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A CLASS OF RUSCHEWEYH - TYPE HARMONIC UNIVALENT FUNCTIONS WITH VARYING ARGUMENTS

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Abstract. A comprehensive class of complex-valued harmonic univalent functions with varying arguments defined by Ruscheweyh derivatives is introduced. Necessary and sufficient coefficient bounds are given for functions in this class to be starlike. Distortion bounds and extreme points are also obtained.

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1. Introduction

A continuous function $f = u + iv$ is a complex- valued harmonic function in a complex domain G if both u and v are real and harmonic in G. In any simply connected domain $D \subset G$ we can write $f = h + \overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|h'(z)| > |g'(z)|$ in D (see [2]).

Denote by H the family of functions $f = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disc $U = \{z : |z| < 1\}$ for which $f(0) = h(0) = 0 = f_z(0) - 1$. Thus for $f = h + \overline{g}$ in H we may express the analytic functions for h and g as

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \ \ g(z) = b_1 z + \sum_{m=2}^{\infty} b_m z^m \ \ (0 \le b_1 < 1). \tag{1}$$

Note that the family H of orientation preserving,normalized harmonic univalent functions reduces to S the class of normalized analytic univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero that is $g \equiv 0$.

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For $f = h + \overline{g}$ given by (1) and $n > -1$, we define the Ruscheweyh derivative of the harmonic function $f = h + \overline{g}$ in H by

$$D^n f(z) = D^n h(z) + \overline{D^n g(z)} \tag{2}$$

where D the Ruscheweyh derivative (see[5]) of a power series $\phi(z) = z + \sum\limits_{m=2}^{\infty} \phi_m z^m$ is given by

$$D^n \phi(z) = \frac{z}{(1-z)^{n+1}} * \phi(z) = z + \sum_{m=2}^{\infty} C(n,m)\phi_m z^m$$

where

$$C(n,m) = \frac{(n+1)_{m-1}}{(m-1)!} = \frac{(n+1)(n+2)\ldots(n+m-1)}{(m-1)!}.$$

The operator $*$ stands for the hadamard product or convolution product of two power series

$$\phi(z) = \sum_{m=1}^{\infty} \phi_m z^m \ \ and \ \ \psi(z) = \sum_{m=1}^{\infty} \psi_m z^m$$

defined by

$$(\phi * \psi)(z) = \phi(z) * \psi(z) = \sum_{m=1}^{\infty} \phi_m \psi_m z^m.$$

For fixed values of $n (n > -1)$, let $R_H(n,\alpha)$ denote the family of harmonic functions $f = h + \overline{g}$ of the form (1) such that

$$\frac{\partial}{\partial\theta}(\arg D^n f(z)) \geq \alpha, \ \ 0 \leq \alpha < 1, \ \ |z| = r < 1. \tag{3}$$

We also let $V_{\overline{H}}(n,\alpha) = R_H(n,\alpha) \cap V_H$, where V_H[3], the class of harmonic functions $f = h + \overline{g}$ for which h and g are of the form (1) and their exists ϕ so that , $\mod 2\pi$,

$$\beta_m + (m-1)\phi \equiv \pi, \ \ \delta_m + (m-1)\phi \equiv 0, \ \ m \geq 2, \tag{4}$$

where $\beta_m = \arg(a_m)$ and $\delta_m = \arg(b_m)$.

Note that $R_H(0,\alpha) = SH(\alpha)$ [4], is the class of orientation preserving harmonic univalent functions f which are starlike of order α in U, that is $\frac{\partial}{\partial\theta}(argf(re^{i\theta})) > \alpha$ where $z = re^{i\theta}$ in U. In [1] , it is proved that the coefficient condition

$$\sum_{m=2}^{\infty} m\left(|a_m| + |b_m|\right) \leq 1 - b_1$$

is sufficient for functions $f = h + \overline{g}$ and of the form (1) to be in $SH(0)$. Recently Jahangiri and Silverman [3] gave the sufficient and necessary conditions for functions $f = h + \overline{g}$ of the form (1) to be in $V_H(\alpha)$ where $0 \leq \alpha < 1$. Further note that if $n = 0$ and the co-analytic part of $f = h + \overline{g}$ is zero, then the class $V_{\overline{H}}(n,\alpha)$ reduces to the class studied in [6].

In this paper, we will give the sufficient condition for $f = h + \overline{g}$ given by (1) to be in the class $R_H(n, \alpha)$, and it is shown that these coefficient condition is also necessary for functions in the class $V_{\overline{H}}(n, \alpha)$. Finally we obtain distortion theorems and characterize the extreme points for functions in $V_{\overline{H}}(n, \alpha)$.

2.Coefficient Bounds

In our first theorem we obtain a sufficient coefficient bound for harmonic functions in $R_H(n, \alpha)$

Theorem 1. Let $f = h + \overline{g}$ given by (1). If

$$\sum_{m=2}^{\infty}\left(\frac{m-\alpha}{1-\alpha}|a_m| + \frac{m+\alpha}{1-\alpha}|b_m|\right)C(n,m) \leq 1 - \frac{1+\alpha}{1-\alpha}b_1 \tag{5}$$

where $a_1 = 1$ and $0 \leq \alpha \leq 1$, then $f \in R_H(n, \alpha)$.

Proof. To prove $f \in R_H(n, \alpha)$, by definition of $R_H(n, \alpha)$ we only need to show that if (5) holds then the required condition (3) satisfied. For (3) we can write

$$\frac{\partial}{\partial\theta}(\arg D^n f(z)) = Re\left\{\frac{z(D^n h(z))' - \overline{z(D^n g(z))'}}{D^n h(z) - D^n g(z)}\right\} = Re\ \frac{A(z)}{B(z)}.$$

Using the fact that $Re\ w \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0. \tag{6}$$

Substituting for $A(z)$ and $B(z)$ in (6), which yields

$$\begin{aligned}
&|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \\
\geq\ & (2-\alpha)|z| - \sum_{m=2}^{\infty}[mC(n,m) + (1-\alpha)C(n,m)]|a_m||z|^m \\
&\quad - \sum_{m=1}^{\infty}[mC(n,m) - (1-\alpha)C(n,m)]|\overline{b}_m|\ |z|^m - \alpha|z| \\
&\quad - \sum_{m=2}^{\infty}[mC(n,m) - (1+\alpha)C(n,m)]|a_m||z|^m \\
&\quad - \sum_{m=1}^{\infty}[mC(n,m) + (1+\alpha)C(n,m)]|\overline{b}_m|\ |z|^m \\
\geq\ & 2(1-\alpha)|z|\left\{1 - \sum_{m=2}^{\infty}\tfrac{m-\alpha}{1-\alpha}|a_m||z|^{m-1}C(n,m) - \sum_{m=1}^{\infty}\tfrac{m+\alpha}{1-\alpha}|b_m||z|^{m-1}C(n,m)\right\} \\
\geq\ & 2(1-\alpha)|z|\left\{1 - \frac{1+\alpha}{1-\alpha}b_1 - \left(\sum_{m=2}^{\infty}\frac{m-\alpha}{1-\alpha}C(n,m)|a_m| + \sum_{m=2}^{\infty}\frac{m+\alpha}{1-\alpha}C(n,m)|b_m|\right)\right\}.
\end{aligned} \tag{7}$$

The last expression is non negative by (5), and so $f \in R_H(n, \alpha)$.

Now we obtain the necessary and sufficient conditions for function $f = h + \overline{g}$ be given with (4).

Theorem 2. Let $f = h + \overline{g}$ be given by (1). Then $f \in V_{\overline{H}}(n, \alpha)$ if and only if

$$\sum_{m=2}^{\infty} \left\{ \frac{m-\alpha}{1-\alpha}|a_m| + \frac{m+\alpha}{1-\alpha}|b_m| \right\} C(n,m) \leq 1 - \frac{1+\alpha}{1-\alpha} b_1 \tag{8}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$.

Proof. Since $V_{\overline{H}}(n,\alpha) \subset R_H(n,\alpha)$, we only need to prove the "only if" part of the theorem. To this end, for functions $f \in V_{\overline{H}}(n,\alpha)$, we notice that the condition $\frac{\partial}{\partial\theta}(\arg D^n f(z)) \geq \alpha$ is equivalent to

$$\frac{\partial}{\partial\theta}(\arg D^n f(z)) - \alpha = Re\left\{ \frac{z(D^n h(z))' - \overline{z(D^n g(z))'}}{D^n h(z) - D^n g(z)} - \alpha \right\} \geq 0.$$

That is

$$Re\left[\frac{(1-\alpha)z + \left(\sum\limits_{m=2}^{\infty} (m-\alpha)C(n,m)|a_m|z^m - \sum\limits_{m=1}^{\infty} (m+\alpha)C(n,m)|b_m|\overline{z}^m \right)}{z + \sum\limits_{m=2}^{\infty} C(n,m)|a_m|z^m + \sum\limits_{m=1}^{\infty} C(n,m)|b_m|\overline{z}^m} \right] \geq 0. \tag{9}$$

The above condition must hold for all values of z in U. Upon choosing ϕ according to (4) we must have

$$\frac{(1-\alpha) - (1+\alpha)b_1 - \left(\sum\limits_{m=2}^{\infty} (m-\alpha)C(n,m)|a_m|r^{m-1} + \sum\limits_{m=2}^{\infty} (m+\alpha)C(n,m)|b_m|r^{m-1} \right)}{1 + |b_1| + \left(\sum\limits_{m=2}^{\infty} C(n,m)|a_m| + \sum\limits_{m=2}^{\infty} C(n,m)|b_m| \right) r^{m-1}} \geq 0. \tag{10}$$

If the condition (8) does not hold then the numerator in (10) is negative for r sufficiently close to 1. Hence there exist a $z_0 = r_0$ in (0,1) for which quotient of (10) is negative. This contradicts the fact $f \in V_{\overline{H}}(n,\alpha)$ and so proof is complete.

Corollary 1. A necessary and sufficient condition for $f = h + \overline{g}$ satisfying (8) to be starlike is that $\arg(a_m) = \pi - 2(m-1)\pi/k$, and $\arg(b_m) = 2\pi - 2(m-1)\pi/k$, $(k = 1, 2, 3, \dots)$.

Our next theorem on distortion bounds for functions in $V_{\overline{H}}(n,\alpha)$ which yields a covering result for the family $V_{\overline{H}}(n,\alpha)$.

Theorem 3. If $f \in V_{\overline{H}}(n,\alpha)$ then

$$|f(z)| \leq (1+|b_1|)r + \frac{1}{C(n,2)} \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|b_1| \right) r^2, \ |z| = r < 1$$

and

$$|f(z)| \geq (1+|b_1|)r - \frac{1}{C(n,2)} \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2+\alpha}|b_1| \right) r^2, \ |z| = r < 1. \tag{11}$$

Proof. We will only prove the right hand inequality in (11). The argument for the left hand inequality is similar. Let $f \in V_{\overline{H}}(n,\alpha)$ taking the absolute value of f, we obtain

$$\begin{aligned} |f(z)| &\le ((1+|b_1|)|r| + \sum_{m=2}^{\infty}(|a_m|+|b_m|)|r|^m \\ &\le \ (1+b_1)r + r^2\sum_{m=2}^{\infty}(|a_m|+|b_m| \end{aligned}$$

That is

$$\begin{aligned} |f(z)| &\le (1+|b_1|)r + \frac{1-\alpha}{C(n,2)(2-\alpha)}\left(\sum_{m=2}^{\infty}\frac{(2-\alpha)C(n,2)}{1-\alpha}|a_m| + \frac{(2-\alpha)C(n,2)}{1-\alpha}|b_m|\right)r^2 \\ &\le \ (1+|b_1|)r + \frac{1-\alpha}{C(n,2)(2-\alpha)}\left[1-\frac{1+\alpha}{1-\alpha}|b_1|\right]r^2 \\ &\le \ (1+|b_1|)r + \frac{1}{C(n,2)}\left(\frac{1-\alpha}{2-\alpha}-\frac{1+\alpha}{2-\alpha}|b_1|\right)r^2. \end{aligned}$$

Corollary 2. Let f of the form (1) be so that $f \in V_{\overline{H}}(n,\alpha)$. Then

$$\left\{w : |w| < \frac{2C(n,2)-1-[C(n,2)-1]\alpha}{(2-\alpha)C(n,2)} - \frac{2C(n,2)-1-[C(n,2)-1]\alpha}{(2+\alpha)C(n,2)}b_1\right\} \subset f(U). \tag{12}$$

We use the coefficient bounds to examine the extreme points for $V_{\overline{H}}(n,\alpha)$ and determine extreme points of $V_{\overline{H}}(n,\alpha)$.

Theorem 4. Set $\lambda_m = \frac{(1-\alpha)}{(m-\alpha)C(n,m)}$ and $\mu_m = \frac{1+\alpha}{(m+\alpha)C(n,m)}$. For b_1 fixed, the extreme points for $V_{\overline{H}}(n,\alpha)$ are

$$\{z+\lambda_m x z^m + \overline{b_1 z}\} \cup \{z + \overline{b_1 z + \mu_m x z^m}\} \tag{13}$$

where $m \ge 2$ and $|x| = 1-|b_1|$.

Proof. Any function f in $V_{\overline{H}}(n,\alpha)$ may expressed as

$$f(z) = z + \sum_{m=2}^{\infty}|a_m|e^{i\beta_m}z^m + \overline{b_1 z} + \overline{\sum_{m=2}^{\infty}|b_m|e^{i\delta_m}z^m},$$

where the coefficients satisfy the inequality (5). Set

$$h_1(z) = z,\ g_1(z) = b_1 z,\ h_m(z) = z+\lambda_m e^{i\beta_m}z^m,\ g_m(z) = b_1 z+\mu_m e^{i\delta_m}z^m\ \ for\ m = 2,3,\ldots.$$

Writing $X_m = \frac{|a_m|}{\lambda_m}$, $Y_m = \frac{|b_m|}{\mu_m}$, $m = 2,3,\ldots$ and $X_1 = 1-\sum_{m=2}^{\infty}X_m$; $Y_1 = 1-\sum_{m=2}^{\infty}Y_m$ we have,

$$f(z) = \sum_{m=1}^{\infty}(X_m h_m(z) + Y_m g_m(z)).$$

In particular, setting

$$f_1(z) = z+\overline{b_1 z} \text{ and } f_m(z) = z+\lambda_m x z^m+\overline{b_1 z}+\overline{\mu_m y z^m},\ (m \geq 2, |x|+|y| = 1-|b_1|)$$

we see that extreme points of $V_{\overline{H}}(n, \alpha)$ are contained in $\{f_m(z)\}$.

To see that $f_1(z)$ is not an extreme point, note that $f_1(z)$ may be written as

$$f_1(z) = \frac{1}{2}\{f_1(z) + \lambda_2(1 - |b_1|)z^2\} + \frac{1}{2}\{f_1(z) - \lambda_2(1 - |b_1|)z^2\},$$

a convex linear combination of functions in $V_{\overline{H}}(n, \alpha)$.

To see that is not an extreme point if both $|x| \neq 0$ and $|y| \neq 0$, we will show that it can then also be expressed as a convex linear combinations of functions in $V_{\overline{H}}(n, \alpha)$. Without loss of generality, assume $|x| \geq |y|$. Choose $\epsilon > 0$ small enough so that $\epsilon < \frac{|x|}{|y|}$. Set $A = 1 + \epsilon$ and $B = 1 - |\frac{\epsilon x}{y}|$. We then see that both

$$t_1(z) = z + \lambda_m A x z^m + \overline{b_1 z + \mu_m y B z^m}$$

and

$$t_2(z) = z + \lambda_m(2 - A)xz^m + \overline{b_1 z + \mu_m y(2 - B)z^m},$$

are in $V_{\overline{H}}(n, \alpha)$ and note that

$$f_n(z) = \frac{1}{2}\{t_1(z) + t_2(z)\}.$$

The extremal coefficient bounds shows that functions of the form (13) are the extreme points for $V_{\overline{H}}(n, \alpha)$, and so the proof is complete.

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