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WEAKLY STARLIKE LOGHARMONIC MAPPINGS

Zayid Abdulhadi

Abstract. We consider the class of all weakly starlike p-valent logharmonic mappings. A complete characterization of these mappings is given.

1991 A.M.S. (MOS) Subject Classification Codes. Primary 30C55, Secondary 30C62, 49Q05

Key Words and Phrases. logharmonic mappings

1. Introduction

Let $H(U)$ be the linear space of all analytic functions defined in the unit disc $U = \{z = x + iy;\ |z| < 1\}$ and let B be the set of all functions $a \in H(U)$ such that $|a(z)| < 1$ for all $z \in U$. We say a function f is logharmonic if f is a solution of the non-linear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = (a\overline{f}/f)f_z;\ a \in B. \tag{1.1}$$

Observe that nonconstant logharmonic mappings are open and orienation preserving on U. If f has exactly p zeros $\{z_k\}_{k=1}^{p}$ in U (inclusive multiplicity), then f admits a global representation given by

$$f(z) = [\prod_{k=1}^{p}(z - z_k)|z - z_k|^{2\beta_k}]h(z)\overline{g(z)}, \tag{1.2}$$

where
i) $\beta_k = \overline{a(z_k)}\frac{(1+a(z_k))}{(1-|a(z_k)|^2)}$,
ii) h and g are analytic in U, such that $g(z_k) = 1$ and $h(z_k) \neq 0$.
If f is a univalent logharmonic mapping on U, then either $0 \notin f(U)$ and log f is univalent and harmonic on U, or, $f(z_0) = 0$, for a fixed $z_0 \in U$. In the latter

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case, the function $F(\zeta) = \log\ f(e^{\zeta})$ is univalent and harmonic on the half-plane $\{\zeta;\ \Re\,\zeta < 0\}$ and the representation (1.2) becomes

$$f(z) = (z - z_0)|z - z_0|^{2\beta} h(z)\overline{g(z)}.$$

If $f(U)$ is a starlike domain, then F is closely related with the theory of minimal surfaces over domains of the form $-\infty < u < u_0(v), u_0(v + 2\pi) = u_0(v)$. For details see e.g. [A1],A2], [AB], [AH1], [AH2], [AH3] , [AH4] and [AH5].

Definition. *let f be a nonconstant complex-valued function defined in U. We say that f is weakly starlike if*

$$\lim_{r\to 1} \inf[\min_{|z|=r} \frac{\partial \arg f(re^{it}, z_k)}{\partial t}] \geq 0. \tag{1.3}$$

Generalizations from univalent starlike analytic functions to weakly starlike analytic functions have been introduced by Hummel(see [H]). He has shown:

Theorem A. *ϕ is a weakly starlike analytic function with exactly p zeros in U if and only if there exists a normalized starlike univalent analytic ϕ such that*

$$\phi(z) = (\phi_0(z))^p \prod_{k=1}^{p} \psi_0(z, z_k)$$

where $z_1, z_2, ..., z_p$ are the zeros of ϕ, and

$$\psi_0(z, z_k) = \frac{(z - z_k)(1 - \overline{z_k} z)}{z}.$$

Note that $\psi_0(z, z_k)$ is a univaelnt meromorphic fuction which satisfies (1.3) and maps U onto the exterior of the line segment whose end points are $(1 \pm |z_k|^2)^2$.

In this paper, we shall consider the class of weakly starlike p-valent logharmonic mappings. In Section 2 we shall include a characterization theorem for class of weakly starlike p-valent logharmonic mappings and also, a representation theorem for these mappings is given which is the analouge of Theorem A.

2. Representation theorems .

We shall need the following lemma:

Lemma 2.1. *Let f be a weakly starlike and logharmonic mapping with respect to a given $a \in B$. Then f is p-valent if and only if f has exactly p zeros (inclusive multiplicity).*

Proof. First, we suppose that f has p zeros (inclusive multiplicity). In order to show that f is p-valent, we shall adapt the method of Hummel (see [H]). Let $\triangle_r = \{\theta |\ \frac{\partial \arg\ f(re^{i\theta})}{\partial\theta} < 0\}$, where $0 < r < 1$. Given any $\epsilon > 0$, there exists a $\rho \in (0, 1)$ such that for all r with $\rho < r < 1$ we have

$$\int_{\triangle_r} \frac{\partial \arg\ f(re^{i\theta})}{\partial\theta} d\theta > -2\pi\epsilon. \tag{2.1}$$

Now we suppose that f is not p-valent then there is some $w_0(\neq 0)$ such that the value w_0 is taken on $p+1$ or more times by $f(z)$. Then there exists N points $z_1, ...z_N$, $N \geq p+1$, such that $g(z_i) = w_0$ for each i. Some of the points may be multiple. Since f is an open mapping there exists a disk W with center w_0, not containing the origin, and a real number r_0, $0 < r_0 < 1$, such that every point $w \in W$ is taken on N or more times by $f(z)$ in $\{z| \, |z| < r_0\}$. Let α be the angle subtended by W at the origin, then $0 < \alpha < \pi$. Let $L_\phi = \{w| \text{ arg } w = \phi\}$ be any ray from the origin which intersects W. If r is such that $r_0 \leq r \leq 1$, then $f(z)$ assumes every value in W at least N times for $|z| < r$.

Let C_r be the circle $\{|z| = r\}$ and let Γ_r be the image curve given by $f(re^{i\theta})$, $0 \leq \theta \leq 2\pi$. Now $f(z)$ has exactly p zeros in $|z| < r$ (we may assume that r_0 is large enough to insure this). Using the argument principle, we deduce that the winding number of the curve Γ_r (w.r.to the origin) is $n(\Gamma_r, 0) = p$.

On the other hand, if $w_1 \in W \bigcap L_\phi$ such that $w_1 \notin \Gamma_r$, the winding number of Γ_r (w.r.to the w_1) must be greater than N. It follows that Γ_r crosses the line segment joining 0 and w_1 in the direction of decreasing argument at least once. But since Γ_r is an analytic curve, there are only finitely many intervals $[\theta_i, \theta_i'] \subset [0, 2\pi]$ such that $\arg f(re^{it})$ is monotone decreasing in $[\theta_i, \theta_i']$. The union of these intervals makes up the closure of the set $\triangle_r$ above. Since the angle subtended by W is α, we have

$$\int_{\triangle_r} \frac{\partial \arg \; f(re^{i\theta})}{\partial \theta} d\theta < -\alpha$$

which contradicts (2.1).

Conversely, if f is p-valent, then f has at most p zeros. If f has m zeros, $m < p$, then f is m-valent.This completes the proof.

Let f be a logharmonic mapping with respect to a given $a \in B$ and let $z_k \in U$, $1 \leq k \leq p$, be the zeros of f. Using Lemma 2.2 in [AB], it follows that f admits the represenation

$$f(z) = [\prod_{k=1}^{p} (\frac{z - z_k}{1 - \overline{z_k}}) | \frac{z - z_k}{1 - \overline{z_k}} |^{2\beta_k}] h(z).\overline{g(z)} \tag{2.2}$$

where $\beta_k = \overline{a(z_k)} \frac{1 + a(z_k)}{1 - |a(z_k)|^2}$. The functions h and g are nonvanishing and analytic in U and $g(0) = 1$.

Now we associate to f the mapping $\phi(z) = [\prod_{k=1}^{p} (\frac{z - z_k}{1 - \overline{z_k}})] \frac{h(z)}{g(z)}$.

The following theorem gives a relation between f and ϕ:

Theorem 2.2. *a) Let f be a weakly starlike, p-valent and logharmonic mapping with respect to a given $a \in B$. Then ϕ is weakly starlike, p-valent and analytic in U.*

b) Let ϕ be a weakly starlike, p-valent and analytic function. For any given $a \in B$ there are h and g in $H(U)$ uniquely determined such that

i) $0 \notin h.g(U)$; $g(0) = 1$

ii) $\phi(z) = [\prod_{k=1}^{p} (\frac{z - z_k}{1 - \overline{z_k}})] h/g$

iii) f defined in (2.2) is a solution of (1.1) with respect to the given a and is weakly starlike p-valent mapping.

Proof. a) Let f defined in (2.2) be weakly starlike and logharmonic with respect to a given $a \in B$. Then ϕ has exactly p zeros and is weakly starlike analytic since

$$\arg\ f(re^{it}) = \arg\ \phi(re^{it}) + 2\sum_{k=1}^{p} \Im(\beta_k)\log|\frac{z-z_k}{1-\overline{z_k}}|. \tag{2.3}$$

By proposition 6 in [H], it follows that ϕ is p-valent.

b) Let $a \in B$ and let ϕ be a weakly starlike p-valent analytic function.

Put

$$g(z) = exp\ \int_0^z [\frac{a(s)}{(1-a(s)}\frac{\phi'(s)}{\phi(s)} + \sum_{k=1}^{p}\nu(s,z_k)]ds,$$

$$h(z) = \phi(z)g(z)/\prod_{k=1}^{p}[\frac{z-z_k}{1-\overline{z_k}}]$$

and

$$f(z) = f(z) = [\prod_{k=1}^{p}(\frac{z-z_k}{1-\overline{z_k}})|\frac{z-z_k}{1-\overline{z_k}}|^{2\beta_k}]h(z).\overline{g(z)}$$

$$= \phi(z)[\prod_{k=1}^{p}|\frac{z-z_k}{1-\overline{z_k}}|^{2\beta_k}]|g(z)|^2,$$

where $\nu(z,z_k) = \frac{(a(z\beta_k-\overline{\beta_k})(1-|z_k|^2)}{(1-a(z))(z-z_k)(1-\overline{z_k}z)}$ and $\beta_k = \overline{a(z_k)}\frac{1+a(z_k)}{1-|a(z_k)|^2}$. Then h and g are nonvanishing analytic functions defined on U, normalized by $g(0)=1$, and f is a weakly starlike logharmonic mapping. Moreover, f has exactly p zeros and by Lemma 2.1, f is p-valent.

Next we introduce a class of fuctions which appear naturally in the represenation theorem.

Definition. *Let z_k be a point in U, and let $a \in B$ be given. We define*

$$\psi(z,z_k) = \frac{\psi_0(z,z_k)|\frac{z-z_k}{1-\overline{z_k}}|^{2\beta_k}}{|z|^{2\beta_0}}\exp\ 2\Re\,\sigma(z,z_k), \tag{2.4}$$

where

a) $\beta_k = \overline{a(z_k)}\frac{(1+a(z_k))}{(1-|a(z_k)|^2)}$,

b) $\beta_0 = \overline{a(0)}\frac{(1+a(0))}{(1-|a(0)|^2)}$,

c) $\psi_0(z,z_k) = \frac{(z-z_k)(1-\overline{z_k}z}{z}$,

d) $\sigma(z,z_k) = \int_0^z[\frac{(a(s)\psi_0'}{(1-a(s)\psi_0(s))} - \frac{a(s)\beta_0-\overline{\beta_0}}{s(1-a(s))} + \nu(s,z_k)]$,

where

$\nu(s,z_k) = \frac{(a(z)\beta_k-\overline{\beta_k})(1-|z_k|^2)}{(1-a(z)(z-z_k)(1-\overline{z_k}z)}$.

As a direct consequence of the represenation (2.4), we have the following lemma:

Lemma 2.3. *For any given $z_k \in U$ and $a \in B$, the function $\psi(z, z_k)$ is continuous, real and positive on $|z| = 1$. Furthermore, $\psi(z, z_k)$ is a univalent meromorphic function which satisfies (1.1).*

Proof. Using (2.4), we have

$$\arg \psi(z, z_k) = \arg \psi_0(z, z_k) + 2\Im(\beta_k) \log |\frac{z - z_k}{1 - \overline{z_k} z}| - 2\Im(\beta_0) \log |z|$$

and since ψ_0 maps U conformally onto the exterior of a line segment joining the points $(1 \pm |z_k|^2)$, we deduce that ψ maps U also onto the exterior of a line segment I joining two points lying on the positive real axis. Moreover, we have

$$\lim_{r \to 1} inf[\min_{|z|=r} \frac{\partial \arg \psi(re^{it}, z_k)}{\partial t}] = \lim_{r \to 1} inf[\min_{|z|=r} \frac{\partial \arg \psi_0(re^{it}, z_k)}{\partial t}] = 0.$$

Thus, ψ satisfies (2.1) and direct calculations shows that ψ satisfies (1.1) with respect to the same a. It is left to show that ψ is univalent. Let C_r be the circle $|z| = r$ and let Γ_r be the image curve given by $\psi(re^{it}, z_k)$, $0 \leq \theta \leq 2\pi$. First let $w^\circ \in \mathbb{C} \setminus \mathbb{R}^+$. If we choose r sufficiently close to one, then the winding number of the curve Γ_r with respect to w° is $n(\Gamma_r, w^\circ) = 0$. But $n(\Gamma_r, w^\circ) = N_r(\psi - w^\circ) - P_r(\psi)$, where $N_r(\psi - w^\circ)$ is the number of zeros of $\psi - w^\circ$ in $|z| \leq r$ and $P_r(\psi)$ is the number of poles of ψ in $|z| \leq r$. Since $P_r(\psi) = 1$, we conclude that $N_r(\psi - w^\circ) = 1$. Next, if $w^\circ \in \mathbb{R}^+ \setminus I$, then there exists a sequence w_n in $\mathbb{C} \setminus \mathbb{R}^+$ such that w_n converges to w°. Now, there is an $r \in (0,1)$ sufficiently close to 1 such that $n(\Gamma_r, w_n) = 0$ for all $n \in \mathbf{N}$. Since $\psi - w_n$ converges uniformly to $\psi - w^\circ$ on C_r, we get that $n(\gamma_r, w^\circ) = 0$.

We are now ready to state and prove the main result of this paper:

Theorem 2.4 (Representation Theorem). *Let f be a logharmonic function with respect to a given $a \in B$. Then a function f defined as in (2.2) is weakly starlike if and only if*

$$f(z) = (f_0(z))^p \prod_{k=1}^{p} \psi(z, z_k)$$

where $z_1, z_2, ..., z_p$ are in U and f_0 is a starlike univalent logharmonic mapping with respect to the same a.

Proof. Let $a \in B$ be given and let f be logharmonic with respect to a such that $f(z) = (f_0(z))^p \prod_{k=1}^p \psi(z, z_k)$, where $z_1, z_2, ..., z_p \in U$ and let f_0 be a starlike univalent logharmonic mapping. Then f has exactly p zeros and we have

$$\lim_{r \to 1} \inf[\min_{|z|=r} \frac{\partial \arg f(re^{it}, z_k)}{\partial t}] =$$

$$p \lim_{r \to 1} \inf[\min_{|z|=r} \frac{\partial \arg f_0(re^{it}, z_k)}{\partial t}] + \lim_{r \to 1} \inf[\min_{|z|=r} \frac{\partial \arg \psi(re^{it}, z_k)}{\partial t}] \geq 0.$$

Using Theorem 2.2, it follows that f is weakly starlike and p-valent and has the representation (2.2).
Conversely, let f be a logharmonic mapping with respect to a given $a \in B$. We

suppose that f has the represenation (2.2) and is weakly starlike p-valent. By Theorem 2.2, we conclude that

$$\phi(z) = [\prod_{k=1}^{p} \frac{z - z_k}{1 - \overline{z_k} z}] \frac{h(z)}{g(z)}$$

is a weakly starlike p-valent analytic function in U. Theorem A implies that there exists a a starlike univalent analytic function ϕ_0 such that $\phi(z) = (\phi_0(z))^p \prod_{k=1}^{p} \psi_0(z, z_k)$ and therefore, $\phi_0(z) = z(\frac{h(z)}{g(z)\prod_{k=1}^{p}(1-\overline{z_k}z)^2})^{\frac{1}{p}}$. From Theorem 2.1 in [AH2] (with $\alpha = 0$), it follows that $f_0(z) = \phi_0(z)|z|^{2\beta_0}|g_1(z)|^2$ is a starlike univalent logharmonic, where $g_1(z) = \exp \int_0^z [\frac{a(s)\phi_0'(s)}{(1-a(s))\phi_0(s)} + \frac{a(s)\beta_0 - \overline{\beta_0}}{s(1-a(s))}]ds$ and $\beta_0 = \frac{\overline{a(0)}(1+a(0))}{(1-|a(0)|^2}$. Put

$$f_1(z) = (f_0(z))^p \prod_{k=1}^{p} \psi(z, z_k).$$

It follows that f_1 is a solution of (1.1) with respect to a and, furthermore, f_1 is a weakly starlike p-valent mapping. Straightforward calculations show that ϕ corresponds in Theorem 2.2 to the mapping f_1 and therefore, we have $f = f_1$.

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CHAIN OF DENDRITES WITHOUT OPEN INFIMUM

Pavel Podbrdský
Pavel Pyrih

Abstract. We consider an ordering with respect to open mappings on the class of all dendrites and construct a bounded chain of dendrites which does not have an infimum answering a question of its existence.

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Key Words and Phrases. Dendrite, open mapping, chain

Flowers and flower-beds

All spaces considered in this paper are assumed to be metric. A *continuum* means a nonempty compact connected space. A *simple closed curve* is any space which is homeomorphic to the unit circle. A *dendrite* means a locally connected continuum containing no simple closed curve.

A *mapping* means a continuous function. A surjective mapping $f : X \to Y$ is said to be *open* provided that for each open subset U of X its image $f(U)$ is an open subset of Y.

In [3, p. 7] J. J. Charatonik, W. J. Charatonik and J. R. Prajs introduced a quasiordering $\leq_O$ on the class of all dendrites. We recall the definition. If X, Y are dendrites, then $Y \leq_O X$ if there is a surjective open mapping $f : X \to Y$. This quasiordering is not an ordering (i.e. $X \leq_O Y$ & $Y \leq_O X$ does not imply X is homeomorphic to Y). To guarantee this they said that X and Y are $\mathbb{O}$-*equivalent* iff $X \leq_O Y$ and $Y \leq_O X$ and then considered the quotient of the class of all dendrites. The quasiordering $\leq_O$ induces the ordering $\leq_{\mathbb{O}}$ on the quotient.

J. J. Charatonik, W. J. Charatonik and J. R. Prajs posed a question if every chain bounded with respect to this ordering has an infimum (see [3, §7, Q4(b)$\mathbb{O}$,

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p. 51]). In Theorem we give a negative answer to this question by constructing a bounded sequence of dendrites with no open infimum.

We shall use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [4, §51, I, p. 274], and we denote by $\operatorname{ord}(p, X)$ the order of the continuum X at a point $p \in X$ or just $\operatorname{ord}(p)$ if there is no risk of confusion. Points of order 1 in a continuum X are called *end points* of X; the set of all end points of X is denoted by $E(X)$. Points of order 2 are called *ordinary points* of X. Points of order at least 3 are called *ramification points* of X.

Let X be a space. A sequence of subsets of X is said to be a *null-sequence* if the diameters of its members tend to zero. Let X be a connected space, and let $p \in X$. If $X \setminus \{p\}$ is not connected, then p is called a *cut point of* X.

An *arc* is any space which is homeomorphic to $I = [0, 1]$. We denote by pq an arc with end points p and q. An arc pq in a space X is said to be *free* if $pq \setminus \{p, q\}$ is an open subset of the space X. Denote by S_n the n-od, i.e. the dendrite where the only ramification point is of order n.

Fact. (See [6, Chapter 8, (7.31), p. 147].) *The order of a point is never increased under an open mapping.*

Proposition. *Let X be a dendrite.*

(A) *Each point of X is either a cut point or an end point.*
(B) *For each point $x \in X$ we have* $\operatorname{ord}(x) = c(x)$, *where $c(x)$ is the number of components of $X \setminus \{x\}$.*
(C) *For every subcontinuum $Z \subset X$ every component of $X \setminus Z$ is open in X.*
(D) *Let $f : X \to Y$ be a surjective open mapping. Then Y is a dendrite.*
(E) *Let $f : X \to Y$ be a surjective open mapping. Then the image under f of a free arc in X is a free arc in Y.*
(F) *Let $f : X \to Y$ be a surjective open mapping. Then $f^{-1}(E(Y)) \setminus E(X)$ is finite.*

See [5, Chapter X, Theorem 10.7, p. 168], [5, Chapter X, Theorem 10.13, p. 170], [5, 5.22(a), p. 83], [5, Corollary 13.41, p. 297], [3, Corollary 6.7, p. 23]. and [3, Proposition 6.16, p. 25].

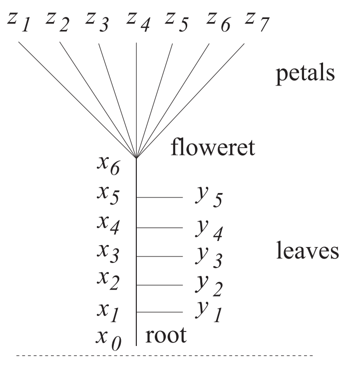

Figure 1 – a flower $F = F(x_0, 5, x_6, 7)$

Given $l, p \in \mathbb{N} \cup \{0\}$. Denote $x_j = (0, j), j = 0, \cdots, l+1$, $y_j = (1, j), j = 1, \cdots, l$, $z_i = (i, l+2), i = 1, \cdots, p$. By a *flower* (*with l leaves and p petals*) we denote the union

$$F = x_0x_{l+1} \cup \bigcup_{j=1}^{l} x_jy_j \cup \bigcup_{i=1}^{p} x_{l+1}z_i \quad .$$

For a flower F we say that the point x_0 is the *root* , the arcs x_jy_j are the *leaves* , x_{l+1} is the *floweret* and the arcs $x_{l+1}z_i$ are the *petals.* We denote by $F(r,l,f,p)$ a flower with a root r, l leaves, a floweret f and p petals. Clearly, a flower is a finite union of arcs. See Figure 1.

Set $a = (0,0)$ and $b = (0,2)$. By a *flower-bed* (*with flowers* F_n) we denote the union

$$B = ab \cup \bigcup_{n=1}^{\infty} F_n \quad ,$$

where $\{F_n\}$ is a null-sequence of mutually disjoint flowers each meeting ab in its root $(1/n, 0)$ only. We observe that a flower-bed is a dendrite. See Figure 2.

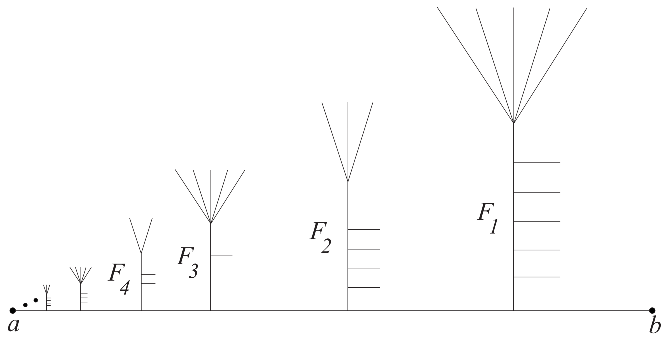

Figure 2 – a flower-bed

Fix a null-sequence of mutually disjoint flowers

$$G_n = ((1/n, 0), n, (1/n, 1/n), n),$$

each meeting ab in its root $(1/n, 0)$ only. We denote by X_n the flower-bed

$$X_n = ab \cup G_n \cup G_{n+1} \cup G_{n+2} \cup \cdots .$$

See Figure 3.

Theorem. *The chain* $X_1 \geq_O X_2 \geq_O X_3 \geq_O \cdots \geq_O I$ *is bounded and has no infimum.*

Proof. Suppose that Y is an infimum of the sequence $\{X_n\}$. We conclude a contradiction in several steps.

(1) Denote by $g_n : X_n \to Y$ an open surjection of X_n onto Y. Then Y is a dendrite due to Proposition D.

(2) $X_1 \geq_O X_2 \geq_O X_3 \geq_O \cdots$.

(Proof. Fix $n \in \mathbb{N}$. We construct a mapping h_n of X_n onto X_{n+1} such that h_n satisfies the following conditions:

(i) h_n is the identity on X_{n+1};

(ii) h_n is piece-wise linear (and nonconstant) elsewhere;

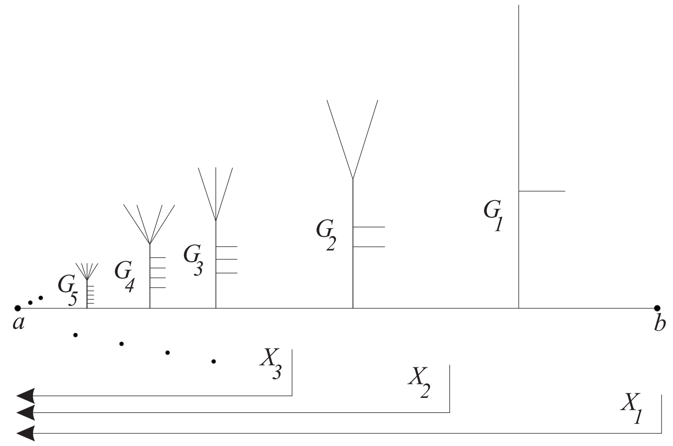

Figure 3 – flower-beds X_1, X_2, $\cdots$

(iii) h_n identifies all end points of X_n contained in $X_n \setminus X_{n+1}$ with b. Obviously, h_n is an open surjection.)

(3) $X_n \geq_O S_m \geq_O I$ for each $n, m \in \mathbb{N}$.

(Proof. Given $m, n \in \mathbb{N}$, fix a ramification point x of order $m' \geq m$ in X_n. The mapping of X_n which (i) identifies on each component of $X_n \setminus \{x\}$ all end points of X_n, and (ii) is piece-wise linear (and nonconstant), is an open surjection of X_n onto $S_{m'}$. Finally, trivial open surjections of $S_{m'}$ onto S_m and of S_m onto I finish the proof.)

(4) For each $m \in \mathbb{N}$, $Y \geq_O S_m$ (see (3), Y is an infimum of $\{X_n\}$).

(5) Ramification points of Y are of finite order (due to (1) and Fact, because the flower-bed X_1 has only points of finite order).

(6) For each $m \in \mathbb{N}$, there is a ramification point in Y of order at least m (see (4) and Fact).

(7) Fix in Y a ramification point y of order $p > 3$ (see (6)).

(8) Find for each $n \in \mathbb{N}$ a ramification point x_n in X_n such that $g_n(x_n) = y$ (see Fact, (1) and (7)).

(9) For each $n \in \mathbb{N}$, the order of x_n in X_n is at least p (see (7), (8) and Fact).

(10) For each $n \in \mathbb{N}$, the point x_n is a floweret of a flower $F_n = F(r_n, l_n, x_n, p_n)$ in X_n with $l_n \geq n$ due to the construction of X_n (see (8)).

(11) For each $n \in \mathbb{N}$, if $g_n(z) = g_n(r_n)$ for some $z \in F_n$, then $z = r_n$.

(Proof. Otherwise $F_n \setminus \{r_n\}$ is an open set (see (10) and Proposition C) with the same image as F_n under g_n. Then $Y = g_n(F_n)$ due to the openness of g_n and the connectedness of Y. Finally, we conclude that Y has only finitely many ramification points due to Fact, contrary to (6).)

(12) For each $n \in \mathbb{N}$, if $g_n(z) = g_n(x_n)$ for some $z \in F_n$, then $z = x_n$ (see (7), (9) and Fact, because the only ramification point of X_n in F_n of order greater than 3 is x_n).

(13) For each $n \in \mathbb{N}$, the image under g_n of a flower F_n is a flower $\hat{F}_n = F(\hat{r}_n, \hat{l}_n, y, \hat{p}_n)$, where $\hat{r}_n = g_n(r_n)$ and $\hat{p}_n = p - 1$ (due to (11), (12), Fact and Propositions C-D).

(14) For each $n \in \mathbb{N}$, the flowers F_n and $\hat{F}_n$ have the same number of leaves, i.e. $\hat{l}_n = l_n (\geq n)$ (due to (10), (13), Fact and Proposition E).

(15) For each $n \in \mathbb{N}$, there is in Y a flower $\hat{F}_n$ rooting from $\hat{r}_n$ containing y with at least n leaves (due to (10) and (14)).
(16) For each $n \in \mathbb{N}$, the flower $\hat{F}_n$ contains at least n end points of Y (due to (15).
(17) Neighborhoods of a contain all but finite number of end points of X_1 due to the construction.
(18) Neighborhoods of $\hat{a} = g_1(a)$ contain all but finite number of end points of $Y = g_1(X_1)$ (due to the openness of g_1, see (1), (17) and Proposition F).
(19) Fix in Y a ramification point $y' \neq y$ of order at least $p(> 3)$ (see (6) and (7)).
(20) The point a is an end point in X_1 due to the construction.
(21) The point $\hat{a}(= g_1(a))$ is an end point in Y (see (1), (18), (20) and Fact).
(22) Fix an open neighborhood $U \subset (Y \setminus \{y, y'\})$ of $\hat{a}$ in Y such that the boundary of U in Y is a singleton $t \neq y, y'$ (see (21)).
(23) Denote by $k \in \mathbb{N}$ the number of end points of Y in $Y \setminus U$ (finite due to (18) and (22)).
(24) Fix $n \in \mathbb{N}$ such that $n > k$. The flower F_n in X_n contains at least $2n$ end points (it contains at least n leaves and at least n petals) and $\hat{F}_n$ contains at least $n + p - 1$ end points of Y (it contains at least n leaves and $p - 1$ petals) (see (13) and (14)).
(25) At least two end points u and v of Y contained in $\hat{F}_n$ are contained also in U (see (7), (23) and (24)).
(26) The root $\hat{r}_n$ of $\hat{F}_n$ is contained in U.

(Proof. Both arcs uy and vy (contained in the flower $\hat{F}_n$) cross the boundary of U in Y at the singleton t (see (25) and (22)). This point is not a floweret of $\hat{F}_n$ (it is y, see (13)). We conclude that there is a ramification point w of $\hat{F}_n$ in U. The arc $\hat{r}_n y$ must contain w (see (13)), and cross the boundary of U in Y at the singleton t only (see (22)). We see that the root $\hat{r}_n$ of $\hat{F}_n$ is contained in U.)

(27) The point t is contained in $\hat{F}_n$ (because the arc $\hat{r}_n y$ contains t, see the proof of (26)).
(28) The component C of $Y \setminus \{\hat{r}_n\}$ containing y is a subset of $\hat{F}_n$ (see (27)).
(29) Both points y and y' are contained in $\hat{F}_n \setminus \{\hat{r}_n\}$ (the component C contains $Y \setminus U$, because $t \in C$, see (22), (27), (28) and Proposition C).
(30) Order of y' in Y is at most 3 (the only ramification point of $\hat{F}_n$ of order greater than 3 is the floweret y, see (13) and (29)).
(31) Items (30) and (19) give a contradiction.

□

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GENERALIZED NONLINEAR PSEUDOCOCOERCIVE VARIATIONAL INEQUALITIES AND GENERAL PROXIMAL POINT METHODS

Ram U. Verma

Abstract. Let $T: K \to H$ be a hemicontinuous, η-pseudococoercive, $\eta - L$-relaxed monotone and d-Hölder continuous mapping from a nonempty closed invex subset K of an infinite-dimensional Hilbert space H into H. Let $f: K \to R$ be proper, invex and lower semicontinuous on K and $h: K \to R$ be continuously Fréchet-differentiable on K with h' (gradient of h) $\eta - \alpha$-strongly monotone and $\eta - \beta$-Lipschitz continuous on K. Suppose that there exist an $x^* \in K$ and numbers $a > 0$, $r \geq 0$, ρ^k $(a < \rho^k < \alpha/L)$ such that for all $t \in [0,1]$, and for all $x \in K^\#$, the set $S^\#$ defined by

$$S^\# = \{(h,\eta): h'(x^* + t(x - x^*))(x - x^*) \geq \langle h'(x^* + t\eta(x, x^*)), \eta(x, x^*)\rangle\}$$

is nonempty, where $K^\# = \{x \in K: \|x - x^*\| \leq r\}$, and $\eta: K \times K \to H$ is λ-Lipschitz continuous with the following assumptions:

(i) $\eta(x,y) + \eta(y,x) = 0$, $\eta(x,y) = \eta(x,z) + \eta(z,y)$, and $\|\eta(x,x^*)\| \leq r$.
(ii) For each fixed $y \in K$, map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.

If in addition, h' is continuous from H equipped with weak topology to H equipped with weak topology, then the sequence $\{x^k\}$ generated by the general proximal point algorithm converges weakly to a solution x^* of the variational inequality problem (VIP), that is,

$$\langle T(x^*), \eta(x, x^*)\rangle + f(x) - f(x^*) \geq 0 \quad \text{for all } x \in K.$$

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1. Introduction

Martinet [11] primarily initiated the proximal point method and then applied it to study convex optimization in Hilbert spaces. It has been followed by extensive generalizations to the method in terms of applications to convex programming, which in turn, led it to a whole range of new multiplier methods. Rockafellar [13] considered mainly three principal applications of proximal point algorithms to convex programming: the primal proximal minimization algorithm, the dual method of multipliers, and the primal-dual proximal method of multipliers. Recently, El Farouq [7] studied, based on general proximal point methods, approximation-solvability of a class of pseudomonotone variational inequalities in a finite-dimensional Hilbert space setting, while convergence of proximal methods for infinite-dimensional case requires a class of strongly pseudomonotone or a class of pseudococoercive mappings with some additional side impositions of hemicontinuity and Hölder continuity instead.

In this paper, we plan, based on the generalized proximal point method, to present the approximation-solvability of variational inequality problems (VIP) involving η-pseudococoercive mappings along with some results on Fréchet-differentiable functions which play a crucial role in the construction of the general framework for the proximal point methods. Results thus obtained complement works of El Farouq [7], Verma [16], and others.

Let H be an infinite-dimensional Hilbert space with the inner produce $\langle , \rangle$ and the norm $\|\cdot\|$ on H. We consider the variational inequality problem (VIP) as follows: determine an element $x^* \in K$ such that

$$\langle T(x^*), \eta(x, x^*)\rangle + f(x) - f(x^*) \geq 0 \quad \text{for all } x \in K, \tag{1.1}$$

where K is a nonempty closed invex subset of H, and $\eta: K \times K \to H$ is any mapping with some side conditions. For $\eta(x, x^*) = x - x^*$, the VIP (1.1) reduces to the VIP: find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle + f(x) - f(x^*) \geq 0 \quad \text{for all } x \in K, \tag{1.2}$$

where K is a nonempty closed convex subset of H.

For $f \equiv 0$ in (1.2), it reduces to: find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle \geq 0 \quad \text{for all } x \in K. \tag{1.3}$$

We need now to define and in some cases to upgrade the existing notions in the literature. Let $\eta: K \times K \to H$ be any mapping.

A mapping $T: K \to H$ is said to be *η-monotone* if

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq 0 \quad \text{for all } x, y \in K.$$

The mapping T is *$\eta - \alpha$-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq \alpha \|x - y\|^2 \quad \text{for all } x, y \in K.$$

The mapping T is *$\eta - \gamma$-cocoercive* if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq \gamma \|T(x) - T(y)\|^2 \quad \text{for all } x, y \in K.$$

A mapping $T: K \to H$ is said to be *η-pseudomonotone* if

$$\langle T(y), \eta(x,y)\rangle \geq 0 \Rightarrow \langle T(x), \eta(x,y)\rangle \geq 0 \quad \text{for all } x, y \in K.$$

$T: K \to H$ is *strongly $\eta - b$-pseudomonotone* if

$$\langle T(y), \eta(x,y)\rangle \geq 0 \Rightarrow \langle T(x), \eta(x,y)\rangle \geq b\|x-y\|^2 \quad \text{for all } x, y \in K.$$

T is *$\eta - c$-pseudococoercive* if there exists a constant $c > 0$ such that

$$\langle T(y), \eta(x,y)\rangle > 0 \Rightarrow \langle T(x), \eta(x,y)\rangle \geq c\|T(x)-T(y)\|^2 \quad \text{for all } x, y \in K.$$

T is *η-quasimonotone* if

$$\langle T(y), \eta(x,y)\rangle > 0 \Rightarrow \langle T(x), \eta(x,y)\rangle \geq 0 \quad \text{for all } x, y \in K.$$

T is *$\eta - L$-relaxed monotone* (also referred to as *weakly monotone*) if there exists a constant $L > 0$ such that

$$\langle T(x) - T(y), \eta(x,y)\rangle \geq (-L)\|x-y\|^2 \quad \text{for all } x, y \in K.$$

T is *η-hemicontinuous* if for all $x, y, w \in K$ if the function

$$t \in [0,1] \to \langle T[y + t\eta(x,y)], w\rangle$$

is continuous.

T is *hemicontinuous* if for all $x, y, w \in K$ if the function

$$t \in [0,1] \to \langle T[y + t(x-y)], w\rangle$$

is continuous.

A subset K of H is said to be *invex* if there exists a function $\eta: K \times K \to H$ such that whenever $x, y \in K$ and $t \in [0,1]$, it follows that $x + t\eta(y,x) \in K$.

A function $f: K \to R$ is called *invex* if whenever $x, y \in K$ and $t \in [0,1]$, it follows that

$$f[x + t\eta(y,x)] \leq (1-t)f(x) + tf(y).$$

When $\eta(x,y) = x - y$, all the above notions reduce to the well-known ones.

2. Auxiliary Results

In this section, we recall some auxiliary results crucial to the approximation-solvability of the VIP (1.1). Let $h: H \to R$ be a continuously Fréchet-differentiable mapping on a Hilbert space H. It follows that $h'(x) \in L(H,R)$—the space of all bounded linear operators from H into R. From now on, we shall denote the real number $h'(x)(y)$ by $\langle h'(x), y\rangle$ for $x, y \in H$. The following result is a modified version of [1, Lemma 4.1]:

Lemma 2.1. *Let H be a Hilbert space and K be a nonempty invex subset of H. Suppose that h', the gradient of $h: K \to R$, is $\eta - \alpha$-strongly monotone on K and the following assumptions hold:*

(i) *There exist an $x^* \in K$ and number $r \geq 0$ such that for all $x \in K^\#$, $t \in [0,1]$, the mapping $\eta: K \times K \to H$ satisfies:*

$$\|\eta(x, x^*)\| \leq r.$$

(ii) *The set $S^\#$ defined by*

$$S^\# = \{(h, \eta): h'(x^* + t(x - x^*))(x - x^*) \geq \langle h'(x^* + t\eta(x, x^*)), \eta(x, x^*)\rangle\}$$

is nonempty, where $h: K \to R$ is a continuously Fréchet-differentiable mapping and h' is the gradient of h.

(iii) *The set*

$$K^\# = \{x \in K: \|x - x^*\| \leq r\}.$$

Then, for all $x \in K^\#$ and $(h, \eta) \in S^\#$, we have

$$h(x) - h(x^*) - \langle h'(x^*), \eta(x, x^*)\rangle \geq (\alpha/2)\|x - x^*\|^2.$$

For $\eta(x, y) = x - y$, in Lemma 2.1, we arrive at

Lemma 2.2. *Let H be a Hilbert space and K be a nonempty convex subset of H. Suppose that h', the gradient of $h: K \to R$, is α-strongly monotone on K, where $h: K \to R$ is a continuously Fréchet-differentiable mapping.*

Then, for all $x, x^* \in K$, we have

$$h(x) - h(x^*) - \langle h'(x^*), x - x^*\rangle \geq (\alpha/2)\|x - x^*\|^2.$$

Lemma 2.3. *Let H be a Hilbert space and K be a nonempty invex subset of H. Suppose that the following assumptions hold:*

(i) *h' is $\eta - \partial$-Lipschitz continuous, where h' is the gradient of $h: K \to R$.*

(ii) *There exist an $x^* \in K$ and number $q \geq 0$ such that for all $x \in K$, $t \in [0,1]$, the mapping $\eta: K \times K \to H$ satisfies:*

$$\|\eta(x, x^*)\| \leq q,$$

where the set

$$K = \{x \in K: \|x - x^*\| \leq q\}.$$

(iii) *The set S_1 defined by*

$$S_1 = \{(h, \eta): h'(x^* + t(x - x^*))(x - x*) \leq \langle h'(x^* + t\eta(x, x^*)), \eta(x, x^*)\rangle\}$$

is nonempty.

Then, for all $x \in K_1$ and $(h, \eta) \in S_1$, the following inequality holds

$$h(x) - h(x^*) - \langle h'(x^*), \eta(x, x^*)\rangle \leq (\partial/2)\|x - x^*\|^2.$$

For $\eta(x, x^*) = x - x^*$ in Lemma 2.3, we have

Lemma 2.4. *Let H be a Hilbert space and K be a nonempty convex subset of H. Suppose that h' (gradient of h) is ∂-Lipschitz continuous, where $h: K \to R$ is continuously differentiable. Then for all $x, x^* \in K$, the following inequality holds*

$$h(x) - h(x^*) - \langle h'(x^*), x - x^*\rangle \leq (\partial/2)\|x - x^*\|^2.$$

The proof is similar to that of Lemma 2.1.

3. General Proximal Algorithm

This section deals with a discussion of the approximation-solvability of the VIP (1.1) based on the general proximal algorithm [7]. For some $x \in H$, the problem of finding $\overline{y}(x)$ such that

$$\langle \rho^k T(\overline{y}(x)) + h'(\overline{y}(x) - h'(x), \eta(y, \overline{y}(x))\rangle + \rho^k[f(y) - f(\overline{y}(x))] \geq 0 \text{ for all } y \in K, \tag{3.1}$$

is called the Proximal Algorithm.

Algorithm 3.1. For a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$)

$$\begin{aligned} &\langle \rho^k T(x^{k+1}) + h'(x^{k+1}) - h'(x^k), \eta(x, x^{k+1})\rangle \\ &\qquad + \rho^k[f(x) - f(x^{k+1})] \geq 0 \quad \text{for all } x \in K, \end{aligned} \tag{3.2}$$

where $h\colon K \to R$ is continuously Fréchet-differentiable, $\{\rho^k\}$ is the sequence of positive numbers, and $\eta\colon K \times K \to H$ is a mapping with some side restrictions.

For $\eta(x, y) = x - y$ in (3.2), we find

Algorithm 3.2. For a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$)

$$\begin{aligned} &\langle \rho^k T(x^{k+1}) + h'(x^{k+1}) - h'(x^k), x - x^{k+1}\rangle \\ &\qquad + \rho^k[f(x) - f(x^{k+1})] \geq 0 \quad \text{for all } x \in K, \end{aligned} \tag{3.3}$$

where K is a nonempty closed convex subset of H.

For $f \equiv 0$ in Algorithm 3.2, we obtain

Algorithm 3.3. For a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$)

$$\langle \rho^k T(x^{k+1}) + h'(x^{k+1}) - h'(x^k), x - x^{k+1}\rangle \geq 0 \quad \text{for all } x \in K, \tag{3.4}$$

where K is a nonempty closed convex subset of H.

We are just about ready to present, based on Algorithm 3.1, the approximation-solvability of the VIP (1.1).

Theorem 3.1. *Let $T\colon K \to H$ be an η-hemicontinuous, $\eta - \gamma$-pseudococoercive, d-Hölder continuous and $\eta - L$-relaxed monotone mapping from a nonempty closed invex subset K of an infinite-dimensional Hilbert space H into H. Let $f\colon K \to R$ be proper, invex and lower semicontinuous on K, $h\colon K \to R$ be continuously Fréchet-differentiable on K with h' (gradient of h) η-hemicontinuous, $\eta - \alpha$-strongly monotone, $\eta - \beta$-Lipschitz continuous, c-Hölder continuous and continuous from H equipped with weak topology to H equipped with weak topology.*

Suppose that there exist an $x' \in K$ and numbers $a > 0$, $r \geq 0$, ρ^k $(a < \rho^k < \alpha/L)$ such that for all $t \in [0, 1]$, and for all $x \in K^\#$, we have

(i) *the set $S^\#$ defined by*

$$S^\# = \{(h, \eta)\colon h'(x^* + t(x - x^*))(x - x^*) \geq \langle h'(x^* + t\eta(x, x^*)), \eta(x, x^*)\rangle\}$$

is nonempty, where $K^\#$ is defined by

$$K^\# = \{x \in K\colon \|x - x'\| \leq r\},$$

and $\eta: K \times K \to H$ *is* λ*-Lipschitz continuous with following assumptions*:

(ii) $\eta(x,y) + \eta(y,x) = 0$ *and* $\eta(x,y) = \eta(x,z) + \eta(z,y)$.

(iii) *For each fixed* $y \in K$, *map* $x \to \eta(y,x)$ *is sequentially continuous from the weak topology to the weak topology.*

(iv) $\|\eta(x,x^*)\| \leq r$.

(v) η *is expanding.*

If in addition, $x^* \in K$ *is a solution of the VIP* (1.1)*, then the sequence* $\{x^k\}$ *generated by Algorithm* 3.1 *converges weakly to* x^*.

Proof. Since T is $\eta - L$-relaxed monotone and h' is $\eta - \alpha$-strongly monotone, it follows that x^{k+1} is a unique solution of the proximal variational problem (3.2) for $\rho^k < \alpha/L$. Assumptions of η-hemicontinuity on T and h' ensure the existence of the solution x^{k+1}. To show that the sequence $\{x^k\}$ converges weakly to x^*, a solution of the VIP (1.1), we need to compute the estimates. Let us define a function Λ^* by

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), \eta(x^*, x)\rangle.$$

Then, on applying Lemma 2.1, we have

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), \eta(x^*, x)\rangle \geq (\alpha/2)\|x^* - x\|^2 \quad \text{for } x \in K, \tag{3.5}$$

where x^* is a solution of the VIP (1.1). It follows that

$$\Lambda^*(x^{k+1}) = h(x^*) - h(x^{k+1}) - \langle h'(x^{k+1}), \eta(x^*, x^{k+1})\rangle. \tag{3.6}$$

Now we can write

(3.7)
$$\begin{aligned}
&\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \\
&\quad = h(x^{k+1}) - h(x^k) - \langle h'(x^k), \eta(x^{k+1}, x^k)\rangle + \langle h'(x^{k+1}) - h'(x^k), \eta(x^*, x^{k+1})\rangle \\
&\quad \geq (\alpha/2)\|x^{k+1} - x^k\|^2 + \langle h'(x^{k+1}) - h'(x^k), \eta(x^*, x^{k+1})\rangle \\
&\quad \geq (\alpha/2)\|x^{k+1} - x^k\|^2 + \rho^k\langle T(x^{k+1}), \eta(x^{k+1}, x^*)\rangle + \rho^k(f(x^{k+1}) - f(x^*))
\end{aligned}$$

for $x = x^*$ in (3.2).

If we replace x by x^{k+1} in (1.1), we obtain

$$\langle (T(x^*), \eta(x^{k+1}, x^*)\rangle + f(x^{k+1}) - f(x^*)) \geq 0. \tag{3.8}$$

Since T is $\eta - \gamma$-pseudococoercive, it implies in light of (3.8) that

$$\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \geq [\alpha/2]\|x^{k+1} - x^k\|^2 + a\gamma\|T(x^{k+1}) - T(x^*)\|^2. \tag{3.9}$$

It follows from (3.9) that the sequence $\{\Lambda^*(x^k)\}$ is strictly decreasing except for $x^{k+1} - x^k$, and in that situation x^k is a solution to (1.1). Since the difference two consecutive terms tends to zero as $k \to \infty$, it implies that

$$\|x^{k+1} - x^k\| \to 0 \;\text{ and }\; \|T(x^{k+1}) - T(x^*)\| \to 0 \;\text{ as } k \to \infty.$$

On the top of that, in light of (3.5), we have

$$\|x^* - x^k\|^2 \leq (2/\alpha)\Lambda^*(x^k),$$

and so the sequence $\{x^k\}$ is bounded. Let x' be a cluster point of the sequence $\{x^k\}$, that is, there exists a subsequence $\{x^{kj}\}$ of the sequence $\{x^k\}$ such that $\{x^{kj}\}$ converges weakly to x'. Since h' is $\eta-\beta$-Lipschitz continuous and $a < \rho^k$, it follows using (3.2) that for some $x \in H$, we have

$$\begin{aligned}\langle \rho^k T(x^{k+1}), \eta(x, x^{k+1})\rangle + \rho^k[f(x) - f(x^{k+1})] &\geq -\langle h'(x^{k+1}) - h'(x^k), \eta(x, x^{k+1})\rangle \\ &\geq (-\beta)\|x^{k+1} - x^k\| \, \|\eta(x, x^{k+1})\|,\end{aligned}$$

or

$$\langle T(x^{k+1}), \eta(x, x^{k+1})\rangle + [f(x) - f(x^{k+1})] \geq (-\beta/a)\|x^{k+1} - x^k\| \, \|\eta(x, x^{k+1})\|. \tag{3.10}$$

Since $T(x^{k+1})$ converges strongly to $T(x^*)$ and $\|x^{kj+1} - x^{kj}\| \to 0$, it follows from (3.10) that

$$\langle T(x^*), \eta(x, x')\rangle + [f(x) - f(x')] \geq 0 \quad \text{for all } x \in K, \tag{3.11}$$

while $\langle T(x'), \eta(x^{kj}, x')\rangle + f(x^{kj}) - f(x')$ tends to zero. At this stage, if $T(x') = 0$, then x' is a solution to (1.1), and if $T(x') \neq 0$ then we can express it in the form

$$\eta(y^{kj}, x^{kj}) = -[\langle T(x'), \eta(x^{kj}, x')\rangle / \|T(x')\|^2] T(x').$$

It follows that

$$\langle T(x'), \eta(y^{kj}, x')\rangle = 0, \tag{3.12}$$

and

$$\|\eta(y^{kj}, x^{kj})\| \to 0 \quad \text{as } k \to \infty.$$

Therefore, y^{kj} converges weakly to x', while the d-Hölder continuity of T implies that $T(y^{kj})$ converges strongly to $T(x^*)$. Since T is $\eta-\gamma$-pseudococoercive, it follows in light of (3.12) that

$$\langle T(y^{kj}, \eta(y^{kj}, x')\rangle \geq \gamma\|T(y^{kj} - T(x')\|^2. \tag{3.13}$$

It clearly follows from (3.13) that $\{T(y^{kj})\}$ converges strongly to $T(x')$, and so $T(x') = T(x^*)$. Hence in light of (3.11) x' is a solution of (1.1).

When $\eta(x, y) = x - y$ in Theorem 3.1, we arrive at

Theorem 3.2. *Let $T: K \to H$ be an η-hemicontinuous, γ-pseudococoercive, d-Hölder continuous and L-relaxed monotone mapping from a nonempty closed invex subset K of an infinite-dimensional Hilbert space H into H. Let $f: K \to R$ be proper, invex and lower semicontinuous on K, $h: K \to R$ be continuously Fréchet-differentiable on K with h' (gradient of h) hemi-continuous, α-strongly monotone, β-Lipschitz continuous, c-Hölder continuous and continuous from H equipped with weak topology to H equipped with weak topology on K. If in addition, $x^* \in K$ is a solution of the VIP* (1.2)*, then the sequence $\{x^k\}$ generated by Algorithm* 3.2 *converges weakly to x^*.*

For $f \equiv 0$ in Theorem 3.2, it reduces to El Farouq [7, Theorem 5.1].

Theorem 3.3. *Let $T: K \to H$ be an η-hemicontinuous, γ-pseoducococoercive, d-Hölder continuous and L-relaxed monotone mapping from a nonepty closed invex subset K of an infinite-dimensional Hilbert space H into H. Let $h: K \to R$ be continuously Fréchet-differentiable on K with h' (gradient of h) hemi-continuous, α-strongly monotone, β-Lipschitz continuous, c-Hölder continuous and continuous from H equipped with weak topology to H equipped with weak topology on K. If in addition, $x^* \in K$ is a solution of the VIP* (1.3)*, then the sequence $\{x^k\}$ generated by Algorithm* 3.3 *converges weakly to x^*.*

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AN ANALYTICAL STUDY OF THE N-PERSON PRISONERS' DILEMMA

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Abstract. An analytical study of the N-person Prisoners' Dilemma game is presented for Pavlovian agents that modify their behavior according to Pavlov's experimental studies formulated by Thorndike's (1911) law of conditioning: if an action is followed by a satisfactory state of affairs, then the tendency to produce that particular action is reinforced. We found that in case of linear payoff functions a maximum 50% cooperation rate can be learned in a large community of agents starting with small cooperation rates. This limit can only be increased if the initial cooperation rate is above 50%. The results are confirmed by computer simulation experiments.

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1. Introduction

Prisoners' Dilemma is usually defined between two players (Rapoport and Chammah, 1965) and within game theory that assumes that the players act rationally. Realistic investigations of collective behavior, however, require a multi-person model of the game (Schelling, 1973). Various aspects of the multi-person Prisoners' Dilemma game have been investigated in the literature (Bixenstine et al. 1966, Weil 1966, Kelley and Grzelak 1972, Hamburger 1973, Anderson 1974, Bonacich et al. 1976, Goehring and Kahan 1976, Dawes 1980, Heckathorn 1988, Liebrand et al. 1992, Huberman and Glance 1993, Schulz et al. 1994, Schroeder 1995, Szilagyi 2000, Szilagyi and Szilagyi 2002).

We will define the N-person Prisoners' Dilemma by the following properties (Szilagyi and Szilagyi, 2000):

1. There are two actions available to each agent, and each agent must choose exactly one action: cooperation or defection.

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2. Regardless of what the other participants do, each participant receives a higher payoff for defecting behavior than for cooperating behavior, and
3. All participants receive a lower payoff if all defect than if all cooperate.

Let us consider the case of N participants with m of them cooperating. If $x = m/N$ is the ratio of cooperators and $C(x)$ and $D(x)$ are the payoff functions to a cooperator and a defector, respectively, then the above conditions can be expressed as

$$D(x) > C(x) \tag{1}$$

and

$$C(1) > D(0). \tag{2}$$

The outcome of the game depends on the personalities of the agents. For example, agents with short-term rationality will always choose defection because of Eq. (1), benevolent agents will ignore their short-term interests and will all cooperate, etc. It is realistic and interesting to consider Pavlovian agents that modify their behavior according to Pavlov's experimental studies formulated by Thorndike's (1911) law of conditioning: if an action is followed by a satisfactory state of affairs, then the tendency to produce that particular action is reinforced. Their response is stochastic but their probability of cooperation p changes by an amount proportional to their reward/punishment from the environment (the coefficient of proportionality is called the learning rate). These agents are primitive enough not to know anything about their rational choices but they have enough 'intelligence' to follow Thorndike's law.

The Prisoners' Dilemma game is an iterative process. Aggregate cooperation proportions are changing in time, i.e., over subsequent iterations. The updated probabilities lead to new decisions in the next iteration.

The payoff (reward/penalty) functions are given as two curves: one (C) for a cooperator and another (D) for a defector. According to Eqs. (1) and (2), the D curve is always above the C curve but $C(1)$ is higher than $D(0)$. The payoff to each agent depends on its choice and on the distribution of other players among cooperators and defectors. The payoff curves are, therefore, functions of the ratio x of cooperators to the total number of agents. To reduce the number of parameters, we will consider linear payoff functions in this study (Figure 1). Even in this very simple case, four parameters specify the environment.

2. Analytical Study

As we came to the conclusion above that the Pavlovian agents are realistic candidates for the investigation of the evolution of the game, let us assume that in a society of N Pavlovian agents the ratio of cooperators is x and the ratio of defectors is $(1 - x)$ at a certain time. It was shown (Szilagyi and Szilagyi, 2002) that when the cooperators receive the same total payoff as the defectors, i.e.,

$$xC(x) = (1 - x)D(x) \tag{3}$$

an equilibrium occurs. This may happen if $C(x)$ and $D(x)$ are both negative or both positive. In the first case, a small number of cooperators are punished big and a large number of defectors are punished little. This leads to a stable equilibrium at this point. In the second case, a large number of cooperators are rewarded slightly and a small number of defectors are rewarded greatly. This point corresponds to an unstable equilibrium.

If C and D are both linear functions of x, then Eq. (3) is a quadratic equation that has up to two real solutions. If both solutions are in the interval $0 < x < 1$, then both equilibria are present. We will call these equilibrium solutions x_1 and x_2, respectively, so that $0 < x_1 < x_2 < 1$. The initial cooperation probability (which is set as a constant and uniform across all the agents) is x_0.

Assume that $D(x) = c_d + b_d x$ and $C(x) = c_c + b_c x$. Relation (1) holds if and only if $D(0) > C(0)$ and $D(1) > C(1)$ which could be written as

$$c_d > c_c \tag{4}$$

and

$$c_d + b_d > c_c + b_c. \tag{5}$$

Inequality (2) has now the new special form

$$c_c + b_c > c_d. \tag{6}$$

The equilibrium condition (3) is a quadratic equation for x:

$$x^2(b_c + b_d) + x(c_c + c_d - b_d) - c_d = 0. \tag{7}$$

Before examining the locations of the equilibria, an important consequence of the above inequalities is derived. By adding equations (4) and (6), and equations (5) and (6) we conclude that both b_c and b_d are positive.

The roots of equation (7) are

$$x_{1,2} = \frac{1}{2(b_c + b_d)} \left[-(c_c + c_d - b_d) \pm \sqrt{(c_e + c_d - b_d)^2 + 4(b_c + b_d)c_d} \right]. \tag{8}$$

For simplified notations let

$$\gamma = -\frac{c_c}{b_c + b_d}, \quad \delta = -\frac{c_d}{b_c + b_d}, \quad b = \frac{b_d}{b_c + b_d},$$

then

$$\frac{b_c}{b_c + b_d} = 1 - b$$

and

$$x_{1,2} = \frac{1}{2} \left[(\gamma + \delta + b) \pm \sqrt{(\gamma + \delta + b)^2 - 4\delta} \right]. \tag{9}$$

Notice that conditions (4)–(6) and the positivity of b_c and b_d can be reformulated as

$$\gamma > \delta \tag{10}$$

$$\gamma + 2b > 1 + \delta \tag{11}$$

$$\gamma + b < 1 + \delta \tag{12}$$

$$0 < b < 1. \tag{13}$$

In our analysis we will consider several cases.

Case 1. Two positive equilibria exist. Both roots are real and positive if and only if

$$\delta > 0 \tag{14}$$

and

$$\gamma + \delta + b > 2\sqrt{\delta}. \tag{15}$$

From the practical point of view those payoff functions are interesting that lead to cooperation even when the initial cooperation rate is very low. Therefore our objective is to maximize the smaller root. This problem is a special nonlinear optimization problem maximizing

$$x_1 = \frac{1}{2}\left[(\gamma + \delta + b) - \sqrt{(\gamma + \delta + b)^2 - 4\delta}\right] \tag{16}$$

subject to constraints (10)–(15). Simple differentiation shows that x_1 strictly decreases in $\gamma + b$, therefore with fixed values of δ, we have to find first the smallest possible value of $\gamma+b$. The problem is a simple two-dimensional linear programming problem, which is solved by a graphical approach. Two cases have to be examined separately.

Assume first that $2\sqrt{\delta} - \delta < \frac{1+\delta}{2}$, which holds if $\delta < \frac{1}{9}$ or $\delta > 1$. Figure 2 shows the corresponding feasible set. The minimal value of $\delta + b$ occurs at the intercept of the lines $\gamma = \delta$ and $\gamma + 2b = 1 + \delta$ giving the solution $b = \frac{1}{2}$ and $\gamma = \delta$ with the optimal value: $\gamma + b = \delta + \frac{1}{2}$.

Assume next that $2\sqrt{\delta} - \delta \geq \frac{1+\delta}{2}$, which occurs when $\frac{1}{9} \leq \delta \leq 1$. Notice first that the intercept of the lines $\gamma + b = 2\sqrt{\delta} - \delta$ and $\gamma + 2b = 1 + \delta$ is given by point P of Figure 3 with coordinates $b = 1 + 2\delta - 2\sqrt{\delta}$ and $\gamma = 4\sqrt{\delta} - 3\delta - 1$. It is easy to see that $4\sqrt{\delta} - 3\delta - 1 \leq \delta$ for all $\delta > 0$ since it is equivalent to the obvious fact $0 \leq (2\sqrt{\delta} - 1)^2$. Hence point P is at or below the horizontal line $\gamma = \delta$, so the optimal value of $\gamma + b$ occurs again at the intercept of lines $\gamma = \delta$ and $\gamma + 2b = 1 + \delta$ with the optimal value of $\gamma + b = \delta + \frac{1}{2}$.

With fixed $\delta > 0$ and optimal $\gamma + b$ the smaller positive root becomes

$$\begin{aligned} x_1 &= \frac{1}{2}\left[\left(2\delta + \frac{1}{2}\right) - \sqrt{\left(2\delta + \frac{1}{2}\right)^2 - 4\delta}\right] \\ &= \frac{1}{2}\left[\left(2\delta + \frac{1}{2}\right) - \left|2\delta - \frac{1}{2}\right|\right] = \left\{\begin{array}{ll} \frac{1}{2} & \text{if } \delta \geq \frac{1}{4} \\ 2\delta & \text{if } \delta < \frac{1}{4} \end{array}\right\} \end{aligned} \tag{17}$$

showing that the maximum value of the smaller positive equilibrium is $\frac{1}{2}$, and this optimum is obtained at $b = \frac{1}{2}$, $\gamma = \delta$ and $\delta \geq \frac{1}{4}$. Notice that this is the case when the lines of $C(x)$ and $D(x)$ coincide, which is not realistic. So in realistic cases the maximum value $\frac{1}{2}$ cannot be reached but it can be approached arbitrarily.

Next we assume that the borderline case is excluded by the stricter constraint

$$\gamma > \delta + \varepsilon \tag{18}$$

with some small positive ε requiring that functions C and D are bounded away from each other at zero. In maximizing $\gamma + b$ we can follow the same procedure as before with the slight modification that the horizontal lines $\gamma = \delta$ of Figures 2 and 3 are replaced by $\gamma = \delta + \varepsilon$. Therefore the optimal value $\gamma + b$ occurs at the intercept of lines $\gamma = \delta + \varepsilon$ and $\gamma + 2b = 1 + \delta$, which is

$$b = \frac{1-\varepsilon}{2} \quad \text{and} \quad \gamma = \delta + \varepsilon$$

with the optimal value of $\gamma + b = \frac{2\delta+\varepsilon+1}{2}$. Then

$$x_1 = \frac{1}{2}\left[\frac{4\delta+\varepsilon+1}{2} - \sqrt{\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - 4\delta}\right]. \tag{19}$$

Simple differentiation shows that

$$\frac{\partial x_1}{\partial \delta} = \frac{1}{\sqrt{\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - 4\delta}}\left[\sqrt{\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - 4\delta} - \left(\frac{4\delta+\varepsilon+1}{2}\right) + 1\right]$$

where the bracketed factor is positive. Hence x_1 is strictly increasing in δ. From (19) we see that

$$\begin{aligned} x_1 &= \frac{\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - \left[\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - 4\delta\right]}{2\left[\frac{4\delta+\varepsilon+1}{2} + \sqrt{\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - 4\delta}\right]} \\ &= \frac{4\delta}{(4\delta+\varepsilon+1) + \sqrt{(4\delta+\varepsilon+1)^2 - 16\delta}} \\ &= \frac{4}{\left(4 + \frac{\varepsilon+1}{\delta}\right) + \sqrt{\left(4 + \frac{\varepsilon+1}{\delta}\right)^2 - \frac{16}{\delta}}} \end{aligned} \tag{20}$$

which implies that $x_1 \to \frac{1}{2}$ as $\delta \to \infty$. Therefore x_1 always remains under $\frac{1}{2}$, and with sufficiently large values of δ this limit can be approached arbitrarily.

Similarly we may require that functions C and D are bounded away from each other at $x = 1$. This condition can be written as

$$-\delta + b > -\gamma + 1 - b + \varepsilon$$

with some small positive ε. In this case constraint (11) is replaced by the more strict condition

$$\gamma + 2b > 1 + \delta + \varepsilon.$$

The minimal value of $\gamma + b$ occurs at $\gamma = \delta$ and $b = \frac{1+\varepsilon}{2}$ with the minimal value of $\gamma + b = \delta + \frac{1+\varepsilon}{2}$. Then x_1 has the same form as given in equation (19), so we get the same conclusions as before.

Next we assume that the lines of C and D are parallel, $b = \frac{1}{2}$, however we assume that $D(x) - C(x)$ is bounded away from zero. This condition can be rewritten as

$$\gamma > \delta + \varepsilon.$$

With the fixed value of δ,

$$x_1 = \frac{1}{2}\left[\left(\gamma+\delta+\frac{1}{2}\right) - \sqrt{\left(\gamma+\delta+\frac{1}{2}\right)^2 - 4\delta}\right]$$

which is strictly decreasing in γ. So the largest value of x_1 occurs at $\gamma = \delta + \varepsilon$, when x_1 becomes

$$x_1 = \frac{1}{2}\left[\left(2\delta+\varepsilon+\frac{1}{2}\right) - \sqrt{\left(2\delta+\varepsilon+\frac{1}{2}\right)^2 - 4\delta}\right].$$

This is the same as equation (19) with ε being replaced by $\frac{\varepsilon}{2}$, so we have the same results as in the previous cases.

Case 2. One positive and one nonpositive equilibrium exist.

From equation (14) we know that this is the case if and only if $\delta < 0$. By introducing the new variable $\Delta = -\delta$,

$$\Delta > 0 \tag{21}$$

and the positive equilibrium has the form

$$x_2 = \frac{1}{2}\left[(\gamma+b-\Delta) + \sqrt{(\gamma+b-\Delta)^2 + 4\Delta}\right]. \tag{22}$$

We want to find the smallest positive value of this root. Simple differentiation shows that x_2 is strictly increasing in $\gamma + b$, therefore with fixed value of Δ, the smallest value of $\gamma+b$ has to be selected subject to the constraints (10)–(13), which can be rewritten as follows:

$$\gamma > -\Delta \tag{23}$$
$$\gamma + 2b > 1 - \Delta \tag{24}$$
$$\gamma + b < 1 - \Delta \tag{25}$$
$$0 < b < 1. \tag{26}$$

This is again a two-dimensional linear programming problem. The graphical solution is illustrated in Figure 4. The smallest value of $\gamma + b$ occurs at the intercept of lines $\gamma = -\Delta$ and $\gamma+2b = 1-\Delta$ which is $\gamma = -\Delta$ and $b = \frac{1}{2}$ with the corresponding optimal value of $\gamma + b = \frac{1}{2} - \Delta$. Then

$$\begin{aligned} x_2 &= \frac{1}{2}\left[\left(\frac{1}{2}-2\Delta\right) + \sqrt{\left(\frac{1}{2}-2\Delta\right)^2 + 4\Delta}\right] \\ &= \frac{1}{2}\left[\left(\frac{1}{2}-2\Delta\right) + \left(\frac{1}{2}+2\Delta\right)\right] \\ &= \frac{1}{2} \end{aligned} \tag{27}$$

regardless of the value of Δ. This optimal case occurs when the lines $C(x)$ and $D(x)$ coincide with positive values at zero.

If we want to exclude the border line case $\gamma = -\Delta = \delta$, then the constraint $\gamma = -\Delta$ has to be replaced by the stricter inequality

$$\gamma > -\Delta + \varepsilon \tag{28}$$

with some small positive ε which bounds $C(0)$ and $D(0)$ away from each other. In optimizing $\gamma + b$ we can use the same method as before, but in this case the horizontal line $\gamma = -\Delta$ has to be replaced by $\gamma = -\Delta + \varepsilon$. The minimal value of $\gamma + b$ occurs at the intercept of lines $\gamma = -\Delta + \varepsilon$ and $\gamma + 2b = 1 - \Delta$ which is $b = \frac{1-\varepsilon}{2}$ and $\gamma = -\Delta + \varepsilon$ with the optimal value of $b + \gamma = \frac{1+\varepsilon-2\Delta}{2}$. Then

$$x_2 = \frac{1}{2}\left[\frac{1+\varepsilon-4\Delta}{2} + \sqrt{\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 + 4\Delta}\right]. \tag{29}$$

Simple differentiation shows that

$$\frac{\partial x_2}{\partial \Delta} = \frac{1}{\sqrt{\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 + 4\Delta}}\left[-\sqrt{\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 + 4\Delta} - \left(\frac{1+\varepsilon-4\Delta}{2}\right) + 1\right]$$

where the bracketed factor is always negative. Hence x_2 is strictly decreasing in Δ. From (29) we see that

$$\begin{aligned} x_2 &= \frac{\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 - \left[\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 + 4\Delta\right]}{2\left[\frac{1+\varepsilon-4\Delta}{2} - \sqrt{\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 + 4\Delta}\right]} \\ &= \frac{-4\Delta}{(1+\varepsilon-4\Delta) - \sqrt{(1+\varepsilon-4\Delta)^2 + 16\Delta}} \\ &= \frac{-4}{\left(-4 + \frac{1+\varepsilon}{\Delta}\right) - \sqrt{\left(\frac{1+\varepsilon}{\Delta} - 4\right)^2 + \frac{16}{\Delta}}} \end{aligned}$$

implying that $x_2 \to \frac{1}{2}$ as $\Delta \to \infty$. Therefore x_2 is always above $\frac{1}{2}$ and with sufficiently large values of Δ this limit can be approached arbitrarily.

If we assume that $D(1)$ and $C(1)$ are bounded away from each other, then condition (24) is modified as

$$\gamma + 2b > 1 + \varepsilon - \Delta.$$

Under this modified constraint the smallest value of $\gamma + b$ becomes $\gamma + b = -\Delta + \frac{1+\varepsilon}{2}$. Then x_2 becomes the same as equation (29) leading to the same results as obtained in the previous case.

And finally we consider the case of parallel lines of $C(x)$ and $D(x)$ but we assume that $D(x) - C(x)$ is bounded away from zero. That is, $b = \frac{1}{2}$ and

$$\gamma > -\Delta + \varepsilon.$$

The smallest value of $\gamma + b$ is $\gamma + b = -\Delta + \varepsilon + \frac{1}{2}$ so

$$x_2 = \frac{1}{2}\left[-2\Delta + \varepsilon + \frac{1}{2} + \sqrt{\left(-2\Delta + \varepsilon + \frac{1}{2}\right)^2 + 4\Delta}\right].$$

Notice that this is the same as equation (29) with ε being replaced by $\frac{\varepsilon}{2}$. So we reach the same results and conclusions as above.

3. Verification by Computer Simulation

We use an agent-based simulation model (Szilagyi and Szilagyi, 2000) for the experimental verification of these results. This model is implemented in the simulation tool called *Dilemma*. It has three distinctive features:

(1) It is a general framework for inquiry in which the properties of the environment as well as those of the agents are user-defined parameters and the number of agents is theoretically unlimited.

(2) The agents have various distinct, user-defined "personalities."

(3) The participating agents are described as stochastic learning cellular automata, i.e., as combinations of cellular automata (Wolfram, 1994) and stochastic learning automata (Narendra and Thathachar, 1989).

The simulation environment is a two-dimensional array of the participating agents. The size and shape of the simulation environment are user-defined variables. The size of the neighborhood is also a user-defined variable. It may be just the immediate neighborhood or any number of layers of agents around the given agent. In the limiting case that is considered in the present study, all agents are neighbors, and they collectively form the environment for each participating agent. In this case, the neighborhood extends to the entire array of agents.

A reward/penalty is the only input that the agents receive from the environment. The probabilities of the agents' actions are updated by this reward/penalty based on their and other agents' actions. New actions are taken according to these probabilities. Behavior is learned by adjusting the action probabilities to the responses of the environment.

The updating scheme may be different for different agents. This means that agents with completely different personalities can be allowed to interact with each other in the same experiment. Agents with various personalities and various initial states and actions can be placed anywhere in the array.

The specific probability updating schemes depend on the agents' personalities. The probability update curve is user-defined. It specifies the change in the probability of choosing the previously chosen action based on a number of factors. Such factors include the reward/penalty received for that action, the history of rewards/penalties received for all actions, the agent's neighbors' actions, etc.

A variety of rational and irrational personality profiles and their arbitrary combinations can be represented, including Pavlovian agents.

The state and action of each agent in the array change with time. One unit of time is called an iteration. With each iteration, the software tool draws the array of agents in a window on the computer's screen, with each agent in the array colored according to its most recent action. The experimenter can view and record the evolution of the society as it changes in time.

We performed a large number of experiments with Pavlovian agents and various linear payoff functions. We found that the aggregate behavior of the society of agents always converges to one of two kinds of equilibrium solutions. As the theoretical study (Szilagyi and Szilagyi, 2002) suggests, the two solutions are different from each other because 1) the solution at x_1 is a stable equilibrium (attractor) while the solution at x_2 is an unstable equilibrium (repulsor) and 2) the solution converges towards x_1 as an oscillation while it stablizes exactly in the $x_2 < x_0$ case.

We found that, in perfect agreement with the analytical study presented above, for the N-person Prisoners' Dilemma game played by Pavlovian agents in case of linear payoff functions a maximum 50% cooperation rate can be learned in a large community of agents starting with small cooperation rates. This limit can only be increased if the initial cooperation rate is above 50%.

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ERROR BOUNDS FOR NEWTON'S METHOD UNDER A WEAK KANTOROVICH-TYPE HYPOTHESIS

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Abstract. We recently showed that the famous Kantorovich hypothesis for the convergence of Newton's method can be weakened. Moreover we proved favorable local and semilocal convergence theorems under this condition involving Lipschitz and center Lipschitz conditions. Here based on these results we obtain even finer error bounds on the distances involved.

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Key Words and Phrases. Banach space, Newton's method, Kantorovich hypothesis, lower/upper error bounds, semilocal convergence theorem.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of equation

$$F(x) = 0 \tag{1}$$

where, $F: D \subseteq X \to Y$ is Fréchet differentiable in an open convex subset $D_0 \subseteq D$ and X, Y are Banach spaces.

The famous Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0) \quad (x_0 \in D) \tag{2}$$

has long played a central role in approximating solution x^* of non-linear equations and systems. Here $F'(x_n)$ denotes the Fréchet-derivative of operator F evaluated at $x = x_n$ $(n \geq 0)$. Under the famous Newton–Kantorovich hypothesis

$$h = 2K_0\eta \leq 1 \tag{3}$$

where K_0, η are defined by (6) and (4) respectively. Several authors have provided semilocal convergence theorems and error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ $(n \geq 0)$ [2]–[4]. Recently we showed that (3) can be weakened (see (7)) [1]. Under (7) we provided error bounds which are finer than the ones using the same information and (3). Here we obtain further finer lower and upper error bounds.

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2. Semilocal Analysis for Newton's Method

We showed in [1]:

Theorem 1. *Let $F\colon D \subseteq X \to Y$ be Fréchet differentiable in an open convex set $D_0 \subseteq D$ and, for some $x_0 \in D_0$, $F'(x_0)^{-1}$ exists. Assume there exist parameters $\eta \geq 0$, $\overline{K}_0 \geq 0$, $K_0 \geq 0$, $\delta \in [0,1]$ such that*

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta, \tag{4}$$

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \overline{K}_0\|x - x_0\|, \tag{5}$$

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq K_0\|x - y\| \tag{6}$$

for all $x, y \in D_0$,

$$h_\delta = (\delta\overline{k}_0 + K_0 + \ell)\eta \leq 1, \tag{7}$$

and

$$\overline{U} = \overline{U}(x_1, t^* - \eta) = \{x \in X \mid \|x - x_1\| \leq t^* - \eta\} \subseteq D_0, \tag{8}$$

where,

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{K_0(t_{n+1} - t_n)^2}{2[1 - \overline{K}_0 t_{n+1}]} \quad (n \geq 0), \tag{9}$$

$$t^* = \lim_{n\to\infty} t_n. \tag{10}$$

Then:

(i) *Newton's method $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in U for all $n \geq 0$ and converges to a solution $x^* \in \overline{U}$ of equation $F(x) = 0$ and $t^* \in [\eta, 2\eta]$.*

(ii) *the solution x^* is unique in $U(x_0, t^{**})$ provided that*

$$U(x_0, t^{**}) \subseteq D_0, \tag{11}$$

$$t^{**} \geq t^* \tag{12}$$

and

$$\overline{K}_0(t^* + t^{**}) \leq 2; \tag{13}$$

Moreover the following error bounds hold for all $n \geq 0$:

$$\|x_{n+2} - x_{n+1}\| \leq \frac{K_0\|x_{n+1} - x_n\|^2}{2[1 - \overline{K}_0\|x_{n+1} - x_0\|]} \leq t_{n+2} - t_{n+1}, \tag{14}$$

$$\|x_n - x^*\| \leq t^* - t_n, \tag{15}$$

$$\frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4a_n\|x_{n+1} - x_n\|}} \leq \|x_n - x^*\| \leq b_n \leq c_n, \tag{16}$$

$$\|x_n - x^*\| \leq d_n, \tag{17}$$

where,

$$a_n = \frac{K_0}{2[1-\overline{K}_0\|x_n - x_0\|]}, \tag{18}$$

$$b_n = \frac{K_0\|x_n - x_{n-1}\|^2}{2\big[1-\frac{\overline{K}_0}{2}(\|x_n - x_0\| + \|x^* - x_0\|)\big]}, \tag{19}$$

$$c_n = \frac{K_0\|x_n - x_{n-1}\|^2}{1-\overline{K}_0\|x_n - x_0\| + \sqrt{(1-\overline{K}_0\|x_n - x_0\|)^2 - \overline{K}_0 K_0\|x_n - x_{n-1}\|^2}}, \tag{20}$$

and

$$d_n = \frac{K_0}{2[1-\overline{K}_0\|x_{n-1} - x_0\|]}\|x_{n-1} - x^*\|^2. \tag{21}$$

Denote by $(0, \delta, F, x_0, \overline{U}_0, K_0, \overline{K}_0)$ the class satisfying the hypotheses of Theorem 1.

Under the hypotheses of Theorem 1, define:

$$\overline{U} = \overline{U}_0 = \overline{U}(x_0, t^*),\ \overline{U}_m = \overline{U}(x_m, t^* - t_m) \quad (m \geq 1), \tag{22}$$

$$K_m = \sup_{\substack{x,y\in\overline{U}_m \\ x\neq y}} \frac{\|F'(x_m)^{-1}[F'(x) - F'(y)]\|}{\|x-y\|}, \tag{23}$$

$$\overline{K}_m = \sup_{\substack{x\neq x_m \\ x\in\overline{U}_m}} \frac{\|F'(x_m)^{-1}[F'(x) - F'(x_m)]\|}{\|x-x_m\|}, \tag{24}$$

$$L_m = \sup_{\substack{x,y\in\overline{U} \\ x\neq y}} \frac{\|F'(x_m)^{-1}[F'(x) - F'(y)]\|}{\|x-y\|}, \tag{25}$$

$$\overline{L}_m = \sup_{\substack{x\neq x_m \\ x\in\overline{U}}} \frac{\|F'(x_m)^{-1}[F'(x) - F'(x_m)\|}{\|x-x_m\|}. \tag{26}$$

It follows from the Banach Lemma on invertible operators [3] and (7)

$$K_m \leq K_0\frac{1}{1-\overline{K}_0\|x_m - x_0\|} \tag{27}$$

$$\overline{K}_m \leq K_0\frac{1}{1-\overline{K}_0\|x_m - x_0\|}. \tag{28}$$

We show:

$$(K_m + \overline{K}_m))\|x_{m+1} - x_m\| \leq 1, \tag{29}$$

or

$$\frac{(1+\delta)K_0}{1-2\overline{K}_0\big[1-\left(\frac{1}{2}\right)^m\big]\eta}\left(\frac{1}{2}\right)^m \eta \leq \delta \tag{30}$$

or

$$\left\{(1+\delta)\left(\frac{1}{2}\right)^m K_0 + 2\delta\left[1-\left(\frac{1}{2}\right)^m\right]\overline{K}_0\right\}\eta \le \delta. \tag{31}$$

By (7) and (31) it suffices to show

$$(1+\delta)\left(\frac{1}{2}\right)^m K_0 + 2\delta\left[1-\left(\frac{1}{2}\right)^m\right]\overline{K}_0 \le K_0 + \overline{K}_0\delta \tag{32}$$

or

$$\overline{K}_0\delta\left[2\left(1-\left(\frac{1}{2}\right)^m\right)-1\right] \le K_0\left[1-(1+\delta)\left(\frac{1}{2}\right)^m\right] \tag{33}$$

or

$$\delta\left[1-\left(\frac{1}{2}\right)^{m-1}\right] \le 1-(1+\delta)\left(\frac{1}{2}\right)^m$$

or

$$\delta\left[1-\left(\frac{1}{2}\right)^m\right] \le 1-\left(\frac{1}{2}\right)^m$$

or

$$\delta \le 1,$$

which is true by hypothesis.

That completes the induction for (29).

Similarly we show:

$$(L_m+\overline{L}_m)\|x_{m+1}-x_m\| \le 1. \tag{34}$$

That is the conclusions of Theorem 1 can pass from $(0, \delta, F, x_0, \overline{U}_0, K_0, \overline{K}_0, L_0, \overline{L}_0)$ to the class $(M, \delta, F, x_m, \overline{U}_m, K_m, \overline{K}_m, L_m, \overline{L}_m)$ $(m \ge 1)$ in order to obtain finer bounds. Indeed, define sequences for all fixed $m \ge 0$ and $n \ge 0$

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$$t^2_{n+m+2} = t^2_{n+m+1} + \frac{K_m\|x_{n+m+1}-x_{n+m}\|^2}{2[1-\overline{K}_m\|x_{n+m+1}-x_m\|]}, \tag{35}$$

$$t^1_{n+m+2} = t^1_{n+m+1} + \frac{K^1_m\|x_{n+m+1}-x_{n+m}\|^2}{2[1-\overline{K}^1_m\|x_{n+m+1}-x_m\|]}, \tag{36}$$

$$K^1_m = \overline{K}^1_m = \frac{K_0}{1-\overline{K}_0\|x_m-x_0\|} \quad (m>0), \tag{37}$$

$$K^1_0 = K_0, \quad \overline{K}^1_0 = \overline{K}_0, \tag{38}$$

$$t_{n+m+2} = t_{n+m+1} + \frac{K^0_m(t_{n+m+1}-t_{n+m})^2}{2[1-K^0_m(t_{n+m+1}-t_{n+m})]}, \tag{39}$$

$$K^0_m = \overline{K}^0_m = \frac{K_0}{1-\overline{K}_0 t_m} \quad (m>0), \tag{40}$$

$$K^0_0 = K_0, \quad \overline{K}^0_0 = \overline{K}_0. \tag{41}$$

Denote by v^2, v^1, v, sequences defined as the t's with "K's" replaced by "L's".

Moreover, we have:

$$x^* \in \overline{U}_m \subseteq \overline{U}_{m-1} \subseteq \cdots \subseteq \overline{U}_1 = \overline{U}_0. \tag{42}$$

Furthermore, using (2) we obtain the approximation

(43)

$$\begin{aligned} x^* - x_{m+1} &= x^* - x_m + F'(x_m)^{-1}F(x_m) \\ &= -\int_0^1 F'(x_m)^{-1}\{F'(x_m + t(x^* - x_m)) - F'(x_m)\}(x^* - x_m)dt, \end{aligned}$$

where, $x_m + t(x^* - x_m)$, x_m belong to $\overline{U}_m$ if $m \geq 1$ and to D_0 if $n = 0$. Hence, we deduce

$$\|x^* - x_{m+1}\| \leq \frac{1}{2}K_m\|x^* - x_m\|^2 \leq \frac{1}{2}L_m\|x^* - x_m\|^2 \ (m \geq 0). \tag{44}$$

Hence, we showed:

Theorem 2. *Under the hypotheses of Theorem 1 the following error bounds hold for all fixed $m \geq 0$ and $n \geq 0$:*

$$\begin{aligned} \|x_{n+m+2} - x_{n+m+1}\| &\leq t^2_{n+m+2} - t^2_{n+m+1} \\ &\leq t^1_{n+m+2} - t^1_{n+m+1} \leq t_{n+m+2} - t_{n+m+1}, \end{aligned} \tag{45}$$

$$\begin{aligned} \|x_{n+m+2} - x_{n+m+1}\| &\leq v^2_{n+m+2} - v^2_{n+m+1} \\ &\leq v^1_{n+m+2} - v^1_{n+m+1} \leq v_{n+m+2} - v_{n+m+1}, \end{aligned} \tag{46}$$

$$t^i_{n+m+2} - t^i_{n+m+1} \leq v^i_{n+m+2} - v^i_{n+m+1} \quad i = 0,1,2, \ (v^{(0)} = v), \tag{47}$$

(48)

$$\begin{aligned} \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4K^1_n\|x_{n+1} - x_n\|}} &\leq \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 2K_n\|x_{n+1} - x_n\|}} \\ &\leq \|x^* - x_n\|, \end{aligned}$$

(49)

$$\begin{aligned} \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4L^1_n\|x_{n+1} - x_n\|}} &\leq \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 2L_n\|x_{n+1} - x_n\|}} \\ &\leq \|x^* - x_n\|, \end{aligned}$$

$$\|x^* - x_n\| \leq \frac{1}{2}K_n\|x^* - x_n\|^2 \leq \frac{K_0\|x^* - x_n\|^2}{2[1 - \overline{K}_0\|x_n - x_0\|]}, \tag{50}$$

and

$$\|x^* - x_n\| \leq \frac{1}{2}L_n\|x^* - x_n\|^2 \leq \frac{L_0\|x^* - x_n\|^2}{2[1 - \overline{L}_0\|x_n - x_0\|]}. \tag{51}$$

Following Yamamoto's suggestion (see [4], Theorem 3.3) as above we get:

Theorem 3. *Let* $F: D \subseteq X \to Y$, $\overset{\circ}{D}$ *be the interior of* D, *and for some* $x_0 \in \overset{\circ}{D}$, $F'(x_0)^{-1}$ *exists. Assume*: $\overline{U}_0' = \overline{U}(x_1, d_0) \subseteq \overset{\circ}{D}$, $d_0 = \|x_1 - x_0\|$ *and* F *is Fréchet-differentiable on* $\overline{U}_0'$.

Define:

$$K_0' = \sup_{\substack{x,y \in \overline{U}_0' \\ x \neq y}} \frac{\|F'(x_0)^{-1}[F'(x) - F'(y)]\|}{\|x - y\|},$$

$$\overline{K}_0' = \sup_{\substack{x \in \overline{U}_0' \\ x \neq x_0}} \frac{\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\|}{\|x - x_0\|}.$$

If $(K_0' + \overline{K}_0')d_0 \le 1$, *then conclusions (i) and (ii) of Theorem 1 hold by replacing* D_0, K_0, $\overline{K}_0$, $\overline{U}$ *by* $\overline{U}_0'$, K_0', $\overline{K}_0'$, $\overline{U}_0'$ *respectively. Furthermore, set* $\overline{U}_n' = \overline{U}(x_n, 2d_n)$, $d_n = \|x_{n+1} - x_n\|$ $(n \ge 1)$ *and*

$$K_n' = \sup_{\substack{x,y \in \overline{U}_n' \\ x \neq y}} \frac{\|F'(x_n)^{-1}[F'(x) - F'(y)]\|}{\|x - y\|} \quad (n \ge 1)$$

$$\overline{K}_n' = \sup_{\substack{x \in \overline{U}_n' \\ x \neq x_n}} \frac{\|F'(x_n)^{-1}[F'(x) - F'(x_n)]\|}{\|x - x_n\|} \quad (n \ge 1).$$

Then,

$$x^* \in \overline{U}_n' \subseteq \overline{U}_{n-1}' \subseteq \cdots \subseteq \overline{U}_1' \subseteq \overline{U}_0'$$

and all bounds involving "K's" hold with the "K's" replaced by "K''s".

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GLOBAL L^∞ BOUNDS FOR A CLASS OF QUASILINEAR PARABOLIC EQUATIONS

Werner Horn

Abstract. In this paper we study quasi-linear parabolic equations which admit solutions which lie in a certain prescribed (possibly infinite) interval. These equations are of the form $u_t - \nabla c'(u)\nabla u = f$ together with appropriate boundary conditions, where $c : (a, b) \to \mathbb{R}$ satisfies certain conditions. We derive sufficient conditions for the solution u to lie in the domain of c for all times $t > 0$.

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Introduction

Many physical systems can be described by a quasilinear parabolic initial-value problem of the form

$$u_t - \nabla c'(u)\nabla u = f \qquad \forall (x,t) \in \Omega \times \mathbb{R}^+ \tag{1}$$

$$u(x,0) = u_0(x) \qquad \forall x \in \Omega \tag{2}$$

together with a flux boundary condition of the form

$$\frac{\partial u}{\partial \nu} = -A(u - u_\Gamma) \qquad \forall (x,t) \in \partial\Omega \times \mathbb{R}^+. \tag{3}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ $(n = 2, 3)$ with smooth boundary $\Gamma = \partial\Omega$, and A is a positive constant. We are interested in the case when

$$c : (a,b) \to \mathbb{R} \tag{4}$$

is a smooth increasing function. The interval (a, b) is either finite or of the form (a, ∞).

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Examples of equations of this form are non-linear heat equations. In these u represents the absolute temperature and the domain of c is $(0, \infty)$. The energy balances in several papers about phase field equations have been of this form (see, e.g. [5 - 7, 9, 11]). Other examples where the solution u has to be positive are diffusion equations, where u is a density. In reaction-diffusion equations u can represent the concentration of a chemical compound. In this case $(a, b) = (0, 1)$.

To get global smooth solutions to the initial-value problem it is necessary to show that for every $T > 0$ there is a compact set $K_T \subset (a, b)$ such that

$$u(x,t) \in K_T \tag{5}$$

for all $(x, t) \in \Omega \times (0, T)$. This has been done for the specific examples mentioned above, but to the best of our knowledge, has not been proven in a more general setting. The intent of this paper is to generalize the techniques used in [5 - 7, 9]. In particular, we will use Moser-type estimates like the ones introduced by Alikakos in [1] to obtain sufficient conditions for global existence.

If c satisfies $c : \mathbb{R}^+ \to \mathbb{R}^-$ with

$$c'(\xi) \geq K_0 \xi^{-\alpha} \tag{6}$$

for all $\xi > 0$ and some $K_0 > 0$ and $\alpha > 0$, we have the following results.

Proposition 1. *Let $T > 0$ be given and assume that $f \in L^\infty(0, T; L^p(\Omega))$ for some $2 < p < \infty$ and*

$$1 < \alpha \leq \min\left\{\frac{2}{9}p + 1, p - 1\right\} \tag{7}$$

$$c'(\xi)\xi^{p-1} \text{ is non-decreasing for } \xi > 0. \tag{8}$$

Furthermore, assume that there are numbers $0 < a_0 \leq b_0 < \infty$ and $0 < a_\Gamma \leq b_\Gamma < \infty$ such that

$$a_0 \leq u_0(x) \leq b_0 \qquad \forall x \in \Omega \tag{9}$$

$$a_\Gamma \leq u_\Gamma(x,t) \leq b_\Gamma \qquad \forall (x,t) \in \partial\Omega \times (0,T). \tag{10}$$

Then there exist constants $0 < a_T \leq b_T < \infty$ such that the solution u of (1)-(3) satisfies

$$a_T \leq u(x,t) \leq b_T \qquad \forall (x,t) \in \Omega_T = \Omega \times (0,T). \tag{11}$$

For the important case of $\alpha = 1$ we have a weaker result:

Proposition 2. *Let u be a classical solution to*

$$u_t - \Delta \log u = f \tag{12}$$

on $\Omega \times (0, T)$ together with boundary condition (3). Furthermore, assume that there are numbers $0 < a_0 \leq b_0 < \infty$ and $0 < a_\Gamma \leq b_\Gamma < \infty$ such that

$$a_0 \leq u_0(x) \leq b_0 \qquad \forall x \in \Omega \tag{13}$$

$$a_\Gamma \leq u_\Gamma(x,t) \leq b_\Gamma \qquad \forall (x,t) \in \partial\Omega \times (0,T), \tag{14}$$

and that $f \in L^\infty(0,T;L^\infty(\Omega))$. Then there exists a constant $0 < b_T < \infty$ such that the solution u of (1)-(3) satisfies

$$u(x,t) \le b_T \qquad \forall (x,t) \in \Omega_T = \Omega \times (0,T). \tag{15}$$

Furthermore, if there is a $q \ge 2$ and constant C such that

$$\left\| \frac{1}{u(t)} \right\|_q \le C, \tag{16}$$

for all $0 < t < T$, there is a constant $a_T > 0$ such that

$$u(x,t) \ge a_T, \qquad \forall (x,t) \in \Omega_T. \tag{17}$$

Observe that we do not require $f > 0$. The equation satisfies an implicit maximum (and minimum) principle.

Remarks:

(1) Propositions 1 and 2 can be easily extended to the case $c: (a,\infty) \to (-\infty, d)$ by using the linear transformations $w(x,t) = u(x,t) - a$ and $\tilde{c}(\xi) = c(\xi - a) - d$ where w and $\tilde{c}$ satisfy the hypotheses of Propositions 1 and 2.

(2) The boundary condition (3) can be changed to

$$c'(u)\frac{\partial u}{\partial \nu} = -A(u - u_\Gamma). \tag{18}$$

The result will still hold and we do not need the condition that $c'(\xi)\xi^{p-1}$ is non-decreasing for $\xi > 0$. The same is true for the result of Proposition 3 below.

(3) The equation in Proposition 2 has been studied extensively for domains in $\mathbb{R}^2$ (see, e.g. [4]).

In Section 3 we investigate the situation, where c is a mapping $c: (a,b) \to \mathbb{R}$. The requirement for $c'(u)$ becomes:

$$c'(\xi) \ge K_a(\xi - a)^{-\alpha} + K_b(b - \xi)^{-\alpha} \tag{19}$$

for all $\xi > 0$, some $K_a, K_b > 0$ and some $\alpha > 1$. We have the following

Proposition 3. *Let $T > 0$ and suppose that $f \in L^\infty(0,T;L^p(\Omega))$ for some $2 \le p < \infty$ and that (15) holds. Further, assume that u satisfies the boundary condition (3) and that there is a $P > 0$ such that*

$$c'(\xi)\xi^P \tag{20}$$

is non-decreasing in ξ. Finally, assume there are numbers $a < a_0 \le b_0 < b$ and $a < a_\Gamma \le b_\Gamma < b$ such that

$$a_0 \le u_0(x) \le b_0 \qquad \forall x \in \Omega \tag{21}$$

$$a_\Gamma \le u_\Gamma(x,t) \le b_\Gamma \qquad \forall (x,t) \in \partial\Omega \times (0,T). \tag{22}$$

Then there exist constants $a < a_T \le b_T < b$ such that the solution u of (1)-(3) satisfies

$$a_T \le u(x,t) \le b_T \qquad \forall (x,t) \in \Omega_T = \Omega \times [0,T]. \tag{23}$$

Remark: Throughout the paper we assume local existence of solutions to these equations. Local existence can be shown for sufficiently smooth c and f. For example, for $c \in C^3((a,b))$, $f \in C^2(\overline{\Omega} \times \mathbb{R})$ and sufficiently smooth u_0, local existence of smooth solutions to (1)-(3) follows from [3: Theorem 16.1]. Global existence of solutions follows from local existence together with the L^∞-bounds proven in this paper (see, e.g., [2, 3]).

Proof of Proposition 1.

We assume that the boundary value problem (1)-(3) admits a unique local solution u. The crucial part of the proof of this theorem is to show that there exist uniform a priori bounds on u. To do this we need the following technical lemma, which is similar to a lemma proven by Ph. Laurençot ([7] and [10: Lemma 4.1.5]).

Lemma 1. *Let $\{\eta_k\}_{k=0}^\infty, \{\beta_k\}_{k=0}^\infty, \{\gamma_k\}_{k=0}^\infty$ be sequences of positive real numbers such that $A2^k \le \eta_k \le B2^k$, $\beta_k \to 2$ as $k \to \infty$ and $\beta_k \le 2\left(1 + \frac{C}{2^k}\right)$, $\gamma_k \uparrow$ and*

$$\gamma_{k+1} \le D\eta_{k+1}^2 \gamma_k^{\beta_k} \tag{24}$$

for positive constants A, B, C, D and all $k \ge 0$. Then there exists a constant $M > 0$ such that $\gamma_k^{\frac{1}{\eta_k}} \le M$ for all $k \ge 0$.

Proof.. The proof is very similar to that of a lemma by Laurençot, which can be found in the references [7] and [10: Lemma 4.1.5]. We only sketch it here. To start we use (24) repeatedly to obtain

$$\begin{aligned}\gamma_{k+1} &\le D\eta_{k+1}^2\gamma_k^{\beta_k} \le D^{1+\beta_k}\eta_{k+1}^2\eta_k^{2\beta_k}\gamma_{k-1}^{\beta_k\beta_{k-1}} \\ &\le D^{1+\beta_k+\beta_k\beta_{k-1}+\cdots+(\beta_k\beta_{k-1}\cdots\beta_2\beta_1)} \\ &\qquad \times \eta_{k+1}^2\eta_k^{2\beta_k}\eta_{k-1}^{2\beta_k\beta_{k-1}}\cdots\eta_1^{2\beta_k\beta_{k-1}\cdots\beta_1}\gamma_0^{\beta_k\beta_{k-1}\cdots\beta_0}.\end{aligned}$$

We continue by taking the η_{k+1}-th root of this last inequality to get

$$\begin{aligned}\gamma_{k+1}^{\frac{1}{\eta_{k+1}}} &= D^{(1+\beta_k+\beta_k\beta_{k-1}+\cdots+(\beta_k\beta_{k-1}\cdots\beta_2\beta_1))/\eta_{k+1}} \\ &\qquad \times \eta_{k+1}^{2/\eta_{k+1}}\eta_k^{2\beta_k/\eta_{k+1}}\eta_{k-1}^{2\beta_k\beta_{k-1}/\eta_{k+1}}\cdots\eta_1^{2\beta_k\beta_{k-1}\cdots\beta_1/\eta_{k+1}}\gamma_0^{\beta_k\beta_{k-1}\cdots\beta_0/\eta_{k+1}}\end{aligned}$$

The right side of this last inequality stays bounded as $k \to \infty$. This can be seen by taking the logarithm of this expression and observing that

$$\frac{\beta_k\beta_{k-1}\cdots\beta_{k_0}}{\eta_{k+1}} \le \frac{2^{k-k_0}}{\eta_{k+1}}\prod_{j=k_0}^{k}\left(1+\frac{C}{2^j}\right) \le K\frac{1}{2^{k_0}}$$

for a suitable constant K. The statement of the lemma follows immediately

The proof of Proposition 1 is now done by the following two lemmas.

Lemma 2. *Assume that the hypotheses of Proposition 1 are satisfied. Then there exists a constant b_T such that $\|u\|_{L^\infty(\Omega_T)} \le b_T$.*

Proof.. For $f \in L^\infty(0,T;L^p(\Omega))$, $2 \le p < \infty$ multiply equation (1) by u^{p-1}. After integration by parts and using the boundary condition (3) we obtain

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + (p-1)\int_\Omega c'(u)u^{p-2}(\nabla u)^2 dx \\ + A\int_{\partial\Omega} c'(u)(u^p - u_\Gamma u^{p-1})\, d\gamma = \int_\Omega f u^{p-1} dx.$$

Using (8) and Hölder's inequality yield

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + (p-1)K_0\int_\Omega u^{p-2-\alpha}(\nabla u)^2\, dx \\ + A\int_{\partial\Omega} c'(u)(u^p - u_\Gamma u^{p-1})\, d\gamma \le \|f\|_{L^p(\Omega)}\|u\|_{L^p(\Omega)}^{p-1}.$$

Note that

$$(p-1)K_0\int_\Omega u^{p-2-\alpha}(\nabla u)^2 dx = K_0\frac{4(p-1)}{(p-\alpha)^2}\int_\Omega \left(\nabla u^{\frac{p-\alpha}{2}}\right)^2 dx.$$

For fixed t the boundary $\Gamma = \partial\Omega$ can be decomposed into two parts depending on $t \in [0,T]$ as follows:

$$\Gamma_+(t) = \{z \in \partial\Omega : u(z,t) \ge 2u_\Gamma(z,t)\}$$
$$\Gamma_-(t) = \{z \in \partial\Omega : u(z,t) < 2u_\Gamma(z,t)\}.$$

On $\Gamma_+(t)$ we have

$$A\int_{\Gamma_+(t)} c'(u)\left(u^p - u_\Gamma u^{p-1}\right)\, d\gamma \ge \frac{A}{2}\int_{\Gamma_+(t)} c'(u)u^p\, d\gamma \\ \ge \frac{A}{2}K_0\int_{\Gamma_+(t)} u^{p-\alpha}\, d\gamma$$

and on $\Gamma_-(t)$ we have

$$A\int_{\Gamma_-(t)} c'(u)\left(u^p - u_\Gamma u^{p-1}\right)\, d\gamma \\ \ge \frac{A}{2}\int_{\Gamma_-(t)} c'(u)u^p\, d\gamma - A\int_{\Gamma_-(t)} c'(u)u^{p-1}u_\Gamma\, d\gamma \\ \ge \frac{A}{2}K_0\int_{\Gamma_-(t)} u^{p-\alpha}\, d\gamma - A\int_{\Gamma_-(t)} c'(u)u^{p-1}u_\Gamma\, d\gamma.$$

Since $c'(\xi)\xi^{p-1}$ is increasing, we get

$$\int_{\Gamma_-(t)} c'(u)u^{p-1}u_\Gamma\, d\gamma \le \int_{\Gamma_-(t)} c'(2u_\Gamma)(2u_\Gamma)^p\, d\gamma \le C.$$

This yields

$$A\int_{\partial\Omega} c'(u)\left(u^p - u_\Gamma u^{p-1}\right)\,d\gamma \geq \frac{A}{2}K_0\int_{\partial\Omega} u^{p-\alpha}\,d\gamma - C.$$

We now get the following inequality

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + K_0\frac{4(p-1)}{(p-\alpha)^2}\int_\Omega\left(\nabla u^{\frac{p-\alpha}{2}}\right)^2\,dx + \frac{A}{2}K_0\int_{\partial\Omega} u^{p-\alpha}\,d\gamma \leq \|f\|_{L^p(\Omega)}\|u\|^{p-1}_{L^p(\Omega)} + C,$$

i.e.

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + K_0\frac{4(p-1)}{(p-\alpha)^2}\|u^{\frac{p-\alpha}{2}}\|^2_{H^1(\Omega)} \leq \|f\|_{L^p(\Omega)}\,\|u\|^{p-1}_{L^p(\Omega)} + C.$$

After integrating this inequality with respect to t it becomes

$$\frac{1}{p}\|u(t)\|_p^p + \int_0^t K_0\frac{4(p-1)}{(p-\alpha)^2}\|u^{\frac{p-\alpha}{2}}(s)\|^2_{H^1(\Omega)} \leq \frac{1}{p}\|u(0)\|_p^p + C_1 t + \int_0^t \|u(s)\|_p^p\,ds + C_2.$$

Applying Gronwall's inequality (see, e.g., [8: Lemma 1.10]) to this last inequality we arrive at

$$\sup_{0\leq s\leq T}\|u\|_{L^p(\Omega)} \leq K_1 \qquad \text{and} \qquad \int_0^T \|u^{\frac{p-\alpha}{2}}\|^2_{H^1(\Omega)}ds \leq K_2$$

for suitable constants K_1 and K_2 depending only on the initial conditions, $\|f\|_p$ and T.

To continue let $\{P_n\}_{n\geq 0}$ be a sequence given by

$$P_0 = p$$
$$P_n = 2P_{n-1} + \alpha \quad (n \geq 1),$$

where α satisfies (7). This implies that $P_n \geq p$ for all $n \geq 0$. We multiply equation (1) by u^{P_n-1} to get after integration by parts and applying (3)

$$\frac{1}{P_n}\frac{d}{dt}\|u\|^{P_n}_{P_n} + (P_n-1)\int_\Omega c'(u)u^{P_n-2}(\nabla u)^2dx + A\int_{\partial\Omega} c'(u)(u^{P_n} - u_\Gamma u^{P_n-1})\,d\gamma = \int_\Omega f u^{P_n-1}dx.$$

The boundary terms are treated as before, and the lower bound on $c'(u)$ is used. After applying Hölder's inequality to the right-hand side we obtain

$$\begin{aligned}\frac{1}{P_n}\frac{d}{dt}\|u\|^{P_n}_{P_n} + K_0\frac{4(P_n-1)}{(P_n-\alpha)^2}\int_\Omega(\nabla u^{\frac{P_n-\alpha}{2}})^2dx &\leq \|f\|_{L^p(\Omega)}\left(\int_\Omega u^{(P_n-1)q}dx\right)^{\frac{1}{q}} + \int_{\Gamma_-(t)} c'(u_\Gamma)u_\Gamma^{P_n}d\gamma \\ &\leq C_1\left(\int_\Omega u^{(P_n-1)q}dx\right)^{\frac{1}{q}} + C_2. \qquad (25)\end{aligned}$$

Next, since $\alpha < 2p-1$ and $P_{n-1} \geq p$, we have $\frac{(P_n-1)q}{P_{n-1}} < 4$ and we can estimate the term on the right-hand side as

$$\left(\int_\Omega u^{(P_n-1)q}\,dx\right)^{\frac{1}{q}} = \left(\int_\Omega \left(u^{P_{n-1}}\right)^{\frac{(P_n-1)q}{P_{n-1}}}\,dx\right)^{\frac{1}{q}} \leq C_3 \left\|u^{P_{n-1}}\right\|_{L^4(\Omega)}^{2+\frac{\alpha-1}{P_{n-1}}}$$

To continue we apply the Nirenberg-Gagliardo inequality followed by Young's inequality to get

$$\begin{aligned} \left\|u^{P_{n-1}}\right\|_{L^4(\Omega)}^{2+\frac{\alpha-1}{P_{n-1}}} &\leq C \left\|u^{P_{n-1}}\right\|_{H^1(\Omega)}^{\frac{9}{10}(2+\frac{\alpha-1}{P_{n-1}})} \left\|u^{P_{n-1}}\right\|_{L^1(\Omega)}^{\frac{1}{10}(2+\frac{\alpha-1}{P_{n-1}})} \\ &\leq \delta_n \left\|u^{P_{n-1}}\right\|_{H^1(\Omega)}^2 + C_4 \frac{1}{\delta_n} \|u\|_{L^{P_{n-1}}(\Omega)}^{P_{n-1}Q_n} \end{aligned}$$

where $Q_n = 2\big(2+\frac{\alpha-1}{P_{n-1}}\big)/\big(2-9\frac{\alpha-1}{P_{n-1}}\big)$. Observe that these last two steps are always possible for the choice of α and p in (7). Using this, (25) becomes

$$\begin{aligned} \frac{1}{P_n}\frac{d}{dt}\|u\|_{P_n}^{P_n} + K_0 \frac{4(P_n-1)^2}{(P_n-\alpha)} \int_\Omega \left\|\nabla u^{P_{n-1}}\right\|^2\,dx \\ \leq \delta_n \left\|u^{P_{n-1}}\right\|_{H^1(\Omega)}^2 + C_4\frac{1}{\delta_n}\|u\|_{L^{P_{n-1}}(\Omega)}^{P_{n-1}Q_n} + C_2 \end{aligned}$$

Choosing δ_n small enough, we can move the H^1-term to the left with the derivative term. Integrating over $(0,t)$ and then take the supremum over $t \in [0,T]$ on both sides of the resulting inequality. We get

$$\sup_{0\leq t\leq T} \|u(t)\|_{P_n}^{P_n} \leq \|u(0)\|_{P_n}^{P_n} + P_n T \left(\sup_{0\leq t\leq T} C_4 \frac{1}{\delta_n} \|u(t)\|_{L^{P_{n-1}}(\Omega)}^{P_{n-1}Q_n} + C_2\right).$$

We now apply Lemma 1 with

$$\eta_n = P_n, \qquad \beta_n = Q_n, \qquad \gamma_n = \max\left\{\max_{k<n}\gamma_k, 1, \sup_{0\leq t\leq T}\|u(t)\|_{P_n}^{P_n}\right\}.$$

It is easy to check that these sequences satisfy the hypothesis of Lemma 2.1. We conclude that $\sup_{0\leq t\leq T}\|u(t)\|_{L^P(\Omega)} \leq b_T$ for all $P \in [p,\infty)$, where the constant b_T is independent of P. This immediately implies the lemma.

To get the lower bound for u in Proposition 1 let $w = \frac{1}{u}$. Then w satisfies the boundary-value problem

$$\frac{w_t}{w^2} - \nabla \frac{c'(1/w)}{w^2}\nabla w = -f \tag{26}$$

$$\frac{1}{w^2}\frac{\partial w}{\partial \nu} = A\Big(\frac{1}{w} - u_\Gamma\Big) \qquad \forall (x,t) \in \partial\Omega \times [0,T]. \tag{27}$$

For this new variable, we have

Lemma 3. *Assume that the hypotheses of Proposition 1 hold. Then there is a constant $C_T > 0$ such that the solution w of (26)-(27) satisfies*

$$\max_{0\le s\le T} \|w(s)\|_{L^\infty(\Omega)} \le C_T.$$

Proof.. The proof of this assertion again relies on a Moser-type argument. To start we multiply equation (26) by w^{r_0+1} for some $r_0 > \alpha$ and integrate over Ω to get

$$\frac{1}{r_0}\frac{d}{dt}\|w\|^{r_0}_{L^{r_0}(\Omega)} + (r_0+1)\int_\Omega w^{r_0}\frac{c'(\frac{1}{w})}{w^2}|\nabla w|^2\,dx \\ -A\int_{\partial\Omega} c'(\frac{1}{w})(w^{r_0} - u_\Gamma w^{r_0+1})\,d\gamma = -\int_\Omega f w^{r_0+1}dx.$$

For the boundary terms we again split the boundary into two parts:

$$\Gamma_+(t) = \left\{z\in\partial\Omega : w(z,t) \ge \frac{2}{u_\Gamma(z,t)}\right\}$$
$$\Gamma_-(t) = \left\{z\in\partial\Omega : w(z,t) < \frac{2}{u_\Gamma(z,t)}\right\}.$$

On $\Gamma_+(t)$ we have

$$\begin{aligned} A\int_{\Gamma_+(t)} c'\big(\frac{1}{w}\big)(u_\Gamma w^{r_0+1} - w^{r_0})\,d\gamma &\ge A\int_{\Gamma_+(t)} c'\big(\frac{1}{w}\big)(2w^{r_0} - w^{r_0})\,d\gamma \\ &= A\int_{\Gamma_+(t)} c'\big(\frac{1}{w}\big)w^{r_0}d\gamma \\ &\ge K_0 A\int_{\Gamma_+(t)} w^{r_0+\alpha}d\gamma \end{aligned}$$

and on $\Gamma_-(t)$ we have

$$\begin{aligned} &A\int_{\Gamma_-(t)} c'\big(\frac{1}{w}\big)(u_\Gamma w^{r_0+1} - w^{r_0})\,d\gamma \\ &\quad\ge Aa_\Gamma\int_{\Gamma_-(t)} c'\big(\frac{1}{w}\big)u_\Gamma w^{r_0+1}d\gamma - A\int_{\Gamma_-(t)} c'\big(\frac{1}{w}\big)w^{r_0}d\gamma \\ &\quad\quad\ge K_0 Aa_\Gamma\int_{\Gamma_-(t)} w^{r_0+\alpha}d\gamma - A\int_{\Gamma_-(t)} c'\big(\frac{1}{w}\big)w^{r_0}d\gamma. \end{aligned}$$

For the second term on the right we employ Lemma 2 to observe that $\frac{u_\Gamma}{2} < \frac{1}{w} < \|u\|_{L^\infty(\Omega\times\mathbb{R}^+)}$ and therefore

$$A\int_{\Gamma_-(t)} c'\big(\frac{1}{w}\big)w^{r_0}d\gamma \le C$$

for a suitable constant C. We combine these estimates to get

$$A\int_{\partial\Omega} c'\big(\frac{1}{w}\big)(u_\Gamma w^{r_0+1} - w^{r_0})\,d\gamma \ge C_1\int_{\partial\Omega} w^{r_0+\alpha}d\gamma - C.$$

The first term on the right of this last inequality and the term with the gradient can be combined in the H^1-norm. After moving the constant C to the right-hand side we get

$$(28)\quad \frac{1}{r_0}\frac{d}{dt}\|w\|^{r_0}_{L^{r_0}(\Omega)} + C_1(r_0)\|w^{\frac{r_0+\alpha}{2}}\|^2_{H^1(\Omega)} \leq \|f\|_{L^p(\Omega)}\left\|w^{\frac{r_0+\alpha}{2}}\right\|^{\frac{2(r_0+1)}{r_0+\alpha}}_{L^{\frac{2q(r_0+1)}{r_0+\alpha}}(\Omega)} + C.$$

Observe that $\frac{2q(r_0+1)}{r_0+\alpha} \leq 4$ and $\frac{r_0+1}{r_0+\alpha} < 1$. We can therefore use the Sobolev imbedding theorem and Young's inequality to get

$$\begin{aligned}\|f\|_{L^p(\Omega)}\left\|w^{\frac{r_0+\alpha}{2}}\right\|^{\frac{2(r_0+1)}{r_0+\alpha}}_{L^{\frac{2q(r_0+1)}{r_0+\alpha}}(\Omega)} &\leq \|f\|_{L^p(\Omega)}\|w^{\frac{r_0+\alpha}{2}}\|^{\frac{2(r_0+1)}{r_0+\alpha}}_{H^1(\Omega)}\\ &\leq \delta_0\frac{r_0+1}{r_0+\alpha}\|w^{\frac{r_0+\alpha}{2}}\|^2_{H^1(\Omega)} + \left(\frac{1}{\delta_0}\right)^{\frac{r_0+1}{\alpha-1}}\frac{\alpha-1}{r_0+\alpha}\|f\|^{\frac{r_0+\alpha}{\alpha-1}}_{L^p(\Omega)}.\end{aligned}$$

Choosing δ_0 sufficiently small, we can use this to estimate the right-hand-side in (28). Integrating the resulting inequality over $(0,t)$ yields

$$\begin{aligned}(29)\quad \|w(t)\|^{r_0}_{L^{r_0}(\Omega)} + C(r_0)\int_0^t \|w^{\frac{r_0+\alpha}{2}}(s)\|^2_{H^1(\Omega)}ds&\\ \leq Ct + C_0 + r_0 t\frac{1}{\delta_1}\frac{\alpha-1}{r_0+\alpha}\|f\|^{\frac{r_0+\alpha}{\alpha-1}}_{L^p(\Omega)}&.\end{aligned}$$

To continue let $\{r_n\}$ be a sequence of real numbers with r_0 as above and

$$r_n = 2r_{n-1} - \alpha.$$

After multiplying (26) with w^{r_n+1} we can repeat the above steps to arrive at.

$$(30)\quad \frac{1}{r_n}\frac{d}{dt}\|w\|^{r_n}_{L^{r_n}(\Omega)} + C_1(r_n)\|w^{r_{n-1}}\|^2_{H^1(\Omega)} \leq \|f\|_{L^p(\Omega)}\left\|w^{\frac{r_n+\alpha}{2}}\right\|^{\frac{2(r_n+1)}{r_n+\alpha}}_{L^{\frac{2q(r_n+1)}{r_n+\alpha}}(\Omega)} + C,$$

where

$$C_1(r_n) = \frac{4K_0(r_n+1)}{(r_n+\alpha)^2}.$$

Again we have $\frac{2q(r_0+1)}{r_0+\alpha} \leq 4$ and $\frac{r_0+1}{r_0+\alpha} < 1$, and we continue by applying the Nirenberg-Gagliardo inequality:

$$\begin{aligned}\left\|w^{\frac{r_n+\alpha}{2}}\right\|^{\frac{2(r_n+1)}{r_n+\alpha}}_{L^{\frac{2q(r_n+1)}{r_n+\alpha}}(\Omega)} &\leq \left\|w^{\frac{r_n+\alpha}{2}}\right\|^{\frac{2(r_n+1)}{r_n+\alpha}}_{L^4(\Omega)}\\ &\leq C\,\|w^{r_{n-1}}\|^{\frac{9(r_n+1)}{10r_{n-1}}}_{H^1(\Omega)}\,\|w^{r_{n-1}}\|^{\frac{r_n+1}{10r_{n-1}}}_{L(\Omega)}.\end{aligned}$$

Applying Young's inequality yields

$$\begin{aligned}&C\,\|w^{r_{n-1}}\|^{\frac{9(r_n+1)}{10r_{n-1}}}_{H^1(\Omega)}\,\|w^{r_{n-1}}\|^{\frac{r_n+1}{10r_{n-1}}}_{L(\Omega)}\\ &\quad\leq \delta_n\frac{9(r_n+1)}{r_{n-1}}\|w^{r_{n-1}}\|^2_{H^1(\Omega)} + \left(\frac{1}{\delta_n}\right)^{\frac{9(r_n+1)}{r_n+10\alpha}}\frac{r_n+10\alpha}{10r_n+10\alpha}\|w\|^{\frac{r_n+1}{r_n+10\alpha}}_{L^{r_{n-1}}(\Omega)}.\end{aligned}$$

We next choose $\delta_n = \frac{r_{n-1}}{18(r_n+1)} C_1(r_n) = \frac{4K_0}{9(r_n+1)}$, to estimate the H^1 term with the same term on the left. After integrating the resulting inequality over $(0,T)$, we get.

$$\frac{1}{r_n} \sup_{s\in(0,t)} \|w(s)\|^{r_n}_{L^r_n(\Omega 0} \leq C_2 t + A_n \left(\sup_{s\in(0,t)} \|w(s)\|_{L^r_{n-1}(\Omega)} \right)^{\frac{r_n+1}{r_n+10\alpha}}, \tag{31}$$

where

$$A_n = \left(\frac{9(r_n+1)}{4K_0} \right)^{\frac{9(r_n+1)}{r_n+10\alpha}}$$

We can now apply Lemma 1 to get the desired result.

Remarks:

(1) The proof of Lemma 3 has very few restrictions on the size of α, we only need $\alpha > 1$.

(2) The boundary conditions (18) significantly simplify the proof. In this case $c'(u)$ is automatically taken care of in the boundary conditions and the estimates for the boundary terms become less technical.

Proof of Proposition 2..

This proof follows largely the same lines as the proofs of Lemma 2 and 3. First we multiply (12) by u^p where p is sufficiently large. Integration over Ω yields.

$$\frac{1}{p+1}\frac{d}{dt}\|u\|^{p+1}_{p+1} + p\int_\Omega u^{p-2}(\nabla u)^2\,dx + A\int_{\partial\Omega} u^{p+1}\,d\gamma - A\int_{\partial\Omega} u_\Gamma u^p\,d\gamma = \int_\Omega f u^p\,dx.$$

Using Young's inequality on the negative boundary term we get

$$\int_{\partial\Omega} u_\Gamma u^p\,d\gamma \leq \frac{1}{2}\frac{p}{p+1}\int_{\partial\Omega} u^{p+1}\,d\gamma + \frac{1}{2(p+1}\int_{\partial\Omega} u_\Gamma^{p+1}\,d\gamma.$$

For the right-hand-side we get:

$$\left|\int_\Omega f u^p\,dx\right| \leq C\|u\|^{p+1}_{p+1}\|f\|_\infty.$$

Combining these and integrating over $(0,t)$ we get:

$$\|u(t)\|^{p+1}_{p+1} \leq \|u_0\|^{p+1}_{p+1} + (p+1)C_1\int_0^t \|u(s)\|^{p+1}_{p+1}\,ds + C_2 t,$$

where C_1 and C_2 only depend on f and the boundary values. Applying Gronwall's inequality we get

$$\|u(t)\|^{p+1}_{p+1} \leq C_2 t\,\|u_0\|^{p+1}_{p+1} e^{C_1(p+1)t}.$$

Taking the $p+1$ root yields

$$\|u(t)\|_{p+1} \leq \|u_0\|_{p+1}(C_2 T)^{1/(p+1)} e^{C_1 T}.$$

We can now take the supremum over $(0,T)$. The resulting inequality holds for all p and its right-hand-side is uniformly bounded in p by

$$\|u_0\|_\infty e^{C_1 T}.$$

Hence, we have

$$\|u(t)\|_\infty \le \|u_0\|_\infty \, e^{C_1 T}.$$

For the second part of Proposition 2 let $w = 1/u$, then w satisfies:

$$w_t - w^2 \nabla \frac{\nabla w}{w} = -f w^2 \tag{32}$$

$$\frac{\partial w}{\partial \nu} = Aw - Aw^2 u_\Gamma \tag{33}$$

As in the proof of Lemma 3 let $r_n = 2r_{n-1} - 1$, with $r_0 = p$ which is given in the hypothesis of the Proposition. We can now follow exactly the steps of the proof of Lemma 3 to get a lower bound.

Remark: Following the proofs of Lemma 2 and 3 one sees that lower bounds become easier as α increases and upper bounds become more difficult as α increases. It seems that $\alpha = 1$ is a limit case.

Proof of Proposition 3..

We prove the case when $a = 0$ and $b = 1$. The general case easily follows using the linear transformation $w = \frac{u-a}{b-a}$. The proof uses a change of the dependent variable. Proposition 3 can then be applied to the problem in the new variable.

To prove the theorem for $a = 0$ and $b = 1$ we change variables by $u = \frac{\phi}{1+\phi}$. Then $\frac{du}{d\phi} = \frac{1}{(1+\phi)^2}$ and ϕ satisfies the parabolic equation

$$\frac{1}{(1+\phi)^2}\phi_t - \nabla\Big(\frac{dc}{d\phi}\nabla\phi\Big) = f \tag{34}$$

and the boundary condition

$$\frac{1}{(1+\phi)^2}\frac{\partial\phi}{\partial\nu} = -A\Big(\frac{\phi}{1+\phi} - u_\Gamma\Big). \tag{35}$$

We now show that ϕ satisfies the a priori bound

$$\|\phi\|_{L^\infty(\Omega_T)} \le B_T \tag{36}$$

for some constant B_T. First observe that (19) implies

$$\frac{dc}{d\phi} \ge \left(K_0\Big(\frac{\phi}{1+\phi}\Big)^{-\alpha} + K_1\Big(\frac{1}{1+\phi}\Big)^{-\alpha}\right)\frac{1}{(1+\phi)^2} \ge K_1\frac{1}{(1+\phi)^{2-\alpha}}$$
$$\frac{dc}{d\phi} \ge \left(K_0\Big(\frac{\phi}{1+\phi}\Big)^{-\alpha} + K_1\Big(\frac{1}{1+\phi}\Big)^{-\alpha}\right)\frac{1}{(1+\phi)^2} \ge K_0\frac{1}{(1+\phi)^{2-\alpha}\phi^\alpha}.$$

To start, we multiply (34) by $(1+\phi)^2\phi^{r-1}$, where $r = r_n$, $n \ge 0$ as in the proof of Lemma 3. We make sure that $r_0 > P$, the exponent in (20). We get after integration by parts

$$\frac{1}{r}\frac{d}{dt}\|\phi\|^r_{L^r(\Omega)} + \int_\Omega \nabla\left((1+\phi)^2\phi^{r-1}\right)\frac{dc}{d\phi}\nabla\phi\,dx$$
$$+ A\int_{\partial\Omega}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}\Big(\frac{\phi}{1+\phi} - u_\Gamma\Big)d\gamma \le \int_\Omega f(1+\phi)^2\phi^{r-1}dx.$$

For the term involving the gradients observe that

$$
\begin{aligned}
&\int_\Omega \nabla\left((1+\phi)^2\phi^{r-1}\right)\frac{dc}{d\phi}\nabla\phi\,dx \\
&\quad= 2\int_\Omega (1+\phi)\phi^{r-1}\frac{dc}{d\phi}\left|\nabla\phi\right|^2 dx + (r-1)\int_\Omega (1+\phi)^2\phi^{r-2}\frac{dc}{d\phi}\left|\nabla\phi\right|^2 dx \\
&\quad\geq K_1 2\int_\Omega (1+\phi)^{\alpha-1}\phi^{r-1}\left|\nabla\phi\right|^2 dx + (r-1)K_1\int_\Omega (1+\phi)^{\alpha}\phi^{r-2}\left|\nabla\phi\right|^2 dx \\
&\quad\geq (r+1)K_1\int_\Omega \phi^{r+\alpha-2}|\nabla\phi|^2 dx \\
&\quad= K_1\frac{4(r+1)}{(r+\alpha)^2}\int_\Omega |\nabla\phi^{\frac{r+\alpha}{2}}|^2 dx
\end{aligned}
$$

To treat the boundary terms let

$$
\begin{aligned}
\Gamma_1(t) &= \left\{z\in\partial\Omega : \frac{\phi(z,t)}{1+\phi(z,t)} \geq c_1 = \frac{1+b_\Gamma}{2}\right\} \\
\Gamma_2(t) &= \left\{z\in\partial\Omega : \frac{\phi(z,t)}{1+\phi(z,t)} < c_1\right\}.
\end{aligned}
$$

On $\Gamma_1(t)$ we have

$$
\begin{aligned}
A\int_{\Gamma_1(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}\Big(\frac{\phi}{1+\phi}-u_\Gamma\Big)d\gamma &\geq A\frac{1-b_\Gamma}{2}K_1\int_{\Gamma_1(t)}(1+\phi)^{2+\alpha}\phi^{r-1}d\gamma \\
&\geq A\frac{1-b_\Gamma}{2}K_1\int_{\Gamma_1(t)}\phi^{r+1-\alpha}d\gamma \\
&\geq C_1\int_{\Gamma_1(t)}|\phi^{\frac{r+\alpha}{2}}|^2 d\gamma
\end{aligned}
$$

for a suitable constant C_1 and on $\Gamma_2(t)$ we have

$$
\begin{aligned}
&A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}\Big(\frac{\phi}{1+\phi}-u_\Gamma\Big)d\gamma \\
&\qquad= A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}\frac{\phi}{1+\phi}\,d\gamma - A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}u_\Gamma d\gamma \\
&\qquad\geq AK_1\int_{\Gamma_2(t)}\phi^{r+1+\alpha}d\gamma - A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}u_\Gamma d\gamma \\
&\qquad\geq C_2\int_{\Gamma_2(t)}|\phi^{\frac{r+\alpha}{2}}|^2 d\gamma - A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}u_\Gamma d\gamma
\end{aligned}
$$

for a suitable constant C_2. For the last term observe that (16) implies $\frac{dc}{du}u^P \leq \frac{dc}{du}\big|_{u=c_1} c_1^P \leq C_3$ and therefore

$$
(1+\phi)^4\phi^{r-1}\frac{dc}{d\phi} \leq C_3\phi^{r-1-P}(1+\phi)^{P-2} \leq C_3 M^{P+r+1}
$$

for a suitable constant M. Hence we have

$$A\int_{\Gamma_2(t)} (1+\phi)^4 \frac{dc}{d\phi}\phi^{r-1} u_\Gamma d\gamma \le C_4 M^{P+r+1}.$$

We can combine the boundary terms with the gradient term to get the inequality:

$$\frac{1}{r}\,\|\phi\|^r_{L^r(\Omega)} + C_5\|\phi^{\frac{r+\alpha}{2}}\|^2_{H^1(\Omega)} \le \int_\Omega f(1+\phi)^2\phi^{r-1}\,dx + C_4 M^{P+r+1}.$$

From here one follows exactly the steps in the proof of Lemma 3, i.e. one first proves a bound for the r_0-norm of ϕ and then uses this to get a sequence of bounds for the r_n norms. Application of Lemma 1. implies that ϕ is uniformly bounded by B_T. It follows that

$$u(x,t) \le \frac{B_T}{1+B_T} = b_T < 1 \qquad \forall (x,t)\in \Omega_T.$$

To get the lower bound for u a different change of dependent variable is used. We let $u = \frac{1}{\psi}$. Then ψ satisfies

$$\frac{1}{(\psi)^2}\psi_t - \nabla\Big(\frac{c'(1/\psi)}{\psi^2}\nabla\psi\Big) = -f$$

and the boundary condition

$$\frac{1}{\psi^2}\frac{\partial\psi}{\partial\nu} = A\Big(\frac{1}{\psi} - u_\Gamma\Big).$$

This is exactly equation (26). Lemma 3 can be applied to get an upper bound A_T for ψ. It follows that $u(x,t) \ge \frac{1}{A_T} = a_T > 0$ for all $(x,t)\in\Omega_T$. This concludes the proof of Proposition 3.

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GENERALIZED JENSEN'S EQUATIONS IN BANACH MODULES OVER A UNITAL C^*-ALGEBRA

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Abstract. We prove the Hyers-Ulam-Rassias stability of generalized Jensen's equations in Banach modules over a unital C^*-algebra. It is applied to show the stability of generalized Jensen's equations in a Hilbert module over a unital C^*-algebra. Moreover, we prove the stability of linear operators in a Hilbert module over a unital C^*-algebra.

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Key Words and Phrases. Banach module over C^*-algebra, generalized Jensen's equation, stability, Hilbert module over C^*-algebra, real rank 0

1. Generalized Jensen's equations

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G, \omega)$. $C^*(\mathbb{Z}^m, \omega)$ is said to be a *noncommutative torus of rank m* and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and the multiplier is called *totally skew* if the symmetry group S_ω is trivial. And A_ω is called *completely irrational* if ω is totally skew (see [2]). It was shown in [2] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G, \omega)$ is a simple C^*-algebra. It was shown in [3, Theorem 1.5] that if A_ω is a simple noncommutative torus then A_ω has real rank 0, where *"real rank* 0" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [3, 5]).

Let E_1 and E_2 be Banach spaces. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$||f(x+y) - f(x) - f(y)|| \leq \epsilon(||x||^p + ||y||^p)$$

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX for all $x, y \in E_1$. Th.M. Rassias [13] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2-2^p}||x||^p$$

for all $x \in E_1$.

Lemma A. *Let V and W be vector spaces, and let r, s, t be positive integers. A mapping $f : V \to W$ with $f(0) = 0$ is a solution of the equation*

$$rf(\frac{sx+ty}{r}) = sf(x) + tf(y) \tag{A}$$

for all $x, y \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Proof. Assume that $f : V \to W$ satisfies the equation (A). Then

$$\begin{aligned} rf(\frac{s}{r}x) &= rf(\frac{sx+t\cdot 0}{r}) = sf(x) + tf(0) = sf(x), \\ rf(\frac{t}{r}x) &= rf(\frac{s\cdot 0+tx}{r}) = sf(0) + tf(x) = tf(x) \end{aligned}$$

for all $x \in V$. So

$$f(\frac{s}{r}x) = \frac{s}{r}f(x) \quad \& \quad f(\frac{t}{r}x) = \frac{t}{r}f(x)$$

for all $x \in V$. And

$$\begin{aligned} f(x) &= f(\frac{s}{r}\cdot\frac{r}{s}x) = \frac{s}{r}f(\frac{r}{s}x), \\ f(x) &= f(\frac{t}{r}\cdot\frac{r}{t}x) = \frac{t}{r}f(\frac{r}{t}x) \end{aligned}$$

for all $x \in V$. So

$$f(\frac{r}{s}x) = \frac{r}{s}f(x) \quad \& \quad f(\frac{r}{t}x) = \frac{r}{t}f(x)$$

for all $x \in V$. Thus

$$\begin{aligned} f(x+y) &= \frac{1}{r}\cdot rf(\frac{s}{r}\cdot\frac{r}{s}x + \frac{t}{r}\cdot\frac{r}{t}y) = \frac{1}{r}(sf(\frac{r}{s}x) + tf(\frac{r}{t}y)) \\ &= \frac{1}{r}(s\cdot\frac{r}{s}f(x) + t\cdot\frac{r}{t}f(y)) = f(x) + f(y) \end{aligned}$$

for all $x, y \in V$.

Conversely, assume that $f : V \to W$ satisfies the additive Cauchy equation. Since $f(sx) = rf(\frac{s}{r}x)$ for all $x \in V$,

$$f(\frac{s}{r}x) = \frac{1}{r}f(sx) = \frac{s}{r}f(x)$$

for all $x \in V$. Similarly, one can show that

$$f(\frac{t}{r}y) = \frac{1}{r}f(ty) = \frac{t}{r}f(y)$$

for all $y \in V$. So $f(\frac{sx+ty}{r}) = f(\frac{s}{r}x) + f(\frac{t}{r}y) = \frac{s}{r}f(x) + \frac{t}{r}f(y)$ for all $x, y \in V$. Thus

$$rf(\frac{sx+ty}{r}) = sf(x) + tf(y)$$

for all $x, y \in V$. $\square$

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$, A_{in} the set of invertible elements in A, A_{sa} the set of self-adjoint elements in A, $A_1 = \{a \in A \mid |a| = 1\}$, and A_1^+ the set of positive elements in A_1. And let ${}_A\mathcal{B}$ and ${}_A\mathcal{C}$ be left Banach A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. Let s, t be different positive integers, r a positive integer, and $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ a mapping such that

$$\text{(i)} \qquad \widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} (\frac{t}{s})^{2k} \varphi((\frac{s}{t})^{2k}x, (\frac{s}{t})^{2k}y) < \infty$$

for all $x, y \in {}_A\mathcal{B}$.

In this paper, we prove the Hyers-Ulam-Rassias stability of the functional equation (A) in Banach modules over a unital C^*-algebra.

2. Stability of generalized Jensen's equations in Banach modules over a C^*-algebra

Theorem 2.1. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\text{(ii)} \qquad \|rf(\frac{sx+ty}{r}) - sf(x) - tf(y)\| \le \varphi(x, y)$$

for all $x, y \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\text{(iii)} \qquad \|f(x) - T(x)\| \le \frac{1}{s}\widetilde{\varphi}(x, -\frac{s}{t}x) + \frac{t}{s^2}\widetilde{\varphi}(-\frac{s}{t}x, (\frac{s}{t})^2 x)$$

for all $x \in {}_A\mathcal{B}$. Furthermore, if $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, then the additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is $\mathbb{R}$-linear.

Proof. Let $x \in {}_A\mathcal{B}$. For $y = -\frac{s}{t}x$, the inequality (ii) implies

$$(1) \qquad \|sf(x) + tf(-\frac{s}{t}x)\| \le \varphi(x, -\frac{s}{t}x).$$

Replacing x by $-\frac{s}{t}x$ and y by $(\frac{s}{t})^2x$, the inequality (ii) implies

$$(2) \qquad \|sf(-\frac{s}{t}x) + tf((\frac{s}{t})^2x)\| \le \varphi(-\frac{s}{t}x, (\frac{s}{t})^2x).$$

From (1) and (2), we get

$$\|f(x) - (\frac{t}{s})^2 f((\frac{s}{t})^2x)\| \le \frac{1}{s}\varphi(x, -\frac{s}{t}x) + \frac{t}{s^2}\varphi(-\frac{s}{t}x, (\frac{s}{t})^2x).$$

Applying the induction argument on n, we obtain

$$\begin{aligned}\|f(x)-(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)\| \le \sum_{k=0}^{n-1}(&\frac{1}{s}(\frac{t}{s})^{2k}\varphi((\frac{s}{t})^{2k}x,-(\frac{s}{t})^{2k+1}x)\\ &+\frac{1}{s}(\frac{t}{s})^{2k+1}\varphi(-(\frac{s}{t})^{2k+1}x,(\frac{s}{t})^{2k+2}x)).\end{aligned} \tag{3}$$

We claim that the sequence $\{(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)\}$ is a Cauchy sequence. Indeed, for $n>m$, we have

$$\begin{aligned}&\|(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)-(\frac{t}{s})^{2m}f((\frac{s}{t})^{2m}x)\|\\ &\quad\le\sum_{k=m}^{n-1}\|(\frac{t}{s})^{2k+2}f((\frac{s}{t})^{2k+2}x)-(\frac{t}{s})^{2k}f((\frac{s}{t})^{2k}x)\|\\ &\quad\le\sum_{k=m}^{n-1}(\frac{1}{s}(\frac{t}{s})^{2k}\varphi((\frac{s}{t})^{2k}x,-(\frac{s}{t})^{2k+1}x)+\frac{1}{s}(\frac{t}{s})^{2k+1}\varphi(-(\frac{s}{t})^{2k+1}x,(\frac{s}{t})^{2k+2}x)).\end{aligned}$$

From (i), it follows that

$$\lim_{m\to\infty}\sum_{k=m}^{n-1}(\frac{1}{s}(\frac{t}{s})^{2k}\varphi((\frac{s}{t})^{2k}x,-(\frac{s}{t})^{2k+1}x)+\frac{1}{s}(\frac{t}{s})^{2k+1}\varphi(-(\frac{s}{t})^{2k+1}x,(\frac{s}{t})^{2k+2}x))=0.$$

Since ${}_A\mathcal{D}$ is a Banach space, the sequence $\{(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)\}$ converges. Define

$$T(x)=\lim_{n\to\infty}(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)$$

for all $x\in {}_A\mathcal{B}$. Taking the limit in (3) as $n\to\infty$, we obtain

$$\|f(x)-T(x)\|\le\frac{1}{s}\widetilde{\varphi}(x,-\frac{s}{t}x)+\frac{t}{s^2}\widetilde{\varphi}(-\frac{s}{t}x,(\frac{s}{t})^2x)$$

for all $x\in {}_A\mathcal{B}$. This completes the proof of the inequality (iii). From the definition of T, we get

$$(\frac{s}{t})^{2n}T(x)=T((\frac{s}{t})^{2n}x)\text{ and }T(0)=0. \tag{4}$$

From (i), (ii), and the definition of T,

$$\begin{aligned}&\|rT(\frac{sx+ty}{r})-sT(x)-tT(y)\|\\ &\qquad=\lim_{n\to\infty}(\frac{t}{s})^{2n}\|rf((\frac{s}{t})^{2n}\frac{sx+ty}{r})-sf((\frac{s}{t})^{2n}x)-tf((\frac{s}{t})^{2n}y)\|\\ &\qquad\le\lim_{n\to\infty}(\frac{t}{s})^{2n}\varphi((\frac{s}{t})^{2n}x,(\frac{s}{t})^{2n}y)=0\end{aligned}$$

for all $x,y\in {}_A\mathcal{B}$. So

$$rT(\frac{sx+ty}{r})=sT(x)+tT(y)$$

for all $x, y \in {}_A\mathcal{B}$. By Lemma A, T is additive.

If $F : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is another additive mapping satisfying (iii), then it follows from (iii), (4) and the proof of Lemma A that

$$\|T(x) - F(x)\| = \|(\frac{t}{s})^{2n}T((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2n}F((\frac{s}{t})^{2n}x)\|$$
$$\leq \|(\frac{t}{s})^{2n}T((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)\| + \|(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2n}F((\frac{s}{t})^{2n}x)\|$$
$$\leq 2(\frac{t}{s})^{2n}(\frac{1}{s}\widetilde{\varphi}((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}(-\frac{s}{t})x) + \frac{t}{s^2}\widetilde{\varphi}((\frac{s}{t})^{2n}(-\frac{s}{t})x), (\frac{s}{t})^{2n}(\frac{s}{t})^2x)),$$

which tends to zero as $n \to \infty$ by (i). Thus we conclude that

$$T(x) = F(x)$$

for all $x \in {}_A\mathcal{B}$. This completes the uniqueness of T.

Assume that $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$. By the assumption, $\frac{s}{t}$ is a rational number which is not an integer. The additive mapping T given above is similar to the additive mapping T given in the proof of [13, Theorem]. By the same reasoning as the proof of [13, Theorem], the additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is $\mathbb{R}$-linear. □

Corollary 2.2. *Let $0 < p < 1$ and $t < s$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|rf(\frac{sx + ty}{r}) - sf(x) - tf(y)\| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \leq \frac{s^{2(1-p)}}{s^{2(1-p)} - t^{2(1-p)}}(\frac{1}{s} + \frac{1}{t^p s^{1-p}} + \frac{t^{1-p}}{s^{2-p}} + \frac{t^{1-2p}}{s^{2-2p}})||x||^p$$

for all $x \in {}_A\mathcal{B}$.

Proof. Define $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ by $\varphi(x, y) = ||x||^p + ||y||^p$, and apply Theorem 2.1. □

Corollary 2.3. *Let $p > 1$ and $t > s$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|rf(\frac{sx + ty}{r}) - sf(x) - tf(y)\| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \leq \frac{t^{2(p-1)}}{t^{2(p-1)} - s^{2(p-1)}}(\frac{1}{s} + \frac{s^{p-1}}{t^p} + \frac{s^{p-1}}{t^{p-1}} + \frac{s^{2-2p}}{t^{2p-1}})||x||^p$$

for all $x \in {}_A\mathcal{B}$.

Proof. Define $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ by $\varphi(x, y) = ||x||^p + ||y||^p$, and apply Theorem 2.1. □

Theorem 2.4. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|raf(\frac{sx+ty}{r}) - sf(ax) - tf(ay)\| \le \varphi(x,y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

Proof. Let $a = 1 \in A_1^+$. By Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii). By the continuity and the uniform convergence, the $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is continuous.

By the assumption, for each $a \in A_1^+ \cup \{i\}$,

$$\begin{aligned}
&\|raf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}x) - (s+t)f((\frac{s}{t})^{2n}ax)\| \le \varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}x),\\
&\|rf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}ax) - (s+t)f((\frac{s}{t})^{2n}ax)\| \le \varphi((\frac{s}{t})^{2n}ax, (\frac{s}{t})^{2n}ax)
\end{aligned}$$

for all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned}
\|rf(\frac{(s+t)}{r}&(\frac{s}{t})^{2n}ax) - raf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}x)\|\\
&\le \|rf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}ax) - (s+t)f((\frac{s}{t})^{2n}ax)\|\\
&\quad + \|raf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}x) - (s+t)f((\frac{s}{t})^{2n}ax)\|\\
&\le \varphi((\frac{s}{t})^{2n}ax, (\frac{s}{t})^{2n}ax) + \varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}x)
\end{aligned}$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. Thus

$$(\frac{t}{s})^{2n}\|rf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}ax) - raf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}x)\| \to 0$$

as $n \to \infty$ for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. Hence

$$\begin{aligned}
rT(\frac{s+t}{r}ax) &= \lim_{n\to\infty}(\frac{t}{s})^{2n} rf(\frac{(s+t)a}{r}(\frac{s}{t})^{2n}x) = \lim_{n\to\infty} raf(\frac{s+t}{r}(\frac{s}{t})^{2n}x)\\
&= raT(\frac{s+t}{r}x)
\end{aligned}$$

for each $a \in A_1^+ \cup \{i\}$. So

$$T(ax) = \frac{r}{s+t}T(\frac{s+t}{r}ax) = \frac{r}{s+t}aT(\frac{s+t}{r}x) = aT(x)$$

for each $a \in A_1^+ \cup \{i\}$.

For any element $a \in A$, $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, and $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = (\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-$, where $(\frac{a+a^*}{2})^+$,

$(\frac{a+a^*}{2})^-$, $(\frac{a-a^*}{2i})^+$, and $(\frac{a-a^*}{2i})^-$ are positive elements (see [4, Lemma 38.8]). So

$$\begin{aligned}
T(ax) =&T((\frac{a+a^*}{2})^+x - (\frac{a+a^*}{2})^-x + i(\frac{a-a^*}{2i})^+x - i(\frac{a-a^*}{2i})^-x)\\
=&(\frac{a+a^*}{2})^+T(x) + (\frac{a+a^*}{2})^-T(-x) + (\frac{a-a^*}{2i})^+T(ix) + (\frac{a-a^*}{2i})^-T(-ix)\\
=&(\frac{a+a^*}{2})^+T(x) - (\frac{a+a^*}{2})^-T(x) + i(\frac{a-a^*}{2i})^+T(x) - i(\frac{a-a^*}{2i})^-T(x)\\
=&((\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-)T(x)\\
=&aT(x)
\end{aligned}$$

for all $a \in A$ and all $x \in {}_A\mathcal{B}$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$, as desired. □

Theorem 2.5. *Let A be a unital C^*-algebra of real rank 0. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|raf(\frac{sx+ty}{r}) - sf(ax) - tf(ay)\| \le \varphi(x, y)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

Proof. By the same reasoning as the proof of Theorem 2.4, there exists a unique continuous $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii), and

$$T(ax) = aT(x) \tag{1}$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

Let $b \in A_1^+ \setminus A_{in}$. Since $A_{in} \cap A_{sa}$ is dense in A_{sa}, there exists a sequence $\{b_m\}$ in $A_{in} \cap A_{sa}$ such that $b_m \to b$ as $m \to \infty$. Put $c_m = \frac{1}{|b_m|} b_m$. Then $c_m \to \frac{1}{|b|} b = b$ as $m \to \infty$. Put $a_m = \sqrt{c_m^* c_m}$. Then $a_m \to b$ as $m \to \infty$ and $a_m \in A_1^+ \cap A_{in}$. Thus there exists a sequence $\{a_m\}$ in $A_1^+ \cap A_{in}$ such that $a_m \to b$ as $m \to \infty$, and by the continuity of T

$$\lim_{m\to\infty} T(a_m x) = T(\lim_{m\to\infty} a_m x) = T(bx) \tag{2}$$

for all $x \in {}_A\mathcal{B}$. By (1),

$$\|T(a_m x) - bT(x)\| = \|a_m T(x) - bT(x)\| \to \|bT(x) - bT(x)\| = 0 \tag{3}$$

as $m \to \infty$. By (2) and (3),

$$\begin{aligned}
\|T(bx) - bT(x)\| \le&\|T(bx) - T(a_m x)\| + \|T(a_m x) - bT(x)\|\\
\to&0 \text{ as } m \to \infty
\end{aligned} \tag{4}$$

for all $x \in {}_A\mathcal{B}$. By (1) and (4), $T(ax) = aT(x)$ for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 2.4. □

Theorem 2.6. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|raf(\frac{sx+ty}{r}) - sf(ax) - tf(ay)\| \le \varphi(x,y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. If $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

Proof. Let $a = 1 \in A_1^+$. By Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

The rest of the proof is similar to the proof of Theorem 2.4. □

Theorem 2.7. *Let A be a unital C^*-algebra of real rank 0. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|raf(\frac{sx+ty}{r}) - sf(ax) - tf(ay)\| \le \varphi(x,y)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. Assume that $f(ax)$ is continuous in $a \in A_1 \cup \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, and that the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n} ax)$ converges uniformly on A_1 for each fixed $x \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

Proof. By the same reasoning as the proof of Theorem 2.4, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii), and

$$T(ax) = aT(x)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. By the continuity and the uniform convergence, one can show that $T(ax)$ is continuous in $a \in A_1$ for each $x \in {}_A\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 2.5. □

Theorem 2.8. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|raf(\frac{sx+ty}{r}) - sf(ax) - tf(ay)\| \le \varphi(x,y)$$

for all $a \in A_1^+ \cup \{i\} \cup \mathbb{R}$ and all $x, y \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

Proof. By the same reasoning as the proof of Theorem 2.1, there exist unique additive mappings $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iii).

By the same reasoning as the proof of Theorem 2.4,

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\} \cup \mathbb{R}$ and all $x \in {}_A\mathcal{B}$. So the mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is $\mathbb{R}$-linear, and satisfies

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 2.4. □

3. Stability of generalized Jensen's equations in a Hilbert module over a C^*-algebra

In this section, let $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be mappings with $h(0) = 0$ and $g(0) = 0$ such that $h((\frac{s}{t})^{2n}x) = (\frac{s}{t})^{2n}h(x)$ and $g((\frac{s}{t})^{2n}x) = (\frac{s}{t})^{2n}g(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$.

We are going to prove the Hyers-Ulam-Rassias stability of generalized Jensen's equations in a Hilbert module over a unital C^*-algebra.

Theorem 3.1. *Let $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be continuous mappings such that*

$$\|rah(\frac{sx+ty}{r}) - sh(ax) - th(ay)\| \le \varphi(x, y)$$
$$\|rag(\frac{sx+ty}{r}) - sg(ax) - tg(ay)\| \le \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Then the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded A-linear operators. Furthermore,

(1) *if the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfy the inequalities*

$$\|h \circ g(x) - x\| \le \varphi(x, x),$$
$$\|g \circ h(x) - x\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping g is the inverse of the mapping h,

(2) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|h(x) - h^*(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a self-adjoint operator,

(3) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|h \circ h^*(x) - h^* \circ h(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a normal operator,

(4) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequalities*

$$\|h \circ h^*(x) - x\| \le \varphi(x, x),$$
$$\|h^* \circ h(x) - x\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a unitary operator, and

(5) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequalities*

$$\|h \circ h(x) - h(x)\| \le \varphi(x, x),$$
$$\|h^*(x) - h(x)\| \le \varphi(x, x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a projection.

Proof. Let $a = 1 \in A_1^+$. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\text{(iv)} \qquad \|g(x) - L(x)\| \le \frac{1}{s}\widetilde{\varphi}(x, -\frac{s}{t}x) + \frac{t}{s^2}\widetilde{\varphi}(-\frac{s}{t}x, (\frac{s}{t})^2 x)$$

for all $x \in {}_A\mathcal{H}$. And the mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ can be defined by

$$L(x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} g((\frac{s}{t})^{2n} x).$$

But the sequence $(\frac{t}{s})^{2n} g((\frac{s}{t})^{2n} x)$ converges uniformly on ${}_A\mathcal{H}$. By the same reasoning as the proof of Theorem 2.4, the mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is continuous.

The sequence $(\frac{t}{s})^{2n} h((\frac{s}{t})^{2n} x)$ converges uniformly on ${}_A\mathcal{H}$. By Theorem 2.4, there exists a unique continuous A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iii). By the assumption,

$$\begin{aligned} T(x) &= \lim_{n\to\infty} (\frac{t}{s})^{2n} h((\frac{s}{t})^{2n} x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} (\frac{s}{t})^{2n} h(x) = h(x) \\ L(x) &= \lim_{n\to\infty} (\frac{t}{s})^{2n} g((\frac{s}{t})^{2n} x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} (\frac{s}{t})^{2n} g(x) = g(x) \end{aligned}$$

for all $x \in {}_A\mathcal{H}$, where the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given in the proof of Theorem 2.1. Hence the A-linear operators T and L are the mappings h and g, respectively. So the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are continuous A-linear operators. Thus the A-linear operators $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded (see [6, Proposition II.1.1]).

(1) By the assumption,

$$\begin{aligned} \|h \circ g((\frac{s}{t})^{2n} x) - (\frac{s}{t})^{2n} x\| &\le \varphi((\frac{s}{t})^{2n} x, (\frac{s}{t})^{2n} x), \\ \|g \circ h((\frac{s}{t})^{2n} x) - (\frac{s}{t})^{2n} x\| &\le \varphi((\frac{s}{t})^{2n} x, (\frac{s}{t})^{2n} x) \end{aligned}$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$\begin{aligned} &(\frac{t}{s})^{2n} \|h \circ g((\frac{s}{t})^{2n} x) - (\frac{s}{t})^{2n} x\| \to 0, \\ &(\frac{t}{s})^{2n} \|g \circ h((\frac{s}{t})^{2n} x) - (\frac{s}{t})^{2n} x\| \to 0 \end{aligned}$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$\begin{aligned} h \circ g(x) &= \lim_{n\to\infty} (\frac{t}{s})^{2n} h \circ g((\frac{s}{t})^{2n} x) = x, \\ g \circ h(x) &= \lim_{n\to\infty} (\frac{t}{s})^{2n} g \circ h((\frac{s}{t})^{2n} x) = x \end{aligned}$$

for all $x \in {}_A\mathcal{H}$. So the mapping g is the inverse of the mapping h.

The proofs of the others are similar to the proof of (1). □

Theorem 3.2. *Let A be a unital C^*-algebra of real rank 0. Let $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be continuous mappings such that*

$$\begin{aligned} \|rah(\frac{sx + ty}{r}) - sh(ax) - th(ay)\| &\le \varphi(x, y) \\ \|rag(\frac{sx + ty}{r}) - sg(ax) - tg(ay)\| &\le \varphi(x, y) \end{aligned}$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Then the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded A-linear operators. Furthermore, the properties, given in the statement of Theorem 3.1, hold.

Proof. The sequence $(\frac{t}{s})^{2n}h((\frac{s}{t})^{2n}x)$ converges uniformly on ${}_A\mathcal{H}$. By the same reasoning as the proof of Theorem 2.5, there exists a unique continuous $\mathbb{R}$-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iii), and

$$T(ax) = aT(x)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{H}$. By the same method as the proof of Theorem 2.5, one can show that

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{H}$. By the same reasoning as the proof of Theorem 2.4, the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is A-linear.

Similarly, one can show that there exists a unique $\mathbb{R}$-linear operator $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iv) and that the mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is A-linear.

The rest of the proof is the same as the proof of Theorem 3.1. So the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded A-linear operators, and the properties, given in the statement of Theorem 3.1, hold. □

So the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are elements of the C^*-algebra $\mathcal{L}({}_A\mathcal{H})$ of all bounded A-linear operators on ${}_A\mathcal{H}$.

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EIGENFUNCTIONS OF LAPLACIAN BY BOUNDEDNESS CONDITIONS IN BOUNDED HYPERFUNCTIONS AND DISTRIBUTIONS

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Abstract. R. S. Strichartz has proved the following theorem: "If $\{f_j(x)\}_{j\in\mathbb{Z}}$ is a doubly infinite sequence of functions on $\mathbb{R}^n$ satisfying (i) $\triangle f_j = f_{j+1}$ and (ii) there exists a constant M such that $|f_j(x)| \le M$ for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$, then $\triangle f_0 = -f_0$" in (*Characterization of eigenfunctions of the Laplacian by boundedness conditions*, Trans. Amer. Math. Soc. **338** (1993), 971–979). We generalize it in the space of bounded hyperfunctions via heat kernel and in the space of distributions via regularization of distributions.

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1. INTRODUCTION

If a function and all its derivatives and integrals are absolutely uniformly bounded, then the function is a sine function with period 2π. This is Roe's theorem [3]. In place of derivatives and antiderivatives, Strichartz [4] has given an n-dimensional generalization using powers of the Laplacian operator $\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. More precisely if $\{f_j(x)\}_{j\in\mathbb{Z}}$ is a doubly infinite sequence of functions on $\mathbb{R}^n$ satisfying (i) $\triangle f_j = f_{j+1}$ and (ii) $|f_j(x)| \le M$ for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$, then $\triangle f_0 = -f_0$. We give an analogue result in the space of bounded hyperfunctions and distributions. We first give the following definitions :

(i) We denote by $A_{L^1}(\mathbb{R}^n)$ the space of all $\phi \in C^\infty(\mathbb{R}^n)$ satisfying

$$\|\phi\|_{A_{L^1,h}} = \sup_{\alpha\in\mathbb{N}_0^n} \frac{\|\partial^\alpha \phi\|_{L^1}}{h^{|\alpha|}\ \alpha!} < \infty$$

for some constant $h > 0$. We say that $\phi_i \to 0$ as $i \to \infty$ if $\|\phi_i\|_{A_{L^1,h}} \to 0$ as $i \to \infty$. We denote by $B_{L^\infty}(\mathbb{R}^n)$ the dual of $A_{L^1}(\mathbb{R}^n)$. Every element in $B_{L^\infty}(\mathbb{R}^n)$ is called

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a bounded hyperfunction. For more details of bounded hyperfunctions, we refer to [1].

(ii) We denote by $E(x,t)$ the n-dimensional heat kernel

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2}\exp(-|x|^2/4t), & t>0, \\ 0, \quad t \le 0. \end{cases}$$

Note that $E(x,t) \in A_{L^1}(\mathbb{R}^n)$ for each $t>0$. So for every $u \in B_{L^\infty}(\mathbb{R}^n)$, we define $U(x,t) := \langle u(y), E(x-y,t)\rangle$ in $\mathbb{R}^{n+1}_+ = \{(x,t)|x\in\mathbb{R}^n, t>0\}$ and call it the *defining function* of u.

2. MAIN RESULTS

Theorem 2.1. *Let $u_j \in B_{L^\infty}(\mathbb{R}^n)$, $j\in\mathbb{Z}$. Suppose that there exist positive constants h and $M := M(h)$ such that*

(i) $\triangle u_j = u_{j+1}$ *in the sense that* $\langle \triangle u_j, \phi\rangle = \langle u_{j+1},\phi\rangle$ *for all* $\phi \in A_{L^1}(\mathbb{R}^n)$.

(ii) $\|u_j * \phi\|_{L^\infty} \le M\|\phi\|_{A_{L^1,h}}$ *for all* $j\in\mathbb{Z}$ *and all* $\phi\in A_{L^1}(\mathbb{R}^n)$.

Then $\triangle u_0 = -u_0$.

Proof. For fixed $t_0>0$, the defining function $U_j(x,t_0) := \langle u_j(y), E(x-y,t_0)\rangle \in C^\infty(\mathbb{R}^n)$. Since $E(x-y,t_0)\in A_{L^1}(\mathbb{R}^n)$, it follows from the hypothesis (i) that

$$\triangle_x U_j(x,t_0) = \langle \triangle_y u_j(y), E(x-y,t_0)\rangle = U_{j+1}(x,t_0) \tag{2.1}$$

Applying the hypothesis (ii) and the following estimate from (p.2426, [1])

$$\|\partial_x^\alpha E(x,t)\|_{L^1(\mathbb{R}^n_x)} \le C^{|\alpha|}{t_0}^{-(n+|\alpha|)/2}\alpha!^{1/2}\|\exp(-|x|^2/8t_0)\|_{L^1(\mathbb{R}^n_x)}$$

for some constant $C>0$, we have

$$\begin{aligned} |U_j(x,t_0)| &\le M \sup_{\alpha\in\mathbb{N}_0^n} \frac{\|\partial_y^\alpha E(y,t_0)\|_{L^1(\mathbb{R}^n_y)}}{h^{|\alpha|}\ \alpha!} \\ &\le M(8\pi)^{n/2} \sup_{\alpha\in\mathbb{N}_0^n}\frac{C^{|\alpha|}\ {t_0}^{-|\alpha|/2}}{h^{|\alpha|}\ \alpha!^{1/2}} \\ &\le M(8\pi)^{n/2}\prod_{i=1}^n \sup_{\alpha_i\in\mathbb{N}_0}\frac{C^{\alpha_i}{t_0}^{-\alpha_i/2}}{h^{\alpha_i}\ {\alpha_i}!^{1/2}}. \end{aligned} \tag{2.2}$$

But in view of the inequalities $\alpha_i! \le {\alpha_i}^{\alpha_i} \le \alpha_i!\ e^{\alpha_i}$ for each $i=1,\dots,n$

$$\sup_{\alpha_i\in\mathbb{N}_0}\frac{C^{\alpha_i}\ {t_0}^{-\alpha_i/2}}{h^{\alpha_i}\ {\alpha_i}!^{1/2}} = \sup_{\alpha_i\in\mathbb{N}_0}\frac{C^{\alpha_i}\ {t_0}^{-\alpha_i/2}}{h^{\alpha_i}\ {\alpha_i}^{\alpha_i/2}e^{-\alpha_i/2}} = \exp\left(C^2 2^{-1}h^{-2}\ {t_0}^{-1}\right)$$

So (2.2) reduces to

$$|U_j(x,t_0)| \le M(8\pi)^{n/2}\exp\left(nC^2 2^{-1}h^{-2}{t_0}^{-1}\right) \tag{2.3}$$

for all $j\in\mathbb{N}_0$ and all $x\in\mathbb{R}^n$. Since $t_0>0$ is fixed, the right hand side of (2.3) is a constant independent of j and x. So in view of (2.1), (2.3) and Theorem 1.1 of [4], we have

$$\triangle U_0(x,t_0) = -U_0(x,t_0). \tag{2.4}$$

It then follows from (2.4) and Theorem 3.5 of [1] that

$$\langle -u_0, \phi\rangle = \lim_{t_0\to 0^+}\langle -U_0(x,t_0), \phi(x)\rangle = \lim_{t_0\to 0^+}\langle \triangle U_0(x,t_0), \phi(x)\rangle = \langle \triangle u_0, \phi\rangle$$

for all $\phi \in A_{L_1}(\mathbb{R}^n)$. Hence $\triangle u_0 = -u_0$. ||

In the space of distributions $D^{'}(\mathbb{R}^n)$, we use the regularization of distributions to obtain the analogue of Theorem 2.1.

Theorem 2.2. *Let $u_j \in D^{'}(\mathbb{R}^n)$, $j \in \mathbb{Z}$. Suppose that for every compact subset K of $\mathbb{R}^n$, there exist a nonnegative integer N and a positive constant $M := M(K,N)$ such that*

(i) $\triangle u_j = u_{j+1}$ *in the sense that* $\langle \triangle u_j, \phi\rangle = \langle u_{j+1}, \phi\rangle$ *for all* $\phi \in D(\mathbb{R}^n)$.

(ii) $\|u_j * \phi\|_{L^\infty} \le M \ \sum_{|\alpha|\le N} \sup_{x\in K} |\partial^\alpha \phi(x)|$ *for all* $j \in \mathbb{Z}$ *and all* $\phi \in D_K(\mathbb{R}^n)$.

Then $\triangle u_0 = -u_0$.

Proof. We can choose some $\psi \in D(\mathbb{R}^n)$ such that $\int \psi dx = 1$, and set $\psi_i(x) = i^n\psi(ix), i = 1, 2, \ldots$. Then $\psi_i \to \delta$ in $D^{'}(\mathbb{R}^n)$, and *supp* $\psi_i \subset$ *supp* ψ for all i. For each j, $u_j * \psi_i \in C^\infty(\mathbb{R}^n)$ which is a regularization of u_j and $u_j * \psi_i \to u_j$ in $D^{'}(\mathbb{R}^n)$ as $i \to \infty$. Let $g_{j,i} := u_j * \psi_i$. Then for $K :=$ *supp* ψ and all $j \in \mathbb{Z}$, it follows from the hypotheses (i) and (ii) that

$$\begin{cases} \triangle g_{j,i} = u_{j+1} * \psi_i = g_{j+1,i} \\ \|g_{j,i}\|_{L^\infty} \le \tilde{M} \end{cases} \tag{2.5}$$

where

$$\tilde{M} := M \sum_{|\alpha|\le N} \sup_{x\in K} |\partial^\alpha \psi_i(x)|$$

is a positive constant. It then follows from (2.5) and Theorem 1.1 of [4] that $\triangle g_{0,i} = -g_{0,i}$. Letting $i \to \infty$, we obtain that $\triangle u_0 = -u_0$. ||

Replacing the Laplacian by a d'Alembertian

$$\Box = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \cdots - \frac{\partial^2}{\partial x_n^2}, \qquad 1 \le p \le n-1,$$

we have the following generalization of previous theorems.

Theorem 2.3. *Let $u_j \in B_{L^\infty}(\mathbb{R}^n)$, $j \in \mathbb{Z}$. Suppose that there exist positive constants h and $M := M(h)$ such that*

(i) $\Box u_j = u_{j+1}$ *in the sense that* $\langle \Box u_j, \phi\rangle = \langle u_{j+1}, \phi\rangle$ *for all* $\phi \in A_{L^1}(\mathbb{R}^n)$.

(ii) $\|u_j * \phi\|_{L^\infty} \le M\|\phi\|_{A_{L^1,h}}$ *for all* $j \in \mathbb{Z}$ *and all* $\phi \in A_{L^1}(\mathbb{R}^n)$.

Then $u_0 = u_+ + u_-$ *where* $u_\pm = \frac{1}{2}(u_0 \pm \Box u_0)$ *and* $\Box u_\pm = \pm u_\pm$.

Proof. Obviously as in (2.1), we obtain

$$\Box U_j(x, t_0) = U_{j+1}(x, t_0). \tag{2.6}$$

It then follows from (2.6), (2.3) and Theorem 3.1 of [4] that

$$U_0(x,t_0) = U_+(x,t_0) + U_-(x,t_0) \tag{2.7}$$

where $U_\pm(x,t_0)$ are bounded solutions of $\Box U_\pm(x,t_0) = \pm U_\pm(x,t_0)$. Also from (p.211, [2])

$$\Box^2 U_0(x,t_0) = U_0(x,t_0), \quad U_\pm(x,t_0) = \frac{1}{2}\left(U_0(x,t_0) \pm \Box U_0(x,t_0)\right). \tag{2.8}$$

Then for every $\phi \in A_{L^1}(\mathbb{R}^n)$, it follows from (2.7), (2.8) and Theorem 3.5 of [1] that

$$\begin{aligned}
\langle u_0, \phi\rangle &= \lim_{t_0\to 0^+}\langle U_0(x,t_0), \phi(x)\rangle \\
&= \lim_{t_0\to 0^+}\left\langle \frac{1}{2}\left(U_0(x,t_0) + \Box U_0(x,t_0)\right) + \frac{1}{2}\left(U_0(x,t_0) - \Box U_0(x,t_0)\right), \phi(x)\right\rangle \\
&= \left\langle \frac{1}{2}\left(u_0 + \Box u_0\right) + \frac{1}{2}\left(u_0 - \Box u_0\right), \phi(x)\right\rangle \\
&= \langle u_+ + u_-, \phi\rangle
\end{aligned}$$

where $u_\pm := \frac{1}{2}\left(u_0 \pm \Box u_0\right)$. Using Theorem 3.5 of [1] and $\Box^2 U_0(x,t_0) = U_0(x,t_0)$ from (2.8), we have

$$\langle \Box^2 u_0, \phi\rangle = \lim_{t_0\to 0^+}\langle U_0(x,t_0), \Box^2\phi\rangle = \lim_{t_0\to 0^+}\langle U_0(x,t_0), \phi\rangle = \langle u_0, \phi\rangle \tag{2.9}$$

for every $\phi \in A_{L^1}(\mathbb{R}^n)$. Then $\Box u_\pm = \pm u_\pm$ due to (2.9). ||

Theorem 2.4. *Let $u_j \in D'(\mathbb{R}^n)$, $j \in \mathbb{Z}$. Suppose that for every compact subset K of $\mathbb{R}^n$, there exist a nonnegative integer N and a positive constant $M := M(K,N)$ such that*

(i) $\Box u_j = u_{j+1}$ *in the sense that* $\langle \Box u_j, \phi\rangle = \langle u_{j+1}, \phi\rangle$ *for all* $\phi \in D(\mathbb{R}^n)$.

(ii) $\|u_j * \phi\|_{L^\infty} \le M \sum_{|\alpha|\le N} \sup_{x\in K} |\partial^\alpha \phi(x)|$ *for all* $j \in \mathbb{Z}$ *and all* $\phi \in D_K(\mathbb{R}^n)$.

Then $u_0 = u_+ + u_-$ *where* $u_\pm = \frac{1}{2}\left(u_0 \pm \Box u_0\right)$ *and* $\Box u_\pm = \pm u_\pm$.

Proof. Applying the same technique, (2.5) holds even for the operator $\Box$. So using Theorem 3.1 of [4], we have $g_{0,i} = g^+_{0,i} + g^-_{0,i}$ where $g^\pm_{0,i}$ are bounded solutions of $\Box g^\pm_{0,i} = \pm g^\pm_{0,i}$. Also from (p.211, [2])

$$\Box^2 g_{0,i} = g_{0,i}\ , \quad g^\pm_{0,i} = \frac{1}{2}\left(g_{0,i} \pm \Box g_{0,i}\right). \tag{2.10}$$

Using regularization of u_j, we obtain that $u_0 = u_+ + u_-$ where $u_\pm = \frac{1}{2}\left(u_0 \pm \Box u_0\right)$. In view of (2.10), it is easy to see that $\Box^2 u_0 = u_0$ and hence $\Box u_\pm = \pm u_\pm$. ||

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ON SEMILINEAR ELLIPTIC EIGENVALUE PROBLEM WITH EIGENPARAMETER IN THE BOUNDARY CONDITIONS

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Abstract. The object of this paper is to establish the expansion theorem for a semilinear elliptic eigenvalue problem of a simply connected bounded domain in $R^n, (n \geq 2)$, where the eigenvalue parameter λ is contained in the semilinear elliptic partial differential equation and in the boundary conditions. We associated with this problem a semilinear self-adjoint operator S in a suitably defined Hilbert space H and then we develop an associated eigenfunction expansion theorem.

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1. Introduction

Our study is mainly motivated by some results for the semilinear elliptic problem :

$$\left.\begin{aligned} -\Delta u(x) &= g(u(x)) + h(x) \qquad in\ \Omega, \\ u(x) &= 0 \qquad on\ \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

recently obtained in [6], [8], [9] and [11].

Regular right-definite eigenvalue problem of elliptic operator in $R^m, (m \geq 2)$ with an eigenparameter λ in the elliptic equation and in the mixed boundary conditions have been studied by many authors, see for example, Ibrahim [15], Grunau and Sweers [12], Alama and Pino [1] and Gupta [14].

In the present paper, we shall study a semilinear elliptic eigenvalue problem in $R^n, (n \geq 2)$ with an eigenparameter λ in the semilinear elliptic equation (1.2) and in the boundary conditions (1.3) which does not seem to have been done elsewhere.

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The problem to be discussed here can be formulated as follows : Let Ω be a simply connected bounded domain in $R^n, (n \geq 2)$ with smooth boundary $\partial\Omega$. We consider the following semilinear elliptic eigenvalue equation :

$$\frac{1}{r}(\ -\Delta_n u + h(x)b(u(x))\ ) = \lambda u \qquad in\ \Omega \tag{1.2}$$

with the boundary conditions

$$u_\nu = \lambda u \qquad on\ \partial\Omega \tag{1.3}$$

where we assume throughout that :

(i) $\Delta_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in R^n, $\quad (n \geq 2)$.

(ii) $u_\nu = \sum_{i=1}^{n} u_{x_i}(x)\nu_i(x)$ denotes differentiation of $u(x)$ along the outward unit normal $\nu(x) = (\nu_1(x), \nu_2(x), ..., \nu_n(x))$ on the boundary $\partial\Omega$, where $x = (x_1, x_2, ..., x_n)$ is a generic point in the Euclidean space R^n.

(iii) The weight function $r(x) > 0$ is a real-valued function such that $r(x) \in C(\overline{\Omega})$, $\overline{\Omega} = \Omega \bigcup \partial\Omega$ and the potential function $h(x)$ is an integrable function on Ω .

(iv) The nonlinear function $b(u(x))$ is a real-valued continuous function in Ω.

(v) The parameter λ is a complex number.

The nonlinearity of our problem (1.2)-(1.3) refers to the function $b(u(x))$ which is interfere with the eigenvalues of the associated semilinear operator S. Our problem (1.2)-(1.3) is a more generalization of Pino and Felmer [16] and Gossez and Omari [11].

This paper is organized as follows. In section 2, we formulate a self-adjoint semilinear operator S in a suitably Hilbert space H. In section 3, we recall some boundedness and regularity results for eigenfunction for problem (1.2)-(1.3). In section 4, we also recall some results about the existence of sequences of eigenvalues for problem (1.2)-(1.3). We give some basic properties of eigenvalues for our problem (1.2)-(1.3) and then we develop an associated eigenfunction expansion theorem.

2. Hilbert Space Formulation

Let $L_r^2(\Omega)$ and $L^2(\partial\Omega)$ be complex Hilbert spaces of Lebesgue measurable functions $f(x)$ in Ω and on $\partial\Omega$ respectively, satisfying :

(2.1)
$$\left.\begin{array}{l} (i) \displaystyle\int_\Omega r(x) \mid f(x) \mid^2 dx < \infty, \\ (ii) \displaystyle\int_{\partial\Omega} \mid f(x) \mid^2 ds < \infty, \\ (iii) lim_{x\to-\infty} sup \dfrac{f(x)}{x} << \lambda_i \quad and \quad lim_{x\to+\infty} inf \dfrac{f(x)}{x} > \lambda_i, \qquad for fixed\ i \end{array}\right\}$$

where $dx = dx_1 dx_2 \cdots dx_n$ is the volume element corresponding to Ω while ds is the surface element corresponding to the boundary $\partial\Omega$ respectively.

Definition 2.1. We define a Hilbert space H of two component vectors by

$$H = \ L_r^2(\Omega) \ \oplus \ L^2(\partial\Omega) \tag{2.2}$$

with inner product

$$[f,\ g]_H \ = < f_1, g_1 >_\Omega + < f_2, g_2 >_{\partial\Omega}$$

(2.3)

$$= \int_\Omega r(x) \left(\nabla f_1(x).\overline{\nabla g_1(x)} + h(x)b(f_1(x)).\overline{b(g_1(x))} \right) \ dx + \int_{\partial\Omega} f_2(x)\overline{g_2(x)} \ ds,$$

and the norm

(2.4)

$$\| f \|_H^2 = \int_\Omega r(x) \left(\mid \nabla f_1(x) \mid^2 + h(x) \mid b(f_1(x)) \mid^2 \right) \ dx + \int_{\partial\Omega} \mid f_2(x) \mid^2 \ ds$$

for each $f = (f_1, f_2)$ and $g = (g_1, g_2)$ in H.

Definition 2.2. Let H_0 be the set of all elements $f = (f_1, f_2)$ satisfying

(2.5)

$$\left.\begin{aligned} &A_1 : f_1 \in C^1(\overline{\Omega}) \bigcap C^2(\Omega), \\ &A_2 : \frac{1}{r}\left(-\Delta_n f_1 + h(x)b(f_1(x)) \right) \in L_r^2(\Omega), \\ &A_3 : lim_{x\to\infty} \frac{f(x)}{x} = c > 0, \\ &A_4 : lim_{x\to 0} \frac{f(x)}{x} = 0. \end{aligned}\right\}$$

Because of the nonlinearity of problem (1.2)-(1.3), the function $f = (f_1, f_2)$ satisfies the conditions A_3 and A_4 (see [2]).

We define a semilinear operator $S : D(S) \longrightarrow H$ by

$$Sf := (-\Delta_n f_1 + h(x)b(f_1(x)), f_{1\nu}) \,; \tag{2.6}$$

for each $f = (f_1, f_2) \in D(S)$, in which the domain $D(S)$ of S is defined as follows:

$$D(S) := \{(f\mid_\Omega, f\mid_{\partial\Omega}) \in H : f \in H_0\} \tag{2.7}$$

where $f\mid_\Omega$ and $f\mid_{\partial\Omega}$ are restrictions of f on Ω and on $\partial\Omega$ respectively.

Remark (2.1): The parameter λ is an eigenvalue and f_1 is a corresponding eigenfunction of the problem (1.2)-(1.3) if and only if

$$Sf = \lambda f \qquad forall\ f = (f_1, f_2) \in D(S). \tag{2.8}$$

Therefore, the eigenvalues and the eigenfunctions of the problem (1.2)-(1.3) are equivalent to the eigenvalues and the eigenfunctions of the semilinear operator S in H.

Remark (2.2): The semilinear operator S is compact and self-adjoint operator such that $Sf = \lambda f$.

Now, the variational characterization of the eigenvalues of the operator S yields

$$\lambda_n << \lambda_{n+1} \qquad\qquad \forall\ n \geq 1 \tag{2.9}$$

3. The Boundedness Of Eigenfunctions of The Operator S

In this section we shall discuss some results about the regularity and boundedness of the eigenfunctions of the problem (1.2)-(1.3). Making use of [13],[18] and [3], it is clear the regularity of eigenfunctions of the operator S; by using " strong maximum principle ", see [10]. Denote the set of eigenfunctions $\Phi_n(x)_{n=1}^{\infty}$ which forms a complete fundamental system in H.

Because of the variationally problem (1.2)-(1.3) and using [16] and [3], the variational arguments are applied to the functional :
(3.1)

$$J_\lambda(\Phi_n) = \int_\Omega \left(\mid \nabla\Phi_n \mid^2 - \lambda_n r(x) \mid \Phi_n \mid^2\right)\, dx + \frac{1}{2}\int_\Omega h(x) \mid b(\Phi_n) \mid^2\, dx + \int_{\partial\Omega} \mid \Phi_{n\nu} \mid^2\, ds < \infty$$

This functional $J_\lambda(\Phi_n)$ is of class $C^2(\overline{\Omega}) \bigcap C^2(\Omega)$, on the Hilbert space H and that its critical points correspond exactly to the solutions of the problem (1.2)-(1.3). We should remark that the positive solutions of problem (1.2)-(1.3) correspond precisely to global minimizers of this functional $J_\lambda(\Phi_n)$ as established in [17].

Lemma 3.1. The set of eigenfunctions $\{\Phi_n\}_{n=1}^{\infty}$ of operator S is bounded from above .

Proof: Without loss of generality, we may assume that

$$v_n = \frac{\Phi_n}{\parallel \Phi_n \parallel_H} \to v_0 \tag{3.2}$$

in strongly Hilbert space H.We claim that $v_0 \neq 0$.
Indeed, dividing

$$\mid J_\lambda(\Phi_n) \mid \leq c \tag{3.3}$$

by $\parallel \Phi_n \parallel_H$ yields

$$\int_\Omega r(x)\left(\mid \nabla\Phi_{n1} \mid^2 + h(x) \mid b(\Phi_{n1}) \mid^2\right)\, dx + \int_{\partial\Omega} \mid \Phi_{n2} \mid^2\, ds \to 0 \qquad \diamondsuit \tag{3.4}$$

Definition 3.1. Let the function b : $\Omega \times C^2(\overline{\Omega}) \to R$ be a continuous function and radially symmetric function; that is , $b(x,u) = b(\mid x \mid, u)$, $\quad forall \quad x \in \Omega$ and

$$B(x,u) := \int_0^u b(x,z)\, dz. \tag{3.5}$$

be the primitive function of the nonlinear function $b(x,u)$ with respect to the spectrum of the associated semilinear operator S.

Lemma 3.2. The set of eigenfunctions $\{\Phi_n\}_{n=1}^{\infty}$ of operator S is bounded from below ; i.e. There exists $\alpha, \rho > 0$ such that $J_\lambda(\Phi_n) \geq \alpha$ for $\parallel \Phi_n \parallel_H = \rho$.

Proof: From the condition A_4, it is given $\varepsilon > 0$, there exists $\delta > 0$, such that; (using [2]);

$$F(x) \leq \frac{\varepsilon x^2}{2}, \qquad \mid x \mid \leq \delta, \tag{3.6}$$

Thus

$$\int_\Omega F(\Phi_n)\ dx \le \frac{\varepsilon}{2}\int_\Omega \Phi_n^2\ dx \qquad \| \Phi_n \|_H < \rho, \tag{3.7}$$

ρ small enought.
And

$$\int_\Omega B(x,\Phi_n)\ dx \le \int_\Omega F(\Phi_n)\ dx \le \frac{\varepsilon}{2}\int_\Omega \Phi_n^2\ dx \qquad \| \Phi_n \|_H = \rho. \tag{3.8}$$

Now, using (3.5),(3.7) and (3.8), we find the critical points on $D(S)$ of the $C^2(\overline{\Omega})$ functional as follows :

$$J_\lambda(\Phi_n) = \int_\Omega \left(| \nabla\Phi_n |^2 - \lambda_n r(x) \mid \Phi_n |^2\right)\ dx + \frac{1}{2}\int_\Omega h(x) \mid b(\Phi_n) \mid^2\ dx$$

$$-\int_\Omega B(x,\Phi_n)\ dx + \int_{\partial\Omega} \mid \Phi_{n\nu} \mid^2\ ds \tag{3.9}$$

$$\ge \int_\Omega \left(| \nabla\Phi_n |^2 - \lambda_n r(x) \mid \Phi_n |^2\right)\ dx + \frac{1}{2}\int_\Omega h(x) \mid b(\Phi_n) \mid^2\ dx$$

$$-\int_\Omega F(\Phi_n)\ dx + \int_{\partial\Omega} \mid \Phi_{n\nu} \mid^2\ ds$$

$$\ge \int_\Omega \left(| \nabla\Phi_n |^2 - \lambda_n r(x) \mid \Phi_n |^2\right)\ dx + \frac{1}{2}\int_\Omega h(x) \mid b(\Phi_n) \mid^2\ dx$$

$$+\int_{\partial\Omega} \mid \Phi_{n\nu} \mid^2\ ds - \frac{\varepsilon}{2}\int_\Omega \Phi_n^2\ dx \tag{3.10}$$

$$\ge c_0 \| \Phi_n \|_H^2 - \frac{\varepsilon}{2}\int_\Omega \Phi_n^2\ dx \tag{3.11}$$

Recalling that

$$\int_\Omega \Phi_n^2\ dx \le c_1 \Big[\quad \int_\Omega \left(| \nabla\Phi_n |^2 - \lambda_n r(x) \mid \Phi_n |^2\right)\ dx$$

$$+\frac{1}{2}\int_\Omega h(x) \mid b(\Phi_n) \mid^2\ dx + \int_{\partial\Omega} \mid \Phi_{n\nu} \mid^2\ ds \quad \Big] \tag{3.12}$$

From (3.11) and (3.12), we have

$$J_\lambda(\Phi_n) \ge c_2 \| \Phi_n \|_H^2 \qquad for \qquad \| \Phi_n \|_H = \rho. \tag{3.13}$$

where c_0, c_1 and c_2 are positive constants. $\diamondsuit$

Remark (3.1): The set of eigenfunctions $\{\Phi_n\}_{n=1}^\infty$ of the operator S is bounded.

4. Properties Of Eigenvalues of The Operator S

The necessary and sufficient condition for the problem (1.2)-(1.3)to be solvable for any eigenfunction $\Phi_n(x) = (\Phi_{n1}, \Phi_{n2}) \in H$ is that the range of $b(x)$ be all of R. Here

$$\lambda_1 = 0 < \lambda_2 << \lambda_3 << .... << \lambda_n << \lambda_{n+1} << ...... \tag{4.1}$$

denote the distinct eigenvalues of the problem (1.2)-(1.3), where the value

$$\lambda_n :=^{def} \ inf \ (\ max_{\Phi_n \in H_0} J_\lambda(\Phi_n) \ ) \tag{4.2}$$

is an eigenvalue of the problem (1.2)-(1.3). Moreover,

$$lim_{n \to +\infty} \lambda_n = +\infty \tag{4.3}$$

Let us take ; by using [11]; the primitive function

$$B(x) := \int_0^x b(t) \ dt, \qquad x \in \overline{\Omega} \tag{4.4}$$

and ; by using [3] ; the condition
(4.5)

$$lim_{x \to \pm\infty} sup \frac{b(x)}{x} << \lambda_{n+1} \qquad and \qquad lim_{x \to +\infty \ (or \ x \to -\infty)} sup \frac{2B(x)}{x^2} > \lambda_n.$$

This new sequence of eigenvalues was introduced in the case of $r(x) = 1$ by [7] and was used to resonance and nonresonance results associated to the n-Laplacian operator. More recently, this sequence appears to be of some interest in the study of Fučik spectrum of the n-Laplacian made by [4] and [5].
Remark (4.1): It is a trivial fact that

$$\lambda_2 = inf\{ \int_\Omega \left(| \nabla \Phi_{n1} |^2 + h(x) \mid b(\Phi_{n1}) |^2 \right) \ dx \quad , \quad \int_\Omega r(x) \mid \Phi_{n1} |^2 \ dx = 1$$

$$and \quad \int_{\partial\Omega} | \Phi_{n2} |^2 \ ds = 1 \quad : \quad \Phi_n \in H\} \tag{4.6}$$

which is the first eigenvalue of the problem (1.2)-(1.3).

Condition (4.5) controls the possible interference of the nonlinearity $b(u(x))$ with higher part of the spectrum of the associated semilinear operator S. The nonlinearity $b(u(x))$ in the problem (1.2)-(1.3) is such that $\frac{b(x)}{x}$ lies asymptotically between two higher eigenvalues λ_n and λ_{n+1}. One obtains in the main results of [6].
Theorem 4.1: For some $n > 1$, we have

$$\lambda_n < lim_{x \to \pm\infty} \ inf \frac{b(x)}{x} \ < lim_{x \to \pm\infty} \ sup \frac{b(x)}{x} \ < \lambda_{n+1} \tag{4.7}$$

and that

(4.8)
$$\lambda_n < lim_{x\to+\infty\ (or\ x\to-\infty)}\ sup\frac{2B(x)}{x^2} < lim_{x\to+\infty\ (or\ x\to-\infty)}\ inf\frac{2B(x)}{x^2} < \lambda_{n+1}.$$

Proof: Making use of (4.2), (4.1) and (2.9), we get
(4.9)
$$\lambda_n := inf\,(\ max_{\Phi_n\in H_0}J_\lambda(\Phi_n)\ ) << \lambda_{n+1} := inf\,\big(\ max_{\Phi_{n+1}\in H_0}J_\lambda(\Phi_{n+1})\ \big)\,.$$

This leads us to
(4.10)
$$inf\,(\ max_{\Phi_n\in H_0}J_\lambda(\Phi_n)\ ) < lim_{x\to\pm\infty}\ inf\frac{b(x)}{x} < lim_{x\to\pm\infty}\ sup\frac{b(x)}{x} < inf\,\big(\ max_{\Phi_{n+1}\in H_0}J_\lambda(\Phi_{n+1})\ \big)\,.$$

This gives us (4.7).
Using the primitive function in(4.4), we assume for λ_n and λ_{n+1} that

$$lim_{x\to+\infty\ (or\ x\to-\infty)}\ inf\frac{2B(x)}{x^2} < \lambda_{n+1} \tag{4.11}$$

And

$$lim_{x\to+\infty\ (or x\to-\infty\ )}\ sup\frac{2B(x)}{x^2} > \lambda_n \tag{4.12}$$

Making use (4.11), (4.12) and (4.5), we obtain (4.8). ♢

The eigenvalues $\lambda_n,\quad n\in N$ of operator S are real if and only if there exists $\Phi_n=(\Phi_{n1},\Phi_{n2})\in H_0$ such that

$$J'_\lambda(\Phi_n(x))+\lambda_n r(x)\mid\Phi_n(x)\mid^2=(\|\ \Phi_n(x)\ \|_H^2)' \tag{4.13}$$

Therefore, we get

$$(\|\ \Phi_n(x)\ \|_H^2)'-J'_\lambda(\Phi_n(x))=\lambda_n r(x)\mid\Phi_n(x)\mid^2\,. \tag{4.14}$$

Let us denote by $\overline{J_\lambda}(\Phi_n)$ the restriction of $J'_\lambda(\Phi_n)$ to H_0. We recall the value σ is a critical value of $\overline{J_\lambda}(\Phi_n)$ if
(4.15)
$$J'_\lambda(\Phi_n(x))\mid_{H_0}=0 \qquad and \qquad J_\lambda(\Phi_n(x))=\sigma \qquad\quad forsome \qquad \Phi_n(x)\in H_0.$$

It follows from standard arguments that positive eigenvalues of operator S correspond to positive critical values of $\overline{J_\lambda}(\Phi_n)$.

The results of our investigations are summarized in the following expansion theorem:

Theorem 4.2. With the assumptions stated above, the spectrum of the operator S in H consists of an unbounded sequence of real eigenvalues of finite multiplicity without an accumulation point in $(-\infty,\infty)$. Denoting them by

$$\lambda_1=0<\lambda_2<<\lambda_3<<\ldots<<\lambda_n<<\lambda_{n+1}<<\ldots\longrightarrow\infty\quad as\quad n\longrightarrow\infty$$

and the corresponding eigenfunctions by $\Phi_1, \Phi_2, ..., \Phi_k, ...$, we have $\{\Phi_n\}_{n=1}^{\infty}$ forms a complete fundamental system in H and for every $f(x) \in H$, we have the expansion formula

$$f = \sum_{n=1}^{\infty} [f, \Phi_n]_H \Phi_n, \tag{4.16}$$

in the sense of strong convergence in H.

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SALAGEAN-TYPE HARMONIC UNIVALENT FUNCTIONS

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Abstract. A sufficient coefficient bound is determined for the Salagean-type class of complex-valued harmonic univalent functions consisting of functions of the form $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=1}^{\infty} \bar{b}_m \bar{z}^m$, $|z| < 1$. It is shown that this coefficient bound is also necessary if $f(z) = z - \sum_{m=2}^{\infty} a_m z^m + (-1)^n \sum_{m=1}^{\infty} b_m \bar{z}^m$ where $a_m \geq 0$, $b_m \geq 0$, and n is a whole number. Furthermore, extreme points, distortion theorem, convolution condition, and comnvex combinations for this latter class are also determined.

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1. Introduction. A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain C if both u and v are real harmonic in C. In any simply connected domain $D \subset C$ we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|g'(z)| < |h'(z)|$ in D. See Clunie and Sheil-Small [2].

Denote by H the family of functions $f = h + \bar{g}$ which are harmonic univalent and orientation preserving in the open unit disk $U = \{z : |z| < 1\}$ so that $f = h + \bar{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Therefore, for $f = h + \bar{g} \in H$ we can express the analytic functions h and g by the following power series expansion

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m, \quad |b_1| < 1. \tag{1}$$

Note that the family H of orientation preserving, normalized harmonic univalent functions reduces to the class S of normalized analytic univalent functions if the co-analytic part of $f = h + \bar{g}$ is identically zero.

For $f = h + \bar{g}$ given by (1), we define the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)} \tag{2}$$

where $D^n h(z) = z + \sum_{m=2}^{\infty} m^n a_m z^m$ and $D^n g(z) = \sum_{m=1}^{\infty} m^n b_m z^m$.

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For $0 \leq \alpha < 1$, we let $H(n,\alpha)$ denote the family of harmonic functions f of the form (1) such that

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) \geq \alpha, \quad 0 \leq \alpha < 1 \tag{3}$$

where $D^n f$ is defined by (2).

If the co-analytic part of $f = h + \bar{g}$ is identically zero, then the family $H(n,\alpha)$ turns out to be the class $S(n,\alpha)$ introduced by Salagean [5] for the analytic case.

We let the subclass $\overline{H}(n,\alpha)$ consist of harmonic functions $f_n = h + \overline{g_n}$ in $H(n,\alpha)$ so that h and g_n are of the form

$$h(z) = z - \sum_{m=2}^{\infty} a_m z^m, \quad g_n(z) = (-1)^n \sum_{m=1}^{\infty} b_m z^m, \quad a_m \geq 0, \ b_m \geq 0. \tag{4}$$

The class $H(n,\alpha)$ includes a variety of well-known subclasses of H. For example, $H(0,\alpha)\equiv HS(\alpha)$ is the class of orientation preserving, harmonic univalent functions f which are starlike of order α in U, that is, $\frac{\partial}{\partial\theta}\{arg(f(re^{i\theta}))\}\geq\alpha$, and $H(1,\alpha)\equiv HK(\alpha)$ is the class of orientation preserving, harmonic univalent functions f which are convex of order α in U, that is, $\frac{\partial}{\partial\theta}\{arg(\frac{\partial}{\partial\theta}f(re^{i\theta}))\}\geq\alpha$ where $z = re^{i\theta} \in U$.

For the harmonic functions f of the form (1) with $b_1 = 0$, Avci and Zlotkiewicz [1] showed that if $\sum_{m=2}^{\infty} m(|a_m|+|b_m|)\leq 1$ then $f \in HS(0)$ and if $\sum_{m=2}^{\infty} m^2(|a_m|+|b_m|)\leq 1$ then $f \in HK(0)$. Silverman [6] proved that the above two coefficient conditions are also necessary if $f = h + \bar{g}$ has negative coefficients. Later, Silverman and Silvia [7] improved the results of [1] and [6] to the case b_1 not necessarily zero. Jahangiri [3] and [4] generalized the results of [1], [6], and [7] by proving that $f \in HS(\alpha)$ if

$$\sum_{m=2}^{\infty}(m-\alpha)|a_m| + \sum_{m=1}^{\infty}(m+\alpha)|b_m| \leq 1-\alpha \tag{5}$$

and $f \in HK(\alpha)$ if

$$\sum_{m=2}^{\infty} m(m-\alpha)|a_m| + \sum_{m=1}^{\infty} m(m+\alpha)|b_m| \leq 1-\alpha. \tag{6}$$

In [3] and [4] is proved that the conditions (5) and (6) are also necessary if $f = h + \bar{g}$ has negative coefficients.

In this paper, the coefficient conditions given in [1, 3, 4, 6, 7] for the classes $HS(\alpha)$ and $HK(\alpha)$ are extended to the class $H(n,\alpha)$ of the forms (3) above. Furthermore, we determine extreme points, distortion theorem, convolution conditions and convex combinations for the functions in $\overline{H}(n,\alpha)$.

Main Results. In our first theorem, we introduce a sufficient coefficient bound for harmonic functions in $H(n,\alpha)$.

Theorem 1. Let $f = h + \bar{g}$ be given by (1). If

$$\sum_{m=1}^{\infty} m^n\left[(m-\alpha)|a_m| + (m+\alpha)|b_m|\right] \leq 2(1-\alpha) \tag{7}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$, then f is orientation preserving, harmonic univalent in U, and $f \in H(n,\alpha)$.

Proof. If the inequality (7) holds for the coefficients of $f = h + \bar{g}$, then by (5), f is orientation preserving and harmonic univalent in U. It remains to show that $\Re\{(D^{n+1}f)/(D^n f)\}\geq\alpha$. According to (2) and (3) we have

$$\Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} = \Re\left\{\frac{D^{n+1}h(z) - (-1)^n\overline{D^{n+1}g(z)}}{D^n h(z) + (-1)^n\overline{D^n g(z)}}\right\} \geq \alpha, \ 0 \leq \alpha < 1.$$

Using the fact that $\Re\{\omega\}\geq\alpha$ if and only if $|1-\alpha+\omega|\geq|1+\alpha-\omega|$ it suffices to show that

$$|D^{n+1}f(z)+(1-\alpha)D^n f(z)|-|D^{n+1}f(z)-(1+\alpha)D^n f(z)|\geq 0. \tag{8}$$

Substituting for $D^n f$ and $D^{n+1}f$ in (8) yields

$$\begin{aligned}
&|D^{n+1}f(z)+(1-\alpha)D^n f(z)|-|D^{n+1}f(z)-(1+\alpha)D^n f(z)|\\
&=|z+\textstyle\sum_{m=2}^{\infty}m^{n+1}a_m z^m-(-1)^n\sum_{m=1}^{\infty}m^{n+1}\bar{b}_m\bar{z}^m\\
&\qquad+(1-\alpha)z+(1-\alpha)\textstyle\sum_{m=2}^{\infty}m^n a_m z^m+(-1)^n(1-\alpha)\sum_{m=1}^{\infty}m^n\bar{b}_m\bar{z}^m|\\
&\quad-|z+\textstyle\sum_{m=2}^{\infty}m^{n+1}a_m z^m-(-1)^n\sum_{m=1}^{\infty}m^{n+1}\bar{b}_m\bar{z}^m\\
&\qquad-(1+\alpha)z-(1+\alpha)\textstyle\sum_{m=2}^{\infty}m^n a_m z^m-(-1)^n(1+\alpha)\sum_{m=1}^{\infty}m^n\bar{b}_m\bar{z}^m|\\
&=\left|(2-\alpha)z+\textstyle\sum_{m=2}^{\infty}m^n(m+1-\alpha)a_m z^m-(-1)^n\sum_{m=1}^{\infty}m^n(m-1+\alpha)\bar{b}_m\bar{z}^m\right|\\
&\quad-\left|-\alpha z+\textstyle\sum_{m=2}^{\infty}m^n(m-1-\alpha)a_m z^m-(-1)^n\sum_{m=1}^{\infty}m^n(m+1+\alpha)\bar{b}_m\bar{z}^m\right|\\
&\geq(2-\alpha)|z|-\textstyle\sum_{m=2}^{\infty}m^n(m+1-\alpha)|a_m||z|^m-\sum_{m=1}^{\infty}m^n(m-1+\alpha)|b_m||z|^m\\
&\qquad-\alpha|z|-\textstyle\sum_{m=2}^{\infty}m^n(m-1-\alpha)|a_m||z|^m-\sum_{m=1}^{\infty}m^n(m+1+\alpha)|b_m|z|^m\\
&=2(1-\alpha)|z|\left\{1-\textstyle\sum_{m=2}^{\infty}\frac{m^n(m-\alpha)}{1-\alpha}|a_m||z|^{m-1}-\sum_{m=1}^{\infty}\frac{m^n(m+\alpha)}{1-\alpha}|b_m||z|^{m-1}\right\}\\
&\geq 2(1-\alpha)|z|\left\{1-\textstyle\sum_{m=2}^{\infty}\frac{m^n(m-\alpha)}{1-\alpha}|a_m|-\sum_{m=1}^{\infty}\frac{m^n(m+\alpha)}{1-\alpha}|b_m|\right\}.
\end{aligned}$$

This last expression is non-negative by (7), and so the proof is complete.

The starlike harmonic function

$$f(z)=z+\sum_{m=2}^{\infty}\frac{1-\alpha}{m^n(m-\alpha)}x_m z^m+\sum_{m=1}^{\infty}\frac{1-\alpha}{m^n(m+\alpha)}\bar{y}_m\bar{z}^m \tag{9}$$

where $\sum_{m=2}^{\infty}|x_m|+\sum_{m=1}^{\infty}|y_m|=1$ shows that the coefficient bound given by (7) is sharp.

The functions of the form (9) are in $H(n,\alpha)$ because

$$\sum_{m=1}^{\infty}\left(\frac{m^n(m-\alpha)}{1-\alpha}|a_m|+\frac{m^n(m+\alpha)}{1-\alpha}|b_m|\right)=1+\sum_{m=2}^{\infty}|x_m|+\sum_{m=1}^{\infty}|y_m|=2.$$

In the following theorem, it is shown that the condition (7) is also necessary for functions $f_n=h+\overline{g_n}$ where h and g_n are of the form (4).

Theorem 2. Let $f_n=h+\bar{g}_n$ be given by (4). Then $f_n\in\overline{H}(n,\alpha)$ if and only if

$$\sum_{m=1}^{\infty}m^n\left[(m-\alpha)a_m+(m+\alpha)b_m\right]\leq 2(1-\alpha). \tag{10}$$

Proof. Since $\overline{H}(n,\alpha)\subset H(n,\alpha)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_n of the form (4), we notice that the condition $\Re[(D^{n+1}f_n(z))/(D^n f_n(z))]\geq\alpha$ is equivalent to

$$\Re\left\{\frac{(1-\alpha)z-\sum_{m=2}^{\infty}m^n(m-\alpha)a_m z^m-(-1)^{2n}\sum_{m=1}^{\infty}m^n(m+\alpha)b_m\bar{z}^m}{z-\sum_{m=2}^{\infty}m^n a_m z^m+(-1)^{2n}\sum_{m=1}^{\infty}m^n b_m\bar{z}^m}\right\}\geq 0. \tag{11}$$

The above required condition (11) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\frac{1-\alpha-\sum_{m=2}^{\infty}m^n(m-\alpha)a_m r^{m-1}-\sum_{m=1}^{\infty}m^n(m+\alpha)b_m r^{m-1}}{1-\sum_{m=2}^{\infty}m^n a_m r^{m-1}+\sum_{m=1}^{\infty}m^n b_m r^{m-1}}\geq 0. \tag{12}$$

If the condition (10) does not hold, then the numerator in (12) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0,1) for which the quotient in (12) is negative. This contradicts the required condition for $f_n \in \overline{H}(n, \alpha)$ and so the proof is complete.

Next we determine the extreme points of closed convex hulls of $\overline{H}(n, \alpha)$ denoted by *clco* $\overline{H}(n, \alpha)$.

Theorem 3. Let f_n be given by (4). Then $f_n \in \overline{H}(n, \alpha)$ if and only if

$$f_n(z) = \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_{n_m}(z)) \tag{13}$$

where $h_1(z) = z$, $h_m(z) = z - \frac{1-\alpha}{m^n(m-\alpha)} z^m$ $(m = 2, 3, ...)$, $g_{n_m}(z) = z + (-1)^n \frac{1-\alpha}{m^n(m+\alpha)} \bar{z}^m$ $(m = 1, 2, 3, ...,$ $\sum_{m=1}^{\infty}(X_m + Y_m) = 1$, $X_m \geq 0$, $Y_m \geq 0$. In particular, the extreme points of $\overline{H}(n, \alpha)$ are $\{h_m\}$ and $\{g_{n_m}\}$.

Proof. For functions f_n of the form (13) we have

$$\begin{aligned} f_n(z) &= \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_{n_m}(z)) \\ &= \sum_{m=1}^{\infty} (x_m + Y_m) z - \sum_{m=2}^{\infty} \frac{1-\alpha}{m^n(m-\alpha)} X_m z^m + (-1)^n \sum_{m=1}^{\infty} \frac{1-\alpha}{m^n(m+\alpha)} Y_m \bar{z}^m. \end{aligned}$$

Then

$$\sum_{m=2}^{\infty} \frac{m^n(m-\alpha)}{1-\alpha} a_m + \sum_{m=1}^{\infty} \frac{m^n(m+\alpha)}{1-\alpha} b_m = \sum_{m=2}^{\infty} X_m + \sum_{m=1}^{\infty} Y_m = 1 - X_1 \leq 1$$

and so $f_n \in$ *clco* $\overline{H}(n, \alpha)$.

Conversely, suppose that $f_n \in$ *clco* $\overline{H}(n, \alpha)$. Setting

$$X_m = \frac{m^n(m-\alpha)}{1-\alpha} a_m; \ (m = 2, 3, ...), \quad Y_m = \frac{m^n(m+\alpha)}{1-\alpha} b_m; \ (m = 1, 2, 3, ...)$$

where $\sum_{m=1}^{\infty}(X_m + Y_m) = 1$ we obtain $f_n(z) = \sum_{m=1}^{\infty}(X_m h_m(z) + Y_m g_{n_m}(z))$ as required.

The following theorem gives the distortion bounds for functions in $\overline{H}(n, \alpha)$ which yields a covering result for this class.

Theorem 4. Let $f_n \in \overline{H}(n, \alpha)$. Then for $|z| = r < 1$ we have

$$(1 - b_1)r - \frac{1}{2^n}\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha} b_1\right) r^2 \leq |f_n(z)| \leq (1 + b_1)r + \frac{1}{2^n}\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha} b_1\right) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_n \in \overline{H}(n, \alpha)$. Taking the absolute value of f_n we obtain

$$\begin{aligned} |f_n(z)| &\leq (1 + b_1)r + \sum_{m=2}^{\infty} (a_m + b_m) r^m \\ &\leq (1 + b_1)r + \sum_{m=2}^{\infty} (a_m + b_m) r^2 \\ &\leq (1 + b_1)r + \frac{1-\alpha}{2^n(2-\alpha)} \sum_{m=2}^{\infty} \left(\frac{2^n(2-\alpha)}{1-\alpha} a_m + \frac{2^n(2-\alpha)}{1-\alpha} b_m\right) r^2 \\ &\leq (1 + b_1)r + \frac{1-\alpha}{2^n(2-\alpha)} \left(1 - \frac{1+\alpha}{1-\alpha} b_1\right) r^2 \\ &= (1 + b_1)r + \frac{1}{2^n}\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha} b_1\right) r^2. \end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 4.

Corollary. Let f_n of the form (4) be so that $f_n \in \overline{H}(n,\alpha)$. Then

$$\left\{\omega : |\omega| < \frac{2^{n+1}-1-(2^n-1)\alpha}{2^n(2-\alpha)} - \frac{2^{n+1}-1-(2^n-1)\alpha}{2^n(2+\alpha)}b_1\right\} \subset f_n(U).$$

Remark. For n=1 and n=0 the above covering result coincides with those covering results given in [3] and [4], respectively.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form $f_n(z) = z - \sum_{m=2}^{\infty} a_m z^m + (-1)^n \sum_{m=1}^{\infty} b_m \bar{z}^m$ and $F_n(z) = z - \sum_{m=2}^{\infty} A_m z^m + (-1)^n \sum_{m=1}^{\infty} B_m \bar{z}^m$ we define the convolution of f_n and F_n as

$$(f_n * F_n)(z) = f_n(z) * F_n(z) = z - \sum_{m=2}^{\infty} a_m A_m z^m + (-1)^n \sum_{m=1}^{\infty} b_m B_m \bar{z}^m. \tag{14}$$

Theorem 5. For $0 \le \beta \le \alpha < 1$, let $f_n \in \overline{H}(n,\alpha)$ and $F_n \in \overline{H}(n,\beta)$. Then the convolution $f_n * F_n \in \overline{H}(n,\alpha) \subset \overline{H}(n,\beta)$.

Proof. For f_n and F_n as in Theorem 5, write $f_n(z) = z - \sum_{m=2}^{\infty} a_m z^m + (-1)^n \sum_{m=1}^{\infty} b_m \bar{z}^m$ and $F_n(z) = z - \sum_{m=2}^{\infty} A_m z^m + (-1)^n \sum_{m=1}^{\infty} B_m \bar{z}^m$. Then the convolution $f_n * F_n$ is given by (14). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2. For $F_n \in \overline{H}(n,\beta)$ we note that $|A_m| \le 1$ and $|B_m| \le 1$. Now, for the convolution function $f_n * F_n$ we obtain

$$\begin{aligned}
\sum_{m=2}^{\infty} \frac{m^n(m-\beta)}{1-\beta} a_m A_m + \sum_{m=1}^{\infty} \frac{m^n(m+\beta)}{1-\beta} b_m B_m &\le \sum_{m=2}^{\infty} \frac{m^n(m-\beta)}{1-\beta} a_m + \sum_{m=1}^{\infty} \frac{m^n(m+\beta)}{1-\beta} b_m \\
&\le \sum_{m=2}^{\infty} \frac{m^n(m-\alpha)}{1-\alpha} a_m + \sum_{m=1}^{\infty} \frac{m^n(m+\alpha)}{1-\alpha} b_m \\
&\le 1
\end{aligned}$$

since $0 \le \beta \le \alpha < 1$ and $f_n \in \overline{H}(n,\alpha)$. Therefore $f_n * F_n \in \overline{H}(n,\alpha) \subset \overline{H}(n,\beta)$.

Finally, we discuss the convex combinations of the class $\overline{H}(n,\alpha)$.

Theorem 6. The family $\overline{H}(n,\alpha)$ is closed under convex combination.

Proof. For $i = 1, 2, \ldots$, suppose that $f_{n_i} \in \overline{H}(n,\alpha)$ where

$$f_{n_i}(z) = z - \sum_{m=2}^{\infty} a_{i_m} z^m + (-1)^n \sum_{m=1}^{\infty} b_{i_m} \bar{z}^m.$$

Then, by Theorem 2,

$$\sum_{m=2}^{\infty} \frac{m^n(m-\alpha)}{1-\alpha} a_{i_m} + \sum_{m=1}^{\infty} \frac{m^n(m+\alpha)}{1-\alpha} b_{i_m} \le 1. \tag{15}$$

For $\sum_{m=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{m=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i_m}\right) z^m + (-1)^n \sum_{m=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{i_m}\right) \bar{z}^m.$$

Then, by (15),

$\sum_{m=2}^{\infty} \frac{m^n(m-\alpha)}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i a_{i_m}\right) + \sum_{m=2}^{\infty} \frac{m^n(m+\alpha)}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i b_{i_m}\right)$

$= \sum_{i=1}^{\infty} t_i \left(\sum_{m=2}^{\infty} \frac{m^n(m-\alpha)}{1-\alpha} a_{i_m} + \sum_{m=2}^{\infty} \frac{m^n(m+\alpha)}{1-\alpha} b_{i_m}\right) \le \sum_{i=1}^{\infty} t_i = 1$

and therefore $\sum_{i=1}^{\infty} t_i f_{n_i} \in \overline{H}(n,\alpha)$.

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