be closed, the proposition follows. □

2. A compatibility criterion

Although smoothers can be useful in several areas, the aim of this paper is its application in differential equations in which only those smoothers being compatible with the associated vectors are useful. To build a compatibility criterion the following result is a powerful tool.

Theorem 2.0.4. *Let $(M_n, \mathcal{A}_n)$ be a smooth manifold and (U, φ) a chart. Let $h_t : U \to U$ stand for the one-parameter group associated to a smooth vector field $\boldsymbol{X}$ and*

$$\mathfrak{S} = \big(\big((U, \mathcal{T}), \sigma_{(U,\mathcal{T})}\big), \big((\mathbb{R}^n, T), \mathfrak{K}_{(\mathbb{R}^n,T)}\big)\big)$$

a smoother. If the following relation holds

$$\exists \delta > 0, \forall t < \delta : \quad \mathfrak{K}_{(\mathbb{R}^n,T)}\Big(\sigma_{(U,\mathcal{T})}\big(h_t(p)\big), \varphi\Big) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}(p), \varphi \circ h_t\right) \tag{2.0.17}$$

then $\mathfrak{S}$ is compatible with $\boldsymbol{X}$.

Proof. First, from

$$\boldsymbol{y}(q,t) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_t(p)\right), \varphi\right)$$

we obtain that

$$\varphi(q) = \boldsymbol{y}(q, t_0) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \varphi\right) = \varphi\left(\mathfrak{S}_\varphi(p)\right)$$

Now, it is not difficult to see that

$$\frac{d}{dt}\boldsymbol{y}\Big|_q = \lim_{t\to 0} \frac{\mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_t(p)\right), \varphi\right) - \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_0(p)\right), \varphi\right)}{t} \tag{2.0.18}$$

and using (2.0.17) the former equation becomes

$$\begin{aligned} \frac{d}{dt}\boldsymbol{y}\Big|_q &= \lim_{t\to 0} \frac{\mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \varphi \circ h_t\right) - \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \varphi\right)}{t} = \\ &\lim_{t\to 0} \frac{\mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \varphi \circ h_t - \varphi\right)}{t} = \\ &\lim_{t\to 0} \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \frac{\varphi \circ h_t - \varphi}{t}\right) \end{aligned} \tag{2.0.19}$$

and by continuity

$$\lim_{t\to 0} \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \frac{\varphi \circ h_t - \varphi}{t}\right) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \lim_{t\to 0}\frac{\varphi \circ h_t - \varphi}{t}\right) \tag{2.0.20}$$

therefore, taking into account (1.1.1) and (1.1.4),

$$\frac{d}{dt}\boldsymbol{y}\Big|_q = \lim_{t\to 0} \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \omega_{\boldsymbol{X}}\right) \tag{2.0.21}$$

accordingly, if $\varphi\big(h_t(p)\big) = (x^1(p,t), x^2(p,t)\dots)$ is solution of (1.1.6) for the initial value p, then

$$(y^1(q,t), y^2(q,t)\dots) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_t(p)\right), \varphi\right)$$

is solution of the equation (1.1.7) for the initial value $q = \mathfrak{S}_\varphi(p)$, being

$$(Y^1, Y^2 \dots) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \omega_{\boldsymbol{X}}\right)$$

□

Corollary 2.0.5. *With the same conditions as in the preceding theorem, if p is a fixed point for $\mathfrak{S}_\varphi$ and $\boldsymbol{x}(p,t) = \left(x^1(p,t), x^2(p,t)\dots\right)$ is solution of the initial value problem*

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{x}(p,t) = \boldsymbol{X}\left(\boldsymbol{x}(p,t)\right) \\ \boldsymbol{x}(p,t_0) = \varphi(p) \end{cases} \tag{2.0.22}$$

then $\boldsymbol{y}(p,t) = \left(y^1(p,t), y^2(p,t)\dots\right) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_t(p)\right), \varphi\right)$ is solution for the initial value problem

$$\begin{cases} \dfrac{d}{dt}\boldsymbol{y}(p,t) = \boldsymbol{Y}\left(\boldsymbol{y}(p,t)\right) = \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right), \omega_{\boldsymbol{X}}\right) \\ \boldsymbol{y}(p,t_0) = \varphi(p) \end{cases} \tag{2.0.23}$$

Remark 2.0.6. The smoother defined in (2.0.39) satisfies the conditions of the former corollary, because each point (x,y) of $\mathbb{R}^2$ is a fixed-point. However, the smoother

$$\mathfrak{S} = \left(\left((U,\mathcal{T}), \sigma_{(U,\mathcal{T})}\right), \left((\mathbb{R}^n,T), \mathfrak{K}_{(\mathbb{R}^n,T)}\right)\right)$$

such that the law $\sigma_{(U,\mathcal{T})}$ sends each point $(x_0,y_0) \in \mathbb{R}^2$ into the subset

$$A_{(x_0,y_0)} = \left\{(x,y) \in \mathbb{R}^2 \mid x_0 \le x \le x_0 + 1 - \frac{y}{y_0}, \quad 0 \le y \le e^{x_0}\right\}$$

the associated transform of which is

$$\begin{aligned} \mathfrak{K}_{(\mathbb{R}^2,T)} : f(x,y) \mapsto \frac{2}{3y} \iint_{\sigma_{(U,\mathcal{T})}(x,y)} f(x+u, y+v)\, dudv = \\ \frac{2}{3y}\int_{-y}^{0}\int_{0}^{1-\frac{v}{y}} f(x+u,y+v)\, dudv \end{aligned} \tag{2.0.24}$$

is compatible with the vector $\binom{1}{y}$ and sends the point (x,y) into $(x+\frac{7}{9}, \frac{4}{9}y)$, this is to say, $\mathfrak{S}_\varphi(x,y) = (x+\frac{7}{9}, \frac{4}{9}y)$, Thus, there is no fixed point for $\mathfrak{S}_\varphi$. Of course, this smoother transforms the solution $(t + x_0, y_0e^t)$ of the equation

$$\begin{cases} \dfrac{d}{dt}x(t) = 1 \\ \dfrac{d}{dt}y(t) = y(t) \end{cases} \tag{2.0.25}$$

for the initial value (x_0, y_0) at $t = 0$, into the solution $(t + x_0 + \frac{7}{9}, \frac{4}{9}y_0e^t)$ of the same equation for the initial value $(x_0 + \frac{7}{9}, \frac{4}{9}y_0)$.

Definition 2.0.7. Given a smoother

$$\mathfrak{S} = \left(\left((U,\mathcal{T}), \sigma_{(U,\mathcal{T})}\right), \left((\mathbb{R}^n,T), \mathfrak{K}_{(\mathbb{R}^n,T)}\right)\right)$$

defined over a chart (U,φ) of a smooth manifold $(M,\mathcal{A}_n)$, and a smooth vector field $\boldsymbol{X}$, define the derivative $\nabla_{\boldsymbol{X}}\mathfrak{S}$ by the following expression.

$$\begin{aligned} &\nabla_{\boldsymbol{X}}\mathfrak{S}|_p = \\ &\lim_{t\to 0}\frac{\mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(h_t(p)\right),\varphi\right) - \mathfrak{K}_{(\mathbb{R}^n,T)}\left(\sigma_{(U,\mathcal{T})}\left(p\right),\varphi\circ h_t\right)}{t} \end{aligned} \tag{2.0.26}$$

Corollary 2.0.8. *If $\nabla_{\boldsymbol{X}}\mathfrak{S} = 0$, then $\mathfrak{S}$ is compatible with $\boldsymbol{X}$.*

Proof. Obviously, taking into account Definition 2.0.7, from $\nabla_{\boldsymbol{X}}\mathfrak{S} = 0$, the statement (2.0.17) follows. □

Definition 2.0.9. Given a smooth vector field $\boldsymbol{X}$, the associated one-parameter group of which is $h_t : U \to U$, say a measure-field $\{\mu_p \mid p \in U\}$ to be invariant with respect to $\boldsymbol{X}$, provided that for every $p \in U$ and each measurable subset E of $\sigma_{(U,\mathcal{T})}(p)$ the following relation holds

$$\forall t \in \mathbb{R} : \qquad \mu_{h_t(p)}\left(\mathfrak{N}h_t(E)\right) = \mu_p(E) \tag{2.0.27}$$

accordingly the measure $\mu_p(E)$ remains unaltered under the one-parameter group $h_t : U \to U$ associated to $\boldsymbol{X}$.

Remark 2.0.10. In [4] it is shown that, for a wide class of vector fields, each differentiable-map $\psi : U \to \mathbb{C}$ satisfying the equation

$$\boldsymbol{X}^{\smile}\psi(p) = \left(\sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}\, \psi(x^1, x^2 \dots)\right)^{\smile} = 0 \tag{2.0.28}$$

satisfies also the equation

$$\frac{d}{dt}\,\psi(h_t(p)) = 0 \tag{2.0.29}$$

accordingly, $\psi(h_t(p))$ does not depend upon the parameter t; where for every continuous function f, $f^{\smile}$ denotes the maximal extension by continuity. Thus, an invariance criterion can consist of proving the existence of a differentiable map $\psi_E : U \to \mathbb{R}$, for each measurable subset $E \subseteq U$, such that

$$\begin{cases} \psi_E(p) = \mu_p(E) \\ \boldsymbol{X}^{\smile}\psi_E(p) = 0 \end{cases} \tag{2.0.30}$$

Theorem 2.0.11. *If a measure field $\{\mu_p \mid p \in U\}$ is invariant with respect to $\boldsymbol{X}$ and, for each $t \in \mathbb{R}$, the member of corresponding one-parameter group $h_t : U \to U$ is a $\boldsymbol{cAlg}(\mathfrak{N})$-morphism, then the smoother*

$$\mathfrak{S} = \left(\left((U,\mathcal{T}), \sigma_{(U,\mathcal{T})}\right), \left((\mathbb{R}^n, T), \mathfrak{K}_{(\mathbb{R}^n,T)}\right)\right)$$

such that

$$\mathfrak{K}_{(V,T)}\left(\sigma_{(U,\mathcal{T})}(p), \varphi\right) = \int \cdots \int_{\sigma_{(U,\mathcal{T})}(p)} \varphi \, d\mu_p \tag{2.0.31}$$

is compatible with $\boldsymbol{X}$.

Proof. First, because, for each $t \in \mathbb{R}$, the map $h_t : U \to U$ is assumed to be a $\boldsymbol{cAlg}(\mathfrak{N})$-morphism, then by virtue of (1.0.4) we have that

$$\begin{aligned} &\mathfrak{K}_{(V,T)}\left(\sigma_{(U,\mathcal{T})}(h_t(p)), \varphi\right) = \\ &\qquad \mathfrak{K}_{(V,T)}\left(\mathfrak{N}h_t\big(\sigma_{(U,\mathcal{T})}(p)\big), \varphi\right) = \int \cdots \int_{\mathfrak{N}h_t\left(\sigma_{(U,\mathcal{T})}(p)\right)} \varphi \, d\mu_p \end{aligned} \tag{2.0.32}$$

and because

$$\mathfrak{N}h_t\left(\sigma_{(U,\mathcal{T})}(p)\right) = \left\{h_t(q) \mid q \in \sigma_{(U,\mathcal{T})}(p)\right\} \tag{2.0.33}$$

then

$$\underset{\mathfrak{N}h_t\left(\sigma_{(U,\mathcal{T})}(p)\right)}{\int\cdots\int} \varphi\, d\mu_p = \underset{\sigma_{(U,\mathcal{T})}(p)}{\int\cdots\int} \varphi \circ h_t\, d\mu_{h_t(p)} \tag{2.0.34}$$

therefore, since the invariance of $\{\mu_p \mid p \in U\}$ with respect to $\boldsymbol{X}$ is assumed, then for every subset $E \subseteq \sigma_{(U,\mathcal{T})}(p)$ the following relation holds

$$\mu_{h_t(p)}\big(\mathfrak{N}h_t(E)\big) = \mu_p(E)$$

therefore from (2.0.34) it follows that

$$\underset{\sigma_{(U,\mathcal{T})}(p)}{\int\cdots\int} \varphi \circ h_t\, d\mu_{h_t(p)} = \underset{\sigma_{(U,\mathcal{T})}(p)}{\int\cdots\int} \varphi \circ h_t\, d\mu_p \tag{2.0.35}$$

consequently,

$$\begin{aligned}&\mathfrak{K}_{(V,\mathcal{T})}\left(\sigma_{(U,\mathcal{T})}(h_t(p)), \varphi\right) = \\ &\qquad \underset{\sigma_{(U,\mathcal{T})}(p)}{\int\cdots\int} \varphi \circ h_t\, d\mu_p = \mathfrak{K}_{(V,\mathcal{T})}\left(\sigma_{(U,\mathcal{T})}(p), \varphi \circ h_t\right)\end{aligned} \tag{2.0.36}$$

accordingly, the smoother $\mathfrak{S}$ satisfies the conditions of the preceding theorem. □

Example 2.0.12. Consider the initial value problem

$$\begin{cases} \dfrac{d}{dt}x(t) = 1 \\ \dfrac{d}{dt}y(t) = 0.1\cos\big(x(t)\big) \\ \big(x(0), y(0)\big) = (x_0, y_0) \end{cases} \tag{2.0.37}$$

the solution of which is

$$\begin{cases} x(t) = x_0 + t \\ y(t) = y_0 + 0.1\big(\sin(t + x_0) - \sin(x_0)\big) \end{cases} \tag{2.0.38}$$

where we are assuming the function $0.1\cos\big(x(t)\big)$, in the second equation of (2.0.37), to be the consequence of a perturbation working over the vector field $\binom{1}{0}$. The map σ, sending each $(x, y) \in \mathbb{R}^2$ into the closed set $[x - \pi, x + \pi] \times [y - 1, y + 1]$, together with the operator $\mathfrak{K}$ defined as follows

(2.0.39)

$$\begin{aligned}\mathfrak{K} : \big(f_1(x, y), f_2(x, y)\big) \mapsto \Big(&\frac{1}{4\pi}\int_{-1}^{1}\int_{-\pi}^{\pi} f_1(x + u, y + v)dudv, \\ &\frac{1}{4\pi}\int_{-1}^{1}\int_{-\pi}^{\pi} f_2(x + u, y + v)dudv\Big)\end{aligned}$$

form a smoother such that $\mathfrak{K}$ transforms the vector $\begin{pmatrix} 1 \\ 0.1\cos(x) \end{pmatrix}$ of the equation (2.0.37) into the perturbation-free vector $\binom{1}{0}$, therefore it transforms also (2.0.37) into the following initial value problem,

$$\begin{cases} \dfrac{d}{dt}\,x(t) = 1 \\ \dfrac{d}{dt}\,y(t) = 0 \\ \big(x(0), y(0)\big) = (x_0, y_0) \end{cases} \tag{2.0.40}$$

Now, it is not difficult to see $\mathfrak{K}$ to be compatible with the vector field of the equation (2.0.37), therefore $\mathfrak{K}$ transforms also the solution (2.0.38) of (2.0.37) into the solution of (2.0.40), as one can see in the following equality

$$\begin{cases} \dfrac{1}{4\pi}\displaystyle\int_{-1}^{1}\int_{-\pi}^{\pi}(x_0 + u + t)\colon dudv = x_0 + t \\ \dfrac{1}{4\pi}\displaystyle\int_{-1}^{1}\int_{-\pi}^{\pi}\big(y_0 + v + 0.1\big(\sin(x_0 + u + t) - \\ \qquad\qquad\qquad \sin(x_0 + u)\big)\big)dudv = y_0 \end{cases} \tag{2.0.41}$$

and

$$\begin{cases} x(t) = x_0 + t \\ y(t) = y_0 \end{cases} \tag{2.0.42}$$

is nothing but the general solution of (2.0.40). Thus, $\mathfrak{K}$ sends (2.0.37) into (2.0.40) and also sends the general solution of (2.0.37) into the perturbation-free solution (2.0.42) of (2.0.40).

References

1. Adámek, J., Herrlich and H., Strecker, G. E., *Abstract and concrete categories. The joy of cats. Pure and Applied Mathematics*, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, xiv+482 pp., 1990.
2. Kobayashi, S., and Nomizu, K., *Foundations of Differential Geometry*, Interscience Publish, 1963.
3. Köthe, G., *Topological vector spaces*, Springer-Verlag, New York, 1969.
4. Palomar Tarancón, J. E., *Vector spaces over function fields. Vector spaces over analytic function fields being associated to ordinary differential equations*, Southwest J. Pure Appl. Math., no. 2, pp. 60-87 (2000).