Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 2, December 2002, pp. 32–37.
Submitted: April 12, 2002. Published: December 31 2002.

ERROR BOUNDS FOR NEWTON'S METHOD UNDER A WEAK KANTOROVICH-TYPE HYPOTHESIS

Ioannis K. Argyros

Abstract. We recently showed that the famous Kantorovich hypothesis for the convergence of Newton's method can be weakened. Moreover we proved favorable local and semilocal convergence theorems under this condition involving Lipschitz and center Lipschitz conditions. Here based on these results we obtain even finer error bounds on the distances involved.

1991 A.M.S. (MOS) Subject Classification Codes. 65B05, 65G99, 65J15, 47H17, 49M15, CR:G1.5.

Key Words and Phrases. Banach space, Newton's method, Kantorovich hypothesis, lower/upper error bounds, semilocal convergence theorem.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of equation

$$F(x) = 0 \tag{1}$$

where, $F\colon D \subseteq X \to Y$ is Fréchet differentiable in an open convex subset $D_0 \subseteq D$ and X, Y are Banach spaces.

The famous Newton's method

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad (n \geq 0) \quad (x_0 \in D) \tag{2}$$

has long played a central role in approximating solution x^* of non-linear equations and systems. Here $F'(x_n)$ denotes the Fréchet-derivative of operator F evaluated at $x = x_n$ $(n \geq 0)$. Under the famous Newton–Kantorovich hypothesis

$$h = 2K_0\eta \leq 1 \tag{3}$$

where K_0, η are defined by (6) and (4) respectively. Several authors have provided semilocal convergence theorems and error bounds on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^*\|$ $(n \geq 0)$ [2]–[4]. Recently we showed that (3) can be weakened (see (7)) [1]. Under (7) we provided error bounds which are finer than the ones using the same information and (3). Here we obtain further finer lower and upper error bounds.

Department of Mathematics, Cameron University, Lawton, OK 73505, U.S.A.
E-mail Address: ioannisa@cameron.edu

2. Semilocal Analysis for Newton's Method

We showed in [1]:

Theorem 1. *Let $F\colon D \subseteq X \to Y$ be Fréchet differentiable in an open convex set $D_0 \subseteq D$ and, for some $x_0 \in D_0$, $F'(x_0)^{-1}$ exists. Assume there exist parameters $\eta \geq 0$, $\overline{K}_0 \geq 0$, $K_0 \geq 0$, $\delta \in [0,1]$ such that*

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta, \tag{4}$$

$$\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \overline{K}_0\|x - x_0\|, \tag{5}$$

$$\|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq K_0\|x - y\| \tag{6}$$

for all $x, y \in D_0$,

$$h_\delta = (\delta\overline{k}_0 + K_0 + \ell)\eta \leq 1, \tag{7}$$

and

$$\overline{U} = \overline{U}(x_1, t^* - \eta) = \{x \in X \mid \|x - x_1\| \leq t^* - \eta\} \subseteq D_0, \tag{8}$$

where,

$$t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{K_0(t_{n+1} - t_n)^2}{2[1 - \overline{K}_0 t_{n+1}]} \quad (n \geq 0), \tag{9}$$

$$t^* = \lim_{n\to\infty} t_n. \tag{10}$$

Then:

(i) *Newton's method $\{x_n\}$ $(n \geq 0)$ generated by (2) is well defined, remains in U for all $n \geq 0$ and converges to a solution $x^* \in \overline{U}$ of equation $F(x) = 0$ and $t^* \in [\eta, 2\eta]$.*

(ii) *the solution x^* is unique in $U(x_0, t^{**})$ provided that*

$$U(x_0, t^{**}) \subseteq D_0, \tag{11}$$

$$t^{**} \geq t^* \tag{12}$$

and

$$\overline{K}_0(t^* + t^{**}) \leq 2; \tag{13}$$

Moreover the following error bounds hold for all $n \geq 0$:

$$\|x_{n+2} - x_{n+1}\| \leq \frac{K_0\|x_{n+1} - x_n\|^2}{2[1 - \overline{K}_0\|x_{n+1} - x_0\|]} \leq t_{n+2} - t_{n+1}, \tag{14}$$

$$\|x_n - x^*\| \leq t^* - t_n, \tag{15}$$

$$\frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4a_n\|x_{n+1} - x_n\|}} \leq \|x_n - x^*\| \leq b_n \leq c_n, \tag{16}$$

$$\|x_n - x^*\| \leq d_n, \tag{17}$$

where,

$$a_n = \frac{K_0}{2[1-\overline{K}_0\|x_n - x_0\|]}, \tag{18}$$

$$b_n = \frac{K_0\|x_n - x_{n-1}\|^2}{2\big[1-\frac{\overline{K}_0}{2}(\|x_n - x_0\| + \|x^* - x_0\|)\big]}, \tag{19}$$

$$c_n = \frac{K_0\|x_n - x_{n-1}\|^2}{1-\overline{K}_0\|x_n - x_0\| + \sqrt{(1-\overline{K}_0\|x_n - x_0\|)^2 - \overline{K}_0 K_0\|x_n - x_{n-1}\|^2}}, \tag{20}$$

and

$$d_n = \frac{K_0}{2[1-\overline{K}_0\|x_{n-1} - x_0\|]}\|x_{n-1} - x^*\|^2. \tag{21}$$

Denote by $(0, \delta, F, x_0, \overline{U}_0, K_0, \overline{K}_0)$ the class satisfying the hypotheses of Theorem 1.

Under the hypotheses of Theorem 1, define:

$$\overline{U} = \overline{U}_0 = \overline{U}(x_0, t^*),\ \overline{U}_m = \overline{U}(x_m, t^* - t_m) \quad (m \geq 1), \tag{22}$$

$$K_m = \sup_{\substack{x,y \in \overline{U}_m \\ x \neq y}} \frac{\|F'(x_m)^{-1}[F'(x) - F'(y)]\|}{\|x - y\|}, \tag{23}$$

$$\overline{K}_m = \sup_{\substack{x \neq x_m \\ x \in \overline{U}_m}} \frac{\|F'(x_m)^{-1}[F'(x) - F'(x_m)]\|}{\|x - x_m\|}, \tag{24}$$

$$L_m = \sup_{\substack{x,y \in \overline{U} \\ x \neq y}} \frac{\|F'(x_m)^{-1}[F'(x) - F'(y)]\|}{\|x - y\|}, \tag{25}$$

$$\overline{L}_m = \sup_{\substack{x \neq x_m \\ x \in \overline{U}}} \frac{\|F'(x_m)^{-1}[F'(x) - F'(x_m)\|}{\|x - x_m\|}. \tag{26}$$

It follows from the Banach Lemma on invertible operators [3] and (7)

$$K_m \leq K_0 \frac{1}{1-\overline{K}_0\|x_m - x_0\|} \tag{27}$$

$$\overline{K}_m \leq K_0 \frac{1}{1-\overline{K}_0\|x_m - x_0\|}. \tag{28}$$

We show:

$$(K_m + \overline{K}_m)\|x_{m+1} - x_m\| \leq 1, \tag{29}$$

or

$$\frac{(1+\delta)K_0}{1-2\overline{K}_0\big[1-\left(\frac{1}{2}\right)^m\big]\eta}\left(\frac{1}{2}\right)^m \eta \leq \delta \tag{30}$$

or

$$\left\{(1+\delta)\left(\frac{1}{2}\right)^{m} K_0 + 2\delta\left[1-\left(\frac{1}{2}\right)^{m}\right]\overline{K}_0\right\}\eta \le \delta. \tag{31}$$

By (7) and (31) it suffices to show

$$(1+\delta)\left(\frac{1}{2}\right)^{m} K_0 + 2\delta\left[1-\left(\frac{1}{2}\right)^{m}\right]\overline{K}_0 \le K_0 + \overline{K}_0\delta \tag{32}$$

or

$$\overline{K}_0\delta\left[2\left(1-\left(\frac{1}{2}\right)^{m}\right)-1\right] \le K_0\left[1-(1+\delta)\left(\frac{1}{2}\right)^{m}\right] \tag{33}$$

or

$$\delta\left[1-\left(\frac{1}{2}\right)^{m-1}\right] \le 1-(1+\delta)\left(\frac{1}{2}\right)^{m}$$

or

$$\delta\left[1-\left(\frac{1}{2}\right)^{m}\right] \le 1-\left(\frac{1}{2}\right)^{m}$$

or

$$\delta \le 1,$$

which is true by hypothesis.

That completes the induction for (29).

Similarly we show:

$$(L_m + \overline{L}_m)\|x_{m+1}-x_m\| \le 1. \tag{34}$$

That is the conclusions of Theorem 1 can pass from $(0,\, \delta,\, F,\, x_0,\, \overline{U}_0,\, K_0,\, \overline{K}_0,\, L_0,\, \overline{L}_0)$ to the class $(M,\, \delta,\, F,\, x_m,\, \overline{U}_m,\, K_m,\, \overline{K}_m,\, L_m,\, \overline{L}_m)$ $(m \ge 1)$ in order to obtain finer bounds. Indeed, define sequences for all fixed $m \ge 0$ and $n \ge 0$

That completes the induction for (29).

Similarly we show:

$$(L_m + \overline{L}_m)\|x_{m+1}-x_m\| \le 1. \tag{34}$$

That is the conclusions of Theorem 1 can pass from $(0,\, \delta,\, F,\, x_0,\, \overline{U}_0,\, K_0,\, \overline{K}_0,\, L_0,\, \overline{L}_0)$ to the class $(M,\, \delta,\, F,\, x_m,\, \overline{U}_m,\, K_m,\, \overline{K}_m,\, L_m,\, \overline{L}_m)$ $(m \ge 1)$ in order to obtain finer bounds. Indeed define sequences for all fixed $m \ge 0$ and $n \ge 0$

$$t^2_{n+m+2} = t^2_{n+m+1} + \frac{K_m\|x_{n+m+1}-x_{n+m}\|^2}{2[1-\overline{K}_m\|x_{n+m+1}-x_m\|]}, \tag{35}$$

$$t^1_{n+m+2} = t^1_{n+m+1} + \frac{K^1_m\|x_{n+m+1}-x_{n+m}\|^2}{2[1-\overline{K}^1_m\|x_{n+m+1}-x_m\|]}, \tag{36}$$

$$K^1_m = \overline{K}^1_m = \frac{K_0}{1-\overline{K}_0\|x_m-x_0\|} \quad (m>0), \tag{37}$$

$$K^1_0 = K_0, \quad \overline{K}^1_0 = \overline{K}_0, \tag{38}$$

$$t_{n+m+2} = t_{n+m+1} + \frac{K^0_m(t_{n+m+1}-t_{n+m})^2}{2[1-K^0_m(t_{n+m+1}-t_{n+m})]}, \tag{39}$$

$$K^0_m = \overline{K}^0_m = \frac{K_0}{1-\overline{K}_0 t_m} \quad (m>0), \tag{40}$$

$$K^0_0 = K_0, \quad \overline{K}^0_0 = \overline{K}_0. \tag{41}$$

Denote by v^2, v^1, v, sequences defined as the t's with "K's" replaced by "L's".

Moreover, we have:

$$x^* \in \overline{U}_m \subseteq \overline{U}_{m-1} \subseteq \cdots \subseteq \overline{U}_1 = \overline{U}_0. \tag{42}$$

Furthermore, using (2) we obtain the approximation

$$\begin{aligned} x^* - x_{m+1} &= x^* - x_m + F'(x_m)^{-1}F(x_m) \\ &= -\int_0^1 F'(x_m)^{-1}\{F'(x_m + t(x^* - x_m)) - F'(x_m)\}(x^* - x_m)dt, \end{aligned} \tag{43}$$

where, $x_m + t(x^* - x_m)$, x_m belong to $\overline{U}_m$ if $m \geq 1$ and to D_0 if $n = 0$. Hence, we deduce

$$\|x^* - x_{m+1}\| \leq \frac{1}{2}K_m\|x^* - x_m\|^2 \leq \frac{1}{2}L_m\|x^* - x_m\|^2 \ (m \geq 0). \tag{44}$$

Hence, we showed:

Theorem 2. *Under the hypotheses of Theorem 1 the following error bounds hold for all fixed $m \geq 0$ and $n \geq 0$:*

$$\begin{aligned} \|x_{n+m+2} - x_{n+m+1}\| &\leq t^2_{n+m+2} - t^2_{n+m+1} \\ &\leq t^1_{n+m+2} - t^1_{n+m+1} \leq t_{n+m+2} - t_{n+m+1}, \end{aligned} \tag{45}$$

$$\begin{aligned} \|x_{n+m+2} - x_{n+m+1}\| &\leq v^2_{n+m+2} - v^2_{n+m+1} \\ &\leq v^1_{n+m+2} - v^1_{n+m+1} \leq v_{n+m+2} - v_{n+m+1}, \end{aligned} \tag{46}$$

$$t^i_{n+m+2} - t^i_{n+m+1} \leq v^i_{n+m+2} - v^i_{n+m+1} \quad i = 0, 1, 2, \ (v^{(0)} = v), \tag{47}$$

$$\begin{aligned} \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4K^1_n\|x_{n+1} - x_n\|}} &\leq \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 2K_n\|x_{n+1} - x_n\|}} \\ &\leq \|x^* - x_n\|, \end{aligned} \tag{48}$$

$$\begin{aligned} \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 4L^1_n\|x_{n+1} - x_n\|}} &\leq \frac{2\|x_{n+1} - x_n\|}{1 + \sqrt{1 + 2L_n\|x_{n+1} - x_n\|}} \\ &\leq \|x^* - x_n\|, \end{aligned} \tag{49}$$

$$\|x^* - x_n\| \leq \frac{1}{2}K_n\|x^* - x_n\|^2 \leq \frac{K_0\|x^* - x_n\|^2}{2[1 - \overline{K}_0\|x_n - x_0\|]}, \tag{50}$$

and

$$\|x^* - x_n\| \leq \frac{1}{2}L_n\|x^* - x_n\|^2 \leq \frac{L_0\|x^* - x_n\|^2}{2[1 - \overline{L}_0\|x_n - x_0\|]}. \tag{51}$$

Following Yamamoto's suggestion (see [4], Theorem 3.3) as above we get:

Theorem 3. *Let* $F\colon D \subseteq X \to Y$, $\overset{\circ}{D}$ *be the interior of* D, *and for some* $x_0 \in \overset{\circ}{D}$, $F'(x_0)^{-1}$ *exists. Assume*: $\overline{U}'_0 = \overline{U}(x_1, d_0) \subseteq \overset{\circ}{D}$, $d_0 = \|x_1 - x_0\|$ *and* F *is Fréchet-differentiable on* $\overline{U}'_0$.

Define:

$$K'_0 = \sup_{\substack{x,y\in\overline{U}'_0\\ x\neq y}} \frac{\|F'(x_0)^{-1}[F'(x) - F'(y)]\|}{\|x-y\|},$$

$$\overline{K}'_0 = \sup_{\substack{x\in\overline{U}'_0\\ x\neq x_0}} \frac{\|F'(x_0)^{-1}[F'(x) - F'(x_0)]\|}{\|x-x_0\|}.$$

If $(K'_0+\overline{K}'_0)d_0 \leq 1$, *then conclusions (i) and (ii) of Theorem 1 hold by replacing* D_0, K_0, $\overline{K}_0$, $\overline{U}$ *by* $\overline{U}'_0$, K'_0, $\overline{K}'_0$, $\overline{U}'_0$ *respectively. Furthermore, set* $\overline{U}'_n = \overline{U}(x_n, 2d_n)$, $d_n = \|x_{n+1} - x_n\|$ $(n \geq 1)$ *and*

$$K'_n = \sup_{\substack{x,y\in\overline{U}'_n\\ x\neq y}} \frac{\|F'(x_n)^{-1}[F'(x) - F'(y)]\|}{\|x-y\|} \quad (n \geq 1)$$

$$\overline{K}'_n = \sup_{\substack{x\in\overline{U}'_n\\ x\neq x_n}} \frac{\|F'(x_n)^{-1}[F'(x) - F'(x_n)]\|}{\|x-x_n\|} \quad (n \geq 1).$$

Then,

$$x^* \in \overline{U}'_n \subseteq \overline{U}'_{n-1} \subseteq \cdots \subseteq \overline{U}'_1 \subseteq \overline{U}'_0$$

and all bounds involving "K's" hold with the "K's" replaced by "K''s".

REFERENCES

[1] Argyros, I.K., On the Newton–Kantorovich hypothesis for solving equations, submitted for publication.

[2] Argyros, I.K. and Szidarovszky, F., *The Theory and Application of Iteration Methods*, CRC Press, Inc., Boca Raton, Florida, U.S.A., 1993.

[3] Kantorovich, L.V. and Akilov, G.P., *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1964.

[4] Yamamoto, T., A method for finding sharp error bounds for Newton's method under the Kantorovich assumptions, *Numer. Math.* **49** (1986), 203–220.