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FIXED POINTS FOR NEAR-CONTRACTIVE TYPE MULTIVALUED MAPPINGS

Abderrahim Mbarki

Abstract. In the present paper we prove some fixed point theorems for near-contractive type multivalued mappings in complete metric spaces. these theorems extend some results in [1], [5], [6] and others

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1 Basic Preliminaries

Let (X, d) be a metric space we put:

$$CB = \{A : \text{ A is a nonempty closed and bounded subset of X }\}$$
$$BN = \{A : \text{ A is a nonempty bounded subset of X }\}$$

If A, B are any nonempty subsets of X we put:

$$D(A,B) = \inf\{d(a,b) : a \in A, b \in B\},$$
$$\delta(A,B) = \sup\{d(a,b) : a \in A, b \in B\},$$
$$H(A,B) = \max\{\ \{\sup\{D(a,B) : a \in A\}, \sup\{D(b,A) : b \in B\}\ \}.$$

If follows immediately from the definitoin that

$$\delta(A,B) = 0 \text{ iff } A = B = \{a\},$$
$$H(a,B) = \delta(a,B),$$
$$\delta(A,A) = diam A,$$
$$\delta(A,B) \leq \delta(A,C) + \delta(A,C),$$
$$D(a,A) = 0 \text{ if } a \in A,$$

for all A, B, C in $BN(X)$ and a in X.

In general both H and δ may be infinite. But on $BN(X)$ they are finite. Moreover, on $CB(X)$ H is actually a metric (the Hansdorff metric).

Department of Mathematics, Faculty of Science, Oujda University, Morocco
mbarki@sciences.univ-oujda.ac.ma

Definition 1.1. [2] *A sequence $\{A_n\}$ of subsets of X is said to be convergent to a subset A of X if*

(i) *given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$ for $n = 1, 2, ...,$ and $\{a_n\}$ converges to a*

(ii) *given $\varepsilon > 0$ there exists a positive integer N such that $A_n \subseteq A_\varepsilon$ for $n > N$ where A_ε is the union of all open spheres with centers in A and radius ε*

Lemma 1.1. [2,3].*If $\{A_n\}$ and $\{B_n\}$ are seqences in $BN(X)$ converging to A and B in $BN(X)$ respectively, then the sequence $\{\delta(A_n, B_n)\}$ converges to $\delta(A, B)$.*

Lemma 1.2. [3] *Let $\{A_n\}$ be a sequence in $BN(X)$ and x be a point of X such that $\delta(A_n, x) \to 0$. Then the sequence $\{A_n\}$ converges to the set $\{x\}$ in $BN(X)$.*

Definition 1.2. [3] *A set-valued mapping F of X into $BN(X)$ is said to be continuous at $x \in X$ if the sequence $\{Fx_n\}$ in $BN(X)$ converges to Fx whenever $\{x_n\}$ is a sequence in X converging to x in X. F is said continuous on X if it is continuous at every point of X.*

The following Lemma was proved in [3]

Lemma 1.3. *Let $\{A_n\}$ be a sequence in $BN(X)$ and x be a point of X such that*

$$\lim_{n\to\infty} a_n = x,$$

x being independent of the particular choice of $a_n \in A_n$. If a selfmap I of X is continuous, then Ix is the limit of the sequence $\{IA_n\}$.

Definition 1.3. [4]. *The mappings $I : X \to X$ and $F : X \to BN(X)$ are δ-compatible if $\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $IFx_n \in BN(X)$,*

$$Fx_n \to t \quad \text{and} \quad Ix_n \to t$$

for some t in X.

2. Our Results

We establish the following:

2. 1. A Coincidence Point Theorem

Theorem 2.1. *Let $I : X \to X$ and $T : X \to BN(X)$ be two mappings such that $FX \subset IX$ and*

$$(C.1) \quad \phi(\delta(Tx, Ty)) \leq a\phi(d(Ix, Iy)) + b[\phi(H(Ix, Tx)) + \phi(H(Iy, Ty))] \\ + c \min\{\phi(D(Iy, Tx)), \phi(D(Ix, Ty))\},$$

where $x, y \in X$, $\quad \phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \quad$ is continuous and strictly increasing such that $\phi(0) = 0$. a, b, c are nonnegative, $a + 2b < 1$ and $a + c < 1$. Suppose in addition that $\{F, I\}$ are δ-compatible and F or I is continuous. Then I and T have a unique common fixed point z in X and further $Tz = \{z\}$.

Proof. Let $x_0 \in X$ be an arbitrary point in X. Since $TX \subset IX$ we choose a point x_1 in X such that $Ix_1 \in Tx_0 = Y_0$ and for this point x_1 there exists a point x_2 in X such that $Ix_2 \in Tx_1 = Y_1$, and so on. Continuing in this manner we can define a sequence $\{x_n\}$ as follows:

$$Ix_{n+1} \in Tx_n = Y_n$$

For sinplicity, we can put $V_n = \delta(Y_n, Y_{n+1})$, for $n = 0, 1, 2, ...$ By $(C, 1)$ we have

$$\begin{aligned}\phi(V_n) &= \phi(\delta(Y_n, Y_{n+1})) = \phi(\delta(Tx_n, Tx_{n+1}))\\ &\le a\phi(d(Ix_n, Ix_{n+1})) + b[\phi(H(Ix_n, Tx_n)) + \phi(H(Ix_{n+1}, Tx_{n+1}))]\\ &+ c\min\{\phi(D(Ix_{n+1}, Tx_n)), \phi(D(Ix_n, Tx_{n+1}))\}\\ &\le A_1 + A_2 + A_3\end{aligned}$$

Where

$$\begin{aligned}A_1 &= a\phi(\delta(Y_{n-1}, Y_n))\\ A_2 &= b[\phi(\delta(Y_{n-1}, Y_n)) + \phi(\delta(Y_n, Y_{n+1}))],\\ A_3 &= c\phi(D(Ix_{n+1}, Y_n)).\end{aligned}$$

So

$$\phi(V_n) \le a\phi(V_{n-1}) + b[\phi(V_{n-1}) + \phi(V_n)]$$

Hence we have

$$\phi(V_n) \le \frac{a+b}{1-b}\phi(V_{n-1}) < \phi(V_{n-1}) \quad (1)$$

Since ϕ is increasing, $\{V_n\}$ is a decreasing sequence. Let $lim_n V_n = V$, assume that $V > 0$. By letting $n \to \infty$ in (1), Since ϕ is continuous , we have:

$$\phi(V) \le \frac{a+b}{1-b}\phi(V) < \phi(V),$$

which is contradiction , hence $V = 0$.

Let y_n be an arbitrary point in Y_n for $n = 0, 1, 2, ...$ Then

$$\lim_{n\to\infty} d(y_n, y_{n+1}) \le \lim_{n\to\infty} \delta(Y_n, Y_{n+1}) = 0.$$

Now, we wish to show that $\{y_n\}$ is a Cauchy sequence, we proceed by contradiction. Then there exist $\varepsilon > 0$ and two sequences of natural numbers $\{m(i)\}$, $\{n(i)\}$, $m(i) > n(i)$, $n(i) \to \infty$ as $i \to \infty$ such taht

$$\delta(Y_{n(i)}, Y_{m(i)}) > \varepsilon \quad \text{while} \quad \delta(Y_{n(i)}, Y_{m(i)-1}) \le \varepsilon$$

Then we have

$$\begin{aligned}\varepsilon < \delta(Y_{n(i)}, Y_{m(i)}) &\le \delta(Y_{n(i)}, Y_{m(i)-1}) + \delta(Y_{m(i)-1}, Y_{m(i)})\\ &\le \varepsilon + V_{m(i)-1},\end{aligned}$$

since $\{V_n\}$ converges to 0, $\delta(Y_{n(i)}, Y_{m(i)}) \to \varepsilon$. Futhermore, by triangular inequality, it follows that

$$\mid \delta(Y_{n(i)+1}, Y_{m(i)+1}) - \delta(Y_{n(i)}, Y_{m(i)}) \mid \le V_{n(i)} + V_{m(i)},$$

and therefore the sequence $\{\delta(Y_{n(i)+1}, Y_{m(i)+1})\}$ converges to ε

¿From (C. 2), we also deduce:

$$\begin{aligned}\phi(\delta(Y_{n(i)+1}, Y_{m(i)+1})) &= \phi(\delta(Tx_{n(i)+1}, Tx_{m(i)+1}))\\ &\leq C_1 + C_2 + C_3\\ &\leq C_4 + C_5 + C_6 \quad (4)\end{aligned}$$

Where

$$\begin{aligned}&C_1 = a\phi(d(Ix_{n(i)+1}, Ix_{m(i)+1})),\\ &C_2 = b\Big\{\phi(\delta(Ix_{n(i)+1}, Tx_{n(i)+1})) + \phi(\delta(Ix_{m(i)+1}, Tx_{m(i)+1}))\Big\},\\ &C_3 = cmin\{\phi(D(Ix_{n(i)+1}, Y_{m(i)+1}), \phi(D(Ix_{n(i)+1}, Y_{m(i)+1})\},\\ &C_4 = a\phi(\delta(Y_{n(i)}, Y_{m(i)})),\\ &C_5 = [\phi(V_{n(i)}) + \phi(V_{m(i)}],\\ &C_6 = c\phi(\delta(Y_{n(i)}, Y_{m(i)}) + V_{m(i)}).\end{aligned}$$

Letting $i \to \infty$ in (4), we have

$$\phi(\varepsilon) \leq (a + c)\phi(\varepsilon) < \phi(\varepsilon)$$

This is a contradiction. Hence $\{y_n\}$ is a Cauchy sequence in X and it has a limit y in X. So the sequence $\{Ix_n\}$ converge to y and further, the sequence $\{Tx_n\}$ converge to set $\{y\}$. Now supose that I is continuous. Then

$$I^2x_n \to Iy \quad \text{and} \quad ITx_n \to \{Iy\}$$

by Lemma 1.3. Since I and T are δ-compatible. Therefore $TIx_n \to \{Iy\}$. Using inequality $(C.1)$, we have

$$\begin{aligned}\phi(\delta(TIx_n, Tx_n)) \leq\ & a\phi(d(I^2x_n, Ix_n)) + b[\phi(H(Ix_n, Tx_n)) + \phi(H(I^2x_n, TIx_n))]\\ &+ cmin\{\phi(D(Ix_n, TIx_n)), \phi(D(I^2x_n, Tx_n))\},\end{aligned}$$

for $n \geq 0$. As $n \to \infty$ we obtain by Lemma 1.1

$$\phi(d(Iy, y)) \leq a\phi(d(Iy, y)) + c\phi(d(y, Iy)),$$

That is $\phi(d(Iy, y)) = 0$ which implies that $Iy = y$. Further

$$\begin{aligned}\phi(\delta(Ty, Tx_n)) \leq\ & a\phi(d(Iy, Ix_n)) + b[\phi(H(Iy, Ty)) + \phi(H(Ix_n, Tx_n))]\\ &+ cmin\{\phi(D(Ix_n, Ty)), \phi(D(Iy, Tx_n))\},\end{aligned}$$

for $n \geq 0$. As $n \to \infty$ we obtain by Lemma 1.1

$$\phi(\delta(Ty, y)) \leq b\phi(\delta(Ty, y)),$$

which implies that $Ty = y$. Thus y is a coincidence point for T and I. Now suppose that T and I have a second common fixed point z such that $Tz = \{z\} = \{Iz\}$. Then, using inequality $(C.1)$, we obtain

$$\phi(d(y, z)) = \phi(\delta(Ty, Tz)) \leq (a + c)\phi(d(z, y)) < \phi(d(z, y))$$

which is a contradiction. This completes the proof of the Theorem.

Corollary 2.1 ([6.*Theorem*2.1]). *Let (X, d) be a complete metric space, $T : X \longrightarrow CB(X)$ a multi-valued map satisfying the following condition :*

$$\phi(\delta(Tx, Ty)) \leq a\phi(d(x, y)) + b\Big[\phi(\delta(x, Tx)) + \phi(\delta(y, Ty))\Big] + \\ + c \min\Big\{\phi(d(x, Ty)), \phi(d(y, Tx))\Big\} \qquad \forall x, y \in X,$$

where $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$ and a, b, c are three positive constants such that $a + 2b < 1$ and $a + c < 1$, then T has a unique fixed point.

Note that the proof of Theorem 2.1 is another proof of Corollary 2.1 which is of interest in part because it avoids the use of Axiom of choice.

2. 2. A Fixed Point Theorem

Theorem 2.2. *Let (X, d) be a complete metric space. If $F : X \to CB(X)$ is a multi-valued mapping and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$. Furthermore, let a, b, c be three functions from $(0, \infty)$ into $[0, 1)$ such that*
$a + 2b : (0, \infty) \to [0, 1)$ *and* $a + c : (0, \infty) \to [0, 1)$ *are decreasing functions. Suppose that F satisfies the following condition:*

$$(C.3) \quad \phi(\delta(Fx, Fy)) \leq a(d(x, y))\phi(d(x, y)) + b(d(x, y))[\phi(H(x, Fx)) + \phi(H(y, Fy))] \\ + c(d(x, y)) \ min\{\phi(D(y, Fx)), \phi(D(x, Fy))\},$$

then F has a unique fixed point z in X such that $Fz = \{z\}$.

Proof.. First we will establish the existence of a fixed point. Put $p = max\{(a + 2b)^{\frac{1}{2}}, (a+c)^{\frac{1}{2}}\}$, take any x_o in X. Since we may assume that $D(x_0, Fx_0)$ is positive, we can choose $x_1 \in Fx_0$ which satisfies $\phi(d(x_0, x_1)) \geq p(D(x_0, Fx_0))\phi(H(x_0, Fx_0))$, we may assume that $p(d(x_0, x_1))$ is positive. Assuming now that $D(x_1, Fx_1)$ is positive, we choose $x_2 \in Fx_1$ such that $\phi(d(x_1, x_2)) \geq p(d(x_0, x_1))\phi(H(x_1, Fx_1))$ and $\phi(d(x_1, x_2)) \geq p(D(x_1, Fx_1))\phi(d(x_1, Fx_1))$, since $d(x_0, x_1) \geq D(x_0, Fx_0)$ and p is deceasing then we have also
$\phi(d(x_0, x_1)) \geq p(d(x_0, x_1))\phi(H(x_0, Fx_0))$. Now

$$\begin{aligned} \phi(d(x_1, x_2)) &\leq \phi(\delta(Fx_0, Fx_1)) \\ &\leq a(d(x_0, x_1))\phi(d(x_0, x_1)) + b(d(x_0, x_1))[\phi(H(x_0, Fx_0)) + \phi(H(x_1, Fx_1))] \\ &+ c(d(x_0, x_1)) \ min\{\phi(D(Fx_0, x_1)), \phi(D(x_0, Fx_1))\} \\ &\leq ap^{-1}\phi(d(x_0, x_1)) + bp^{-1}[\phi(d(x_0, x_1)) + \phi(d(x_1, x_2))], \end{aligned}$$

which implies

$$\phi(d(x_1, x_2)) \leq q(d(x_0, x_1))\phi(d(x_0, x_1))$$

where

$$q : (0, \infty) \to [0, 1)$$

is defined by

$$q = \frac{a + b}{p - b}.$$

Note that $r \geq t$ implies $q(r) \leq p(t) < 1$. By induction, assumunig that $D(x_i, Fx_i)$ and $p(d(x_{i-1}, x_i))$ are positive, we obtain a sequence $\{x_i\}$ which satisfies $x_i \in Fx_{i-1}$, $\phi(d(x_{i-1}, x_i)) \geq p(d(x_{i-1}, x_i))\phi(H(x_{i-1}, Fx_{i-1}))$,

$$\begin{aligned} \phi(d(x_i, x_{i+1})) &\geq p(d(x_{i-1}, x_i))\phi(H(x_i, Fx_i)), \\ \phi(d(x_i, x_{i+1})) &\leq q(d(x_{i-1}, x_i))\phi(d(x_{i-1}, x_i)) \\ &\leq p(d(x_{i-1}, x_i))\phi(d(x_{i-1}, x_i)) \\ &< \phi(d(x_{i-1}, x_i)). \end{aligned}$$

It is not difficult to verify that $lim_i\ d(x_i, x_{i+1}) = 0$. If $\{x_i\}$ is not Cauchy, there exists $\varepsilon > 0$ and two sequences of natural numbers $\{m(i)\}, \{n(i)\}$,

$m(i) > n(i) > i$ such that $d(x_{m(i)}, x_{n(i)}) > \varepsilon$ while $d(x_{m(i)-1}, x_{n(i)}) \leq \varepsilon$. It is not difficult to verify that

$$d(x_{m(i)}, x_{n(i)}) \to \varepsilon \ \text{ as } \ i \to \infty \ \text{ and } \ d(x_{m(i)+1}, x_{n(i)+1}) \to \varepsilon \ \text{ as } \ i \to \infty.$$

For i sufficiently large $d(x_{m(i)}, x_{m(i)+1}) < \varepsilon$ and $d(x_{n(i)}, x_{n(i)+1}) < \varepsilon$. For these i we have

$$\begin{aligned} \phi(d(x_{m(i)+1}, x_{n(i)+1})) &\leq \phi(\delta(Fx_{m(i)}, Fx_{n(i)})) \\ &\leq a(d(x_{m(i)}, x_{n(i)}))\phi(d(x_{m(i)}, x_{n(i)})) \\ &+ b(d(x_{m(i)}, x_{n(i)}))[\phi(H(x_{m(i)}, Fx_{m(i)})) + \phi(H(x_{n(i)}, Fx_{n(i)}))] \\ &+ c(d(x_{m(i)}, x_{n(i)}))\ min\{\phi(D(x_{m(i)}, Fx_{n(i)})), \phi(D(x_{n(i)}, Fx_{m(i)})\} \\ &\leq a(d(x_{m(i)}, x_{n(i)}))\phi(d(x_{m(i)}, x_{n(i)})) \\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{n(i)}, x_{n(i)+1}))\phi(d(x_{n(i)}, x_{n(i)+1})) \\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{m(i)}, x_{m(i)+1}))\phi(d(x_m(i), x_{m(i)+1})) \\ &+ c(d(x_{n(i)}, x_{m(i)})\phi(d(x_{m(i)}, x_{n(i)+1})) \\ &\leq a(d(x_{m(i)}, x_{n(i)}))\phi(d(x_{m(i)}, x_{n(i)+1}) + d(x_{n(i)+1}, x_{n(i)}) \\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{n(i)}, x_{n(i)+1}))\phi(d(x_{n(i)}, x_{n(i)+1})) \\ &+ b(d(x_{m(i)}, x_{n(i)}))p^{-1}(d(x_{m(i)}, x_{m(i)+1}))\phi(d(x_{m(i)}, x_{m(i)+1})) \\ &+ c(d(x_{n(i)}, x_{m(i)})\phi(d(x_{m(i)}, x_{n(i)+1} + d(x_{n(i)+1}, x_{n(i)})) \\ &\leq [a(\varepsilon) + c(\varepsilon)]\phi(d(x_{m(i)}, x_{n(i)}) + d(x_{n(i)}, x_{n(i)+1})) \\ &+ \phi(d(x_{m(i)}, x_{m(i)+1})) + \phi(d(x_{n(i)}, x_{n(i)+1})) \quad (*) \end{aligned}$$

Letting $i \to \infty$ in $(*)$, we have: $\phi(\varepsilon) \leq [a(\varepsilon) + c(\varepsilon)]\phi(\varepsilon) < \phi(\varepsilon)$. This is contradiction. Hence $\{x_i\}$ is cauchy sequence in a complete metric space X, then there existe a point $x \in X$ such that $x_n \to x$ as $i \to \infty$. This x is a fixed point of F because

$$\begin{aligned} \phi(H(x_{i+1}, Fx)) = \phi(\delta(x_{i+1}, Fx)) &\leq \phi(\delta(Fx_i, Fx)) \\ &\leq a(d(x_i, x))\phi(d(x_i, x)) \\ &+ b(d(x_i, x))[\phi(H(x, Fx)) + \phi(H(x_i, Fx_i))] \\ &+ c(d(x_i, x))\ min\{\phi(D(x_i, Fx)), \phi(D(x, Fx_i))\} \\ &\leq a(d(x_i, x))\phi(d(x_i, x)) \\ &+ b(d(x_i, x))p^{-1}(d(x_i, x_{i+1})\phi(d(x_i, x_{i+1})) \\ &+ b(d(x_i, x))\phi(H(x, Fx)) + c(d(x_i, x))\phi(d(x, x_{i+1}) \quad (**) \end{aligned}$$

Using $b < \frac{1}{2}$, $p^{-1}(d(x_i, x_{i+1})) < p^{-1}(d(x_0, x_1))$ and letting $i \to \infty$ in $(**)$, we have:

$$\phi(\delta(x, Fx)) \leq \frac{1}{2}\phi(H(x, Fx)).$$

That is $\phi(H(x, Fx)) = 0$ and therefore $H(x, Fx) = 0$ i.e, $Fx = x$. $Fx = \{x\}$. We claim that x is unique fixed point of F. For this, we suppose that y $(x \neq y)$ is another fixed point of F such that $Fy = \{y\}$. Then

$$\begin{aligned}\phi(d(y, x)) &\leq \phi(\delta(Fy, Fx)) \\ &\leq a\phi(d(x, y)) + b[\phi(H(x, Fx)) + \phi(H(y, Fy))] \\ &+ c\ min\{\phi(D(x, Fy)), \phi(D(y, Fx))\} \\ &\leq [a + c]\phi(d(x, y)) < \phi(d(x, y)),\end{aligned}$$

a contradiction. This completes the proof of the theorem.

We may establish a common fixed point theorem for a pair of mappings F and G which stisfying the contractive condition corresponding to $(C.1)$, i.e., for all $x, y \in X$

$$\begin{aligned}(C.2)\quad \phi(\delta(Fx, Gy)) &\leq a\phi(d(x, y)) + b[\phi(H(x, Fx)) + \phi(H(y, Gy))] \\ &+ c\ min\{\phi(D(y, Fx)), \phi(D(x, Gy))\},\end{aligned}$$

2. 3 A Common Fixed Point Theorem.

Theorem 2.3. *Let (X, d) be a metric space. Let F and G be two mappings of X into $BN(X)$ and $\phi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous and strictly increasing such that $\phi(0) = 0$. Furthermore, let a, b, c be three nonnegative constants such that $a+2b < 1$ and $a+c < 1$. Suppose that F and G satisfy $(C.2)$. Then F and G have a unique common fixed point. This fixed point satisfies $Fx = Gx = \{x\}$.*

Proof. Put $p = max\{(a+2b)^{\frac{1}{2}},\ c^{\frac{1}{2}}\}$. we may assume that is positive. We define by using the Axiom of choice the two single-valued functions $f, g : X \to X$ by letting $f(x)$ be a point $w_1 \in Fx$ and $g(x)$ be a point $w_2 \in Gx$ such that $\phi(d(x, w_1)) \geq p\phi(H(x, Fx))$ and $\phi(d(x, w_2)) \geq p\phi(H(x, Gx))$. Then for every $x, y \in X$ we have:

$$\begin{aligned}\phi(d(f(x), g(y))) \leq \phi(\delta(Fx, Gy)) &\leq a\phi(d(x, y)) + b[\phi(H(x, Fx)) + \phi(H(y, Gy))] \\ &+ c\ min\{\phi(D(y, Fx)), \phi(D(x, Gy))\} \\ &\leq a\phi(d(x, y)) + p^{-1}b[\phi(d(x, fx)) + \phi(d(y, gy))] \\ &+ c\ min\{\phi(d(y, fx)), \phi(d(x, gy))\}.\end{aligned}$$

Since $a+2p^{-1}b \leq p^{-1}(a+2b) \leq p < 1$, from [7, Theorem 2.1] we conclude that f and g has a common fixed point. That is, there exists a point x such that $0 = d(x, fx) = \phi(d(x, fx)) \geq p\phi(H(x, Fx))$ and $0 = d(x, gx) = \phi(d(x, gx)) \geq p\phi(H(x, Gx))$ which implies $\phi(H(x, Fx)) = 0$ and $\phi(H(x, Gx)) = 0$, then $H(x, Fx) = \delta(x, Fx) = 0$ and $H(x, Gx) = \delta(x, Gx) = 0$ i.e. $Fx = Gx = \{x\}$. Hence F and G have a common fixed point $x \in X$. We claim that x is unique common fixed point of F and G. For this, we suppose that y $(x \neq y$ $)$ is another fixed point of F and G. Since $y \in Fy$ and $y \in Gy$, from $(C.2)$ we have

$$\begin{aligned}\max\{\phi(H(y, Fy)), \phi(H(y, Gy))\} &\leq \phi(\delta(Fy, Gy)) \\ &\leq b[\phi(H(y, Fy)) + \phi(H(y, Gy))] \\ &\leq 2b \max\{\phi(H(y, Fy)), \phi(H(y, Gy))\}\end{aligned}$$

which implies $\delta(Fy, Gy) = 0$, that is $Fy = Gy = \{y\}$. Then

$$\begin{aligned}\phi(d(y,x)) &= \phi(\delta(Fy, Gx)) \\ &\leq a\phi(d(x,y)) + b[\phi(H(x,Gx)) + \phi(H(y,Fy))] \\ &+ c\min\{\phi(D(x,Fy)), \phi(D(y,Gx))\} \\ &\leq [a+c]\phi(d(x,y)) < \phi(d(x,y)),\end{aligned}$$

a contradiction. This completes the proof of the theorem.

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