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EIGENFUNCTIONS OF LAPLACIAN BY BOUNDEDNESS CONDITIONS IN BOUNDED HYPERFUNCTIONS AND DISTRIBUTIONS

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Abstract. R. S. Strichartz has proved the following theorem: "If $\{f_j(x)\}_{j\in\mathbb{Z}}$ is a doubly infinite sequence of functions on $\mathbb{R}^n$ satisfying (i) $\triangle f_j = f_{j+1}$ and (ii) there exists a constant M such that $|f_j(x)| \le M$ for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$, then $\triangle f_0 = -f_0$" in (*Characterization of eigenfunctions of the Laplacian by boundedness conditions*, Trans. Amer. Math. Soc. **338** (1993), 971–979). We generalize it in the space of bounded hyperfunctions via heat kernel and in the space of distributions via regularization of distributions.

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1. Introduction

If a function and all its derivatives and integrals are absolutely uniformly bounded, then the function is a sine function with period 2π. This is Roe's theorem [3]. In place of derivatives and antiderivatives, Strichartz [4] has given an n-dimensional generalization using powers of the Laplacian operator $\triangle = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$. More precisely if $\{f_j(x)\}_{j\in\mathbb{Z}}$ is a doubly infinite sequence of functions on $\mathbb{R}^n$ satisfying (i) $\triangle f_j = f_{j+1}$ and (ii) $|f_j(x)| \le M$ for all $j \in \mathbb{Z}$ and all $x \in \mathbb{R}^n$, then $\triangle f_0 = -f_0$. We give an analogue result in the space of bounded hyperfunctions and distributions. We first give the following definitions :

(i) We denote by $A_{L^1}(\mathbb{R}^n)$ the space of all $\phi \in C^\infty(\mathbb{R}^n)$ satisfying

$$\|\phi\|_{A_{L^1,h}} = \sup_{\alpha\in\mathbb{N}_0^n} \frac{\|\partial^\alpha \phi\|_{L^1}}{h^{|\alpha|}\ \alpha!} < \infty$$

for some constant $h > 0$. We say that $\phi_i \to 0$ as $i \to \infty$ if $\|\phi_i\|_{A_{L^1,h}} \to 0$ as $i \to \infty$. We denote by $B_{L^\infty}(\mathbb{R}^n)$ the dual of $A_{L^1}(\mathbb{R}^n)$. Every element in $B_{L^\infty}(\mathbb{R}^n)$ is called

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a bounded hyperfunction. For more details of bounded hyperfunctions, we refer to [1].

(ii) We denote by $E(x,t)$ the n-dimensional heat kernel

$$E(x,t) = \begin{cases} (4\pi t)^{-n/2}\exp(-|x|^2/4t), & t > 0, \\ 0, & t \le 0. \end{cases}$$

Note that $E(x,t) \in A_{L^1}(\mathbb{R}^n)$ for each $t > 0$. So for every $u \in B_{L^\infty}(\mathbb{R}^n)$, we define $U(x,t) := \langle u(y), E(x-y,t)\rangle$ in $\mathbb{R}^{n+1}_+ = \{(x,t)|x \in \mathbb{R}^n, t > 0\}$ and call it the *defining function* of u.

2. MAIN RESULTS

Theorem 2.1. *Let $u_j \in B_{L^\infty}(\mathbb{R}^n)$, $j \in \mathbb{Z}$. Suppose that there exist positive constants h and $M := M(h)$ such that*

(i) $\triangle u_j = u_{j+1}$ *in the sense that* $\langle \triangle u_j, \phi\rangle = \langle u_{j+1}, \phi\rangle$ *for all* $\phi \in A_{L^1}(\mathbb{R}^n)$.

(ii) $\|u_j * \phi\|_{L^\infty} \le M\|\phi\|_{A_{L^1,h}}$ *for all* $j \in \mathbb{Z}$ *and all* $\phi \in A_{L^1}(\mathbb{R}^n)$.

Then $\triangle u_0 = -u_0$.

Proof. For fixed $t_0 > 0$, the defining function $U_j(x,t_0) := \langle u_j(y), E(x-y,t_0)\rangle \in C^\infty(\mathbb{R}^n)$. Since $E(x-y,t_0) \in A_{L^1}(\mathbb{R}^n)$, it follows from the hypothesis (i) that

$$\triangle_x U_j(x,t_0) = \langle \triangle_y u_j(y), E(x-y,t_0)\rangle = U_{j+1}(x,t_0) \tag{2.1}$$

Applying the hypothesis (ii) and the following estimate from (p.2426, [1])

$$\|\partial_x^\alpha E(x,t)\|_{L^1(\mathbb{R}^n_x)} \le C^{|\alpha|}{t_0}^{-(n+|\alpha|)/2}\alpha!^{1/2}\|\exp(-|x|^2/8t_0)\|_{L^1(\mathbb{R}^n_x)}$$

for some constant $C > 0$, we have

$$\begin{aligned} |U_j(x,t_0)| &\le M \sup_{\alpha\in\mathbb{N}_0^n} \frac{\|\partial_y^\alpha E(y,t_0)\|_{L^1(\mathbb{R}^n_y)}}{h^{|\alpha|}\ \alpha!} \\ &\le M(8\pi)^{n/2} \sup_{\alpha\in\mathbb{N}_0^n} \frac{C^{|\alpha|}\ {t_0}^{-|\alpha|/2}}{h^{|\alpha|}\ \alpha!^{1/2}} \\ &\le M(8\pi)^{n/2}\prod_{i=1}^n \sup_{\alpha_i\in\mathbb{N}_0} \frac{C^{\alpha_i}{t_0}^{-\alpha_i/2}}{h^{\alpha_i}\ {\alpha_i}!^{1/2}}. \end{aligned} \tag{2.2}$$

But in view of the inequalities $\alpha_i! \le {\alpha_i}^{\alpha_i} \le \alpha_i!\ e^{\alpha_i}$ for each $i = 1,\dots,n$

$$\sup_{\alpha_i\in\mathbb{N}_0} \frac{C^{\alpha_i}\ {t_0}^{-\alpha_i/2}}{h^{\alpha_i}\ {\alpha_i}!^{1/2}} = \sup_{\alpha_i\in\mathbb{N}_0} \frac{C^{\alpha_i}\ {t_0}^{-\alpha_i/2}}{h^{\alpha_i}\ {\alpha_i}^{\alpha_i/2}e^{-\alpha_i/2}} = \exp\left(C^2 2^{-1}h^{-2}\ {t_0}^{-1}\right)$$

So (2.2) reduces to

$$|U_j(x,t_0)| \le M(8\pi)^{n/2}\exp\left(nC^2 2^{-1}h^{-2}{t_0}^{-1}\right) \tag{2.3}$$

for all $j \in \mathbb{N}_0$ and all $x \in \mathbb{R}^n$. Since $t_0 > 0$ is fixed, the right hand side of (2.3) is a constant independent of j and x. So in view of (2.1), (2.3) and Theorem 1.1 of [4], we have

$$\triangle U_0(x,t_0) = -U_0(x,t_0). \tag{2.4}$$

It then follows from (2.4) and Theorem 3.5 of [1] that

$$\langle -u_0, \phi\rangle = \lim_{t_0\to 0^+} \langle -U_0(x,t_0), \phi(x)\rangle = \lim_{t_0\to 0^+} \langle \triangle U_0(x,t_0), \phi(x)\rangle = \langle \triangle u_0, \phi\rangle$$

for all $\phi \in A_{L_1}(\mathbb{R}^n)$. Hence $\triangle u_0 = -u_0$. ||

In the space of distributions $D^{'}(\mathbb{R}^n)$, we use the regularization of distributions to obtain the analogue of Theorem 2.1.

Theorem 2.2. *Let $u_j \in D^{'}(\mathbb{R}^n)$, $j \in \mathbb{Z}$. Suppose that for every compact subset K of $\mathbb{R}^n$, there exist a nonnegative integer N and a positive constant $M := M(K,N)$ such that*

(i) $\triangle u_j = u_{j+1}$ *in the sense that* $\langle \triangle u_j, \phi\rangle = \langle u_{j+1}, \phi\rangle$ *for all* $\phi \in D(\mathbb{R}^n)$.

(ii) $\|u_j * \phi\|_{L^\infty} \le M \ \sum_{|\alpha|\le N} \sup_{x\in K} |\partial^\alpha \phi(x)|$ *for all* $j \in \mathbb{Z}$ *and all* $\phi \in D_K(\mathbb{R}^n)$.

Then $\triangle u_0 = -u_0$.

Proof. We can choose some $\psi \in D(\mathbb{R}^n)$ such that $\int \psi dx = 1$, and set $\psi_i(x) = i^n\psi(ix), i = 1, 2, \ldots$. Then $\psi_i \to \delta$ in $D^{'}(\mathbb{R}^n)$, and *supp* $\psi_i \subset$ *supp* ψ for all i. For each j, $u_j * \psi_i \in C^\infty(\mathbb{R}^n)$ which is a regularization of u_j and $u_j * \psi_i \to u_j$ in $D^{'}(\mathbb{R}^n)$ as $i \to \infty$. Let $g_{j,i} := u_j * \psi_i$. Then for $K :=$ *supp* ψ and all $j \in \mathbb{Z}$, it follows from the hypotheses (i) and (ii) that

$$\begin{cases} \triangle g_{j,i} = u_{j+1} * \psi_i = g_{j+1,i} \\ \|g_{j,i}\|_{L^\infty} \le \tilde{M} \end{cases} \tag{2.5}$$

where

$$\tilde{M} := M \sum_{|\alpha|\le N} \sup_{x\in K} |\partial^\alpha \psi_i(x)|$$

is a positive constant. It then follows from (2.5) and Theorem 1.1 of [4] that $\triangle g_{0,i} = -g_{0,i}$. Letting $i \to \infty$, we obtain that $\triangle u_0 = -u_0$. ||

Replacing the Laplacian by a d'Alembertian

$$\Box = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \cdots - \frac{\partial^2}{\partial x_n^2}, \qquad 1 \le p \le n-1,$$

we have the following generalization of previous theorems.

Theorem 2.3. *Let $u_j \in B_{L^\infty}(\mathbb{R}^n)$, $j \in \mathbb{Z}$. Suppose that there exist positive constants h and $M := M(h)$ such that*

(i) $\Box u_j = u_{j+1}$ *in the sense that* $\langle \Box u_j, \phi\rangle = \langle u_{j+1}, \phi\rangle$ *for all* $\phi \in A_{L^1}(\mathbb{R}^n)$.

(ii) $\|u_j * \phi\|_{L^\infty} \le M\|\phi\|_{A_{L^1,h}}$ *for all* $j \in \mathbb{Z}$ *and all* $\phi \in A_{L^1}(\mathbb{R}^n)$.

Then $u_0 = u_+ + u_-$ *where* $u_\pm = \frac{1}{2}(u_0 \pm \Box u_0)$ *and* $\Box u_\pm = \pm u_\pm$.

Proof. Obviously as in (2.1), we obtain

$$\Box U_j(x,t_0) = U_{j+1}(x,t_0). \tag{2.6}$$

It then follows from (2.6), (2.3) and Theorem 3.1 of [4] that

$$U_0(x,t_0) = U_+(x,t_0) + U_-(x,t_0) \tag{2.7}$$

where $U_\pm(x,t_0)$ are bounded solutions of $\Box U_\pm(x,t_0) = \pm U_\pm(x,t_0)$. Also from (p.211, [2])

$$\Box^2 U_0(x,t_0) = U_0(x,t_0), \quad U_\pm(x,t_0) = \frac{1}{2}\left(U_0(x,t_0) \pm \Box U_0(x,t_0)\right). \tag{2.8}$$

Then for every $\phi \in A_{L^1}(\mathbb{R}^n)$, it follows from (2.7), (2.8) and Theorem 3.5 of [1] that

$$\begin{aligned}
\langle u_0, \phi\rangle &= \lim_{t_0\to 0^+} \langle U_0(x,t_0), \phi(x)\rangle \\
&= \lim_{t_0\to 0^+} \left\langle \frac{1}{2}\left(U_0(x,t_0) + \Box U_0(x,t_0)\right) + \frac{1}{2}\left(U_0(x,t_0) - \Box U_0(x,t_0)\right), \phi(x)\right\rangle \\
&= \left\langle \frac{1}{2}\left(u_0 + \Box u_0\right) + \frac{1}{2}\left(u_0 - \Box u_0\right), \phi(x)\right\rangle \\
&= \langle u_+ + u_-, \phi\rangle
\end{aligned}$$

where $u_\pm := \frac{1}{2}(u_0 \pm \Box u_0)$. Using Theorem 3.5 of [1] and $\Box^2 U_0(x,t_0) = U_0(x,t_0)$ from (2.8), we have

$$\langle \Box^2 u_0, \phi\rangle = \lim_{t_0\to 0^+} \langle U_0(x,t_0), \Box^2\phi\rangle = \lim_{t_0\to 0^+} \langle U_0(x,t_0), \phi\rangle = \langle u_0, \phi\rangle \tag{2.9}$$

for every $\phi \in A_{L^1}(\mathbb{R}^n)$. Then $\Box u_\pm = \pm u_\pm$ due to (2.9). ||

Theorem 2.4. *Let $u_j \in D'(\mathbb{R}^n)$, $j \in \mathbb{Z}$. Suppose that for every compact subset K of $\mathbb{R}^n$, there exist a nonnegative integer N and a positive constant $M := M(K,N)$ such that*

(i) $\Box u_j = u_{j+1}$ *in the sense that* $\langle \Box u_j, \phi\rangle = \langle u_{j+1}, \phi\rangle$ *for all* $\phi \in D(\mathbb{R}^n)$.

(ii) $\|u_j * \phi\|_{L^\infty} \le M \sum_{|\alpha|\le N} \sup_{x\in K} |\partial^\alpha \phi(x)|$ *for all* $j \in \mathbb{Z}$ *and all* $\phi \in D_K(\mathbb{R}^n)$.

Then $u_0 = u_+ + u_-$ *where* $u_\pm = \frac{1}{2}(u_0 \pm \Box u_0)$ *and* $\Box u_\pm = \pm u_\pm$.

Proof. Applying the same technique, (2.5) holds even for the operator $\Box$. So using Theorem 3.1 of [4], we have $g_{0,i} = g_{0,i}^+ + g_{0,i}^-$ where $g_{0,i}^\pm$ are bounded solutions of $\Box g_{0,i}^\pm = \pm g_{0,i}^\pm$. Also from (p.211, [2])

$$\Box^2 g_{0,i} = g_{0,i}\ , \quad g_{0,i}^\pm = \frac{1}{2}\left(g_{0,i} \pm \Box g_{0,i}\right). \tag{2.10}$$

Using regularization of u_j, we obtain that $u_0 = u_+ + u_-$ where $u_\pm = \frac{1}{2}(u_0 \pm \Box u_0)$. In view of (2.10), it is easy to see that $\Box^2 u_0 = u_0$ and hence $\Box u_\pm = \pm u_\pm$. ||

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