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AN ANALYTICAL STUDY OF THE N-PERSON PRISONERS' DILEMMA

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Abstract. An analytical study of the N-person Prisoners' Dilemma game is presented for Pavlovian agents that modify their behavior according to Pavlov's experimental studies formulated by Thorndike's (1911) law of conditioning: if an action is followed by a satisfactory state of affairs, then the tendency to produce that particular action is reinforced. We found that in case of linear payoff functions a maximum 50% cooperation rate can be learned in a large community of agents starting with small cooperation rates. This limit can only be increased if the initial cooperation rate is above 50%. The results are confirmed by computer simulation experiments.

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1. Introduction

Prisoners' Dilemma is usually defined between two players (Rapoport and Chammah, 1965) and within game theory that assumes that the players act rationally. Realistic investigations of collective behavior, however, require a multi-person model of the game (Schelling, 1973). Various aspects of the multi-person Prisoners' Dilemma game have been investigated in the literature (Bixenstine et al. 1966, Weil 1966, Kelley and Grzelak 1972, Hamburger 1973, Anderson 1974, Bonacich et al. 1976, Goehring and Kahan 1976, Dawes 1980, Heckathorn 1988, Liebrand et al. 1992, Huberman and Glance 1993, Schulz et al. 1994, Schroeder 1995, Szilagyi 2000, Szilagyi and Szilagyi 2002).

We will define the N-person Prisoners' Dilemma by the following properties (Szilagyi and Szilagyi, 2000):

1. There are two actions available to each agent, and each agent must choose exactly one action: cooperation or defection.

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2. Regardless of what the other participants do, each participant receives a higher payoff for defecting behavior than for cooperating behavior, and
3. All participants receive a lower payoff if all defect than if all cooperate.

Let us consider the case of N participants with m of them cooperating. If $x = m/N$ is the ratio of cooperators and $C(x)$ and $D(x)$ are the payoff functions to a cooperator and a defector, respectively, then the above conditions can be expressed as

$$D(x) > C(x) \tag{1}$$

and

$$C(1) > D(0). \tag{2}$$

The outcome of the game depends on the personalities of the agents. For example, agents with short-term rationality will always choose defection because of Eq. (1), benevolent agents will ignore their short-term interests and will all cooperate, etc. It is realistic and interesting to consider Pavlovian agents that modify their behavior according to Pavlov's experimental studies formulated by Thorndike's (1911) law of conditioning: if an action is followed by a satisfactory state of affairs, then the tendency to produce that particular action is reinforced. Their response is stochastic but their probability of cooperation p changes by an amount proportional to their reward/punishment from the environment (the coefficient of proportionality is called the learning rate). These agents are primitive enough not to know anything about their rational choices but they have enough 'intelligence' to follow Thorndike's law.

The Prisoners' Dilemma game is an iterative process. Aggregate cooperation proportions are changing in time, i.e., over subsequent iterations. The updated probabilities lead to new decisions in the next iteration.

The payoff (reward/penalty) functions are given as two curves: one (C) for a cooperator and another (D) for a defector. According to Eqs. (1) and (2), the D curve is always above the C curve but $C(1)$ is higher than $D(0)$. The payoff to each agent depends on its choice and on the distribution of other players among cooperators and defectors. The payoff curves are, therefore, functions of the ratio x of cooperators to the total number of agents. To reduce the number of parameters, we will consider linear payoff functions in this study (Figure 1). Even in this very simple case, four parameters specify the environment.

2. Analytical Study

As we came to the conclusion above that the Pavlovian agents are realistic candidates for the investigation of the evolution of the game, let us assume that in a society of N Pavlovian agents the ratio of cooperators is x and the ratio of defectors is $(1 - x)$ at a certain time. It was shown (Szilagyi and Szilagyi, 2002) that when the cooperators receive the same total payoff as the defectors, i.e.,

$$xC(x) = (1 - x)D(x) \tag{3}$$

an equilibrium occurs. This may happen if $C(x)$ and $D(x)$ are both negative or both positive. In the first case, a small number of cooperators are punished big and a large number of defectors are punished little. This leads to a stable equilibrium at this point. In the second case, a large number of cooperators are rewarded slightly and a small number of defectors are rewarded greatly. This point corresponds to an unstable equilibrium.

If C and D are both linear functions of x, then Eq. (3) is a quadratic equation that has up to two real solutions. If both solutions are in the interval $0 < x < 1$, then both equilibria are present. We will call these equilibrium solutions x_1 and x_2, respectively, so that $0 < x_1 < x_2 < 1$. The initial cooperation probability (which is set as a constant and uniform across all the agents) is x_0.

Assume that $D(x) = c_d + b_d x$ and $C(x) = c_c + b_c x$. Relation (1) holds if and only if $D(0) > C(0)$ and $D(1) > C(1)$ which could be written as

$$c_d > c_c \tag{4}$$

and

$$c_d + b_d > c_c + b_c. \tag{5}$$

Inequality (2) has now the new special form

$$c_c + b_c > c_d. \tag{6}$$

The equilibrium condition (3) is a quadratic equation for x:

$$x^2(b_c + b_d) + x(c_c + c_d - b_d) - c_d = 0. \tag{7}$$

Before examining the locations of the equilibria, an important consequence of the above inequalities is derived. By adding equations (4) and (6), and equations (5) and (6) we conclude that both b_c and b_d are positive.

The roots of equation (7) are

$$x_{1,2} = \frac{1}{2(b_c + b_d)} \left[-(c_c + c_d - b_d) \pm \sqrt{(c_e + c_d - b_d)^2 + 4(b_c + b_d)c_d} \right]. \tag{8}$$

For simplified notations let

$$\gamma = -\frac{c_c}{b_c + b_d}, \quad \delta = -\frac{c_d}{b_c + b_d}, \quad b = \frac{b_d}{b_c + b_d},$$

then

$$\frac{b_c}{b_c + b_d} = 1 - b$$

and

$$x_{1,2} = \frac{1}{2} \left[(\gamma + \delta + b) \pm \sqrt{(\gamma + \delta + b)^2 - 4\delta} \right]. \tag{9}$$

Notice that conditions (4)–(6) and the positivity of b_c and b_d can be reformulated as

$$\gamma > \delta \tag{10}$$

$$\gamma + 2b > 1 + \delta \tag{11}$$

$$\gamma + b < 1 + \delta \tag{12}$$

$$0 < b < 1. \tag{13}$$

In our analysis we will consider several cases.

Case 1. Two positive equilibria exist. Both roots are real and positive if and only if

$$\delta > 0 \tag{14}$$

and

$$\gamma + \delta + b > 2\sqrt{\delta}. \tag{15}$$

From the practical point of view those payoff functions are interesting that lead to cooperation even when the initial cooperation rate is very low. Therefore our objective is to maximize the smaller root. This problem is a special nonlinear optimization problem maximizing

$$x_1 = \frac{1}{2}\left[(\gamma + \delta + b) - \sqrt{(\gamma + \delta + b)^2 - 4\delta}\right] \tag{16}$$

subject to constraints (10)–(15). Simple differentiation shows that x_1 strictly decreases in $\gamma + b$, therefore with fixed values of δ, we have to find first the smallest possible value of $\gamma + b$. The problem is a simple two-dimensional linear programming problem, which is solved by a graphical approach. Two cases have to be examined separately.

Assume first that $2\sqrt{\delta} - \delta < \frac{1+\delta}{2}$, which holds if $\delta < \frac{1}{9}$ or $\delta > 1$. Figure 2 shows the corresponding feasible set. The minimal value of $\delta + b$ occurs at the intercept of the lines $\gamma = \delta$ and $\gamma + 2b = 1 + \delta$ giving the solution $b = \frac{1}{2}$ and $\gamma = \delta$ with the optimal value: $\gamma + b = \delta + \frac{1}{2}$.

Assume next that $2\sqrt{\delta} - \delta \geq \frac{1+\delta}{2}$, which occurs when $\frac{1}{9} \leq \delta \leq 1$. Notice first that the intercept of the lines $\gamma + b = 2\sqrt{\delta} - \delta$ and $\gamma + 2b = 1 + \delta$ is given by point P of Figure 3 with coordinates $b = 1 + 2\delta - 2\sqrt{\delta}$ and $\gamma = 4\sqrt{\delta} - 3\delta - 1$. It is easy to see that $4\sqrt{\delta} - 3\delta - 1 \leq \delta$ for all $\delta > 0$ since it is equivalent to the obvious fact $0 \leq (2\sqrt{\delta} - 1)^2$. Hence point P is at or below the horizontal line $\gamma = \delta$, so the optimal value of $\gamma + b$ occurs again at the intercept of lines $\gamma = \delta$ and $\gamma + 2b = 1 + \delta$ with the optimal value of $\gamma + b = \delta + \frac{1}{2}$.

With fixed $\delta > 0$ and optimal $\gamma + b$ the smaller positive root becomes

$$\begin{aligned} x_1 &= \frac{1}{2}\left[\left(2\delta + \frac{1}{2}\right) - \sqrt{\left(2\delta + \frac{1}{2}\right)^2 - 4\delta}\right] \\ &= \frac{1}{2}\left[\left(2\delta + \frac{1}{2}\right) - \left|2\delta - \frac{1}{2}\right|\right] = \left\{\begin{array}{ll} \frac{1}{2} & \text{if } \delta \geq \frac{1}{4} \\ 2\delta & \text{if } \delta < \frac{1}{4} \end{array}\right\} \end{aligned} \tag{17}$$

showing that the maximum value of the smaller positive equilibrium is $\frac{1}{2}$, and this optimum is obtained at $b = \frac{1}{2}$, $\gamma = \delta$ and $\delta \geq \frac{1}{4}$. Notice that this is the case when the lines of $C(x)$ and $D(x)$ coincide, which is not realistic. So in realistic cases the maximum value $\frac{1}{2}$ cannot be reached but it can be approached arbitrarily.

Next we assume that the borderline case is excluded by the stricter constraint

$$\gamma > \delta + \varepsilon \tag{18}$$

with some small positive ε requiring that functions C and D are bounded away from each other at zero. In maximizing $\gamma + b$ we can follow the same procedure as before with the slight modification that the horizontal lines $\gamma = \delta$ of Figures 2 and 3 are replaced by $\gamma = \delta + \varepsilon$. Therefore the optimal value $\gamma + b$ occurs at the intercept of lines $\gamma = \delta + \varepsilon$ and $\gamma + 2b = 1 + \delta$, which is

$$b = \frac{1-\varepsilon}{2} \quad \text{and} \quad \gamma = \delta + \varepsilon$$

with the optimal value of $\gamma + b = \frac{2\delta+\varepsilon+1}{2}$. Then

$$x_1 = \frac{1}{2}\left[\frac{4\delta+\varepsilon+1}{2} - \sqrt{\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - 4\delta}\,\right]. \tag{19}$$

Simple differentiation shows that

$$\frac{\partial x_1}{\partial \delta} = \frac{1}{\sqrt{\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - 4\delta}}\left[\sqrt{\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - 4\delta} - \left(\frac{4\delta+\varepsilon+1}{2}\right) + 1\right]$$

where the bracketed factor is positive. Hence x_1 is strictly increasing in δ. From (19) we see that

$$\begin{aligned} x_1 &= \frac{\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - \left[\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - 4\delta\right]}{2\left[\frac{4\delta+\varepsilon+1}{2} + \sqrt{\left(\frac{4\delta+\varepsilon+1}{2}\right)^2 - 4\delta}\right]} \\ &= \frac{4\delta}{(4\delta+\varepsilon+1) + \sqrt{(4\delta+\varepsilon+1)^2 - 16\delta}} \\ &= \frac{4}{\left(4 + \frac{\varepsilon+1}{\delta}\right) + \sqrt{\left(4+\frac{\varepsilon+1}{\delta}\right)^2 - \frac{16}{\delta}}} \end{aligned} \tag{20}$$

which implies that $x_1 \to \frac{1}{2}$ as $\delta \to \infty$. Therefore x_1 always remains under $\frac{1}{2}$, and with sufficiently large values of δ this limit can be approached arbitrarily.

Similarly we may require that functions C and D are bounded away from each other at $x = 1$. This condition can be written as

$$-\delta + b > -\gamma + 1 - b + \varepsilon$$

with some small positive ε. In this case constraint (11) is replaced by the more strict condition

$$\gamma + 2b > 1 + \delta + \varepsilon.$$

The minimal value of $\gamma + b$ occurs at $\gamma = \delta$ and $b = \frac{1+\varepsilon}{2}$ with the minimal value of $\gamma + b = \delta + \frac{1+\varepsilon}{2}$. Then x_1 has the same form as given in equation (19), so we get the same conclusions as before.

Next we assume that the lines of C and D are parallel, $b = \frac{1}{2}$, however we assume that $D(x) - C(x)$ is bounded away from zero. This condition can be rewritten as

$$\gamma > \delta + \varepsilon.$$

With the fixed value of δ,

$$x_1 = \frac{1}{2}\left[\left(\gamma+\delta+\frac{1}{2}\right) - \sqrt{\left(\gamma+\delta+\frac{1}{2}\right)^2 - 4\delta}\right]$$

which is strictly decreasing in γ. So the largest value of x_1 occurs at $\gamma = \delta + \varepsilon$, when x_1 becomes

$$x_1 = \frac{1}{2}\left[\left(2\delta+\varepsilon+\frac{1}{2}\right) - \sqrt{\left(2\delta+\varepsilon+\frac{1}{2}\right)^2 - 4\delta}\right].$$

This is the same as equation (19) with ε being replaced by $\frac{\varepsilon}{2}$, so we have the same results as in the previous cases.

Case 2. One positive and one nonpositive equilibrium exist.

From equation (14) we know that this is the case if and only if $\delta < 0$. By introducing the new variable $\Delta = -\delta$,

$$\Delta > 0 \tag{21}$$

and the positive equilibrium has the form

$$x_2 = \frac{1}{2}\left[(\gamma+b-\Delta) + \sqrt{(\gamma+b-\Delta)^2 + 4\Delta}\right]. \tag{22}$$

We want to find the smallest positive value of this root. Simple differentiation shows that x_2 is strictly increasing in $\gamma + b$, therefore with fixed value of Δ, the smallest value of $\gamma + b$ has to be selected subject to the constraints (10)–(13), which can be rewritten as follows:

$$\gamma > -\Delta \tag{23}$$
$$\gamma + 2b > 1 - \Delta \tag{24}$$
$$\gamma + b < 1 - \Delta \tag{25}$$
$$0 < b < 1. \tag{26}$$

This is again a two-dimensional linear programming problem. The graphical solution is illustrated in Figure 4. The smallest value of $\gamma + b$ occurs at the intercept of lines $\gamma = -\Delta$ and $\gamma + 2b = 1 - \Delta$ which is $\gamma = -\Delta$ and $b = \frac{1}{2}$ with the corresponding optimal value of $\gamma + b = \frac{1}{2} - \Delta$. Then

$$\begin{aligned} x_2 &= \frac{1}{2}\left[\left(\frac{1}{2} - 2\Delta\right) + \sqrt{\left(\frac{1}{2} - 2\Delta\right)^2 + 4\Delta}\right] \\ &= \frac{1}{2}\left[\left(\frac{1}{2} - 2\Delta\right) + \left(\frac{1}{2} + 2\Delta\right)\right] \\ &= \frac{1}{2} \end{aligned} \tag{27}$$

regardless of the value of Δ. This optimal case occurs when the lines $C(x)$ and $D(x)$ coincide with positive values at zero.

If we want to exclude the border line case $\gamma = -\Delta = \delta$, then the constraint $\gamma = -\Delta$ has to be replaced by the stricter inequality

$$\gamma > -\Delta + \varepsilon \tag{28}$$

with some small positive ε which bounds $C(0)$ and $D(0)$ away from each other. In optimizing $\gamma + b$ we can use the same method as before, but in this case the horizontal line $\gamma = -\Delta$ has to be replaced by $\gamma = -\Delta + \varepsilon$. The minimal value of $\gamma + b$ occurs at the intercept of lines $\gamma = -\Delta + \varepsilon$ and $\gamma + 2b = 1 - \Delta$ which is $b = \frac{1-\varepsilon}{2}$ and $\gamma = -\Delta + \varepsilon$ with the optimal value of $b + \gamma = \frac{1+\varepsilon-2\Delta}{2}$. Then

$$x_2 = \frac{1}{2}\left[\frac{1+\varepsilon-4\Delta}{2} + \sqrt{\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 + 4\Delta}\right]. \tag{29}$$

Simple differentiation shows that

$$\frac{\partial x_2}{\partial \Delta} = \frac{1}{\sqrt{\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 + 4\Delta}}\left[-\sqrt{\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 + 4\Delta} - \left(\frac{1+\varepsilon-4\Delta}{2}\right) + 1\right]$$

where the bracketed factor is always negative. Hence x_2 is strictly decreasing in Δ. From (29) we see that

$$\begin{aligned} x_2 &= \frac{\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 - \left[\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 + 4\Delta\right]}{2\left[\frac{1+\varepsilon-4\Delta}{2} - \sqrt{\left(\frac{1+\varepsilon-4\Delta}{2}\right)^2 + 4\Delta}\right]} \\ &= \frac{-4\Delta}{(1+\varepsilon-4\Delta) - \sqrt{(1+\varepsilon-4\Delta)^2 + 16\Delta}} \\ &= \frac{-4}{\left(-4 + \frac{1+\varepsilon}{\Delta}\right) - \sqrt{\left(\frac{1+\varepsilon}{\Delta} - 4\right)^2 + \frac{16}{\Delta}}} \end{aligned}$$

implying that $x_2 \to \frac{1}{2}$ as $\Delta \to \infty$. Therefore x_2 is always above $\frac{1}{2}$ and with sufficiently large values of Δ this limit can be approached arbitrarily.

If we assume that $D(1)$ and $C(1)$ are bounded away from each other, then condition (24) is modified as

$$\gamma + 2b > 1 + \varepsilon - \Delta.$$

Under this modified constraint the smallest value of $\gamma + b$ becomes $\gamma + b = -\Delta + \frac{1+\varepsilon}{2}$. Then x_2 becomes the same as equation (29) leading to the same results as obtained in the previous case.

And finally we consider the case of parallel lines of $C(x)$ and $D(x)$ but we assume that $D(x) - C(x)$ is bounded away from zero. That is, $b = \frac{1}{2}$ and

$$\gamma > -\Delta + \varepsilon.$$

The smallest value of $\gamma + b$ is $\gamma + b = -\Delta + \varepsilon + \frac{1}{2}$ so

$$x_2 = \frac{1}{2}\left[-2\Delta + \varepsilon + \frac{1}{2} + \sqrt{\left(-2\Delta + \varepsilon + \frac{1}{2}\right)^2 + 4\Delta}\right].$$

Notice that this is the same as equation (29) with ε being replaced by $\frac{\varepsilon}{2}$. So we reach the same results and conclusions as above.

3. Verification by Computer Simulation

We use an agent-based simulation model (Szilagyi and Szilagyi, 2000) for the experimental verification of these results. This model is implemented in the simulation tool called *Dilemma*. It has three distinctive features:

(1) It is a general framework for inquiry in which the properties of the environment as well as those of the agents are user-defined parameters and the number of agents is theoretically unlimited.

(2) The agents have various distinct, user-defined "personalities."

(3) The participating agents are described as stochastic learning cellular automata, i.e., as combinations of cellular automata (Wolfram, 1994) and stochastic learning automata (Narendra and Thathachar, 1989).

The simulation environment is a two-dimensional array of the participating agents. The size and shape of the simulation environment are user-defined variables. The size of the neighborhood is also a user-defined variable. It may be just the immediate neighborhood or any number of layers of agents around the given agent. In the limiting case that is considered in the present study, all agents are neighbors, and they collectively form the environment for each participating agent. In this case, the neighborhood extends to the entire array of agents.

A reward/penalty is the only input that the agents receive from the environment. The probabilities of the agents' actions are updated by this reward/penalty based on their and other agents' actions. New actions are taken according to these probabilities. Behavior is learned by adjusting the action probabilities to the responses of the environment.

The updating scheme may be different for different agents. This means that agents with completely different personalities can be allowed to interact with each other in the same experiment. Agents with various personalities and various initial states and actions can be placed anywhere in the array.

The specific probability updating schemes depend on the agents' personalities. The probability update curve is user-defined. It specifies the change in the probability of choosing the previously chosen action based on a number of factors. Such factors include the reward/penalty received for that action, the history of rewards/penalties received for all actions, the agent's neighbors' actions, etc.

A variety of rational and irrational personality profiles and their arbitrary combinations can be represented, including Pavlovian agents.

The state and action of each agent in the array change with time. One unit of time is called an iteration. With each iteration, the software tool draws the array of agents in a window on the computer's screen, with each agent in the array colored according to its most recent action. The experimenter can view and record the evolution of the society as it changes in time.

We performed a large number of experiments with Pavlovian agents and various linear payoff functions. We found that the aggregate behavior of the society of agents always converges to one of two kinds of equilibrium solutions. As the theoretical study (Szilagyi and Szilagyi, 2002) suggests, the two solutions are different from each other because 1) the solution at x_1 is a stable equilibrium (attractor) while the solution at x_2 is an unstable equilibrium (repulsor) and 2) the solution converges towards x_1 as an oscillation while it stablizes exactly in the $x_2 < x_0$ case.

We found that, in perfect agreement with the analytical study presented above, for the N-person Prisoners' Dilemma game played by Pavlovian agents in case of linear payoff functions a maximum 50% cooperation rate can be learned in a large community of agents starting with small cooperation rates. This limit can only be increased if the initial cooperation rate is above 50%.

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