Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 2, December, 2003, pp. 26–35.
Submitted: December 24, 2002. Published: December 31 2003.

# NON-AUTONOMOUS INHOMOGENEOUS BOUNDARY CAUCHY PROBLEMS AND RETARDED EQUATIONS

M. Filali and M. Moussi

Abstract. In this paper we prove the existence and the uniqueness of classical solution of non-autonomous inhomogeneous boundary Cauchy problems, this solution is given by a variation of constants formula. Then, we apply this result to show the existence of solution of a retarded equation.

**A.M.S. (MOS) Subject Classification Codes.** 34G10; 47D06

**Key Words and Phrases.** Boundary Cauchy problem, Evolution families, Classical solution, Wellposedness, Variation of constants formula, Retarded equation

## 1 Introduction

Consider the following Cauchy problem with boundary conditions

$$(IBCP)\begin{cases} \dfrac{d}{dt}u(t) = A(t)u(t), & 0 \le s \le t \le T, \\ L(t)u(t) = \phi(t)u(t) + f(t), & 0 \le s \le t \le T, \\ u(s) = u_0. \end{cases}$$

This type of problems presents an abstract formulation of several natural equations such as retarded differential equations, retarded (difference) equations, dynamical population equations and neutral differential equations.

In the autonomous case ($A(t) = A, L(t) = L, \phi(t) = \phi$) the Cauchy problem ($IBCP$) was studied by Greiner [2,3]. He used a perturbation of domain of generator of semigroups, and showed the existence of classical solutions of ($IBCP$) via variation of constants formula. In the homogeneous case ($f = 0$), Kellermann [6] and Nguyen Lan [8] have showed the existence of an evolution family $(U(t,s))_{t\ge s\ge 0}$ as the classical solution of the problem ($IBCP$).

Department of Mathematics, Faculty of Sciences, Oujda, Morocco
E-mail Address: filali@sciences.univ-oujda.ac.ma
moussi@sciences.univ-oujda.ac.ma

The aim of this paper is to show well-posedness in the general case ($f \neq 0$) and apply this result to get a solution of a retarded equation. In Section 2 we prove the existence and uniqueness of the classical solution of $(IBCP)$. For that purpose, we transform $(IBCP)$ into an ordinary Cauchy problem and prove the equivalence of the two problems. Moreover, the solution of $(IBCP)$ is explicitly given by a variation of constants formula similar to the one given in [3] in the autonomous case. We note that the operator matrices method was also used in [4, 8, 9] for the investigation of inhomogeneous Cauchy problems without boundary conditions.

Section 3 is devoted to an application to the retarded equation

$$(RE)\begin{cases} v(t) = K(t)v_t + f(t), & t \geq s \geq 0, \\ v_s = \varphi. \end{cases}$$

We introduce now the following basic definitions which will be used in the sequel. A family of linear (unbounded) operators $(A(t))_{0\leq t\leq T}$ on a Banach space $X$ is called a stable family if there are constants $M \geq 1$, $\omega \in \mathbb{R}$ such that $]\omega, \infty[\subset \rho(A(t))$ for all $0 \leq t \leq T$ and

$$\left\| \prod_{i=1}^{k} R(\lambda, A(t_i)) \right\| \leq M(\lambda - \omega)^{-k} \text{ for } \lambda > \omega$$

for any finite sequence $0 \leq t_1 \leq ... \leq t_k \leq T$.
A family of bounded linear operators $(U(t,s))_{0\leq s\leq t}$ on $X$ is said an evolution family if

**(1)** $U(t,t) = I_d$ and $U(t,r)U(r,s) = U(t,s)$ for all $0 \leq s \leq r \leq t$,
**(2)** the mapping $\left\{(t,s) \in \mathbb{R}^2_+ : t \geq s\right\} \ni (t,s) \longmapsto U(t,s)$ is strongly continuous.

For evolution families and their applications to non-autonomous Cauchy problems we refer to [1,5,10].

## 2 Well-posedness of Cauchy problem with boundary coditions

Let $D$, $X$ and $Y$ be Banach spaces, $D$ densely and continuously embedded in $X$, consider families of operators $A(t) \in L(D,X)$, $L(t) \in L(D,Y)$ and $\phi(t) \in L(X,Y), 0 \leq t \leq T$. In this section we will use the operator matrices method in order to prove the existence of classical solution for the non-autonomous Cauchy problem with inhomogeneous boundary conditions

$$(IBCP)\begin{cases} \dfrac{d}{dt}u(t) = A(t)u(t), & 0 \leq s \leq t \leq T, \\ L(t)u(t) = \phi(t)u(t) + f(t), & 0 \leq s \leq t \leq T, \\ u(s) = u_0, \end{cases}$$

it means that we will transform this Cauchy problem into an ordinary homogeneous one.

In all this section we consider the following hypotheses :

$(H_1)$ $t \longmapsto A(t)x$ is continuously differentiable for all $x \in D$;
$(H_2)$ the family $(A^0(t))_{0\leq t\leq T}$, $A^0(t) := A(t)|_{ker L(t)}$, is stable, with $(M_0, \omega_0)$ constants of stability;
$(H_3)$ the operator $L(t)$ is surjective for every $t \in [0,T]$ and $t \longmapsto L(t)x$ is continuously differentiable for all $x \in D$;

$(H_4)$ $t \longmapsto \phi(t)x$ is continuously differentiable for all $x \in X$;
$(H_5)$ there exist constants $\gamma > 0$ and $\omega \in \mathbb{R}$ such that

$$||L(t)x||_Y \geq \gamma^{-1}(\lambda - \omega)||x||_X \text{ for } x \in ker(\lambda - A(t)), \lambda > \omega \text{ and } t \in [0, T].$$

**Definition 2.1.** *A function $u : [s, T] \longrightarrow X$ is called a classical solution of $(IBCP)$ if it is continuously differentiable, $u(t) \in D$, $\forall t \in [s, T]$ and $u$ satisfies $(IBCP)$. If $(IBCP)$ has a classical solution, we say that it is well-posed.*

We recall the following results which will be used in the sequel.

**Lemma 2.1.** [6,7] *For $t \in [0, T]$ and $\lambda \in \rho(A^0(t))$ we have the following properties*
*i) $D = D(A^0(t)) \oplus ker(\lambda - A(t))$.*
*ii) $L(t)|_{ker(\lambda - A(t))}$ is an isomorphism from $ker(\lambda - A(t))$ onto $Y$.*
*iii) $t \longmapsto L_{\lambda,t} := (L(t)|_{ker(\lambda - A(t))})^{-1}$ is strongly continuously differentiable.*

As consequences of this lemma we have $L(t)L_{\lambda,t} = I_{d_Y}$, $L_{\lambda,t}L(t)$ and $(I - L_{\lambda,t}L(t))$ are the projections in $D$ onto $ker(\lambda - A(t))$ and $D(A^0(t))$ respectively.
In order to get the homogenization of $(IBCP)$, we introduce the Banach space $E := X \times C^1([0, T], Y) \times Y$, where $C^1([0, T], Y)$ is the space of continuously differentiable functions from $[0, T]$ into $Y$ equipped with the norm $\|g\| := \|g\|_\infty + \|g'\|_\infty$, for $g \in C^1([0, T], Y)$.
Let $\mathcal{A}^\phi(t)$ be a matrix operator defined on $E$ by

$$\mathcal{A}^\phi(t) := \begin{pmatrix} A(t) & 0 & 0 \\ 0 & 0 & 0 \\ L(t) - \phi(t) & -\delta_t & 0 \end{pmatrix}, \ D(\mathcal{A}^\phi(t)) := D \times C^1([0, T], Y) \times \{0\}, \ t \in [0, T],$$

here $\delta_t : C([0, T], Y) \longrightarrow Y$ is such that $\delta_t(g) = g(t)$.
To the family $\mathcal{A}^\phi(\cdot)$ we associate the homogeneous Cauchy problem

$$(NCP) \begin{cases} \dfrac{d}{dt}\mathcal{U}(t) = \mathcal{A}^\phi(t)\mathcal{U}(t), \quad 0 \leq s \leq t \leq T, \\ \mathcal{U}(s) = \begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix}. \end{cases}$$

In the following proposition we give an equivalence between solutions of $(IBCP)$ and those of $(NCP)$.

**Proposition 2.1.** *Let $\begin{pmatrix} u_0 \\ f \end{pmatrix} \in D \times C^1([0, T], Y)$.*
*(i) If the function $t \longmapsto \mathcal{U}(t) := \begin{pmatrix} u_1(t) \\ u_2(t) \\ 0 \end{pmatrix}$ is a classical solution of $(NCP)$ with initial value $\begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix}$. Then $t \longmapsto u_1(t)$ is a classical solution of $(IBCP)$ with initial value $u_0$.*
*(ii) Let $u$ be a classical solution of $(IBCP)$ with initial value $u_0$. Then, the function $t \longmapsto \mathcal{U}(t) = \begin{pmatrix} u(t) \\ f \\ 0 \end{pmatrix}$ is a classical solution of $(NCP)$ with initial value $\begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix}$.*

**Proof..** *i*) Since $\mathcal{U}$ is a classical solution, then, from Definition 2.1, $u_1$ is continuously differentiable and $u_1(t) \in D$, for $t \in [s,T]$. Moreover we have

$$\begin{aligned}\mathcal{U}'(t) &= \begin{pmatrix} u_1'(t) \\ u_2'(t) \\ 0 \end{pmatrix} \\ &= \mathcal{A}^{\phi}(t)\mathcal{U}(t) \\ &= \begin{pmatrix} A(t)u_1(t) \\ 0 \\ L(t)u_1(t)-\phi(t)u_1(t)-\delta_t u_2(t) \end{pmatrix}. \qquad (2.1)\end{aligned}$$

Therefore

$$u_1'(t) = A(t)u_1(t) \text{ and } u_2'(t) = 0.$$

This implies that $u_2(t) = u_2(s) = f, \forall t \in [s,T]$, hence the equation (2.1) yields to

$$L(t)u_1(t) = \phi(t)u_1(t) + f(t),\ 0 \leq s \leq t \leq T.$$

The initial value condition is obvious.
The assertion $(ii)$ is obvious. □

Now we return to the study of the Cauchy problem $(NCP)$. For that aim, we recall the following result.

**Theorem 2.1.** *([11], Theorem 1.3) Let $(A(t))_{0\leq t\leq T}$ be a stable family of linear operators on a Banach space $X$ such that*
*i) the domain $D := (D(A(t)), \|\cdot\|_D)$ is a Banach space independent of $t$,*
*ii) the mapping $t \longmapsto A(t)x$ is continuously differentiable in $X$ for every $x \in D$. Then there is an evolution family $(U(t,s))_{0\leq s\leq t\leq T}$ on $\overline{D}$. Moreover $U(t,s)$ has the following properties :*
**(1)** *$U(t,s)D(s) \subset D(t)$ for all $0 \leq s \leq t \leq T$, where $D(r)$ is defined by*

$$D(r) := \left\{x \in D : A(r)x \in \overline{D}\right\},$$

**(2)** *the mapping $t \longmapsto U(t,s)x$ is continuously differentiable in $X$ on $[s,T]$ and*

$$\frac{d}{dt}U(t,s)x = A(t)U(t,s)x \text{ for all } x \in D(s) \text{ and } t \in [s,T].$$

In order to apply Theorem 2.1, we need the following lemma.

**Lemma 2.2.** *The family of operators $(\mathcal{A}^{\phi}(t))_{0\leq t\leq T}$ is stable.*

**Proof..** For $t \in [0,T]$, we write $\mathcal{A}^{\phi}(t)$ as

$$\mathcal{A}^{\phi}(t) = \mathcal{A}(t) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\phi(t) & -\delta_t & 0 \end{pmatrix},$$

where $\mathcal{A}(t) = \begin{pmatrix} A(t) & 0 & 0 \\ 0 & 0 & 0 \\ L(t) & 0 & 0 \end{pmatrix}$, with domain $D(\mathcal{A}(t)) = D(\mathcal{A}^{\phi}(t))$.

Since $\mathcal{A}^{\phi}(t)$ is a perturbation of $\mathcal{A}(t)$ by a linear bounded operator on $E$, hence, in view of a perturbation result ([10], Thm. 5.2.3) it is sufficient to show the stability of $(\mathcal{A}(t))_{0\leq t\leq T}$.

Let $\lambda > \omega_0$ and $\begin{pmatrix} x \\ f \\ y \end{pmatrix}$, we have

$$(\lambda-\mathcal{A}(t))\begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ f \\ y \end{pmatrix} = \begin{pmatrix} (\lambda - A(t))R(\lambda, A^0(t))x - (\lambda - A(t))L_{\lambda,t}y \\ f \\ -L(t)R(\lambda, A^0(t))x + L(t)L_{\lambda,t}y \end{pmatrix}$$

Since $R(\lambda, A^0(t))x \in D(A^0(t)) = ker(L(t))$, $L_{\lambda,t}y \in ker(\lambda - A(t))$ and $L(t)L_{\lambda,t} = I_{d_Y}$, we obtain

$$(\lambda-\mathcal{A}(t))\begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix} = I_{d_E}. \qquad (2.2)$$

On the other hand, for $\begin{pmatrix} x \\ f \\ 0 \end{pmatrix} \in D(\mathcal{A}(t))$, we have

$$\begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}(\lambda-\mathcal{A}(t))\begin{pmatrix} x \\ f \\ 0 \end{pmatrix} = \begin{pmatrix} R(\lambda, A^0(t))(\lambda - A(t))x + L_{\lambda,t}L(t)x \\ f \\ 0 \end{pmatrix}.$$

¿From Lemma 2.1, let $x_1 \in D(A^0(t))$ and $x_2 \in ker(\lambda - A(t))$ such that $x = x_1 + x_2$. Then

$$\begin{aligned} R(\lambda, A^0(t))(\lambda - A(t))x + L_{\lambda,t}L(t)x &= R(\lambda, A^0(t))(\lambda - A(t))(x_1 + x_2) \\ &\quad + L_{\lambda,t}L(t)(x_1 + x_2) \\ &= R(\lambda, A^0(t))(\lambda - A(t))x_1 + L_{\lambda,t}L(t)x_2 \\ &= x_1 + x_2 \\ &= x. \end{aligned}$$

As a consequence, we get

$$\begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}(\lambda-\mathcal{A}(t)) = I_{D(\mathcal{A}(t))}.$$

¿From (2.2) and (2.3) , we obtain that the resolvent of $\mathcal{A}(t)$ is given by

$$R(\lambda, \mathcal{A}(t)) = \begin{pmatrix} R(\lambda, A^0(t)) & 0 & -L_{\lambda,t} \\ 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, by a direct computation one can obtain, for a finite sequence $0 \leq t_1 \leq ... \leq t_k \leq T$,

$$\prod_{i=1}^{k} R(\lambda, \mathcal{A}(t_i)) = \begin{pmatrix} \prod_{i=1}^{k} R(\lambda, A^0(t_i)) & 0 & -\prod_{i=2}^{k} R(\lambda, A^0(t_i))L_{\lambda,t_1} \\ 0 & \frac{1}{\lambda^k} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

¿From the hypothesis $(H_5)$, we conclude that $||L_{\lambda,t}|| \leq \gamma(\lambda-\omega)^{-1}$ for all $t \in [0,T]$ and $\lambda > \omega$. Define $\omega_1 = \max(\omega_0, \omega)$. Therefore, by using $(H_2)$, we obtain for $\begin{pmatrix} x \\ f \\ y \end{pmatrix} \in E$

$$\begin{aligned}\left\| \prod_{i=1}^{k} R(\lambda, \mathcal{A}(t_i)) \begin{pmatrix} x \\ f \\ y \end{pmatrix} \right\| &= \left\| \prod_{i=1}^{k} R(\lambda, A^0(t_i))x - \prod_{i=2}^{k} R(\lambda, A^0(t_i)) L_{\lambda,t_1} y + \frac{1}{\lambda^k} f \right\| \\ &\leq M(\lambda-\omega_1)^{-k}||x|| + M(\lambda-\omega_1)^{-(k-1)}\gamma(\lambda-\omega_1)^{-1}||y|| \\ &\quad + (\lambda-\omega_1)^{-k}||f|| \\ &\leq M^{'}(\lambda-\omega_1)^{-k} \left\| \begin{pmatrix} x \\ f \\ y \end{pmatrix} \right\|,\end{aligned}$$

where $M^{'} := \max(M, M\gamma)$. Thus the lemma is proved. □

Now we are ready to state the main result.

**Theorem 2.2.** *Let $f$ be a continuously differentiable function on $[0,T]$ onto $Y$. Then, for all initial value $u_0 \in D$, such that $L(s)u_0 = \phi(s)u_0 + f(s)$, the Cauchy problem $(IBCP)$ has a unique classical solution $u$. Moreover, $u$ is given by the variation of constants formula*

$$\begin{aligned} u(t) &= U(t,s)(I - L_{\lambda,s}L(s))u_0 + L_{\lambda,t} f(t, u(t)) \\ &\quad + \int_s^t U(t,r) \left[ \lambda L_{\lambda,r} f(r, u(r)) - (L_{\lambda,r} f(r, u(r)))^{'} \right] dr, \end{aligned} \tag{2.4}$$

*where $(U(t,s))_{t \geq s \geq 0}$ is the evolution family generated by $A^0(t)$ and $f(t, u(t)) := \phi(t)u(t) + f(t)$.*

**Proof.** First, for the existence of $U(t,s)$ we refer to [7]. Since $(\mathcal{A}^\phi(t))_{0 \leq t \leq T}$ is stable and from assumptions $(H_1)$, $(H_3)$ and $(H_4)$, $(\mathcal{A}^\phi(t))_{0 \leq t \leq T}$ satisfies all conditions of Theorem 2.1, then there exists an evolution family $\mathcal{U}^\phi(t,s)$ such that, for all initial value $\begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix} \in D(\mathcal{A}^\phi(s))$, the function $\begin{pmatrix} u_1(t) \\ u_2(t) \\ 0 \end{pmatrix} := \mathcal{U}^\phi(t,s) \begin{pmatrix} u_0 \\ f \\ 0 \end{pmatrix}$ is a classical solution of $(NCP)$. Therefore, from $(i)$ of Proposition 2.1, $u_1$ is a classical solution of $(IBCP)$. The uniqueness of $u_1$ comes from the uniqueness of the solution of (NCP) and Proposition 2.1.

Let $u$ be a classical solution of $(IBCP)$, at first, we assume that $\phi(t) = 0$, then

$$\begin{aligned} u_2(t) &:= L_{\lambda,t} L(t) u(t) \\ &= L_{\lambda,t} f(t), \end{aligned}$$

and $u_1(t) := (I - L_{\lambda,t}L(t))u(t)$ are differentiable on $t$ and we have

$$\begin{aligned} u_1'(t) &= u'(t) - u_2'(t) \\ &= A(t)(u_1(t) + u_2(t)) - (L_{\lambda,t} f(t))^{'} \\ &= A^0(t)u_1(t) + \lambda L_{\lambda,t} f(t) - (L_{\lambda,t} f(t))'. \end{aligned}$$

If we define $\tilde{f}(t) := \lambda L_{\lambda,t} f(t) - (L_{\lambda,t} f(t))^{'}$, we obtain

$$u_1(t) = U(t,s)u_1(s) + \int_s^t U(t,r)\tilde{f}(r)\,dr.$$

Replacing $u_1(s)$ by $(I - L_{\lambda,s}L(s))u_0$, we obtain

$$u_1(t) = U(t,s)(I - L_{\lambda,s}L(s))u_0 + \int_s^t U(t,r)\tilde{f}(r)\,dr,$$

consequently,

$$u(t) = U(t,s)(I - L_{\lambda,s}L(s))u_0 + L_{\lambda,t}f(t) + \int_s^t U(t,r)\left[\lambda L_{\lambda,r}f(r) - (L_{\lambda,r}f(r))^{'}\right]\,dr. \quad (2.5)$$

Now in the case $\Phi(t) \neq 0$, since $f(\cdot, u(\cdot))$ is continuously differentiable, similar arguments are used to obtain the formula (2.5) for $f(\cdot) := f(\cdot, u(\cdot))$ which is exactly (2.4). $\square$

## 3 Retarded equation

On the Banach space $C_E^1 := C^1([-r,0],E)$, where $r > 0$ and $E$ is a Banach space, we consider the retarded equation

$$(RE)\begin{cases} v(t) = K(t)v_t + f(t), \quad 0 \leq s \leq t \leq T, \\ v_s = \varphi \in C_E^1. \end{cases}$$

Where $v_t(\tau) := v(t+\tau)$, for $\tau \in [-r,0]$, and $f : [0,T] \longrightarrow E$.

**Definition 3.1.** *A function $v : [s-r,T] \longrightarrow E$ is said a solution of $(RE)$, if it is continuously differentiable, $K(t)v_t$ is well defined, $\forall t \in [0,T]$ and $v$ satisfies $(RE)$.*

In this section we are interested in the study of the retarded equation $(RE)$, we will apply the abstract result obtained in the previous section in order to get a solution of $(RE)$. As a first step, we show that this problem can be written as a boundary Cauchy one. More precisely, we show in the following theorem that solutions of $(RE)$ are equivalent to those of the boundary Cauchy problem

$$(IBCP)'\begin{cases} \dfrac{d}{dt}u(t) = A(t)u(t), \quad 0 \leq s \leq t \leq T, & (3.1) \\ L(t)u(t) = \phi(t)u(t) + f(t), & (3.2) \\ u(s) = \varphi. & (3.3) \end{cases}$$

Where $A(t)$ is defined by

$$\begin{cases} A(t)u := u' \\ D := D(A(t)) = C^1([-r,0],E), \end{cases}$$

$L(t) : D \longrightarrow E : L(t)\varphi = \varphi(0)$ and $\phi(t) : C([-r,0],E) \longrightarrow E : \phi(t)\varphi = K(t)\varphi$.
Note that the spaces $D, X$ and $Y$ in section 2, are given here by $D := C^1([-r,0],E)$, $X := C([-r,0],E)$ and $Y := E$.
We have the following theorem

**Theorem.** *i) If $u$ is a classical solution of $(IBCP)'$, then the function $v$ defined by*

$$v(t) := \begin{cases} u(t)(0), & s \leq t \leq T, \\ \varphi(t-s), & -r+s \leq t \leq s, \end{cases}$$

*is a solution of $(RE)$.*
*ii) If $v$ is a solution of $(RE)$, then $t \longmapsto u(t) := v_t$ is a classical solution of $(IBCP)'$.*

**Proof.** *i)* Let $u$ be a classical solution of $(IBCP)'$, then from Definition 2.1, $v$ is continuously differentiable. On the other hand, (3.1) and (3.3) implies that $u$ verifies the translation property

$$u(t)(\tau) = \begin{cases} u(t+\tau)(0), & s \leq t+\tau \leq T \\ \varphi(t+\tau-s), & -r+s \leq t+\tau \leq s, \end{cases}$$

which implies that $v_t(\cdot) = u(t)(\cdot)$. Therefore, from (3.2), we obtain

$$\begin{aligned} v(t) &= u(t)(0) \\ &= L(t)u(t)(\cdot) + f(t) \\ &= K(t)u(t)(\cdot) + f(t) \\ &= K(t)v_t(\cdot) + f(t). \end{aligned}$$

Hence $v$ satisfies $(RE)$.
*ii)* Now, let $v$ be a solution of $(RE)$. From Definition 3.1, $u(t) \in C^1([-r,0],E) = D(A(t))$, for $s \leq t \leq T$. Moreover,

$$\begin{aligned} L(t)u(t) &= u(t)(0) \\ &= v(t) \\ &= K(t)v(t) + f(t) \\ &= \phi(t)u(t) + f(t). \end{aligned}$$

The equation (3.1) is obvious. $\square$

This theorem allows us to concentrate our self on the problem $(IBCP)'$. So, it remains to show that the hypotheses $(H_1) - (H_5)$ are satisfied.
The hypotheses $(H_1)$, $(H_3)$ are obvious and $(H_4)$ can be obtained from the assumptions on the operator $K(t)$.
For $(H_2)$, let $\varphi \in D(A^0(t)) = \{\varphi \in C^1([-r,0],E) \,;\, \varphi(0) = 0\}$ and $f \in C([-r,0],E)$ such that $(\lambda - A^0(t))\varphi = f$, for $\lambda > 0$. Then

$$\varphi(\tau) = e^{\lambda\tau}\varphi(0) + \int_\tau^0 e^{\lambda(\tau-\sigma)} f(\sigma)\, d\sigma, \quad \tau \in [-r,0].$$

Since $\varphi(0) = 0$, we get

$$(R(\lambda, A^0(t))f)(\tau) = \int_\tau^0 e^{\lambda(\tau-\sigma)} f(\sigma)\, d\sigma.$$

By induction , we can show that

$$\left(\prod_{i=1}^{k} R(\lambda, A^0(t_i))f\right)(\tau) = \frac{1}{(k-1)!}\int_{\tau}^{0} (\sigma-\tau)^{k-1} e^{\lambda(\tau-\sigma)} f(\sigma)\, d\sigma,$$

for a finite sequence $0 \leq t_1 \leq ... \leq t_k \leq T$. Hence

$$\begin{aligned}\left\|\left(\prod_{i=1}^{k} R(\lambda, A^0(t_i))f\right)(\tau)\right\| &\leq \frac{1}{(k-1)!}\int_{\tau}^{0} (\sigma-\tau)^{k-1} e^{\lambda(\tau-\sigma)}\, d\sigma \|f\| \\ &= e^{\lambda\tau}\sum_{i=k}^{\infty} \frac{\lambda^{i-k}(-\tau)^i}{i!}\|f\| \\ &= \frac{e^{\lambda\tau}}{\lambda^k}\sum_{i=k}^{\infty} \frac{(-\lambda\tau)^i}{i!}\|f\| \\ &\leq \frac{1}{\lambda^k}\|f\|, \quad \text{for } \tau \in [-r, 0].\end{aligned}$$

Therefore

$$\left\|\prod_{i=1}^{k} R(\lambda, A^0(t_i))f\right\| \leq \frac{1}{\lambda^k}\|f\|, \quad \lambda > 0.$$

This proves the stability of $A^0(t))_{t\in[0,T]}$.
Now, if $\varphi \in ker(\lambda - A(t))$, then $\varphi(\tau) = e^{\lambda\tau}\varphi(0)$, for $\tau \in [-r, 0]$. Hence

$$\begin{aligned}\|L(t)\varphi\| &= \|\varphi(0)\| \\ &= \|e^{-\lambda\tau}\varphi(\tau)\|,\end{aligned}$$

since $\lim\limits_{\lambda\to+\infty} \dfrac{e^{-\lambda\cdot}}{\lambda} = +\infty$, in $C_E$, we can take $c > 0$ such that $\frac{e^{-\lambda\cdot}}{\lambda} \geq c$, therefore

$$\|L(t)\varphi\| \geq c\lambda\|\varphi\|, \ \forall t \in [0, T].$$

So $(H_5)$ holds. As a conclusion, we get the following corollary

**Corollary 3.1.** *Let $f$ be a continuously differentiable function from $[0, T]$ onto $E$, then for all $\varphi \in C_E^1$ such that, $\varphi(0) = K(s)\varphi + f(s)$, the retarded equation $(RE)$ has a solution $v$, moreover, $v$ satisfies*

$$v_t = T(t-s)(\varphi - e^{\lambda\cdot}\varphi(0)) + e^{\lambda\cdot} f(t, v_t) + \int_s^t T(t-r)e^{\lambda\cdot}\left[\lambda f(r, v_r) - (f(r, v_r))'\right] dr,$$

*where $(T(t))_{t\geq 0}$ is the $c_0$-semigroup generated by $A^0(t)$.*

## References


1. K.J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations, Graduate Texts in Mathematics* **194**, Springer-Verlag, 2000.
2. G. Greiner, *Perturbing the boundary conditions of a generator*, Houston J. Math. vol 13, 213-229 (1987).

3. G. Greiner, *Semilinear boundary conditions for evolution equations of hyperbolic type*, Lecture Notes in Pure and Appl. Math. vol 116 Dekker, New York, 127-153 (1989).
4. R. Grimmer and E. Sinestrari, *Maximum norm in one-dimensional hyperbolic problems*, Diff. Integ. Equat. vol 5, 421-432 (1992).
5. T. Kato,, *Linear evolution equations of "hyperbolic" type*, J. Fac. Sci. Tokyo. vol 17, 241-258 (1970).
6. H. Kellermann, *Linear evolution equations with time dependent domain*, Tübingerbericht Funktionalanalysis, 15-44 (1986-1987).
7. N. T. Lan, *On nonautonomous Functional Differential Equations*, J. Math. Anal. Appl. vol 239, 158-174 (1999).
8. N. T. Lan, G. Nickel, *Time-Dependent operator matrices and inhomogeneous Cauchy Problems*, Rendiconti Del circolo matematico di Palermo. XLVII, 5-24 (1998).
9. R. Nagel and E. Sinestrari, *Inhomogeneous Volterra integrodifferential equations for Hille-Yosida operators*, Marcel Dekker, Lecture Notes Pure Appl. Math. vol 150, 51-70 (1994).
10. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, New york Springer, 1983.
11. N. Tanaka, *Quasilinear evolution equations with non-densely defined operators*, Diff. Integral Eq. vol 9, 1067-1106 (1996).