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ON SEMILINEAR ELLIPTIC EIGENVALUE PROBLEM WITH EIGENPARAMETER IN THE BOUNDARY CONDITIONS

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Abstract. The object of this paper is to establish the expansion theorem for a semilinear elliptic eigenvalue problem of a simply connected bounded domain in $R^n, (n \geq 2)$, where the eigenvalue parameter λ is contained in the semilinear elliptic partial differential equation and in the boundary conditions. We associated with this problem a semilinear self-adjoint operator S in a suitably defined Hilbert space H and then we develop an associated eigenfunction expansion theorem.

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1. Introduction

Our study is mainly motivated by some results for the semilinear elliptic problem :

$$\left.\begin{aligned} -\Delta u(x) &= g(u(x)) + h(x) \qquad in\ \Omega, \\ u(x) &= 0 \qquad on\ \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

recently obtained in [6], [8], [9] and [11].

Regular right-definite eigenvalue problem of elliptic operator in $R^m, (m \geq 2)$ with an eigenparameter λ in the elliptic equation and in the mixed boundary conditions have been studied by many authors, see for example, Ibrahim [15], Grunau and Sweers [12], Alama and Pino [1] and Gupta [14].

In the present paper, we shall study a semilinear elliptic eigenvalue problem in $R^n, (n \geq 2)$ with an eigenparameter λ in the semilinear elliptic equation (1.2) and in the boundary conditions (1.3) which does not seem to have been done elsewhere.

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The problem to be discussed here can be formulated as follows : Let Ω be a simply connected bounded domain in $R^n, (n \geq 2)$ with smooth boundary $\partial\Omega$. We consider the following semilinear elliptic eigenvalue equation :

$$\frac{1}{r}\,(\ -\Delta_n u + h(x)b(u(x))\ ) = \lambda u \qquad in\ \Omega \tag{1.2}$$

with the boundary conditions

$$u_\nu = \lambda u \qquad on\ \partial\Omega \tag{1.3}$$

where we assume throughout that :

(i) $\Delta_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator in R^n, $\quad (n \geq 2)$.

(ii) $u_\nu = \sum_{i=1}^{n} u_{x_i}(x)\nu_i(x)$ denotes differentiation of $u(x)$ along the outward unit normal $\nu(x) = (\nu_1(x), \nu_2(x), ..., \nu_n(x))$ on the boundary $\partial\Omega$, where $x = (x_1, x_2, ..., x_n)$ is a generic point in the Euclidean space R^n.

(iii) The weight function $r(x) > 0$ is a real-valued function such that $r(x) \in C(\overline{\Omega})$, $\overline{\Omega} = \Omega \bigcup \partial\Omega$ and the potential function $h(x)$ is an integrable function on Ω .

(iv) The nonlinear function $b(u(x))$ is a real-valued continuous function in Ω.

(v) The parameter λ is a complex number.

The nonlinearity of our problem (1.2)-(1.3) refers to the function $b(u(x))$ which is interfere with the eigenvalues of the associated semilinear operator S. Our problem (1.2)-(1.3) is a more generalization of Pino and Felmer [16] and Gossez and Omari [11].

This paper is organized as follows. In section 2, we formulate a self-adjoint semilinear operator S in a suitably Hilbert space H. In section 3, we recall some boundedness and regularity results for eigenfunction for problem (1.2)-(1.3). In section 4, we also recall some results about the existence of sequences of eigenvalues for problem (1.2)-(1.3). We give some basic properties of eigenvalues for our problem (1.2)-(1.3) and then we develop an associated eigenfunction expansion theorem.

2. Hilbert Space Formulation

Let $L_r^2(\Omega)$ and $L^2(\partial\Omega)$ be complex Hilbert spaces of Lebesgue measurable functions $f(x)$ in Ω and on $\partial\Omega$ respectively, satisfying :

(2.1)
$$\left.\begin{array}{l}
(i) \displaystyle\int_\Omega r(x) \mid f(x) \mid^2 dx < \infty, \\
(ii) \displaystyle\int_{\partial\Omega} \mid f(x) \mid^2 ds < \infty, \\
(iii) lim_{x\to-\infty} sup \dfrac{f(x)}{x} << \lambda_i \quad and \quad lim_{x\to+\infty} inf \dfrac{f(x)}{x} > \lambda_i, \qquad for fixed\ i
\end{array}\right\}$$

where $dx = dx_1 dx_2 \cdots dx_n$ is the volume element corresponding to Ω while ds is the surface element corresponding to the boundary $\partial\Omega$ respectively.

Definition 2.1. We define a Hilbert space H of two component vectors by

$$H = \ L_r^2(\Omega) \ \oplus \ L^2(\partial\Omega) \tag{2.2}$$

with inner product

$$[f,\ g]_H \ = < f_1, g_1 >_\Omega + < f_2, g_2 >_{\partial\Omega}$$

(2.3)

$$= \int_\Omega r(x) \left(\nabla f_1(x).\overline{\nabla g_1(x)} + h(x) b(f_1(x)).\overline{b(g_1(x))} \ \right) \ dx + \int_{\partial\Omega} f_2(x)\overline{g_2(x)} \ ds,$$

and the norm

(2.4)

$$\| \ f \ \|_H^2 = \int_\Omega r(x) \left(\ | \ \nabla f_1(x) \ |^2 + h(x) \ | \ b(f_1(x)) \ |^2 \ \right) \ dx + \int_{\partial\Omega} | \ f_2(x) \ |^2 \ \ ds$$

for each $f = (f_1, f_2)$ and $g = (g_1, g_2)$ in H.

Definition 2.2. Let H_0 be the set of all elements $f = (f_1, f_2)$ satisfying

(2.5)

$$\left. \begin{array}{l} A_1 : f_1 \in C^1(\overline{\Omega}) \bigcap C^2(\Omega), \\ A_2 : \dfrac{1}{r}(\ -\Delta_n f_1 + h(x) b(f_1(x)) \ ) \in L_r^2(\Omega), \\ A_3 : lim_{x\to\infty} \dfrac{f(x)}{x} = c > 0, \\ A_4 : lim_{x\to 0} \dfrac{f(x)}{x} = 0. \end{array} \right\}$$

Because of the nonlinearity of problem (1.2)-(1.3), the function $f = (f_1, f_2)$ satisfies the conditions A_3 and A_4 (see [2]).

We define a semilinear operator $S : D(S) \longrightarrow H$ by

$$Sf := (-\Delta_n f_1 + h(x) b(f_1(x)), f_{1\nu}) \,; \tag{2.6}$$

for each $f = (f_1, f_2) \in D(S)$, in which the domain $D(S)$ of S is defined as follows:

$$D(S) := \{(f \mid_\Omega, f \mid_{\partial\Omega}) \in H : f \in H_0\} \tag{2.7}$$

where $f \mid_\Omega$ and $f \mid_{\partial\Omega}$ are restrictions of f on Ω and on $\partial\Omega$ respectively.

Remark (2.1): The parameter λ is an eigenvalue and f_1 is a corresponding eigenfunction of the problem (1.2)-(1.3) if and only if

$$Sf = \lambda f \qquad forall \ f = (f_1, f_2) \in D(S). \tag{2.8}$$

Therefore, the eigenvalues and the eigenfunctions of the problem (1.2)-(1.3) are equivalent to the eigenvalues and the eigenfunctions of the semilinear operator S in H.

Remark (2.2): The semilinear operator S is compact and self-adjoint operator such that $Sf = \lambda f$.

Now, the variational characterization of the eigenvalues of the operator S yields

$$\lambda_n << \lambda_{n+1} \qquad\qquad \forall \ n \geq 1 \tag{2.9}$$

3. The Boundedness Of Eigenfunctions of The Operator S

In this section we shall discuss some results about the regularity and boundedness of the eigenfunctions of the problem (1.2)-(1.3). Making use of [13],[18] and [3], it is clear the regularity of eigenfunctions of the operator S; by using " strong maximum principle ", see [10]. Denote the set of eigenfunctions $\Phi_n(x)_{n=1}^{\infty}$ which forms a complete fundamental system in H.

Because of the variationally problem (1.2)-(1.3) and using [16] and [3], the variational arguments are applied to the functional :

(3.1)

$$J_\lambda(\Phi_n) = \int_\Omega \left(\mid \nabla\Phi_n \mid^2 - \lambda_n r(x) \mid \Phi_n \mid^2\right)\, dx + \frac{1}{2}\int_\Omega h(x) \mid b(\Phi_n) \mid^2\, dx + \int_{\partial\Omega} \mid \Phi_{n\nu} \mid^2\, ds < \infty$$

This functional $J_\lambda(\Phi_n)$ is of class $C^2(\overline{\Omega}) \bigcap C^2(\Omega)$, on the Hilbert space H and that its critical points correspond exactly to the solutions of the problem (1.2)-(1.3). We should remark that the positive solutions of problem (1.2)-(1.3) correspond precisely to global minimizers of this functional $J_\lambda(\Phi_n)$ as established in [17].

Lemma 3.1. The set of eigenfunctions $\{\Phi_n\}_{n=1}^{\infty}$ of operator S is bounded from above .

Proof: Without loss of generality, we may assume that

$$v_n = \frac{\Phi_n}{\| \Phi_n \|_H} \to v_0 \tag{3.2}$$

in strongly Hilbert space H.We claim that $v_0 \neq 0$.
Indeed, dividing

$$\mid J_\lambda(\Phi_n) \mid \leq c \tag{3.3}$$

by $\| \Phi_n \|_H$ yields

$$\int_\Omega r(x) \left(\mid \nabla\Phi_{n1} \mid^2 + h(x) \mid b(\Phi_{n1}) \mid^2\right)\, dx + \int_{\partial\Omega} \mid \Phi_{n2} \mid^2\, ds \to 0 \qquad \Diamond \tag{3.4}$$

Definition 3.1. Let the function b : $\Omega \times C^2(\overline{\Omega}) \to R$ be a continuous function and radially symmetric function; that is , $b(x,u) = b(\mid x \mid, u), \quad forall \quad x \in \Omega$ and

$$B(x,u) := \int_0^u b(x,z)\, dz. \tag{3.5}$$

be the primitive function of the nonlinear function $b(x,u)$ with respect to the spectrum of the associated semilinear operator S.

Lemma 3.2. The set of eigenfunctions $\{\Phi_n\}_{n=1}^{\infty}$ of operator S is bounded from below ; i.e. There exists $\alpha, \rho > 0$ such that $J_\lambda(\Phi_n) \geq \alpha$ for $\| \Phi_n \|_H = \rho$.

Proof: From the condition A_4, it is given $\varepsilon > 0$, there exists $\delta > 0$, such that; (using [2]);

$$F(x) \leq \frac{\varepsilon x^2}{2}, \qquad \mid x \mid \leq \delta, \tag{3.6}$$

Thus

$$\int_{\Omega} F(\Phi_n)\ dx \le \frac{\varepsilon}{2}\int_{\Omega} \Phi_n^2\ dx \qquad \| \Phi_n \|_H < \rho, \tag{3.7}$$

ρ small enought.
And

$$\int_{\Omega} B(x,\Phi_n)\ dx \le \int_{\Omega} F(\Phi_n)\ dx \le \frac{\varepsilon}{2}\int_{\Omega} \Phi_n^2\ dx \qquad \| \Phi_n \|_H = \rho. \tag{3.8}$$

Now, using (3.5),(3.7) and (3.8), we find the critical points on $D(S)$ of the $C^2(\overline{\Omega})$ functional as follows :

$$J_\lambda(\Phi_n) = \int_{\Omega} \left(| \nabla\Phi_n |^2 - \lambda_n r(x) | \Phi_n |^2 \right)\ dx + \frac{1}{2}\int_{\Omega} h(x) | b(\Phi_n) |^2\ \ dx$$

$$- \int_{\Omega} B(x,\Phi_n)\ dx + \int_{\partial\Omega} | \Phi_{n\nu} |^2\ \ ds \tag{3.9}$$

$$\ge \int_{\Omega} \left(| \nabla\Phi_n |^2 - \lambda_n r(x) | \Phi_n |^2 \right)\ dx + \frac{1}{2}\int_{\Omega} h(x) | b(\Phi_n) |^2\ \ dx$$

$$- \int_{\Omega} F(\Phi_n)\ dx + \int_{\partial\Omega} | \Phi_{n\nu} |^2\ \ ds$$

$$\ge \int_{\Omega} \left(| \nabla\Phi_n |^2 - \lambda_n r(x) | \Phi_n |^2 \right)\ dx + \frac{1}{2}\int_{\Omega} h(x) | b(\Phi_n) |^2\ \ dx$$

$$+ \int_{\partial\Omega} | \Phi_{n\nu} |^2\ \ ds - \frac{\varepsilon}{2}\int_{\Omega} \Phi_n^2\ dx \tag{3.10}$$

$$\ge c_0 \| \Phi_n \|_H^2 - \frac{\varepsilon}{2}\int_{\Omega} \Phi_n^2\ dx \tag{3.11}$$

Recalling that

$$\int_{\Omega} \Phi_n^2\ dx \le c_1 \left[\ \int_{\Omega} \left(| \nabla\Phi_n |^2 - \lambda_n r(x) | \Phi_n |^2 \right)\ dx \right.$$

$$\left. + \frac{1}{2}\int_{\Omega} h(x) | b(\Phi_n) |^2\ \ dx + \int_{\partial\Omega} | \Phi_{n\nu} |^2\ \ ds \ \right] \tag{3.12}$$

From (3.11) and (3.12), we have

$$J_\lambda(\Phi_n) \ge c_2 \| \Phi_n \|_H^2 \qquad for \qquad \| \Phi_n \|_H = \rho. \tag{3.13}$$

where c_0, c_1 and c_2 are positive constants. $\diamondsuit$

Remark (3.1): The set of eigenfunctions $\{\Phi_n\}_{n=1}^{\infty}$ of the operator S is bounded.

4. Properties Of Eigenvalues of The Operator S

The necessary and sufficient condition for the problem (1.2)-(1.3)to be solvable for any eigenfunction $\Phi_n(x) = (\Phi_{n1}, \Phi_{n2}) \in H$ is that the range of $b(x)$ be all of R. Here

$$\lambda_1 = 0 < \lambda_2 << \lambda_3 << .... << \lambda_n << \lambda_{n+1} << ...... \tag{4.1}$$

denote the distinct eigenvalues of the problem (1.2)-(1.3), where the value

$$\lambda_n :=^{def} \ inf \ (\ max_{\Phi_n \in H_0} J_\lambda(\Phi_n) \ ) \tag{4.2}$$

is an eigenvalue of the problem (1.2)-(1.3). Moreover,

$$lim_{n \to +\infty} \lambda_n = +\infty \tag{4.3}$$

Let us take ; by using [11]; the primitive function

$$B(x) := \int_0^x b(t) \ dt, \qquad x \in \overline{\Omega} \tag{4.4}$$

and ; by using [3] ; the condition
(4.5)

$$lim_{x \to \pm\infty} sup \frac{b(x)}{x} << \lambda_{n+1} \qquad and \qquad lim_{x \to +\infty \ (or \ x \to -\infty)} sup \frac{2B(x)}{x^2} > \lambda_n.$$

This new sequence of eigenvalues was introduced in the case of $r(x) = 1$ by [7] and was used to resonance and nonresonance results associated to the n-Laplacian operator. More recently, this sequence appears to be of some interest in the study of Fučik spectrum of the n-Laplacian made by [4] and [5].
Remark (4.1): It is a trivial fact that

$$\lambda_2 = inf\{ \int_\Omega \left(\mid \nabla \Phi_{n1} \mid^2 + h(x) \mid b(\Phi_{n1}) \mid^2 \right) \ dx \quad , \quad \int_\Omega r(x) \mid \Phi_{n1} \mid^2 \ dx = 1$$

$$and \quad \int_{\partial\Omega} \mid \Phi_{n2} \mid^2 \ ds = 1 \quad : \quad \Phi_n \in H \} \tag{4.6}$$

which is the first eigenvalue of the problem (1.2)-(1.3).

Condition (4.5) controls the possible interference of the nonlinearity $b(u(x))$ with higher part of the spectrum of the associated semilinear operator S. The nonlinearity $b(u(x))$ in the problem (1.2)-(1.3) is such that $\frac{b(x)}{x}$ lies asymptotically between two higher eigenvalues λ_n and λ_{n+1}. One obtains in the main results of [6].
Theorem 4.1: For some $n > 1$, we have

$$\lambda_n < lim_{x \to \pm\infty} \ inf \frac{b(x)}{x} \ < lim_{x \to \pm\infty} \ sup \frac{b(x)}{x} \ < \lambda_{n+1} \tag{4.7}$$

and that

(4.8)
$$\lambda_n < lim_{x\to+\infty\ (or\ x\to-\infty)}\ sup\frac{2B(x)}{x^2} < lim_{x\to+\infty\ (or\ x\to-\infty)}\ inf\frac{2B(x)}{x^2} < \lambda_{n+1}.$$

Proof: Making use of (4.2), (4.1) and (2.9), we get
(4.9)
$$\lambda_n := \ inf\ (\ max_{\Phi_n\in H_0} J_\lambda(\Phi_n)\ ) << \lambda_{n+1} := \ inf\ \big(\ max_{\Phi_{n+1}\in H_0} J_\lambda(\Phi_{n+1})\ \big)\ .$$

This leads us to
(4.10)
$$inf\ (\ max_{\Phi_n\in H_0} J_\lambda(\Phi_n)\ ) < lim_{x\to\pm\infty}\ inf\frac{b(x)}{x} < lim_{x\to\pm\infty}\ sup\frac{b(x)}{x} < inf\ \big(\ max_{\Phi_{n+1}\in H_0} J_\lambda(\Phi_{n+1})\ \big)\ .$$

This gives us (4.7).
Using the primitive function in(4.4), we assume for λ_n and λ_{n+1} that

$$lim_{x\to+\infty\ (or\ x\to-\infty)}\ inf\frac{2B(x)}{x^2} < \lambda_{n+1} \tag{4.11}$$

And

$$lim_{x\to+\infty\ (or x\to-\infty\ )}\ sup\frac{2B(x)}{x^2} > \lambda_n \tag{4.12}$$

Making use (4.11), (4.12) and (4.5), we obtain (4.8). $\Diamond$

The eigenvalues $\lambda_n,\quad n\in N$ of operator S are real if and only if there exists $\Phi_n = (\Phi_{n1}, \Phi_{n2}) \in H_0$ such that

$$J'_\lambda(\Phi_n(x)) + \lambda_n r(x) \mid \Phi_n(x) \mid^2 = (\| \Phi_n(x) \|_H^2)' \tag{4.13}$$

Therefore, we get

$$(\| \Phi_n(x) \|_H^2)' - J'_\lambda(\Phi_n(x)) = \lambda_n r(x) \mid \Phi_n(x) \mid^2 . \tag{4.14}$$

Let us denote by $\overline{J_\lambda}(\Phi_n)$ the restriction of $J'_\lambda(\Phi_n)$ to H_0. We recall the value σ is a critical value of $\overline{J_\lambda}(\Phi_n)$ if
(4.15)
$$J'_\lambda(\Phi_n(x)) \mid_{H_0} = 0 \qquad and \qquad J_\lambda(\Phi_n(x)) = \sigma \qquad\qquad for some \qquad \Phi_n(x) \in H_0.$$

It follows from standard arguments that positive eigenvalues of operator S correspond to positive critical values of $\overline{J_\lambda}(\Phi_n)$.

The results of our investigations are summarized in the following expansion theorem:

Theorem 4.2. With the assumptions stated above, the spectrum of the operator S in H consists of an unbounded sequence of real eigenvalues of finite multiplicity without an accumulation point in $(-\infty, \infty)$. Denoting them by

$$\lambda_1 = 0 < \lambda_2 << \lambda_3 << \ldots << \lambda_n << \lambda_{n+1} << \ldots \longrightarrow \infty \quad as \quad n \longrightarrow \infty$$

and the corresponding eigenfunctions by $\Phi_1, \Phi_2, ..., \Phi_k, ...$, we have $\{\Phi_n\}_{n=1}^{\infty}$ forms a complete fundamental system in H and for every $f(x) \in H$, we have the expansion formula

$$f = \sum_{n=1}^{\infty} [f, \Phi_n]_H \Phi_n, \tag{4.16}$$

in the sense of strong convergence in H.

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