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# $\tau$-DISTANCE IN A GENERAL TOPOLOGICAL SPACE $(X, \tau)$ WITH APPLICATION TO FIXED POINT THEORY

M. AAMRI AND D. EL MOUTAWAKIL

ABSTRACT. The main purpose of this paper is to define the notion of a $\tau$-distance function in a general topological space $(X, \tau)$. As application, we get a generalization of the well known Banach's fixed point theorem.

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## 1. Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory, which has been used and extended In many different directions ([2],[3],[4],[6],[9]). On the other hand, it has been observed ([3],[5]) that the distance function used in metric theorems proofs need not satisfy the triangular inequality nor $d(x, x) = 0$ for all $x$. Motivated by this fact, we define the concept of a $\tau$-distance function in a general topological space $(X, \tau)$ and we prove that symmetrizable topological spaces ([5]) and F-type topological spaces introduced in 1996 by Fang [4] (recall that metric spaces, Hausdorff topological vector spaces and Menger probabilistic metric space are all a special case of F-type topological spaces) possess such functions. finally, we give a fixed point theorem for contractive maps in a general topological space $(X, \tau)$ with a $\tau$-distance which gives the Banach's fixed point theorem in a new setting and also gives a generalization of jachymski's fixed point result [3] established in a semi-metric case.

Department of Mathematics and Informatics, University Hassan-II Mohammedia, Casablanca , Morocco
E-mail Address: d.elmoutawakil@math.net

## 2. $\tau$-DISTANCE

Let $(X, \tau)$ be a topological space and $p : X \times X \longrightarrow IR^+$ be a function. For any $\epsilon > 0$ and any $x \in X$, let $B_p(x, \epsilon) = \{y \in X : p(x, y) < \epsilon\}$.

**Definition 2.1.** The function $p$ is said to be a $\tau$-distance if for each $x \in X$ and any neighborhood $V$ of $x$, there exists $\epsilon > 0$ with $B_p(x, \epsilon) \subset V$.

**Example 2.1.** Let $X = \{0; 1; 3\}$ and $\tau = \{\emptyset; X; \{0; 1\}\}$. Consider the function $p : X \times X \longrightarrow IR^+$ defined by

$$p(x, y) = \begin{cases} y, \ x \neq 1 \\ \dfrac{1}{2}y, \ x = 1. \end{cases}$$

We have, $p(1; 3) = \frac{3}{2} \neq p(3; 1) = 1$. Thus, $p$ us not symmetric. Moreover, we have

$$p(0; 3) = 3 > p(0; 1) + p(1; 3) = \frac{5}{2}$$

which implies that $p$ fails the triangular inequality. However, the function $p$ is a $\tau$-distance.

**Example 2.2.** Let $X = IR^+$ and $\tau = \{X, \emptyset\}$. It is well known that the space $(X, \tau)$ is not metrizable. Consider the function $p$ defined on $X \times X$ by $p(x, y) = x$ for all $x, y \in X$. It is easy to see that the function $p$ is a $\tau$-distance.

**Example 2.3.** In [5], Hicks established several important common fixed point theorems for general contractive selfmappings of a symmetrizable (resp. semi-metrizable) topological spaces. Recall that a symmetric on a set $X$ is a nonnegative real valued function $d$ defined on $X \times X$ by

(1) $d(x, y) = 0$ if and only if $x = y$,
(2) $d(x, y) = d(y, x)$

A symmetric function $d$ on a set $X$ is a semi-metric if for each $x \in X$ and each $\epsilon > 0$, $B_d(x, \epsilon) = \{y \in X : d(x, y) \leq \epsilon\}$ is a neighborhood of $x$ in the topology $\tau_d$ defined as follows

$$\tau_d = \{U \subseteq X / \ \forall x \in U, \ B_d(x, \epsilon) \subset U, \ for some \ \epsilon > 0\}$$

A topological space $X$ is said to be symmetrizable (semi-metrizable) if its topology is induced by a symmetric (semi-metric) on $X$. Moreover, Hicks [5] proved that very general probabilistic structures admit a compatible symmetric or semi-metric. For further details on semi-metric spaces (resp. probabilistic metric spaces), see, for example, [8] (resp. [7]). Each symmetric function $d$ on a nonempty set $X$ is a $\tau_d$-distance on $X$ where the topology $\tau_d$ is defined as follows: $U \in \tau_d$ if $\forall x \in U$, $B_d(x, \epsilon) \subset U$, for some $\epsilon > 0$.

**Example 2.4.** Let $X = [0, +\infty[$ and $d(x, y) = |x - y|$ the usual metric. Consider the function $p : X \times X \longrightarrow IR^+$ defined by

$$p(x, y) = e^{|x-y|}, \ \ \forall x, y \in X$$

It is easy to see that the function $p$ is a $\tau$-distance on $X$ where $\tau$ is the usual topology since $\forall x \in X$, $B_p(x, \epsilon) \subset B_d(x, \epsilon)$, $\epsilon > 0$. Moreover, $(X, p)$ is not a symmetric space since for all $x \in X$, $p(x, x) = 1$.

**Example 2.5 - Topological spaces of type (EL).**

**Definition 2.2.** A topological space $(X,\tau)$ is said to be of type $(EL)$ if for each $x \in X$, there exists a neighborhood base $F_x = \{U_x(\lambda,t)/\lambda \in D, t > 0\}$, where $D = (D,\prec)$ denotes a directed set, such that $X = \cup_{t>0} U_x(\lambda,t)$, $\forall \lambda \in D$, $\forall x \in X$.

*remark 2.1.* In [4], Fang introduced the concept of F-type topological space and gave a characterization of the kind of spaces. The usual metric spaces, Hausdorff topological vector spaces, and Menger probabilistic metric spaces are all the special cases of F-type topological Spaces. Furtheremore, Fang established a fixed point theorem in F-type topological spaces which extends Caristi's theorem [2]. We recall the concept of this space as given in [4]

**Definition [4].** A topological space $(X,\theta)$ is said to be F-type topological space if it is Hausdorff and for each $x \in X$, there exists a neighborhood base $F_x = \{U_x(\lambda,t)/\lambda \in D, t > 0\}$, where $D = (D,\prec)$ denotes a directed set, such that

1. If $y \in U_x(\lambda,t)$, then $x \in U_y(\lambda,t)$,
2. $U_x(\lambda,t) \subset U_x(\mu,s)$ for $\mu \prec \lambda, t \leq s$,
3. $\forall \lambda \in D$, $\exists \mu \in D$ such that $\lambda \prec \mu$ and $U_x(\mu,t_1) \cap U_y(\mu,t_2) \neq \emptyset$, implies $y \in U_x(\lambda, t_1 + t_2)$,
4. $X = \cup_{t>0} U_x(\lambda,t)$, $\forall \lambda \in D$, $\forall x \in X$.

It is clear that a topological space of type F is a Hausdorff topological space of type (EL). Therefore The usual metric spaces, Hausdorff topological vector spaces, and Menger probabilistic metric spaces are special cases of a Hausdorff topological Space of type (EL).

**proposition 2.1.** *Let $(X,\tau)$ be a topological space of type (EL). Then, for each $\lambda \in D$, there exists a $\tau$-distance function $p_\lambda$.*

*Proof.* Let $x \in X$ and $\lambda \in D$. Consider the set $E_x = \{U_x(\lambda,t) | \lambda \in D, t > 0\}$ of neighborhoods of $x$ such that $X = \cup_{t>0} U_x(\lambda,t)$. Then for each $y \in X$, there exists $t^* > 0$ such that $y \in U_x(\lambda,t^*)$. Therefore, for each $\lambda \in D$, we can define a function $p_\lambda : X \times X \longrightarrow IR^+$ as follows

$$p_\lambda(x,y) = inf\{t > 0, y \in U_x(\lambda,t)\}.$$

set $B_\lambda(x,t) = \{y \in X | p_\lambda(x,y) < t\}$. let $x \in X$ and $V_x$ a neighborhood of $x$. Then the exists $(\lambda,t) \in D \times IR^+$, such that $U_x(\lambda,t) \subset V_x$. We show that $B_\lambda(x,t) \subset U_x(\lambda,t)$. Indeed, consider $y \in B_\lambda(x,t)$ and suppose that $y \notin U_x(\lambda,t)$. It follows that $p_\lambda(x,y) \geq t$, which implies that $y \notin B_\lambda(x,t)$. A contradiction. Thus $B_\lambda(x,t) \subset V_x$. Therefore $p_\lambda$ is a $\tau$-distance function.

*remark 2.2.* As a consequence of proposition 3.1, we claim that each topological space of type (EL) has a familly of $\tau$-distances $M = \{p_\lambda | \lambda \in D\}$.

## 3. Some properties of $\tau$-distances

**lemma 3.1.** *Let $(X,\tau)$ be a topological space with a $\tau$-distance $p$.*

1. *Let $(x_n)$ be arbitrary sequence in $X$ and $(\alpha_n)$ be a sequence in $IR^+$ converging to 0 such that $p(x,x_n) \leq \alpha_n$ for all $n \in IN$. Then $(x_n)$ converges to $x$ with respect to the topology $\tau$.*

(2) *If* $(X, \tau)$ *is a Hausdorff topological space, then* (2.1) $p(x, y) = 0$ *implies* $x = y$. (2.2) *Given* $(x_n)$ *in* $X$,

$$\lim_{n\to\infty} p(x, x_n) = 0 \; and \; \lim_{n\to\infty} p(y, x_n) = 0$$

*imply* $x = y$.

*Proof.*

(1) Let $V$ be a neighborhood of $x$. Since $\lim p(x, x_n) = 0$, there exists $N \in IN$ such that $\forall n \geq N,\ x_n \in V$. Therefore $\lim x_n = x$ with respect to $\tau$.

(2) (2.1) Since $p(x, y) = 0$, then $p(x, y) < \epsilon$ for all $\epsilon > 0$. Let $V$ be a neighborhood of $x$. Then there exists $\epsilon > 0$ such that $B_p(x, \epsilon) \subset V$, which implies that $y \in V$. Since $V$ is arbitrary, we conclude $y = x$. (2.2) From (2.1), $\lim p(x, x_n) = 0$ and $\lim p(y, x_n) = 0$ imply $\lim x_n = x$ and $\lim x_n = y$ with respect to the topology $\tau$ which is Hausdorff. Thus $x = y$.

Let $(X, \tau)$ be a topological space with a $\tau$-distance $p$. A sequence in $X$ is p-Cauchy if it satisfies the usual metric condition with respect to $p$. There are several concepts of completeness in this setting.

**Definition 3.1.** Let $(X, \tau)$ be a topological space with a $\tau$-distance $p$.

(1) $X$ is S-complete if for every p-Cauchy sequence $(x_n)$, there exists x in X with $\lim p(x, x_n) = 0$.

(2) $X$ is p-Cauchy complete if for every p-Cauchy sequence $(x_n)$, there exists x in X with $\lim x_n = x$ with respect to $\tau$.

(3) $X$ is said to be p-bounded if $sup\{p(x, y)/x, y \in X\} < \infty$.

*remark 3.1.* Let $(X, \tau)$ be a topological space with a $\tau$-distance $p$ and let $(x_n)$ be a p-Cauchy sequence. Suppose that $X$ is S-complete, then there exists $x \in X$ such that $\lim p(x_n, x) = 0$. Lemma 4.1($b$) then gives $\lim x_n = x$ with respect to the topology $\tau$. Therefore S-completeness implies p-Cauchy completeness.

## 4. Fixed point theorem

In what follows, we involve a function $\psi : IR^+ \longrightarrow IR^+$ which satisfies the following conditions

(1) $\psi$ is nondecreasing on $IR^+$ ,

(2) $\lim \psi^n(t) = 0,\ \forall t \in ]0, +\infty[$.

It is easy to see that under the above properties, $\psi$ satisfies also the following condition

$$\psi(t) < t,\ foreach\ t \in ]0, +\infty[$$

**Theorem 4.1.** *Let* $(X, \tau)$ *be a Hausdorff topological space with a* $\tau$*-distance* $p$. *Suppose that* $X$ *is p-bounded and S-complete. Let* $f$ *be a selfmapping of* $X$ *such that*

$$p(fx, fy) \leq \psi(p(x, y)), \quad \forall x, y \in X$$

*Then* $f$ *has a unique fixed point.*

*Proof.* Let $x_0 \in X$. Consider the sequence $(x_n)$ defined by

$$\begin{cases} x_0 \in X, \\ x_{n+1} = f x_n \end{cases}$$

We have

$$\begin{aligned} p(x_n, x_{n+m}) =& p(fx_{n-1}, fx_{n+m-1}) \\ \leq& \psi(p(x_{n-1}, x_{n+m-1})) = \psi(p(fx_{n-2}, fx_{n+m-2})) \\ \leq& \psi^2(p(x_{n-2}, x_{n+m-2})) \\ &.... \\ \leq& \psi^n(p(x_0, x_m)) \leq \psi^n(M) \end{aligned}$$

where $M = sup\{p(x, y)/x, y \in X\}$. Since $\lim \psi^n(M) = 0$, we deduce that the sequence $(x_n)$ is a p-cauchy sequence. $X$ is S-complete, then $\lim p(u, x_n) = 0$, for some $u \in X$, and therefore $\lim p(u, x_{n+1}) = 0$ and $\lim p(fu, fx_n) = 0$. Now, we have $\lim p(fu, x_{n+1}) = 0$ and $\lim p(u, x_{n+1}) = 0$. Therefore, lemma 3.1(2.2) then gives $fu = u$. Suppose that there exists $u, v \in X$ such that $fu = u$ and $fv = v$. If $p(u, v) \neq 0$, then

$$p(u, v) = p(fu, fv) \leq \psi(p(u, v)) < p(u, v)$$

a contradiction. Therefore the fixed point is unique. Hence we have the theorem.

When $\psi(t) = kt,\ k \in [0, 1[$, we get the following result, which gives a generalization of Banach's fixed point theorem in this new setting

**Corollary 4.1.** *Let $(X, \tau)$ be a Hausdorff topological space with a $\tau$-distance $p$. Suppose that $X$ is p-bounded and S-complete. Let $f$ be a selfmapping of $X$ such that*

$$p(fx, fy) \leq kp(x, y),\ k \in [0, 1[,\ \forall x, y \in X$$

*Then $f$ has a unique fixed point.*

Since a symmetric space $(X, d)$ admits a $\tau_d$-distance where $\tau_d$ is the topology defined earlier in example 2.3, corollary 4.1 gives a genaralization of the following known result (Theorem 1[5] for $f = Id_X$ which generalize Proposition 1[3]). Recall that $(W.3)$ denotes the following axiom given by Wilson [8] in a symmetric space $(X, d)$: $(W.3)$ Given $\{x_n\}, x$ and $y$ in $X$, $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$ imply $x = y$. It is clear that $(W.3)$ guarantees the uniqueness of limits of sequences.

**corollary 4.2.** *Let $(X, d)$ be a d-bounded and S-complete symmetric space satisfying $(W.3)$ and $f$ be a selfmapping of $X$ such that*

$$d(fx, fy) \leq kd(x, y),\ \ k \in [0, 1[,\ \ \forall x, y \in X$$

*Then $f$ has a fixed point.*

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