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# CLASSES OF RUSCHEWEYH-TYPE ANALYTIC UNIVALENT FUNCTIONS

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Abstract. A class of univalent functions is defined by making use of the Ruscheweyh derivatives. This class provides an interesting transition from starlike functions to convex functions. In special cases it has close inter-relations with uniformly starlike and uniformly convex functions. We study the effects of certain integral transforms and convolutions on the functions in this class.

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## 1. Introduction

Let $A$ denote the family of functions $f$ that are analytic in the open unit disk $U = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Consider the subclass $T$ consisting of functions $f$ in $A$, which are univalent in $U$ and are of the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, where $a_n \geq 0$. Such functions were first studied by Silverman [6]. For $\alpha \geq 0,\ 0 \leq$ ¡1 and fixed $\lambda > -1$, we let $D(\alpha, \beta, \lambda)$ denote the set of all functions in $T$ for which

$$Re\left(\frac{z(D^\lambda f(z))'}{D^\lambda f(z)}\right) > \alpha \left|\frac{z(D^\lambda f(z))'}{D^\lambda f(z)} - 1\right| + \beta.$$

Here, the operator $D^\lambda f(z)$ is the Ruscheweyh derivative of $f$, (see [4]), given by

$$D^\lambda f(z) = \frac{z}{(1-z)^{1+\lambda}} * f(z) = z - \sum_{n=2}^{\infty} a_n B_n(\lambda) z^n,$$

where $*$ stands for the convolution or Hadamard product of two power series and

$$B_n(\lambda) = \frac{(\lambda+1)(\lambda+2) - (\lambda+n-1)}{(n-1)!}.$$

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The family $D(\alpha, \beta, \lambda)$, which has been studied in [5], is of special interest for it contains many well-known as well as new classes of analytic univalent functions. In particular, for $\alpha = 0$ and $0 \leq \lambda \leq 1$ it provides a transition from starlike functions to convex functions. More specifically, $D(0, \beta, 0)$ is the family of functions starlike of order $\beta$ and $D(0, \beta, 1)$ is the family of functions convex of order $\beta$. For $D(\alpha, 0, 0)$, we obtain the class of uniformly $\alpha$-starlike functions introduced by Kanas and Wisniowski [2], which can be generalized to $D(\alpha, \beta, 0)$, the class of uniformly $\alpha$-starlike functions of order $\beta$. Generally speaking, $D(\alpha, \beta, \lambda)$ consists of functions $F(z) = D^{\lambda} f(z)$ which are uniformly $\alpha$-starlike functions of order $\beta$ in$U$. In Section 2 we study the effects of certain integral operators on the class $D(\alpha, \beta, \lambda)$. Section 3 deals with the convolution properties of the class $D(\alpha, \beta, \lambda)$ in connection with Gaussian hypergeometric functions.

**2. Integral transform of the class $D(\alpha, \beta, \lambda)$.**

For $f \in A$ we define the integral transform

$$V_{\mu}(f)(z) = \int_0^1 \mu(t) \frac{f(tz)}{t} dt,$$

where $\mu$ is a real-valued, non-negative weight function normalized so that $\int_0^1 \mu(t) dt = 1$. Some special cases of $\mu(t)$ are particularly interesting such as $\mu(t) = (1+c)t^c, c > -1$, for which $V_{\mu}$ is known as the Bernardi operator, and

$$\mu(t) = \frac{(c+1)^{\delta}}{\mu(\delta)} \, t^c \left( \log \frac{1}{t} \right)^{\delta - 1}, \; c > -1, \delta \geq 0$$

which gives the Komatu operator. For more details see [3].

First we show that the class $D(\alpha, \beta, \lambda)$ is closed under $V_{\mu}(f)$.

**Theorem 1**. Let $f \in D(\alpha, \beta, \lambda)$. Then $V_{\mu}(f) \in D(\alpha, \beta, \lambda)$.

**Proof**. By definition, we have

$$\begin{aligned} V_{\mu}(f)(z) &= \frac{(c+1)^{\delta}}{\mu(\delta)} \int_0^1 (-1)^{\delta-1} t^c (\log t)^{\delta - 1} \left( z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right) dt \\ &= \frac{(-1)^{\delta-1}(c+1)^{\delta}}{\mu(\delta)} \lim_{\gamma \to 0^+} \left[ \int_{\gamma}^1 t^c (\log t)^{\delta-1} \left( z - \sum_{n=2}^{\infty} a_n z^n t^{n-1} \right) dt \right]. \end{aligned}$$

A simple calculation gives

$$V_{\mu}(f)(z) = z - \sum_{n=2}^{\infty} \left( \frac{c+1}{c+n} \right)^{\delta} a_n z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty} \frac{n(1+\alpha) - (\alpha + \beta)}{1 - \beta} \left( \frac{c+1}{c+n} \right)^{\delta} a_n B_n(\lambda) < 1. \tag{2.1}$$

On the other hand (see [5], Theorem 1), $f \in D(\alpha, \beta, \lambda)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n(1+\alpha) - (\alpha + \beta)}{1 - \beta} a_n B_n(\lambda) < 1.$$

Hence $\frac{c+1}{c+n} < 1$. Therefore (2.1) holds and the proof is complete.

The above theorem yields the following two special cases.

**Corollary 1**. If $f$ is starlike of order $\beta$ then $V_\mu(f)$ is also starlike of order $\beta$.

**Corollary 2**. If $f$ is convex of order $\beta$ then $V_\mu(f)$ is also convex of order $\beta$.

Next we provide a starlikeness condition for functions in $D(\alpha,\beta,\lambda)$ under $V_\mu(f)$.

**Theorem 2**. Let $f \in D(\alpha,\beta,\lambda)$. Then $V_\mu(f)$ is starlike of order $0 \le \gamma < 1$ in $|z| < R_1$ where

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^\delta \frac{(1-\gamma)[n(1+\alpha)-(\alpha+\beta)]}{(n-\gamma)(1-\beta)}\ B_n(\lambda)\right]^{\frac{1}{n-1}}.$$

**Proof**. It is sufficient to prove

$$\left|\frac{z(V_\mu(f)(z))'}{V_\mu(f)(z)} - 1\right| < 1-\gamma. \tag{2.2}$$

For the left hand side of (2.2) we have

$$\left|\frac{z(V_\mu(f)(z))'}{V_\mu(f)(z)} - 1\right| = \left|\frac{\sum_{n=2}^\infty (1-n)\left(\frac{c+1}{c+n}\right)^\delta a_n z^{n-1}}{1-\sum_{n=2}^\infty \left(\frac{c+1}{c+n}\right)^\delta a_n z^{n-1}}\right|$$

$$\le \frac{\sum_{n=2}^\infty (n-1)\left(\frac{c+1}{c+n}\right)^\delta a_n |z|^{n-1}}{1-\sum_{n=2}^\infty \left(\frac{c+1}{c+n}\right)^\delta a_n |z|^{n-1}}.$$

This last expression is less than $1-\gamma$ since.

$$|z|^{n-1} < \left(\frac{c+n}{c+1}\right)\frac{(1-\gamma)[n(1+\alpha)-(\alpha+\beta)]}{(n-\gamma)(1-\beta)} B_n(\lambda).$$

Therefore, the proof is complete.

Using the fact that $f$ is convex if and only if $zf'$ is starlike, we obtain the following

**Theorem 3**. Let $f \in D(\alpha,\beta,\lambda)$. Then $V_\mu(f)$ is convex of order $0 \le \gamma < 1$ in $|z| < R_2$ where

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^\delta \frac{(1-\gamma)[n(1+\alpha)-(\alpha+\beta)]}{n(n-\gamma)(1-\beta)} B_n(\lambda)\right]^{\frac{1}{n-1}}.$$

## 3. A Convolution Operator on $D(\alpha,\beta,\lambda)$.

Denote by $F_1(a,b,c;z)$ the usual Gaussian hypergeometric functions defined by

$$F_1(a,b,c;z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!} z^n,\ |z| < 1, \tag{3.1}$$

where

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)},\ c > b > 0 \textit{ and } c > a+b.$$

It is well known (see [1]) that under the condition $c > b > 0$ and $c > a + b$ we have

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

For every $f \in T$ we define the convolution operator $H_{a,b,c}(f)(z)$ as

$$H_{a,b,c}(f)(z) =_2 F_1(a, b, c; z) * f(z),$$

where ${}_2F_1(a, b, c; z)$ is the Gaussian hypergeometric function defined in (3.1). For determining the resultant of $H_{a,b,c}(f)(z)$ if we set

$$k = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}$$

then we have

$H_{a,b,c}(f)(z) = \left(z + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^{n+1}\right) * \left(z - \sum_{n=1}^{\infty} a_{n+1} z^{n+1}\right)$

$z - \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} a_{n+1} z^{n+1} = z - \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=1}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)\Gamma(n+1)} a_{n+1} z^{n+1}$

$= z - k \sum_{n=1}^{\infty} \left[a_{n+1} z^{n+1} \sum_{n=0}^{\infty} \left[\frac{(c-a)_m(1-a)_m}{(c-a-b+1)_m m!} \int_0^1 t^{b+n-1}(1-t)^{c-a-b+m} dt\right]\right]$

$= z + k \int_0^1 \frac{t^{b-1}}{t} (1-t)^{c-a-b} {}_2F_1(c-a, 1-a, c-a-b+1; 1-t) \left(-\sum_{n=1}^{\infty} a_{n+1} t^{n+1} z^{n+1}\right) dt$

$= z + k \int_0^1 t^{b-1}(1-t)^{c-a-b} {}_2F_1(c-a, 1-a, c-a-b+1; 1-t) \frac{f(tz)-tz}{t} dt$

$= z + k \int_0^1 t^{b-1}(1-t)^{c-a-b} {}_2F_1(c-a, 1-a, c-a-b+1; 1-t) \frac{f(tz)}{t} dt$

$-zk \int_0^1 t^{b-1}(1-t)^{c-a-b} {}_2F_1(c-a, 1-a, c-a-b+1; 1-t) dt.$

If we set

$$\mu(t) = kt^{b-1}(1-t)^{c-a-b} {}_2F_1(c-a, 1-a, c-a-b+1; 1-t) \tag{3.2}$$

then it is easy to see that $\int_0^1 \mu(t)dt = 1$. Consequently

$$H_{a,b,c}(f)(z) = \int_0^1 \mu(t) \frac{f(tz)}{t} dt,$$

where $\mu(t)$ is as in (3.2).

This paves the way to state and prove our next theorem.

**Theorem 4**. Let $f \in D(\alpha, \beta, \lambda)$. Then $H_{a,b,c}(f) \in D(\alpha, \beta, \lambda_1)$ where

$$\lambda_1 \le \inf_n \left[\frac{(c)_{n-1}(n-1)!}{(a)_{n-1}(b)_{n-1}} \; B_n(\lambda) - 1\right].$$

**Proof**. Since

$$H_{a,b,c}(f)(z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n z^n$$

we need to show that

$$\sum_{n=2}^{\infty} \frac{[n(1+\alpha) - (\alpha+\beta)](a)_{n-1}(b)_{n-1}}{(1-\beta)(c)_{n-1}(n-1)!} \; a_n B_n(\lambda_1) < 1. \tag{3.3}$$

The inequality (3.3) holds if

$$\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!}\ B_n(\lambda_1) < B_n(\lambda).$$

Therefore

$$\lambda_1 < \frac{(c)_{n-1}(n-1)!}{(a)_{n-1}(b)_{n-1}}\ B_n(\lambda) - 1$$

which completes the proof.

The starlikeness of the functions in $D(\alpha,\beta,\lambda)$ under $H_{a,b,c}$ is investigated in the following theorem.

**Theorem 5**. Let $f \in D(\alpha,\beta,\lambda)$. Then $H_{a,b,c}(f) \in S^*(\gamma)$ for $|z| < R$ and

$$R = \inf_n \left[\frac{[n(1+\alpha)-(\alpha+\beta)](1-\gamma)(c)_{n-1}(n-1)!}{(n-\gamma)(1-\beta)(a)_{n-1}(b)_{n-1}}\ B_n(\lambda)\right]^{\frac{1}{n-1}} .$$

**Proof**. We need to show that

$$\left|\frac{zH'_{a,b,c}(f)(z)}{H_{a,b,c}(f)(z)} - 1\right| < 1-\gamma. \tag{3.4}$$

For the left hand side of (3.4) we have

$$\left|\frac{zH'_{a,b,c}(f)(z)}{H_{a,b,c}(f)(z)} - 1\right| \leq \frac{\sum_{n=2}^{\infty} \frac{(n-1)(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(n-1)!} a_n |z|^{n-1}} .$$

This last expression is less than $1-\gamma$ if

$$\sum_{n=2}^{\infty} \frac{(n-\gamma)(a)_{n-1}(b)_{n-1}}{(1-\gamma)(c)_{n-1}(n-1)!}\ a_n |z|^{n-1} < 1. \tag{3.5}$$

Using the fact, see [1], that $f \in D(\alpha,\beta,\lambda)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta} a_n B_n(\lambda) < 1,$$

we can say (3.5) is true if

$$\frac{(n-\gamma)(a)_{n-1}(b)_{n-1}}{(1-\gamma)(c)_{n-1}(n-1)!}\ |z|^{n-1} < \frac{n(1+\alpha)-(\alpha+\beta)}{1-\beta}\ B_n(\lambda).$$

Or equivalently

$$|z|^{n-1} < \frac{[n(1+\alpha)-(\alpha+\beta)](1-\gamma)(c)_{n-1}(n-1)!}{(n-\gamma)(1-\beta)(a)_{n-1}(b)_{n-1}}\ B_n(\lambda)$$

which yields the starlikeness of the family $H_{a,b,c}(f)$.

For $\alpha = \lambda = 0$ and $\gamma = \beta$ we obtain the following

**Corollary 3**. Let $f \in S^*(\beta)$. Then $H_{a,b,c}(f) \in S^*(\beta)$ in $|z| < R_1$ for

$$R_1 = \inf_n \left[\frac{(c)_{n-1}(n-1)!}{(a)_{n-1}(b)_{n-1}}\right]^{\frac{1}{n-1}} .$$

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