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GENERALIZED NONLINEAR PSEUDOCOCOERCIVE VARIATIONAL INEQUALITIES AND GENERAL PROXIMAL POINT METHODS

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Abstract. Let $T: K \to H$ be a hemicontinuous, η-pseudococoercive, $\eta - L$-relaxed monotone and d-Hölder continuous mapping from a nonempty closed invex subset K of an infinite-dimensional Hilbert space H into H. Let $f: K \to R$ be proper, invex and lower semicontinuous on K and $h: K \to R$ be continuously Fréchet-differentiable on K with h' (gradient of h) $\eta - \alpha$-strongly monotone and $\eta - \beta$-Lipschitz continuous on K. Suppose that there exist an $x^* \in K$ and numbers $a > 0$, $r \geq 0$, ρ^k $(a < \rho^k < \alpha/L)$ such that for all $t \in [0,1]$, and for all $x \in K^\#$, the set $S^\#$ defined by

$$S^\# = \{(h,\eta): h'(x^* + t(x - x^*))(x - x^*) \geq \langle h'(x^* + t\eta(x,x^*)), \eta(x,x^*)\rangle\}$$

is nonempty, where $K^\# = \{x \in K: \|x - x^*\| \leq r\}$, and $\eta: K \times K \to H$ is λ-Lipschitz continuous with the following assumptions:

(i) $\eta(x,y) + \eta(y,x) = 0$, $\eta(x,y) = \eta(x,z) + \eta(z,y)$, and $\|\eta(x,x^*)\| \leq r$.
(ii) For each fixed $y \in K$, map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.

If in addition, h' is continuous from H equipped with weak topology to H equipped with weak topology, then the sequence $\{x^k\}$ generated by the general proximal point algorithm converges weakly to a solution x^* of the variational inequality problem (VIP), that is,

$$\langle T(x^*), \eta(x,x^*)\rangle + f(x) - f(x^*) \geq 0 \quad \text{for all } x \in K.$$

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1. Introduction

Martinet [11] primarily initiated the proximal point method and then applied it to study convex optimization in Hilbert spaces. It has been followed by extensive generalizations to the method in terms of applications to convex programming, which in turn, led it to a whole range of new multiplier methods. Rockafellar [13] considered mainly three principal applications of proximal point algorithms to convex programming: the primal proximal minimization algorithm, the dual method of multipliers, and the primal-dual proximal method of multipliers. Recently, El Farouq [7] studied, based on general proximal point methods, approximation-solvability of a class of pseudomonotone variational inequalities in a finite-dimensional Hilbert space setting, while convergence of proximal methods for infinite-dimensional case requires a class of strongly pseudomonotone or a class of pseudococoercive mappings with some additional side impositions of hemicontinuity and Hölder continuity instead.

In this paper, we plan, based on the generalized proximal point method, to present the approximation-solvability of variational inequality problems (VIP) involving η-pseudococoercive mappings along with some results on Fréchet-differentiable functions which play a crucial role in the construction of the general framework for the proximal point methods. Results thus obtained complement works of El Farouq [7], Verma [16], and others.

Let H be an infinite-dimensional Hilbert space with the inner produce $\langle , \rangle$ and the norm $\| \cdot \|$ on H. We consider the variational inequality problem (VIP) as follows: determine an element $x^* \in K$ such that

$$\langle T(x^*), \eta(x, x^*)\rangle + f(x) - f(x^*) \geq 0 \quad \text{for all } x \in K, \tag{1.1}$$

where K is a nonempty closed invex subset of H, and $\eta: K \times K \to H$ is any mapping with some side conditions. For $\eta(x, x^*) = x - x^*$, the VIP (1.1) reduces to the VIP: find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle + f(x) - f(x^*) \geq 0 \quad \text{for all } x \in K, \tag{1.2}$$

where K is a nonempty closed convex subset of H.

For $f \equiv 0$ in (1.2), it reduces to: find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle \geq 0 \quad \text{for all } x \in K. \tag{1.3}$$

We need now to define and in some cases to upgrade the existing notions in the literature. Let $\eta: K \times K \to H$ be any mapping.

A mapping $T: K \to H$ is said to be *η-monotone* if

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq 0 \quad \text{for all } x, y \in K.$$

The mapping T is *$\eta - \alpha$-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq \alpha \|x - y\|^2 \quad \text{for all } x, y \in K.$$

The mapping T is *$\eta - \gamma$-cocoercive* if there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), \eta(x, y)\rangle \geq \gamma \|T(x) - T(y)\|^2 \quad \text{for all } x, y \in K.$$

A mapping $T\colon K \to H$ is said to be *η-pseudomonotone* if

$$\langle T(y), \eta(x,y)\rangle \geq 0 \Rightarrow \langle T(x), \eta(x,y)\rangle \geq 0 \quad \text{for all } x, y \in K.$$

$T\colon K \to H$ is *strongly $\eta - b$-pseudomonotone* if

$$\langle T(y), \eta(x,y)\rangle \geq 0 \Rightarrow \langle T(x), \eta(x,y)\rangle \geq b\|x-y\|^2 \quad \text{for all } x, y \in K.$$

T is *$\eta - c$-pseudococoercive* if there exists a constant $c > 0$ such that

$$\langle T(y), \eta(x,y)\rangle > 0 \Rightarrow \langle T(x), \eta(x,y)\rangle \geq c\|T(x)-T(y)\|^2 \quad \text{for all } x, y \in K.$$

T is *η-quasimonotone* if

$$\langle T(y), \eta(x,y)\rangle > 0 \Rightarrow \langle T(x), \eta(x,y)\rangle \geq 0 \quad \text{for all } x, y \in K.$$

T is *$\eta - L$-relaxed monotone* (also referred to as *weakly monotone*) if there exists a constant $L > 0$ such that

$$\langle T(x) - T(y), \eta(x,y)\rangle \geq (-L)\|x-y\|^2 \quad \text{for all } x, y \in K.$$

T is *η-hemicontinuous* if for all $x, y, w \in K$ if the function

$$t \in [0,1] \to \langle T[y + t\eta(x,y)], w\rangle$$

is continuous.

T is *hemicontinuous* if for all $x, y, w \in K$ if the function

$$t \in [0,1] \to \langle T[y + t(x-y)], w\rangle$$

is continuous.

A subset K of H is said to be *invex* if there exists a function $\eta\colon K \times K \to H$ such that whenever $x, y \in K$ and $t \in [0,1]$, it follows that $x + t\eta(y,x) \in K$.

A function $f\colon K \to R$ is called *invex* if whenever $x, y \in K$ and $t \in [0,1]$, it follows that

$$f[x + t\eta(y,x)] \leq (1-t)f(x) + tf(y).$$

When $\eta(x,y) = x - y$, all the above notions reduce to the well-known ones.

2. Auxiliary Results

In this section, we recall some auxiliary results crucial to the approximation-solvability of the VIP (1.1). Let $h\colon H \to R$ be a continuously Fréchet-differentiable mapping on a Hilbert space H. It follows that $h'(x) \in L(H,R)$—the space of all bounded linear operators from H into R. From now on, we shall denote the real number $h'(x)(y)$ by $\langle h'(x), y\rangle$ for $x, y \in H$. The following result is a modified version of [1, Lemma 4.1]:

Lemma 2.1. *Let H be a Hilbert space and K be a nonempty invex subset of H. Suppose that h', the gradient of $h: K \to R$, is $\eta - \alpha$-strongly monotone on K and the following assumptions hold:*

(i) *There exist an $x^* \in K$ and number $r \geq 0$ such that for all $x \in K^{\#}$, $t \in [0,1]$, the mapping $\eta: K \times K \to H$ satisfies:*

$$\|\eta(x, x^*)\| \leq r.$$

(ii) *The set $S^{\#}$ defined by*

$$S^{\#} = \{(h, \eta): h'(x^* + t(x - x^*))(x - x^*) \geq \langle h'(x^* + t\eta(x, x^*)), \eta(x, x^*)\rangle\}$$

is nonempty, where $h: K \to R$ is a continuously Fréchet-differentiable mapping and h' is the gradient of h.

(iii) *The set*

$$K^{\#} = \{x \in K: \|x - x^*\| \leq r\}.$$

Then, for all $x \in K^{\#}$ and $(h, \eta) \in S^{\#}$, we have

$$h(x) - h(x^*) - \langle h'(x^*), \eta(x, x^*)\rangle \geq (\alpha/2)\|x - x^*\|^2.$$

For $\eta(x, y) = x - y$, in Lemma 2.1, we arrive at

Lemma 2.2. *Let H be a Hilbert space and K be a nonempty convex subset of H. Suppose that h', the gradient of $h: K \to R$, is α-strongly monotone on K, where $h: K \to R$ is a continuously Fréchet-differentiable mapping.*

Then, for all $x, x^* \in K$, we have

$$h(x) - h(x^*) - \langle h'(x^*), x - x^*\rangle \geq (\alpha/2)\|x - x^*\|^2.$$

Lemma 2.3. *Let H be a Hilbert space and K be a nonempty invex subset of H. Suppose that the following assumptions hold:*

(i) *h' is $\eta - \partial$-Lipschitz continuous, where h' is the gradient of $h: K \to R$.*

(ii) *There exist an $x^* \in K$ and number $q \geq 0$ such that for all $x \in K$, $t \in [0,1]$, the mapping $\eta: K \times K \to H$ satisfies:*

$$\|\eta(x, x^*)\| \leq q,$$

where the set

$$K = \{x \in K: \|x - x^*\| \leq q\}.$$

(iii) *The set S_1 defined by*

$$S_1 = \{(h, \eta): h'(x^* + t(x - x^*))(x - x*) \leq \langle h'(x^* + t\eta(x, x^*)), \eta(x, x^*)\rangle\}$$

is nonempty.

Then, for all $x \in K_1$ and $(h, \eta) \in S_1$, the following inequality holds

$$h(x) - h(x^*) - \langle h'(x^*), \eta(x, x^*)\rangle \leq (\partial/2)\|x - x^*\|^2.$$

For $\eta(x, x^*) = x - x^*$ in Lemma 2.3, we have

Lemma 2.4. *Let H be a Hilbert space and K be a nonempty convex subset of H. Suppose that h' (gradient of h) is ∂-Lipschitz continuous, where $h: K \to R$ is continuously differentiable. Then for all $x, x^* \in K$, the following inequality holds*

$$h(x) - h(x^*) - \langle h'(x^*), x - x^*\rangle \leq (\partial/2)\|x - x^*\|^2.$$

The proof is similar to that of Lemma 2.1.

3. General Proximal Algorithm

This section deals with a discussion of the approximation-solvability of the VIP (1.1) based on the general proximal algorithm [7]. For some $x \in H$, the problem of finding $\overline{y}(x)$ such that

$$\langle \rho^k T(\overline{y}(x)) + h'(\overline{y}(x) - h'(x), \eta(y, \overline{y}(x))\rangle + \rho^k[f(y) - f(\overline{y}(x))] \geq 0 \text{ for all } y \in K, \tag{3.1}$$

is called the Proximal Algorithm.

Algorithm 3.1. For a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$)

$$\begin{aligned} &\langle \rho^k T(x^{k+1}) + h'(x^{k+1}) - h'(x^k), \eta(x, x^{k+1})\rangle \\ &\qquad + \rho^k[f(x) - f(x^{k+1})] \geq 0 \quad \text{for all } x \in K, \end{aligned} \tag{3.2}$$

where $h: K \to R$ is continuously Fréchet-differentiable, $\{\rho^k\}$ is the sequence of positive numbers, and $\eta: K \times K \to H$ is a mapping with some side restrictions.

For $\eta(x, y) = x - y$ in (3.2), we find

Algorithm 3.2. For a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$)

$$\begin{aligned} &\langle \rho^k T(x^{k+1}) + h'(x^{k+1}) - h'(x^k), x - x^{k+1}\rangle \\ &\qquad + \rho^k[f(x) - f(x^{k+1})] \geq 0 \quad \text{for all } x \in K, \end{aligned} \tag{3.3}$$

where K is a nonempty closed convex subset of H.

For $f \equiv 0$ in Algorithm 3.2, we obtain

Algorithm 3.3. For a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$)

$$\langle \rho^k T(x^{k+1}) + h'(x^{k+1}) - h'(x^k), x - x^{k+1}\rangle \geq 0 \quad \text{for all } x \in K, \tag{3.4}$$

where K is a nonempty closed convex subset of H.

We are just about ready to present, based on Algorithm 3.1, the approximation-solvability of the VIP (1.1).

Theorem 3.1. *Let $T: K \to H$ be an η-hemicontinuous, $\eta - \gamma$-pseudocococoercive, d-Hölder continuous and $\eta - L$-relaxed monotone mapping from a nonempty closed invex subset K of an infinite-dimensional Hilbert space H into H. Let $f: K \to R$ be proper, invex and lower semicontinuous on K, $h: K \to R$ be continuously Fréchet-differentiable on K with h' (gradient of h) η-hemicontinuous, $\eta - \alpha$-strongly monotone, $\eta - \beta$-Lipschitz continuous, c-Hölder continuous and continuous from H equipped with weak topology to H equipped with weak topology.*

Suppose that there exist an $x' \in K$ and numbers $a > 0$, $r \geq 0$, ρ^k $(a < \rho^k < \alpha/L)$ such that for all $t \in [0, 1]$, and for all $x \in K^{\#}$, we have

(i) *the set $S^{\#}$ defined by*

$$S^{\#} = \{(h, \eta): h'(x^* + t(x - x^*))(x - x^*) \geq \langle h'(x^* + t\eta(x, x^*)), \eta(x, x^*)\rangle\}$$

is nonempty, where $K^{\#}$ is defined by

$$K^{\#} = \{x \in K: \|x - x'\| \leq r\},$$

and $\eta: K \times K \to H$ is λ-Lipschitz continuous with following assumptions:

(ii) $\eta(x,y) + \eta(y,x) = 0$ *and* $\eta(x,y) = \eta(x,z) + \eta(z,y)$.

(iii) *For each fixed $y \in K$, map $x \to \eta(y,x)$ is sequentially continuous from the weak topology to the weak topology.*

(iv) $\|\eta(x,x^*)\| \le r$.

(v) η *is expanding.*

If in addition, $x^* \in K$ is a solution of the VIP (1.1)*, then the sequence $\{x^k\}$ generated by Algorithm* 3.1 *converges weakly to x^*.*

Proof. Since T is $\eta - L$-relaxed monotone and h' is $\eta - \alpha$-strongly monotone, it follows that x^{k+1} is a unique solution of the proximal variational problem (3.2) for $\rho^k < \alpha/L$. Assumptions of η-hemicontinuity on T and h' ensure the existence of the solution x^{k+1}. To show that the sequence $\{x^k\}$ converges weakly to x^*, a solution of the VIP (1.1), we need to compute the estimates. Let us define a function Λ^* by

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), \eta(x^*, x)\rangle.$$

Then, on applying Lemma 2.1, we have

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), \eta(x^*, x)\rangle \ge (\alpha/2)\|x^* - x\|^2 \quad \text{for } x \in K, \tag{3.5}$$

where x^* is a solution of the VIP (1.1). It follows that

$$\Lambda^*(x^{k+1}) = h(x^*) - h(x^{k+1}) - \langle h'(x^{k+1}), \eta(x^*, x^{k+1})\rangle. \tag{3.6}$$

Now we can write

(3.7)
$$\begin{aligned}
\Lambda^*(x^k) &- \Lambda^*(x^{k+1}) \\
&= h(x^{k+1}) - h(x^k) - \langle h'(x^k), \eta(x^{k+1}, x^k)\rangle + \langle h'(x^{k+1}) - h'(x^k), \eta(x^*, x^{k+1})\rangle \\
&\ge (\alpha/2)\|x^{k+1} - x^k\|^2 + \langle h'(x^{k+1}) - h'(x^k), \eta(x^*, x^{k+1})\rangle \\
&\ge (\alpha/2)\|x^{k+1} - x^k\|^2 + \rho^k\langle T(x^{k+1}), \eta(x^{k+1}, x^*)\rangle + \rho^k(f(x^{k+1}) - f(x^*))
\end{aligned}$$

for $x = x^*$ in (3.2).

If we replace x by x^{k+1} in (1.1), we obtain

$$\langle (T(x^*), \eta(x^{k+1}, x^*)\rangle + f(x^{k+1}) - f(x^*)) \ge 0. \tag{3.8}$$

Since T is $\eta - \gamma$-pseudococoercive, it implies in light of (3.8) that

$$\Lambda^*(x^k) - \Lambda^*(x^{k+1}) \ge [\alpha/2]\|x^{k+1} - x^k\|^2 + a\gamma\|T(x^{k+1}) - T(x^*)\|^2. \tag{3.9}$$

It follows from (3.9) that the sequence $\{\Lambda^*(x^k)\}$ is strictly decreasing except for $x^{k+1} - x^k$, and in that situation x^k is a solution to (1.1). Since the difference two consecutive terms tends to zero as $k \to \infty$, it implies that

$$\|x^{k+1} - x^k\| \to 0 \ \text{ and } \ \|T(x^{k+1}) - T(x^*)\| \to 0 \ \text{ as } k \to \infty.$$

On the top of that, in light of (3.5), we have

$$\|x^* - x^k\|^2 \le (2/\alpha)\Lambda^*(x^k),$$

and so the sequence $\{x^k\}$ is bounded. Let x' be a cluster point of the sequence $\{x^k\}$, that is, there exists a subsequence $\{x^{kj}\}$ of the sequence $\{x^k\}$ such that $\{x^{kj}\}$ converges weakly to x'. Since h' is $\eta-\beta$-Lipschitz continuous and $a < \rho^k$, it follows using (3.2) that for some $x \in H$, we have

$$\begin{aligned}\langle \rho^k T(x^{k+1}), \eta(x, x^{k+1})\rangle + \rho^k[f(x) - f(x^{k+1})] &\geq -\langle h'(x^{k+1}) - h'(x^k), \eta(x, x^{k+1})\rangle \\ &\geq (-\beta)\|x^{k+1} - x^k\| \, \|\eta(x, x^{k+1})\|,\end{aligned}$$

or

$$\langle T(x^{k+1}), \eta(x, x^{k+1})\rangle + [f(x) - f(x^{k+1})] \geq (-\beta/a)\|x^{k+1} - x^k\| \, \|\eta(x, x^{k+1})\|. \tag{3.10}$$

Since $T(x^{k+1})$ converges strongly to $T(x^*)$ and $\|x^{kj+1} - x^{kj}\| \to 0$, it follows from (3.10) that

$$\langle T(x^*), \eta(x, x')\rangle + [f(x) - f(x')] \geq 0 \quad \text{for all } x \in K, \tag{3.11}$$

while $\langle T(x'), \eta(x^{kj}, x')\rangle + f(x^{kj}) - f(x')$ tends to zero. At this stage, if $T(x') = 0$, then x' is a solution to (1.1), and if $T(x') \neq 0$ then we can express it in the form

$$\eta(y^{kj}, x^{kj}) = -[\langle T(x'), \eta(x^{kj}, x')\rangle / \|T(x')\|^2] T(x').$$

It follows that

$$\langle T(x'), \eta(y^{kj}, x')\rangle = 0, \tag{3.12}$$

and

$$\|\eta(y^{kj}, x^{kj})\| \to 0 \quad \text{as } k \to \infty.$$

Therefore, y^{kj} converges weakly to x', while the d-Hölder continuity of T implies that $T(y^{kj})$ converges strongly to $T(x^*)$. Since T is $\eta-\gamma$-pseudococoercive, it follows in light of (3.12) that

$$\langle T(y^{kj}, \eta(y^{kj}, x')\rangle \geq \gamma\|T(y^{kj} - T(x')\|^2. \tag{3.13}$$

It clearly follows from (3.13) that $\{T(y^{kj})\}$ converges strongly to $T(x')$, and so $T(x') = T(x^*)$. Hence in light of (3.11) x' is a solution of (1.1).

When $\eta(x, y) = x - y$ in Theorem 3.1, we arrive at

Theorem 3.2. *Let $T: K \to H$ be an η-hemicontinuous, γ-pseudococoercive, d-Hölder continuous and L-relaxed monotone mapping from a nonempty closed invex subset K of an infinite-dimensional Hilbert space H into H. Let $f: K \to R$ be proper, invex and lower semicontinuous on K, $h: K \to R$ be continuously Fréchet-differentiable on K with h' (gradient of h) hemi-continuous, α-strongly monotone, β-Lipschitz continuous, c-Hölder continuous and continuous from H equipped with weak topology to H equipped with weak topology on K. If in addition, $x^* \in K$ is a solution of the VIP* (1.2), *then the sequence $\{x^k\}$ generated by Algorithm* 3.2 *converges weakly to x^*.*

For $f \equiv 0$ in Theorem 3.2, it reduces to El Farouq [7, Theorem 5.1].

Theorem 3.3. *Let $T: K \to H$ be an η-hemicontinuous, γ-pseoducococoercive, d-Hölder continuous and L-relaxed monotone mapping from a nonepty closed invex subset K of an infinite-dimensional Hilbert space H into H. Let $h: K \to R$ be continuously Fréchet-differentiable on K with h' (gradient of h) hemi-continuous, α-strongly monotone, β-Lipschitz continuous, c-Hölder continuous and continuous from H equipped with weak topology to H equipped with weak topology on K. If in addition, $x^* \in K$ is a solution of the VIP* (1.3), *then the sequence $\{x^k\}$ generated by Algorithm* 3.3 *converges weakly to x^*.*

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