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GENERALIZED JENSEN'S EQUATIONS IN BANACH MODULES OVER A UNITAL C^*-ALGEBRA

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Abstract. We prove the Hyers-Ulam-Rassias stability of generalized Jensen's equations in Banach modules over a unital C^*-algebra. It is applied to show the stability of generalized Jensen's equations in a Hilbert module over a unital C^*-algebra. Moreover, we prove the stability of linear operators in a Hilbert module over a unital C^*-algebra.

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1. Generalized Jensen's equations

Given a locally compact abelian group G and a multiplier ω on G, one can associate to them the twisted group C^*-algebra $C^*(G,\omega)$. $C^*(\mathbb{Z}^m,\omega)$ is said to be a *noncommutative torus of rank* m and denoted by A_ω. The multiplier ω determines a subgroup S_ω of G, called its *symmetry group*, and the multiplier is called *totally skew* if the symmetry group S_ω is trivial. And A_ω is called *completely irrational* if ω is totally skew (see [2]). It was shown in [2] that if G is a locally compact abelian group and ω is a totally skew multiplier on G, then $C^*(G,\omega)$ is a simple C^*-algebra. It was shown in [3, Theorem 1.5] that if A_ω is a simple noncommutative torus then A_ω has real rank 0, where *"real rank* 0" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [3, 5]).

Let E_1 and E_2 be Banach spaces. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0,1)$ such that

$$||f(x+y) - f(x) - f(y)|| \leq \epsilon(||x||^p + ||y||^p)$$

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for all $x, y \in E_1$. Th.M. Rassias [13] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2-2^p}||x||^p$$

for all $x \in E_1$.

Lemma A. *Let V and W be vector spaces, and let r, s, t be positive integers. A mapping $f : V \to W$ with $f(0) = 0$ is a solution of the equation*

$$rf(\frac{sx+ty}{r}) = sf(x) + tf(y) \tag{A}$$

for all $x, y \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Proof. Assume that $f : V \to W$ satisfies the equation (A). Then

$$\begin{aligned} rf(\frac{s}{r}x) &= rf(\frac{sx + t\cdot 0}{r}) = sf(x) + tf(0) = sf(x), \\ rf(\frac{t}{r}x) &= rf(\frac{s\cdot 0 + tx}{r}) = sf(0) + tf(x) = tf(x) \end{aligned}$$

for all $x \in V$. So

$$f(\frac{s}{r}x) = \frac{s}{r}f(x) \quad \& \quad f(\frac{t}{r}x) = \frac{t}{r}f(x)$$

for all $x \in V$. And

$$\begin{aligned} f(x) &= f(\frac{s}{r}\cdot\frac{r}{s}x) = \frac{s}{r}f(\frac{r}{s}x), \\ f(x) &= f(\frac{t}{r}\cdot\frac{r}{t}x) = \frac{t}{r}f(\frac{r}{t}x) \end{aligned}$$

for all $x \in V$. So

$$f(\frac{r}{s}x) = \frac{r}{s}f(x) \quad \& \quad f(\frac{r}{t}x) = \frac{r}{t}f(x)$$

for all $x \in V$. Thus

$$\begin{aligned} f(x+y) &= \frac{1}{r}\cdot rf(\frac{s}{r}\cdot\frac{r}{s}x + \frac{t}{r}\cdot\frac{r}{t}y) = \frac{1}{r}(sf(\frac{r}{s}x) + tf(\frac{r}{t}y)) \\ &= \frac{1}{r}(s\cdot\frac{r}{s}f(x) + t\cdot\frac{r}{t}f(y)) = f(x) + f(y) \end{aligned}$$

for all $x, y \in V$.

Conversely, assume that $f : V \to W$ satisfies the additive Cauchy equation. Since $f(sx) = rf(\frac{s}{r}x)$ for all $x \in V$,

$$f(\frac{s}{r}x) = \frac{1}{r}f(sx) = \frac{s}{r}f(x)$$

for all $x \in V$. Similarly, one can show that

$$f(\frac{t}{r}y) = \frac{1}{r}f(ty) = \frac{t}{r}f(y)$$

for all $y \in V$. So $f(\frac{sx+ty}{r}) = f(\frac{s}{r}x) + f(\frac{t}{r}y) = \frac{s}{r}f(x) + \frac{t}{r}f(y)$ for all $x, y \in V$. Thus

$$rf(\frac{sx+ty}{r}) = sf(x) + tf(y)$$

for all $x, y \in V$. $\square$

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$, A_{in} the set of invertible elements in A, A_{sa} the set of self-adjoint elements in A, $A_1 = \{a \in A \mid |a| = 1\}$, and A_1^+ the set of positive elements in A_1. And let ${}_A\mathcal{B}$ and ${}_A\mathcal{C}$ be left Banach A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. Let s, t be different positive integers, r a positive integer, and $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ a mapping such that

$$\text{(i)} \qquad \widetilde{\varphi}(x, y) := \sum_{k=0}^{\infty} (\frac{t}{s})^{2k} \varphi((\frac{s}{t})^{2k}x, (\frac{s}{t})^{2k}y) < \infty$$

for all $x, y \in {}_A\mathcal{B}$.

In this paper, we prove the Hyers-Ulam-Rassias stability of the functional equation (A) in Banach modules over a unital C^*-algebra.

2. Stability of generalized Jensen's equations in Banach modules over a C^*-algebra

Theorem 2.1. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\text{(ii)} \qquad \|rf(\frac{sx+ty}{r}) - sf(x) - tf(y)\| \le \varphi(x, y)$$

for all $x, y \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\text{(iii)} \qquad \|f(x) - T(x)\| \le \frac{1}{s}\widetilde{\varphi}(x, -\frac{s}{t}x) + \frac{t}{s^2}\widetilde{\varphi}(-\frac{s}{t}x, (\frac{s}{t})^2x)$$

for all $x \in {}_A\mathcal{B}$. Furthermore, if $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, then the additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is $\mathbb{R}$-linear.

Proof. Let $x \in {}_A\mathcal{B}$. For $y = -\frac{s}{t}x$, the inequality (ii) implies

$$(1) \qquad \|sf(x) + tf(-\frac{s}{t}x)\| \le \varphi(x, -\frac{s}{t}x).$$

Replacing x by $-\frac{s}{t}x$ and y by $(\frac{s}{t})^2x$, the inequality (ii) implies

$$(2) \qquad \|sf(-\frac{s}{t}x) + tf((\frac{s}{t})^2x)\| \le \varphi(-\frac{s}{t}x, (\frac{s}{t})^2x).$$

From (1) and (2), we get

$$\|f(x) - (\frac{t}{s})^2 f((\frac{s}{t})^2x)\| \le \frac{1}{s}\varphi(x, -\frac{s}{t}x) + \frac{t}{s^2}\varphi(-\frac{s}{t}x, (\frac{s}{t})^2x).$$

Applying the induction argument on n, we obtain

$$\begin{aligned}\|f(x) - (\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)\| \le \sum_{k=0}^{n-1} (\frac{1}{s}(\frac{t}{s})^{2k} \varphi((\frac{s}{t})^{2k}x, -(\frac{s}{t})^{2k+1}x) \\ + \frac{1}{s}(\frac{t}{s})^{2k+1} \varphi(-(\frac{s}{t})^{2k+1}x, (\frac{s}{t})^{2k+2}x)).\end{aligned} \tag{3}$$

We claim that the sequence $\{(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)\}$ is a Cauchy sequence. Indeed, for $n > m$, we have

$$\begin{aligned}&\|(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2m} f((\frac{s}{t})^{2m}x)\| \\ &\le \sum_{k=m}^{n-1} \|(\frac{t}{s})^{2k+2} f((\frac{s}{t})^{2k+2}x) - (\frac{t}{s})^{2k} f((\frac{s}{t})^{2k}x)\| \\ &\le \sum_{k=m}^{n-1} (\frac{1}{s}(\frac{t}{s})^{2k} \varphi((\frac{s}{t})^{2k}x, -(\frac{s}{t})^{2k+1}x) + \frac{1}{s}(\frac{t}{s})^{2k+1} \varphi(-(\frac{s}{t})^{2k+1}x, (\frac{s}{t})^{2k+2}x)).\end{aligned}$$

From (i), it follows that

$$\lim_{m\to\infty} \sum_{k=m}^{n-1} (\frac{1}{s}(\frac{t}{s})^{2k} \varphi((\frac{s}{t})^{2k}x, -(\frac{s}{t})^{2k+1}x) + \frac{1}{s}(\frac{t}{s})^{2k+1} \varphi(-(\frac{s}{t})^{2k+1}x, (\frac{s}{t})^{2k+2}x)) = 0.$$

Since ${}_A\mathcal{D}$ is a Banach space, the sequence $\{(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)\}$ converges. Define

$$T(x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)$$

for all $x \in {}_A\mathcal{B}$. Taking the limit in (3) as $n \to \infty$, we obtain

$$\|f(x) - T(x)\| \le \frac{1}{s}\widetilde{\varphi}(x, -\frac{s}{t}x) + \frac{t}{s^2}\widetilde{\varphi}(-\frac{s}{t}x, (\frac{s}{t})^2 x)$$

for all $x \in {}_A\mathcal{B}$. This completes the proof of the inequality (iii). From the definition of T, we get

$$(\frac{s}{t})^{2n} T(x) = T((\frac{s}{t})^{2n}x) \text{ and } T(0) = 0. \tag{4}$$

From (i), (ii), and the definition of T,

$$\begin{aligned}&\|rT(\frac{sx+ty}{r}) - sT(x) - tT(y)\| \\ &\qquad = \lim_{n\to\infty} (\frac{t}{s})^{2n} \|rf((\frac{s}{t})^{2n}\frac{sx+ty}{r}) - sf((\frac{s}{t})^{2n}x) - tf((\frac{s}{t})^{2n}y)\| \\ &\qquad \le \lim_{n\to\infty} (\frac{t}{s})^{2n} \varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}y) = 0\end{aligned}$$

for all $x, y \in {}_A\mathcal{B}$. So

$$rT(\frac{sx+ty}{r}) = sT(x) + tT(y)$$

for all $x, y \in {}_A\mathcal{B}$. By Lemma A, T is additive.

If $F : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is another additive mapping satisfying (iii), then it follows from (iii), (4) and the proof of Lemma A that

$$\begin{aligned}
&\|T(x) - F(x)\| = \|(\frac{t}{s})^{2n}T((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2n}F((\frac{s}{t})^{2n}x)\| \\
&\le \|(\frac{t}{s})^{2n}T((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x)\| + \|(\frac{t}{s})^{2n}f((\frac{s}{t})^{2n}x) - (\frac{t}{s})^{2n}F((\frac{s}{t})^{2n}x)\| \\
&\le 2(\frac{t}{s})^{2n}(\frac{1}{s}\widetilde{\varphi}((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}(-\frac{s}{t})x) + \frac{t}{s^2}\widetilde{\varphi}((\frac{s}{t})^{2n}(-\frac{s}{t})x), (\frac{s}{t})^{2n}(\frac{s}{t})^2 x)),
\end{aligned}$$

which tends to zero as $n \to \infty$ by (i). Thus we conclude that

$$T(x) = F(x)$$

for all $x \in {}_A\mathcal{B}$. This completes the uniqueness of T.

Assume that $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$. By the assumption, $\frac{s}{t}$ is a rational number which is not an integer. The additive mapping T given above is similar to the additive mapping T given in the proof of [13, Theorem]. By the same reasoning as the proof of [13, Theorem], the additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is $\mathbb{R}$-linear. □

Corollary 2.2. *Let $0 < p < 1$ and $t < s$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|rf(\frac{sx + ty}{r}) - sf(x) - tf(y)\| \le \|x\|^p + \|y\|^p$$

for all $x, y \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \le \frac{s^{2(1-p)}}{s^{2(1-p)} - t^{2(1-p)}}(\frac{1}{s} + \frac{1}{t^p s^{1-p}} + \frac{t^{1-p}}{s^{2-p}} + \frac{t^{1-2p}}{s^{2-2p}})||x||^p$$

for all $x \in {}_A\mathcal{B}$.

Proof. Define $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ by $\varphi(x, y) = ||x||^p + ||y||^p$, and apply Theorem 2.1. □

Corollary 2.3. *Let $p > 1$ and $t > s$. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|rf(\frac{sx + ty}{r}) - sf(x) - tf(y)\| \le \|x\|^p + \|y\|^p$$

for all $x, y \in {}_A\mathcal{B}$. Then there exists a unique additive mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ such that

$$\|f(x) - T(x)\| \le \frac{t^{2(p-1)}}{t^{2(p-1)} - s^{2(p-1)}}(\frac{1}{s} + \frac{s^{p-1}}{t^p} + \frac{s^{p-1}}{t^{p-1}} + \frac{s^{2-2p}}{t^{2p-1}})||x||^p$$

for all $x \in {}_A\mathcal{B}$.

Proof. Define $\varphi : {}_A\mathcal{B} \times {}_A\mathcal{B} \to [0, \infty)$ by $\varphi(x, y) = ||x||^p + ||y||^p$, and apply Theorem 2.1. □

Theorem 2.4. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|raf(\frac{sx+ty}{r}) - sf(ax) - tf(ay)\| \le \varphi(x,y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

Proof. Let $a = 1 \in A_1^+$. By Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii). By the continuity and the uniform convergence, the $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is continuous.

By the assumption, for each $a \in A_1^+ \cup \{i\}$,

$$\|raf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}x) - (s+t)f((\frac{s}{t})^{2n}ax)\| \le \varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}x),$$
$$\|rf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}ax) - (s+t)f((\frac{s}{t})^{2n}ax)\| \le \varphi((\frac{s}{t})^{2n}ax, (\frac{s}{t})^{2n}ax)$$

for all $x \in {}_A\mathcal{B}$. So

$$\begin{aligned} \|rf(\frac{(s+t)}{r}&(\frac{s}{t})^{2n}ax) - raf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}x)\| \\ &\le \|rf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}ax) - (s+t)f((\frac{s}{t})^{2n}ax)\| \\ &\quad + \|raf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}x) - (s+t)f((\frac{s}{t})^{2n}ax)\| \\ &\le \varphi((\frac{s}{t})^{2n}ax, (\frac{s}{t})^{2n}ax) + \varphi((\frac{s}{t})^{2n}x, (\frac{s}{t})^{2n}x) \end{aligned}$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. Thus

$$(\frac{t}{s})^{2n}\|rf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}ax) - raf(\frac{(s+t)}{r}(\frac{s}{t})^{2n}x)\| \to 0$$

as $n \to \infty$ for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. Hence

$$\begin{aligned} rT(\frac{s+t}{r}ax) &= \lim_{n\to\infty}(\frac{t}{s})^{2n} rf(\frac{(s+t)a}{r}(\frac{s}{t})^{2n}x) = \lim_{n\to\infty} raf(\frac{s+t}{r}(\frac{s}{t})^{2n}x) \\ &= raT(\frac{s+t}{r}x) \end{aligned}$$

for each $a \in A_1^+ \cup \{i\}$. So

$$T(ax) = \frac{r}{s+t}T(\frac{s+t}{r}ax) = \frac{r}{s+t}aT(\frac{s+t}{r}x) = aT(x)$$

for each $a \in A_1^+ \cup \{i\}$.

For any element $a \in A$, $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, and $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = (\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-$, where $(\frac{a+a^*}{2})^+$,

$(\frac{a+a^*}{2})^-$, $(\frac{a-a^*}{2i})^+$, and $(\frac{a-a^*}{2i})^-$ are positive elements (see [4, Lemma 38.8]). So

$$\begin{aligned}
T(ax) =&T((\frac{a+a^*}{2})^+x - (\frac{a+a^*}{2})^-x + i(\frac{a-a^*}{2i})^+x - i(\frac{a-a^*}{2i})^-x)\\
=&(\frac{a+a^*}{2})^+T(x) + (\frac{a+a^*}{2})^-T(-x) + (\frac{a-a^*}{2i})^+T(ix) + (\frac{a-a^*}{2i})^-T(-ix)\\
=&(\frac{a+a^*}{2})^+T(x) - (\frac{a+a^*}{2})^-T(x) + i(\frac{a-a^*}{2i})^+T(x) - i(\frac{a-a^*}{2i})^-T(x)\\
=&((\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-)T(x)\\
=&aT(x)
\end{aligned}$$

for all $a \in A$ and all $x \in {}_A\mathcal{B}$. Hence

$$T(ax+by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in {}_A\mathcal{B}$, as desired. □

Theorem 2.5. *Let A be a unital C^*-algebra of real rank 0. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a continuous mapping with $f(0) = 0$ such that*

$$\|raf(\frac{sx+ty}{r}) - sf(ax) - tf(ay)\| \le \varphi(x,y)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. If the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n}x)$ converges uniformly on ${}_A\mathcal{B}$, then there exists a unique continuous A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

Proof. By the same reasoning as the proof of Theorem 2.4, there exists a unique continuous $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii), and

$$T(ax) = aT(x) \tag{1}$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

Let $b \in A_1^+ \setminus A_{in}$. Since $A_{in} \cap A_{sa}$ is dense in A_{sa}, there exists a sequence $\{b_m\}$ in $A_{in} \cap A_{sa}$ such that $b_m \to b$ as $m \to \infty$. Put $c_m = \frac{1}{|b_m|} b_m$. Then $c_m \to \frac{1}{|b|} b = b$ as $m \to \infty$. Put $a_m = \sqrt{c_m^* c_m}$. Then $a_m \to b$ as $m \to \infty$ and $a_m \in A_1^+ \cap A_{in}$. Thus there exists a sequence $\{a_m\}$ in $A_1^+ \cap A_{in}$ such that $a_m \to b$ as $m \to \infty$, and by the continuity of T

$$\lim_{m\to\infty} T(a_m x) = T(\lim_{m\to\infty} a_m x) = T(bx) \tag{2}$$

for all $x \in {}_A\mathcal{B}$. By (1),

$$\|T(a_m x) - bT(x)\| = \|a_m T(x) - bT(x)\| \to \|bT(x) - bT(x)\| = 0 \tag{3}$$

as $m \to \infty$. By (2) and (3),

$$\begin{aligned}
\|T(bx) - bT(x)\| \le& \|T(bx) - T(a_m x)\| + \|T(a_m x) - bT(x)\|\\
\to& 0 \text{ as } m \to \infty
\end{aligned} \tag{4}$$

for all $x \in {}_A\mathcal{B}$. By (1) and (4), $T(ax) = aT(x)$ for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 2.4. □

Theorem 2.6. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|raf(\frac{sx+ty}{r}) - sf(ax) - tf(ay)\| \le \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. If $f(\lambda x)$ is continuous in $\lambda \in \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

Proof. Let $a = 1 \in A_1^+$. By Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

The rest of the proof is similar to the proof of Theorem 2.4. □

Theorem 2.7. *Let A be a unital C^*-algebra of real rank 0. Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|raf(\frac{sx+ty}{r}) - sf(ax) - tf(ay)\| \le \varphi(x, y)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{B}$. Assume that $f(ax)$ is continuous in $a \in A_1 \cup \mathbb{R}$ for each fixed $x \in {}_A\mathcal{B}$, and that the sequence $(\frac{t}{s})^{2n} f((\frac{s}{t})^{2n} ax)$ converges uniformly on A_1 for each fixed $x \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

Proof. By the same reasoning as the proof of Theorem 2.4, there exists a unique $\mathbb{R}$-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii), and

$$T(ax) = aT(x)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{B}$. By the continuity and the uniform convergence, one can show that $T(ax)$ is continuous in $a \in A_1$ for each $x \in {}_A\mathcal{B}$.

The rest of the proof is similar to the proof of Theorem 2.5. □

Theorem 2.8. *Let $f : {}_A\mathcal{B} \to {}_A\mathcal{C}$ be a mapping with $f(0) = 0$ such that*

$$\|raf(\frac{sx+ty}{r}) - sf(ax) - tf(ay)\| \le \varphi(x, y)$$

for all $a \in A_1^+ \cup \{i\} \cup \mathbb{R}$ and all $x, y \in {}_A\mathcal{B}$. Then there exists a unique A-linear mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ satisfying (iii).

Proof. By the same reasoning as the proof of Theorem 2.1, there exist unique additive mappings $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iii).

By the same reasoning as the proof of Theorem 2.4,

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\} \cup \mathbb{R}$ and all $x \in {}_A\mathcal{B}$. So the mapping $T : {}_A\mathcal{B} \to {}_A\mathcal{C}$ is $\mathbb{R}$-linear, and satisfies

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{B}$.

The rest of the proof is the same as the proof of Theorem 2.4. □

3. Stability of generalized Jensen's equations in a Hilbert module over a C^*-algebra

In this section, let $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be mappings with $h(0) = 0$ and $g(0) = 0$ such that $h((\frac{s}{t})^{2n}x) = (\frac{s}{t})^{2n}h(x)$ and $g((\frac{s}{t})^{2n}x) = (\frac{s}{t})^{2n}g(x)$ for all positive integers n and all $x \in {}_A\mathcal{H}$.

We are going to prove the Hyers-Ulam-Rassias stability of generalized Jensen's equations in a Hilbert module over a unital C^*-algebra.

Theorem 3.1. *Let $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be continuous mappings such that*

$$\|rah(\frac{sx+ty}{r}) - sh(ax) - th(ay)\| \le \varphi(x,y)$$
$$\|rag(\frac{sx+ty}{r}) - sg(ax) - tg(ay)\| \le \varphi(x,y)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Then the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded A-linear operators. Furthermore,

(1) *if the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfy the inequalities*

$$\|h \circ g(x) - x\| \le \varphi(x,x),$$
$$\|g \circ h(x) - x\| \le \varphi(x,x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping g is the inverse of the mapping h,

(2) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|h(x) - h^*(x)\| \le \varphi(x,x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a self-adjoint operator,

(3) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequality*

$$\|h \circ h^*(x) - h^* \circ h(x)\| \le \varphi(x,x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a normal operator,

(4) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequalities*

$$\|h \circ h^*(x) - x\| \le \varphi(x,x),$$
$$\|h^* \circ h(x) - x\| \le \varphi(x,x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a unitary operator, and

(5) *if the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfies the inequalities*

$$\|h \circ h(x) - h(x)\| \le \varphi(x,x),$$
$$\|h^*(x) - h(x)\| \le \varphi(x,x)$$

for all $x \in {}_A\mathcal{H}$, then the mapping $h : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is a projection.

Proof. Let $a = 1 \in A_1^+$. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ such that

$$\text{(iv)} \qquad \|g(x) - L(x)\| \le \frac{1}{s}\widetilde{\varphi}(x, -\frac{s}{t}x) + \frac{t}{s^2}\widetilde{\varphi}(-\frac{s}{t}x, (\frac{s}{t})^2x)$$

for all $x \in {}_A\mathcal{H}$. And the mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ can be defined by

$$L(x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} g((\frac{s}{t})^{2n} x).$$

But the sequence $(\frac{t}{s})^{2n} g((\frac{s}{t})^{2n} x)$ converges uniformly on ${}_A\mathcal{H}$. By the same reasoning as the proof of Theorem 2.4, the mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is continuous.

The sequence $(\frac{t}{s})^{2n} h((\frac{s}{t})^{2n} x)$ converges uniformly on ${}_A\mathcal{H}$. By Theorem 2.4, there exists a unique continuous A-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iii). By the assumption,

$$T(x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} h((\frac{s}{t})^{2n} x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} (\frac{s}{t})^{2n} h(x) = h(x)$$
$$L(x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} g((\frac{s}{t})^{2n} x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} (\frac{s}{t})^{2n} g(x) = g(x)$$

for all $x \in {}_A\mathcal{H}$, where the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is given in the proof of Theorem 2.1. Hence the A-linear operators T and L are the mappings h and g, respectively. So the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are continuous A-linear operators. Thus the A-linear operators $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded (see [6, Proposition II.1.1]).

(1) By the assumption,

$$\|h \circ g((\frac{s}{t})^{2n} x) - (\frac{s}{t})^{2n} x\| \le \varphi((\frac{s}{t})^{2n} x, (\frac{s}{t})^{2n} x),$$
$$\|g \circ h((\frac{s}{t})^{2n} x) - (\frac{s}{t})^{2n} x\| \le \varphi((\frac{s}{t})^{2n} x, (\frac{s}{t})^{2n} x)$$

for all positive integers n and all $x \in {}_A\mathcal{H}$. Thus

$$(\frac{t}{s})^{2n} \|h \circ g((\frac{s}{t})^{2n} x) - (\frac{s}{t})^{2n} x\| \to 0,$$
$$(\frac{t}{s})^{2n} \|g \circ h((\frac{s}{t})^{2n} x) - (\frac{s}{t})^{2n} x\| \to 0$$

as $n \to \infty$ for all $x \in {}_A\mathcal{H}$. Hence

$$h \circ g(x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} h \circ g((\frac{s}{t})^{2n} x) = x,$$
$$g \circ h(x) = \lim_{n\to\infty} (\frac{t}{s})^{2n} g \circ h((\frac{s}{t})^{2n} x) = x$$

for all $x \in {}_A\mathcal{H}$. So the mapping g is the inverse of the mapping h.

The proofs of the others are similar to the proof of (1). $\square$

Theorem 3.2. *Let A be a unital C^*-algebra of real rank 0. Let $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ be continuous mappings such that*

$$\|rah(\frac{sx + ty}{r}) - sh(ax) - th(ay)\| \le \varphi(x, y)$$
$$\|rag(\frac{sx + ty}{r}) - sg(ax) - tg(ay)\| \le \varphi(x, y)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x, y \in {}_A\mathcal{H}$. Then the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded A-linear operators. Furthermore, the properties, given in the statement of Theorem 3.1, hold.

Proof. The sequence $(\frac{t}{s})^{2n}h((\frac{s}{t})^{2n}x)$ converges uniformly on ${}_A\mathcal{H}$. By the same reasoning as the proof of Theorem 2.5, there exists a unique continuous $\mathbb{R}$-linear operator $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iii), and

$$T(ax) = aT(x)$$

for all $a \in (A_1^+ \cap A_{in}) \cup \{i\}$ and all $x \in {}_A\mathcal{H}$. By the same method as the proof of Theorem 2.5, one can show that

$$T(ax) = aT(x)$$

for all $a \in A_1^+ \cup \{i\}$ and all $x \in {}_A\mathcal{H}$. By the same reasoning as the proof of Theorem 2.4, the mapping $T : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is A-linear.

Similarly, one can show that there exists a unique $\mathbb{R}$-linear operator $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ satisfying (iv) and that the mapping $L : {}_A\mathcal{H} \to {}_A\mathcal{H}$ is A-linear.

The rest of the proof is the same as the proof of Theorem 3.1. So the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are bounded A-linear operators, and the properties, given in the statement of Theorem 3.1, hold. $\square$

So the mappings $h, g : {}_A\mathcal{H} \to {}_A\mathcal{H}$ are elements of the C^*-algebra $\mathcal{L}({}_A\mathcal{H})$ of all bounded A-linear operators on ${}_A\mathcal{H}$.

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