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GLOBAL L^∞ BOUNDS FOR A CLASS OF QUASILINEAR PARABOLIC EQUATIONS

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Abstract. In this paper we study quasi-linear parabolic equations which admit solutions which lie in a certain prescribed (possibly infinite) interval. These equations are of the form $u_t - \nabla c'(u)\nabla u = f$ together with appropriate boundary conditions, where $c : (a,b) \to \mathbb{R}$ satisfies certain conditions. We derive sufficient conditions for the solution u to lie in the domain of c for all times $t > 0$.

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Introduction

Many physical systems can be described by a quasilinear parabolic initial-value problem of the form

$$u_t - \nabla c'(u)\nabla u = f \qquad \forall (x,t) \in \Omega \times \mathbb{R}^+ \tag{1}$$

$$u(x,0) = u_0(x) \qquad \forall x \in \Omega \tag{2}$$

together with a flux boundary condition of the form

$$\frac{\partial u}{\partial \nu} = -A(u - u_\Gamma) \qquad \forall (x,t) \in \partial\Omega \times \mathbb{R}^+. \tag{3}$$

Here Ω is a bounded domain in $\mathbb{R}^n$ $(n = 2,3)$ with smooth boundary $\Gamma = \partial\Omega$, and A is a positive constant. We are interested in the case when

$$c : (a,b) \to \mathbb{R} \tag{4}$$

is a smooth increasing function. The interval (a,b) is either finite or of the form (a,∞).

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Examples of equations of this form are non-linear heat equations. In these u represents the absolute temperature and the domain of c is $(0,\infty)$. The energy balances in several papers about phase field equations have been of this form (see, e.g. [5 - 7, 9, 11]). Other examples where the solution u has to be positive are diffusion equations, where u is a density. In reaction-diffusion equations u can represent the concentration of a chemical compound. In this case $(a,b)=(0,1)$.

To get global smooth solutions to the initial-value problem it is necessary to show that for every $T>0$ there is a compact set $K_T \subset (a,b)$ such that

$$u(x,t) \in K_T \tag{5}$$

for all $(x,t) \in \Omega \times (0,T)$. This has been done for the specific examples mentioned above, but to the best of our knowledge, has not been proven in a more general setting. The intent of this paper is to generalize the techniques used in [5 - 7, 9]. In particular, we will use Moser-type estimates like the ones introduced by Alikakos in [1] to obtain sufficient conditions for global existence.

If c satisfies $c:\mathbb{R}^+ \to \mathbb{R}^-$ with

$$c'(\xi) \geq K_0 \xi^{-\alpha} \tag{6}$$

for all $\xi > 0$ and some $K_0 > 0$ and $\alpha > 0$, we have the following results.

Proposition 1. *Let $T>0$ be given and assume that $f \in L^\infty(0,T;L^p(\Omega))$ for some $2<p<\infty$ and*

$$1 < \alpha \leq \min\left\{\frac{2}{9}p+1, p-1\right\} \tag{7}$$

$$c'(\xi)\xi^{p-1} \text{ is non-decreasing for } \xi > 0. \tag{8}$$

Furthermore, assume that there are numbers $0<a_0\leq b_0<\infty$ and $0<a_\Gamma \leq b_\Gamma < \infty$ such that

$$a_0 \leq u_0(x) \leq b_0 \qquad \forall x \in \Omega \tag{9}$$

$$a_\Gamma \leq u_\Gamma(x,t) \leq b_\Gamma \qquad \forall (x,t) \in \partial\Omega \times (0,T). \tag{10}$$

Then there exist constants $0<a_T\leq b_T<\infty$ such that the solution u of (1)-(3) satisfies

$$a_T \leq u(x,t) \leq b_T \qquad \forall (x,t) \in \Omega_T = \Omega \times (0,T). \tag{11}$$

For the important case of $\alpha = 1$ we have a weaker result:

Proposition 2. *Let u be a classical solution to*

$$u_t - \Delta \log u = f \tag{12}$$

on $\Omega \times (0,T)$ together with boundary condition (3). Furthermore, assume that there are numbers $0<a_0\leq b_0<\infty$ and $0<a_\Gamma\leq b_\Gamma<\infty$ such that

$$a_0 \leq u_0(x) \leq b_0 \qquad \forall x \in \Omega \tag{13}$$

$$a_\Gamma \leq u_\Gamma(x,t) \leq b_\Gamma \qquad \forall (x,t) \in \partial\Omega \times (0,T), \tag{14}$$

and that $f \in L^\infty(0,T;L^\infty(\Omega))$. *Then there exists a constant* $0 < b_T < \infty$ *such that the solution* u *of (1)-(3) satisfies*

$$u(x,t) \le b_T \qquad \forall (x,t) \in \Omega_T = \Omega \times (0,T). \tag{15}$$

Furthermore, if there is a $q \ge 2$ *and constant* C *such that*

$$\left\| \frac{1}{u(t)} \right\|_q \le C, \tag{16}$$

for all $0 < t < T$, *there is a constant* $a_T > 0$ *such that*

$$u(x,t) \ge a_T, \qquad \forall (x,t) \in \Omega_T. \tag{17}$$

Observe that we do not require $f > 0$. The equation satisfies an implicit maximum (and minimum) principle.

Remarks:

(1) Propositions 1 and 2 can be easily extended to the case $c : (a,\infty) \to (-\infty, d)$ by using the linear transformations $w(x,t) = u(x,t) - a$ and $\tilde{c}(\xi) = c(\xi - a) - d$ where w and $\tilde{c}$ satisfy the hypotheses of Propositions 1 and 2.

(2) The boundary condition (3) can be changed to

$$c'(u)\frac{\partial u}{\partial \nu} = -A(u - u_\Gamma). \tag{18}$$

The result will still hold and we do not need the condition that $c'(\xi)\xi^{p-1}$ is non-decreasing for $\xi > 0$. The same is true for the result of Proposition 3 below.

(3) The equation in Proposition 2 has been studied extensively for domains in $\mathbb{R}^2$ (see, e.g. [4]).

In Section 3 we investigate the situation, where c is a mapping $c : (a,b) \to \mathbb{R}$. The requirement for $c'(u)$ becomes:

$$c'(\xi) \ge K_a(\xi - a)^{-\alpha} + K_b(b - \xi)^{-\alpha} \tag{19}$$

for all $\xi > 0$, some $K_a, K_b > 0$ and some $\alpha > 1$. We have the following

Proposition 3. *Let* $T > 0$ *and suppose that* $f \in L^\infty(0,T;L^p(\Omega))$ *for some* $2 \le p < \infty$ *and that (15) holds. Further, assume that* u *satisfies the boundary condition (3) and that there is a* $P > 0$ *such that*

$$c'(\xi)\xi^P \tag{20}$$

is non-decreasing in ξ. *Finally, assume there are numbers* $a < a_0 \le b_0 < b$ *and* $a < a_\Gamma \le b_\Gamma < b$ *such that*

$$a_0 \le u_0(x) \le b_0 \qquad \forall x \in \Omega \tag{21}$$

$$a_\Gamma \le u_\Gamma(x,t) \le b_\Gamma \qquad \forall (x,t) \in \partial\Omega \times (0,T). \tag{22}$$

Then there exist constants $a < a_T \le b_T < b$ such that the solution u of (1)-(3) satisfies

$$a_T \le u(x,t) \le b_T \qquad \forall (x,t) \in \Omega_T = \Omega \times [0,T]. \tag{23}$$

Remark: Throughout the paper we assume local existence of solutions to these equations. Local existence can be shown for sufficiently smooth c and f. For example, for $c \in C^3((a,b))$, $f \in C^2(\overline{\Omega} \times \mathbb{R})$ and sufficiently smooth u_0, local existence of smooth solutions to (1)-(3) follows from [3: Theorem 16.1]. Global existence of solutions follows from local existence together with the L^∞-bounds proven in this paper (see, e.g., [2, 3]).

Proof of Proposition 1.

We assume that the boundary value problem (1)-(3) admits a unique local solution u. The crucial part of the proof of this theorem is to show that there exist uniform a priori bounds on u. To do this we need the following technical lemma, which is similar to a lemma proven by Ph. Laurençot ([7] and [10: Lemma 4.1.5]).

Lemma 1. *Let $\{\eta_k\}_{k=0}^\infty, \{\beta_k\}_{k=0}^\infty, \{\gamma_k\}_{k=0}^\infty$ be sequences of positive real numbers such that $A2^k \le \eta_k \le B2^k$, $\beta_k \to 2$ as $k \to \infty$ and $\beta_k \le 2\big(1 + \frac{C}{2^k}\big)$, $\gamma_k \uparrow$ and*

$$\gamma_{k+1} \le D\eta_{k+1}^2 \gamma_k^{\beta_k} \tag{24}$$

for positive constants A, B, C, D and all $k \ge 0$. Then there exists a constant $M > 0$ such that $\gamma_k^{\frac{1}{\eta_k}} \le M$ for all $k \ge 0$.

Proof.. The proof is very similar to that of a lemma by Laurençot, which can be found in the references [7] and [10: Lemma 4.1.5]. We only sketch it here. To start we use (24) repeatedly to obtain

$$\begin{aligned}\gamma_{k+1} &\le D\eta_{k+1}^2 \gamma_k^{\beta_k} \le D^{1+\beta_k} \eta_{k+1}^2 \eta_k^{2\beta_k} \gamma_{k-1}^{\beta_k\beta_{k-1}} \\ &\le D^{1+\beta_k+\beta_k\beta_{k-1}+\cdots+(\beta_k\beta_{k-1}\cdots\beta_2\beta_1)} \\ &\qquad\qquad \times \eta_{k+1}^2 \eta_k^{2\beta_k} \eta_{k-1}^{2\beta_k\beta_{k-1}} \cdots \eta_1^{2\beta_k\beta_{k-1}\cdots\beta_1} \gamma_0^{\beta_k\beta_{k-1}\cdots\beta_0}.\end{aligned}$$

We continue by taking the η_{k+1}-th root of this last inequality to get

$$\begin{aligned}\gamma_{k+1}^{\frac{1}{\eta_{k+1}}} &= D^{(1+\beta_k+\beta_k\beta_{k-1}+\cdots+(\beta_k\beta_{k-1}\cdots\beta_2\beta_1))/\eta_{k+1}} \\ &\qquad \times \eta_{k+1}^{2/\eta_{k+1}} \eta_k^{2\beta_k/\eta_{k+1}} \eta_{k-1}^{2\beta_k\beta_{k-1}/\eta_{k+1}} \cdots \eta_1^{2\beta_k\beta_{k-1}\cdots\beta_1/\eta_{k+1}} \gamma_0^{\beta_k\beta_{k-1}\cdots\beta_0/\eta_{k+1}}\end{aligned}$$

The right side of this last inequality stays bounded as $k \to \infty$. This can be seen by taking the logarithm of this expression and observing that

$$\frac{\beta_k\beta_{k-1}\cdots\beta_{k_0}}{\eta_{k+1}} \le \frac{2^{k-k_0}}{\eta_{k+1}} \prod_{j=k_0}^{k} \Big(1 + \frac{C}{2^j}\Big) \le K\frac{1}{2^{k_0}}$$

for a suitable constant K. The statement of the lemma follows immediately

The proof of Proposition 1 is now done by the following two lemmas.

Lemma 2. *Assume that the hypotheses of Proposition 1 are satisfied. Then there exists a constant b_T such that $\|u\|_{L^\infty(\Omega_T)} \leq b_T$.*

Proof.. For $f \in L^\infty(0,T;L^p(\Omega))$, $2 \leq p < \infty$ multiply equation (1) by u^{p-1}. After integration by parts and using the boundary condition (3) we obtain

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + (p-1)\int_\Omega c'(u)u^{p-2}(\nabla u)^2 dx \\ + A\int_{\partial\Omega} c'(u)(u^p - u_\Gamma u^{p-1})\,d\gamma = \int_\Omega f u^{p-1} dx.$$

Using (8) and Hölder's inequality yield

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + (p-1)K_0\int_\Omega u^{p-2-\alpha}(\nabla u)^2\,dx \\ + A\int_{\partial\Omega} c'(u)(u^p - u_\Gamma u^{p-1})\,d\gamma \leq \|f\|_{L^p(\Omega)}\|u\|_{L^p(\Omega)}^{p-1}.$$

Note that

$$(p-1)K_0\int_\Omega u^{p-2-\alpha}(\nabla u)^2 dx = K_0\frac{4(p-1)}{(p-\alpha)^2}\int_\Omega \left(\nabla u^{\frac{p-\alpha}{2}}\right)^2 dx.$$

For fixed t the boundary $\Gamma = \partial\Omega$ can be decomposed into two parts depending on $t \in [0,T]$ as follows:

$$\Gamma_+(t) = \{z \in \partial\Omega : u(z,t) \geq 2u_\Gamma(z,t)\}$$
$$\Gamma_-(t) = \{z \in \partial\Omega : u(z,t) < 2u_\Gamma(z,t)\}.$$

On $\Gamma_+(t)$ we have

$$A\int_{\Gamma_+(t)} c'(u)\left(u^p - u_\Gamma u^{p-1}\right)\,d\gamma \geq \frac{A}{2}\int_{\Gamma_+(t)} c'(u)u^p\,d\gamma \\ \geq \frac{A}{2}K_0\int_{\Gamma_+(t)} u^{p-\alpha}\,d\gamma$$

and on $\Gamma_-(t)$ we have

$$A\int_{\Gamma_-(t)} c'(u)\left(u^p - u_\Gamma u^{p-1}\right)\,d\gamma \\ \geq \frac{A}{2}\int_{\Gamma_-(t)} c'(u)u^p\,d\gamma - A\int_{\Gamma_-(t)} c'(u)u^{p-1}u_\Gamma\,d\gamma \\ \geq \frac{A}{2}K_0\int_{\Gamma_-(t)} u^{p-\alpha}\,d\gamma - A\int_{\Gamma_-(t)} c'(u)u^{p-1}u_\Gamma\,d\gamma.$$

Since $c'(\xi)\xi^{p-1}$ is increasing, we get

$$\int_{\Gamma_-(t)} c'(u)u^{p-1}u_\Gamma\,d\gamma \leq \int_{\Gamma_-(t)} c'(2u_\Gamma)(2u_\Gamma)^p\,d\gamma \leq C.$$

This yields

$$A\int_{\partial\Omega} c'(u)\left(u^p - u_\Gamma u^{p-1}\right)\, d\gamma \geq \frac{A}{2}K_0\int_{\partial\Omega} u^{p-\alpha}\, d\gamma - C.$$

We now get the following inequality

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + K_0\frac{4(p-1)}{(p-\alpha)^2}\int_\Omega \left(\nabla u^{\frac{p-\alpha}{2}}\right)^2 dx + \frac{A}{2}K_0\int_{\partial\Omega} u^{p-\alpha}\, d\gamma \\ \leq \|f\|_{L^p(\Omega)}\|u\|_{L^p(\Omega)}^{p-1} + C,$$

i.e.

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + K_0\frac{4(p-1)}{(p-\alpha)^2}\|u^{\frac{p-\alpha}{2}}\|_{H^1(\Omega)}^2 \leq \|f\|_{L^p(\Omega)}\,\|u\|_{L^p(\Omega)}^{p-1} + C.$$

After integrating this inequality with respect to t it becomes

$$\frac{1}{p}\|u(t)\|_p^p + \int_0^t K_0\frac{4(p-1)}{(p-\alpha)^2}\|u^{\frac{p-\alpha}{2}}(s)\|_{H^1(\Omega)}^2 \\ \leq \frac{1}{p}\|u(0)\|_p^p + C_1 t + \int_0^t \|u(s)\|_p^p\, ds + C_2.$$

Applying Gronwall's inequality (see, e.g., [8: Lemma 1.10]) to this last inequality we arrive at

$$\sup_{0\leq s\leq T}\|u\|_{L^p(\Omega)} \leq K_1 \qquad \text{and} \qquad \int_0^T \|u^{\frac{p-\alpha}{2}}\|_{H^1(\Omega)}^2 ds \leq K_2$$

for suitable constants K_1 and K_2 depending only on the initial conditions, $\|f\|_p$ and T.

To continue let $\{P_n\}_{n\geq 0}$ be a sequence given by

$$P_0 = p$$
$$P_n = 2P_{n-1} + \alpha \quad (n \geq 1),$$

where α satisfies (7). This implies that $P_n \geq p$ for all $n \geq 0$. We multiply equation (1) by u^{P_n-1} to get after integration by parts and applying (3)

$$\frac{1}{P_n}\frac{d}{dt}\|u\|_{P_n}^{P_n} + (P_n - 1)\int_\Omega c'(u)u^{P_n-2}(\nabla u)^2 dx \\ + A\int_{\partial\Omega} c'(u)(u^{P_n} - u_\Gamma u^{P_n-1})\, d\gamma = \int_\Omega f u^{P_n-1} dx.$$

The boundary terms are treated as before, and the lower bound on $c'(u)$ is used. After applying Hölder's inequality to the right-hand side we obtain

$$\frac{1}{P_n}\frac{d}{dt}\|u\|_{P_n}^{P_n} + K_0\frac{4(P_n-1)}{(P_n-\alpha)^2}\int_\Omega (\nabla u^{\frac{P_n-\alpha}{2}})^2 dx \\ \leq \|f\|_{L^p(\Omega)}\left(\int_\Omega u^{(P_n-1)q}dx\right)^{\frac{1}{q}} + \int_{\Gamma_-(t)} c'(u_\Gamma)u_\Gamma^{P_n}\, d\gamma \\ \leq C_1\left(\int_\Omega u^{(P_n-1)q}dx\right)^{\frac{1}{q}} + C_2. \tag{25}$$

Next, since $\alpha < 2p-1$ and $P_{n-1} \geq p$, we have $\frac{(P_n-1)q}{P_{n-1}} < 4$ and we can estimate the term on the right-hand side as

$$\left(\int_\Omega u^{(P_n-1)q}\,dx\right)^{\frac{1}{q}} = \left(\int_\Omega \left(u^{P_{n-1}}\right)^{\frac{(P_n-1)q}{P_{n-1}}}\,dx\right)^{\frac{1}{q}} \leq C_3 \left\|u^{P_{n-1}}\right\|_{L^4(\Omega)}^{2+\frac{\alpha-1}{P_{n-1}}}$$

To continue we apply the Nirenberg-Gagliardo inequality followed by Young's inequality to get

$$\begin{aligned}\left\|u^{P_{n-1}}\right\|_{L^4(\Omega)}^{2+\frac{\alpha-1}{P_{n-1}}} &\leq C\left\|u^{P_{n-1}}\right\|_{H^1(\Omega)}^{\frac{9}{10}(2+\frac{\alpha-1}{P_{n-1}})}\left\|u^{P_{n-1}}\right\|_{L^1(\Omega)}^{\frac{1}{10}(2+\frac{\alpha-1}{P_{n-1}})}\\ &\leq \delta_n\left\|u^{P_{n-1}}\right\|_{H^1(\Omega)}^2 + C_4\frac{1}{\delta_n}\|u\|_{L^{P_{n-1}}(\Omega)}^{P_{n-1}Q_n}\end{aligned}$$

where $Q_n = 2\big(2+\frac{\alpha-1}{P_{n-1}}\big)/\big(2-9\frac{\alpha-1}{P_{n-1}}\big)$. Observe that these last two steps are always possible for the choice of α and p in (7). Using this, (25) becomes

$$\begin{aligned}&\frac{1}{P_n}\frac{d}{dt}\|u\|_{P_n}^{P_n} + K_0\frac{4(P_n-1)^2}{(P_n-\alpha)}\int_\Omega\left\|\nabla u^{P_{n-1}}\right\|^2\,dx\\ &\qquad\leq \delta_n\left\|u^{P_{n-1}}\right\|_{H^1(\Omega)}^2 + C_4\frac{1}{\delta_n}\|u\|_{L^{P_{n-1}}(\Omega)}^{P_{n-1}Q_n} + C_2\end{aligned}$$

Choosing δ_n small enough, we can move the H^1-term to the left with the derivative term. Integrating over $(0,t)$ and then take the supremum over $t\in[0,T]$ on both sides of the resulting inequality. We get

$$\sup_{0\leq t\leq T}\|u(t)\|_{P_n}^{P_n} \leq \|u(0)\|_{P_n}^{P_n} + P_nT\left(\sup_{0\leq t\leq T} C_4\frac{1}{\delta_n}\|u(t)\|_{L^{P_{n-1}}(\Omega)}^{P_{n-1}Q_n} + C_2\right).$$

We now apply Lemma 1 with

$$\eta_n = P_n,\qquad \beta_n = Q_n,\qquad \gamma_n = \max\left\{\max_{k<n}\gamma_k, 1, \sup_{0\leq t\leq T}\|u(t)\|_{P_n}^{P_n}\right\}.$$

It is easy to check that these sequences satisfy the hypothesis of Lemma 2.1. We conclude that $\sup_{0\leq t\leq T}\|u(t)\|_{L^P(\Omega)} \leq b_T$ for all $P\in[p,\infty)$, where the constant b_T is independent of P. This immediately implies the lemma.

To get the lower bound for u in Proposition 1 let $w = \frac{1}{u}$. Then w satisfies the boundary-value problem

$$\frac{w_t}{w^2} - \nabla\frac{c'(1/w)}{w^2}\nabla w = -f \tag{26}$$

$$\frac{1}{w^2}\frac{\partial w}{\partial\nu} = A\Big(\frac{1}{w} - u_\Gamma\Big)\qquad \forall(x,t)\in\partial\Omega\times[0,T]. \tag{27}$$

For this new variable, we have

Lemma 3. *Assume that the hypotheses of Proposition 1 hold. Then there is a constant $C_T > 0$ such that the solution w of (26)-(27) satisfies*

$$\max_{0\le s\le T} \|w(s)\|_{L^\infty(\Omega)} \le C_T.$$

Proof.. The proof of this assertion again relies on a Moser-type argument. To start we multiply equation (26) by w^{r_0+1} for some $r_0 > \alpha$ and integrate over Ω to get

$$\frac{1}{r_0}\frac{d}{dt}\|w\|^{r_0}_{L^{r_0}(\Omega)} + (r_0+1)\int_\Omega w^{r_0}\frac{c'(\frac{1}{w})}{w^2}|\nabla w|^2\,dx$$
$$-A\int_{\partial\Omega} c'(\frac{1}{w})(w^{r_0} - u_\Gamma w^{r_0+1})\,d\gamma = -\int_\Omega f w^{r_0+1}dx.$$

For the boundary terms we again split the boundary into two parts:

$$\Gamma_+(t) = \left\{z\in\partial\Omega : w(z,t) \ge \frac{2}{u_\Gamma(z,t)}\right\}$$
$$\Gamma_-(t) = \left\{z\in\partial\Omega : w(z,t) < \frac{2}{u_\Gamma(z,t)}\right\}.$$

On $\Gamma_+(t)$ we have

$$\begin{aligned} A\int_{\Gamma_+(t)} c'(\frac{1}{w})(u_\Gamma w^{r_0+1} - w^{r_0})\,d\gamma &\ge A\int_{\Gamma_+(t)} c'(\frac{1}{w})(2w^{r_0} - w^{r_0})\,d\gamma \\ &= A\int_{\Gamma_+(t)} c'(\frac{1}{w})w^{r_0}d\gamma \\ &\ge K_0 A\int_{\Gamma_+(t)} w^{r_0+\alpha}d\gamma \end{aligned}$$

and on $\Gamma_-(t)$ we have

$$\begin{aligned} &A\int_{\Gamma_-(t)} c'(\frac{1}{w})(u_\Gamma w^{r_0+1} - w^{r_0})\,d\gamma \\ &\qquad\ge A a_\Gamma\int_{\Gamma_-(t)} c'(\frac{1}{w})u_\Gamma w^{r_0+1}d\gamma - A\int_{\Gamma_-(t)} c'(\frac{1}{w})w^{r_0}d\gamma \\ &\qquad\qquad\ge K_0 A a_\Gamma\int_{\Gamma_-(t)} w^{r_0+\alpha}d\gamma - A\int_{\Gamma_-(t)} c'(\frac{1}{w})w^{r_0}d\gamma. \end{aligned}$$

For the second term on the right we employ Lemma 2 to observe that $\frac{u_\Gamma}{2} < \frac{1}{w} < \|u\|_{L^\infty(\Omega\times\mathbb{R}^+)}$ and therefore

$$A\int_{\Gamma_-(t)} c'(\frac{1}{w})w^{r_0}d\gamma \le C$$

for a suitable constant C. We combine these estimates to get

$$A\int_{\partial\Omega} c'(\frac{1}{w})(u_\Gamma w^{r_0+1} - w^{r_0})\,d\gamma \ge C_1\int_{\partial\Omega} w^{r_0+\alpha}d\gamma - C.$$

The first term on the right of this last inequality and the term with the gradient can be combined in the H^1-norm. After moving the constant C to the right-hand side we get

$$(28)\quad \frac{1}{r_0}\frac{d}{dt}\|w\|^{r_0}_{L^{r_0}(\Omega)} + C_1(r_0)\|w^{\frac{r_0+\alpha}{2}}\|^2_{H^1(\Omega)} \le \|f\|_{L^p(\Omega)}\left\|w^{\frac{r_0+\alpha}{2}}\right\|^{\frac{2(r_0+1)}{r_0+\alpha}}_{L^{\frac{2q(r_0+1)}{r_0+\alpha}}(\Omega)} + C.$$

Observe that $\frac{2q(r_0+1)}{r_0+\alpha} \le 4$ and $\frac{r_0+1}{r_0+\alpha} < 1$. We can therefore use the Sobolev imbedding theorem and Young's inequality to get

$$\begin{aligned}\|f\|_{L^p(\Omega)}\left\|w^{\frac{r_0+\alpha}{2}}\right\|^{\frac{2(r_0+1)}{r_0+\alpha}}_{L^{\frac{2q(r_0+1)}{r_0+\alpha}}(\Omega)} &\le \|f\|_{L^p(\Omega)}\|w^{\frac{r_0+\alpha}{2}}\|^{\frac{2(r_0+1)}{r_0+\alpha}}_{H^1(\Omega)}\\ &\le \delta_0\frac{r_0+1}{r_0+\alpha}\|w^{\frac{r_0+\alpha}{2}}\|^2_{H^1(\Omega)} + \left(\frac{1}{\delta_0}\right)^{\frac{r_0+1}{\alpha-1}}\frac{\alpha-1}{r_0+\alpha}\|f\|^{\frac{r_0+\alpha}{\alpha-1}}_{L^p(\Omega)}.\end{aligned}$$

Choosing δ_0 sufficiently small, we can use this to estimate the right-hand-side in (28). Integrating the resulting inequality over $(0,t)$ yields

$$\begin{aligned}(29)\quad \|w(t)\|^{r_0}_{L^{r_0}(\Omega)} + C(r_0)\int_0^t \|w^{\frac{r_0+\alpha}{2}}(s)\|^2_{H^1(\Omega)}ds&\\ \le Ct + C_0 + r_0 t\frac{1}{\delta_1}\frac{\alpha-1}{r_0+\alpha}\|f\|^{\frac{r_0+\alpha}{\alpha-1}}_{L^p(\Omega)}&.\end{aligned}$$

To continue let $\{r_n\}$ be a sequence of real numbers with r_0 as above and

$$r_n = 2r_{n-1} - \alpha.$$

After multiplying (26) with w^{r_n+1} we can repeat the above steps to arrive at.

$$(30)\quad \frac{1}{r_n}\frac{d}{dt}\|w\|^{r_n}_{L^{r_n}(\Omega)} + C_1(r_n)\|w^{r_{n-1}}\|^2_{H^1(\Omega)} \le \|f\|_{L^p(\Omega)}\left\|w^{\frac{r_n+\alpha}{2}}\right\|^{\frac{2(r_n+1)}{r_n+\alpha}}_{L^{\frac{2q(r_n+1)}{r_n+\alpha}}(\Omega)} + C,$$

where

$$C_1(r_n) = \frac{4K_0(r_n+1)}{(r_n+\alpha)^2}.$$

Again we have $\frac{2q(r_0+1)}{r_0+\alpha} \le 4$ and $\frac{r_0+1}{r_0+\alpha} < 1$, and we continue by applying the Nirenberg-Gagliardo inequality:

$$\begin{aligned}\left\|w^{\frac{r_n+\alpha}{2}}\right\|^{\frac{2(r_n+1)}{r_n+\alpha}}_{L^{\frac{2q(r_n+1)}{r_n+\alpha}}(\Omega)} &\le \left\|w^{\frac{r_n+\alpha}{2}}\right\|^{\frac{2(r_n+1)}{r_n+\alpha}}_{L^4(\Omega)}\\ &\le C\|w^{r_{n-1}}\|^{\frac{9(r_n+1)}{10r_{n-1}}}_{H^1(\Omega)}\|w^{r_{n-1}}\|^{\frac{r_n+1}{10r_{n-1}}}_{L(\Omega)}.\end{aligned}$$

Applying Young's inequality yields

$$\begin{aligned}&C\|w^{r_{n-1}}\|^{\frac{9(r_n+1)}{10r_{n-1}}}_{H^1(\Omega)}\|w^{r_{n-1}}\|^{\frac{r_n+1}{10r_{n-1}}}_{L(\Omega)}\\ &\quad\le \delta_n\frac{9(r_n+1)}{r_{n-1}}\|w^{r_{n-1}}\|^2_{H^1(\Omega)} + \left(\frac{1}{\delta_n}\right)^{\frac{9(r_n+1)}{r_n+10\alpha}}\frac{r_n+10\alpha}{10r_n+10\alpha}\|w\|^{\frac{r_n+1}{r_n+10\alpha}}_{L^{r_{n-1}}(\Omega)}.\end{aligned}$$

We next choose $\delta_n = \frac{r_{n-1}}{18(r_n+1)} C_1(r_n) = \frac{4K_0}{9(r_n+1)}$, to estimate the H^1 term with the same term on the left. After integrating the resulting inequality over $(0,T)$, we get.

$$\frac{1}{r_n} \sup_{s\in(0,t)} \|w(s)\|^{r_n}_{L^r_n(\Omega 0} \le C_2 t + A_n \left(\sup_{s\in(0,t)} \|w(s)\|_{L^r_{n-1}(\Omega)} \right)^{\frac{r_n+1}{r_n+10\alpha}}, \tag{31}$$

where

$$A_n = \left(\frac{9(r_n+1)}{4K_0}\right)^{\frac{9(r_n+1)}{r_n+10\alpha}}$$

We can now apply Lemma 1 to get the desired result.

Remarks:

(1) The proof of Lemma 3 has very few restrictions on the size of α, we only need $\alpha > 1$.

(2) The boundary conditions (18) significantly simplify the proof. In this case $c'(u)$ is automatically taken care of in the boundary conditions and the estimates for the boundary terms become less technical.

Proof of Proposition 2..

This proof follows largely the same lines as the proofs of Lemma 2 and 3. First we multiply (12) by u^p where p is sufficiently large. Integration over Ω yields.

$$\frac{1}{p+1}\frac{d}{dt}\|u\|^{p+1}_{p+1} + p\int_\Omega u^{p-2}(\nabla u)^2\,dx + A\int_{\partial\Omega} u^{p+1}\,d\gamma - A\int_{\partial\Omega} u_\Gamma u^p\,d\gamma = \int_\Omega f u^p\,dx.$$

Using Young's inequality on the negative boundary term we get

$$\int_{\partial\Omega} u_\Gamma u^p\,d\gamma \le \frac{1}{2}\frac{p}{p+1}\int_{\partial\Omega} u^{p+1}\,d\gamma + \frac{1}{2(p+1}\int_{\partial\Omega} u_\Gamma^{p+1}\,d\gamma.$$

For the right-hand-side we get:

$$\left|\int_\Omega f u^p\,dx\right| \le C\,\|u\|^{p+1}_{p+1}\|f\|_\infty.$$

Combining these and integrating over $(0,t)$ we get:

$$\|u(t)\|^{p+1}_{p+1} \le \|u_0\|^{p+1}_{p+1} + (p+1)C_1\int_0^t \|u(s)\|^{p+1}_{p+1}\,ds + C_2 t,$$

where C_1 and C_2 only depend on f and the boundary values. Applying Gronwall's inequality we get

$$\|u(t)\|^{p+1}_{p+1} \le C_2 t\,\|u_0\|^{p+1}_{p+1} e^{C_1(p+1)t}.$$

Taking the $p+1$ root yields

$$\|u(t)\|_{p+1} \le \|u_0\|_{p+1}(C_2T)^{1/(p+1)}e^{C_1T}.$$

We can now take the supremum over $(0,T)$. The resulting inequality holds for all p and its right-hand-side is uniformly bounded in p by

$$\|u_0\|_\infty e^{C_1T}.$$

Hence, we have

$$\|u(t)\|_\infty \leq \|u_0\|_\infty e^{C_1 T}.$$

For the second part of Proposition 2 let $w = 1/u$, then w satisfies:

$$w_t - w^2 \nabla \frac{\nabla w}{w} = -f w^2 \tag{32}$$

$$\frac{\partial w}{\partial \nu} = Aw - Aw^2 u_\Gamma \tag{33}$$

As in the proof of Lemma 3 let $r_n = 2r_{n-1} - 1$, with $r_0 = p$ which is given in the hypothesis of the Proposition. We can now follow exactly the steps of the proof of Lemma 3 to get a lower bound.

Remark: Following the proofs of Lemma 2 and 3 one sees that lower bounds become easier as α increases and upper bounds become more difficult as α increases. It seems that $\alpha = 1$ is a limit case.

Proof of Proposition 3..

We prove the case when $a = 0$ and $b = 1$. The general case easily follows using the linear transformation $w = \frac{u-a}{b-a}$. The proof uses a change of the dependent variable. Proposition 3 can then be applied to the problem in the new variable.

To prove the theorem for $a = 0$ and $b = 1$ we change variables by $u = \frac{\phi}{1+\phi}$. Then $\frac{du}{d\phi} = \frac{1}{(1+\phi)^2}$ and ϕ satisfies the parabolic equation

$$\frac{1}{(1+\phi)^2}\phi_t - \nabla\Big(\frac{dc}{d\phi}\nabla\phi\Big) = f \tag{34}$$

and the boundary condition

$$\frac{1}{(1+\phi)^2}\frac{\partial\phi}{\partial\nu} = -A\Big(\frac{\phi}{1+\phi} - u_\Gamma\Big). \tag{35}$$

We now show that ϕ satisfies the a priori bound

$$\|\phi\|_{L^\infty(\Omega_T)} \leq B_T \tag{36}$$

for some constant B_T. First observe that (19) implies

$$\frac{dc}{d\phi} \geq \left(K_0\Big(\frac{\phi}{1+\phi}\Big)^{-\alpha} + K_1\Big(\frac{1}{1+\phi}\Big)^{-\alpha}\right)\frac{1}{(1+\phi)^2} \geq K_1\frac{1}{(1+\phi)^{2-\alpha}}$$
$$\frac{dc}{d\phi} \geq \left(K_0\Big(\frac{\phi}{1+\phi}\Big)^{-\alpha} + K_1\Big(\frac{1}{1+\phi}\Big)^{-\alpha}\right)\frac{1}{(1+\phi)^2} \geq K_0\frac{1}{(1+\phi)^{2-\alpha}\phi^\alpha}.$$

To start, we multiply (34) by $(1+\phi)^2\phi^{r-1}$, where $r = r_n$, $n \geq 0$ as in the proof of Lemma 3. We make sure that $r_0 > P$, the exponent in (20). We get after integration by parts

$$\frac{1}{r}\frac{d}{dt}\|\phi\|^r_{L^r(\Omega)} + \int_\Omega \nabla\left((1+\phi)^2\phi^{r-1}\right)\frac{dc}{d\phi}\nabla\phi\,dx$$
$$+ A\int_{\partial\Omega}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}\Big(\frac{\phi}{1+\phi} - u_\Gamma\Big)d\gamma \leq \int_\Omega f(1+\phi)^2\phi^{r-1}dx.$$

For the term involving the gradients observe that

$$\begin{aligned}
&\int_\Omega \nabla\left((1+\phi)^2\phi^{r-1}\right)\frac{dc}{d\phi}\nabla\phi\,dx\\
&\quad= 2\int_\Omega (1+\phi)\phi^{r-1}\frac{dc}{d\phi}\left|\nabla\phi\right|^2dx+(r-1)\int_\Omega(1+\phi)^2\phi^{r-2}\frac{dc}{d\phi}\left|\nabla\phi\right|^2dx\\
&\quad\geq K_1 2\int_\Omega(1+\phi)^{\alpha-1}\phi^{r-1}\left|\nabla\phi\right|^2dx+(r-1)K_1\int_\Omega(1+\phi)^\alpha\phi^{r-2}\left|\nabla\phi\right|^2dx\\
&\quad\geq (r+1)K_1\int_\Omega \phi^{r+\alpha-2}|\nabla\phi|^2dx\\
&\quad= K_1\frac{4(r+1)}{(r+\alpha)^2}\int_\Omega|\nabla\phi^{\frac{r+\alpha}{2}}|^2dx
\end{aligned}$$

To treat the boundary terms let

$$\begin{aligned}
\Gamma_1(t)&=\left\{z\in\partial\Omega:\frac{\phi(z,t)}{1+\phi(z,t)}\geq c_1=\frac{1+b_\Gamma}{2}\right\}\\
\Gamma_2(t)&=\left\{z\in\partial\Omega:\frac{\phi(z,t)}{1+\phi(z,t)}< c_1\right\}.
\end{aligned}$$

On $\Gamma_1(t)$ we have

$$\begin{aligned}
A\int_{\Gamma_1(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}\Big(\frac{\phi}{1+\phi}-u_\Gamma\Big)d\gamma &\geq A\frac{1-b_\Gamma}{2}K_1\int_{\Gamma_1(t)}(1+\phi)^{2+\alpha}\phi^{r-1}d\gamma\\
&\geq A\frac{1-b_\Gamma}{2}K_1\int_{\Gamma_1(t)}\phi^{r+1-\alpha}d\gamma\\
&\geq C_1\int_{\Gamma_1(t)}|\phi^{\frac{r+\alpha}{2}}|^2d\gamma
\end{aligned}$$

for a suitable constant C_1 and on $\Gamma_2(t)$ we have

$$\begin{aligned}
&A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}\Big(\frac{\phi}{1+\phi}-u_\Gamma\Big)d\gamma\\
&\qquad= A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}\frac{\phi}{1+\phi}\,d\gamma-A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}u_\Gamma d\gamma\\
&\qquad\geq AK_1\int_{\Gamma_2(t)}\phi^{r+1+\alpha}d\gamma-A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}u_\Gamma d\gamma\\
&\qquad\geq C_2\int_{\Gamma_2(t)}|\phi^{\frac{r+\alpha}{2}}|^2d\gamma-A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}u_\Gamma d\gamma
\end{aligned}$$

for a suitable constant C_2. For the last term observe that (16) implies $\frac{dc}{du}u^P\leq \frac{dc}{du}\big|_{u=c_1}c_1^P\leq C_3$ and therefore

$$(1+\phi)^4\phi^{r-1}\frac{dc}{d\phi}\leq C_3\phi^{r-1-P}(1+\phi)^{P-2}\leq C_3M^{P+r+1}$$

for a suitable constant M. Hence we have

$$A\int_{\Gamma_2(t)}(1+\phi)^4\frac{dc}{d\phi}\phi^{r-1}u_\Gamma d\gamma \le C_4M^{P+r+1}.$$

We can combine the boundary terms with the gradient term to get the inequality:

$$\frac{1}{r}\,\|\phi\|^r_{L^r(\Omega)} + C_5\|\phi^{\frac{r+\alpha}{2}}\|^2_{H^1(\Omega)} \le \int_\Omega f(1+\phi)^2\phi^{r-1}\,dx + C_4M^{P+r+1}.$$

From here one follows exactly the steps in the proof of Lemma 3, i.e. one first proves a bound for the r_0-norm of ϕ and then uses this to get a sequence of bounds for the r_n norms. Application of Lemma 1. implies that ϕ is uniformly bounded by B_T. It follows that

$$u(x,t) \le \frac{B_T}{1+B_T} = b_T < 1 \qquad \forall (x,t) \in \Omega_T.$$

To get the lower bound for u a different change of dependent variable is used. We let $u = \frac{1}{\psi}$. Then ψ satisfies

$$\frac{1}{(\psi)^2}\psi_t - \nabla\Big(\frac{c'(1/\psi)}{\psi^2}\nabla\psi\Big) = -f$$

and the boundary condition

$$\frac{1}{\psi^2}\frac{\partial\psi}{\partial\nu} = A\Big(\frac{1}{\psi} - u_\Gamma\Big).$$

This is exactly equation (26). Lemma 3 can be applied to get an upper bound A_T for ψ. It follows that $u(x,t) \ge \frac{1}{A_T} = a_T > 0$ for all $(x,t) \in \Omega_T$. This concludes the proof of Proposition 3.

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