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CHAIN OF DENDRITES WITHOUT OPEN INFIMUM

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Abstract. We consider an ordering with respect to open mappings on the class of all dendrites and construct a bounded chain of dendrites which does not have an infimum answering a question of its existence.

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Key Words and Phrases. Dendrite, open mapping, chain

Flowers and flower-beds

All spaces considered in this paper are assumed to be metric. A *continuum* means a nonempty compact connected space. A *simple closed curve* is any space which is homeomorphic to the unit circle. A *dendrite* means a locally connected continuum containing no simple closed curve.

A *mapping* means a continuous function. A surjective mapping $f : X \to Y$ is said to be *open* provided that for each open subset U of X its image $f(U)$ is an open subset of Y.

In [3, p. 7] J. J. Charatonik, W. J. Charatonik and J. R. Prajs introduced a quasiordering $\leq_O$ on the class of all dendrites. We recall the definition. If X, Y are dendrites, then $Y \leq_O X$ if there is a surjective open mapping $f : X \to Y$. This quasiordering is not an ordering (i.e. $X \leq_O Y$ & $Y \leq_O X$ does not imply X is homeomorphic to Y). To guarantee this they said that X and Y are $\mathbb{O}$-*equivalent* iff $X \leq_O Y$ and $Y \leq_O X$ and then considered the quotient of the class of all dendrites. The quasiordering $\leq_O$ induces the ordering $\leq_{\mathbb{O}}$ on the quotient.

J. J. Charatonik, W. J. Charatonik and J. R. Prajs posed a question if every chain bounded with respect to this ordering has an infimum (see [3, §7, Q4(b)$\mathbb{O}$,

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p. 51]). In Theorem we give a negative answer to this question by constructing a bounded sequence of dendrites with no open infimum.

We shall use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [4, §51, I, p. 274], and we denote by $\operatorname{ord}(p, X)$ the order of the continuum X at a point $p \in X$ or just $\operatorname{ord}(p)$ if there is no risk of confusion. Points of order 1 in a continuum X are called *end points* of X; the set of all end points of X is denoted by $E(X)$. Points of order 2 are called *ordinary points* of X. Points of order at least 3 are called *ramification points* of X.

Let X be a space. A sequence of subsets of X is said to be a *null-sequence* if the diameters of its members tend to zero. Let X be a connected space, and let $p \in X$. If $X \setminus \{p\}$ is not connected, then p is called a *cut point of* X.

An *arc* is any space which is homeomorphic to $I = [0, 1]$. We denote by pq an arc with end points p and q. An arc pq in a space X is said to be *free* if $pq \setminus \{p, q\}$ is an open subset of the space X. Denote by S_n the n-od, i.e. the dendrite where the only ramification point is of order n.

Fact. (See [6, Chapter 8, (7.31), p. 147].) *The order of a point is never increased under an open mapping.*

Proposition. *Let X be a dendrite.*

(A) *Each point of X is either a cut point or an end point.*
(B) *For each point $x \in X$ we have* $\operatorname{ord}(x) = c(x)$, *where $c(x)$ is the number of components of $X \setminus \{x\}$.*
(C) *For every subcontinuum $Z \subset X$ every component of $X \setminus Z$ is open in X.*
(D) *Let $f : X \to Y$ be a surjective open mapping. Then Y is a dendrite.*
(E) *Let $f : X \to Y$ be a surjective open mapping. Then the image under f of a free arc in X is a free arc in Y.*
(F) *Let $f : X \to Y$ be a surjective open mapping. Then $f^{-1}(E(Y)) \setminus E(X)$ is finite.*

See [5, Chapter X, Theorem 10.7, p. 168], [5, Chapter X, Theorem 10.13, p. 170], [5, 5.22(a), p. 83], [5, Corollary 13.41, p. 297], [3, Corollary 6.7, p. 23]. and [3, Proposition 6.16, p. 25].

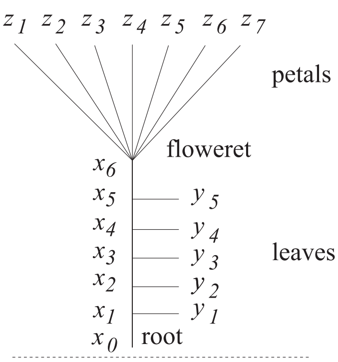

Figure 1 – a flower $F = F(x_0, 5, x_6, 7)$

Given $l, p \in \mathbb{N} \cup \{0\}$. Denote $x_j = (0, j), j = 0, \cdots, l+1$, $y_j = (1, j), j = 1, \cdots, l$, $z_i = (i, l+2), i = 1, \cdots, p$. By a *flower* (*with l leaves and p petals*) we denote the union

$$F = x_0 x_{l+1} \cup \bigcup_{j=1}^{l} x_j y_j \cup \bigcup_{i=1}^{p} x_{l+1} z_i \quad .$$

For a flower F we say that the point x_0 is the *root* , the arcs $x_j y_j$ are the *leaves* , x_{l+1} is the *floweret* and the arcs $x_{l+1} z_i$ are the *petals.* We denote by $F(r, l, f, p)$ a flower with a root r, l leaves, a floweret f and p petals. Clearly, a flower is a finite union of arcs. See Figure 1.

Set $a = (0, 0)$ and $b = (0, 2)$. By a *flower-bed* (*with flowers* F_n) we denote the union

$$B = ab \cup \bigcup_{n=1}^{\infty} F_n \quad ,$$

where $\{F_n\}$ is a null-sequence of mutually disjoint flowers each meeting ab in its root $(1/n, 0)$ only. We observe that a flower-bed is a dendrite. See Figure 2.

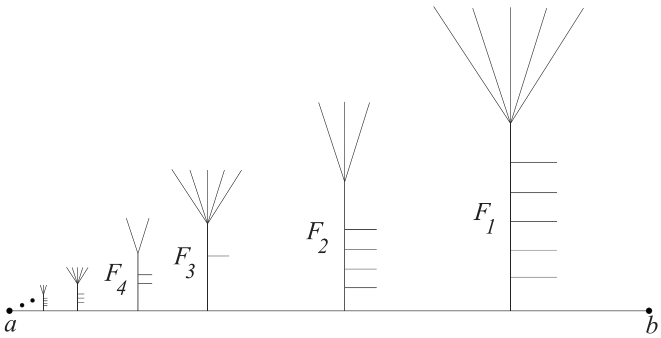

Figure 2 – a flower-bed

Fix a null-sequence of mutually disjoint flowers

$$G_n = ((1/n, 0), n, (1/n, 1/n), n),$$

each meeting ab in its root $(1/n, 0)$ only. We denote by X_n the flower-bed

$$X_n = ab \cup G_n \cup G_{n+1} \cup G_{n+2} \cup \cdots .$$

See Figure 3.

Theorem. *The chain* $X_1 \geq_O X_2 \geq_O X_3 \geq_O \cdots \geq_O I$ *is bounded and has no infimum.*

Proof. Suppose that Y is an infimum of the sequence $\{X_n\}$. We conclude a contradiction in several steps.

(1) Denote by $g_n : X_n \to Y$ an open surjection of X_n onto Y. Then Y is a dendrite due to Proposition D.

(2) $X_1 \geq_O X_2 \geq_O X_3 \geq_O \cdots$.

(Proof. Fix $n \in \mathbb{N}$. We construct a mapping h_n of X_n onto X_{n+1} such that h_n satisfies the following conditions:

(i) h_n is the identity on X_{n+1};

(ii) h_n is piece-wise linear (and nonconstant) elsewhere;

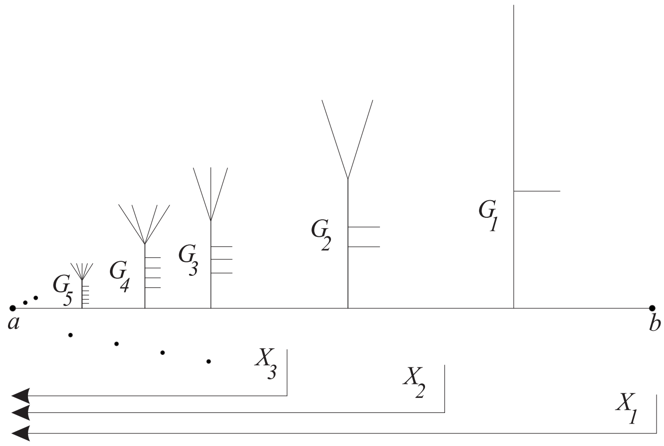

Figure 3 – flower-beds X_1, X_2, $\cdots$

(iii) h_n identifies all end points of X_n contained in $X_n \setminus X_{n+1}$ with b. Obviously, h_n is an open surjection.)

(3) $X_n \geq_O S_m \geq_O I$ for each $n, m \in \mathbb{N}$.

(Proof. Given $m, n \in \mathbb{N}$, fix a ramification point x of order $m' \geq m$ in X_n. The mapping of X_n which (i) identifies on each component of $X_n \setminus \{x\}$ all end points of X_n, and (ii) is piece-wise linear (and nonconstant), is an open surjection of X_n onto $S_{m'}$. Finally, trivial open surjections of $S_{m'}$ onto S_m and of S_m onto I finish the proof.)

(4) For each $m \in \mathbb{N}$, $Y \geq_O S_m$ (see (3), Y is an infimum of $\{X_n\}$).

(5) Ramification points of Y are of finite order (due to (1) and Fact, because the flower-bed X_1 has only points of finite order).

(6) For each $m \in \mathbb{N}$, there is a ramification point in Y of order at least m (see (4) and Fact).

(7) Fix in Y a ramification point y of order $p > 3$ (see (6)).

(8) Find for each $n \in \mathbb{N}$ a ramification point x_n in X_n such that $g_n(x_n) = y$ (see Fact, (1) and (7)).

(9) For each $n \in \mathbb{N}$, the order of x_n in X_n is at least p (see (7), (8) and Fact).

(10) For each $n \in \mathbb{N}$, the point x_n is a floweret of a flower $F_n = F(r_n, l_n, x_n, p_n)$ in X_n with $l_n \geq n$ due to the construction of X_n (see (8)).

(11) For each $n \in \mathbb{N}$, if $g_n(z) = g_n(r_n)$ for some $z \in F_n$, then $z = r_n$.

(Proof. Otherwise $F_n \setminus \{r_n\}$ is an open set (see (10) and Proposition C) with the same image as F_n under g_n. Then $Y = g_n(F_n)$ due to the openness of g_n and the connectedness of Y. Finally, we conclude that Y has only finitely many ramification points due to Fact, contrary to (6).)

(12) For each $n \in \mathbb{N}$, if $g_n(z) = g_n(x_n)$ for some $z \in F_n$, then $z = x_n$ (see (7), (9) and Fact, because the only ramification point of X_n in F_n of order greater than 3 is x_n).

(13) For each $n \in \mathbb{N}$, the image under g_n of a flower F_n is a flower $\hat{F}_n = F(\hat{r}_n, \hat{l}_n, y, \hat{p}_n)$, where $\hat{r}_n = g_n(r_n)$ and $\hat{p}_n = p - 1$ (due to (11), (12), Fact and Propositions C-D).

(14) For each $n \in \mathbb{N}$, the flowers F_n and $\hat{F}_n$ have the same number of leaves, i.e. $\hat{l}_n = l_n (\geq n)$ (due to (10), (13), Fact and Proposition E).

(15) For each $n \in \mathbb{N}$, there is in Y a flower $\hat{F}_n$ rooting from $\hat{r}_n$ containing y with at least n leaves (due to (10) and (14)).
(16) For each $n \in \mathbb{N}$, the flower $\hat{F}_n$ contains at least n end points of Y (due to (15).
(17) Neighborhoods of a contain all but finite number of end points of X_1 due to the construction.
(18) Neighborhoods of $\hat{a} = g_1(a)$ contain all but finite number of end points of $Y = g_1(X_1)$ (due to the openness of g_1, see (1), (17) and Proposition F).
(19) Fix in Y a ramification point $y' \neq y$ of order at least $p(> 3)$ (see (6) and (7)).
(20) The point a is an end point in X_1 due to the construction.
(21) The point $\hat{a}(= g_1(a))$ is an end point in Y (see (1), (18), (20) and Fact).
(22) Fix an open neighborhood $U \subset (Y \setminus \{y, y'\})$ of $\hat{a}$ in Y such that the boundary of U in Y is a singleton $t \neq y, y'$ (see (21)).
(23) Denote by $k \in \mathbb{N}$ the number of end points of Y in $Y \setminus U$ (finite due to (18) and (22)).
(24) Fix $n \in \mathbb{N}$ such that $n > k$. The flower F_n in X_n contains at least $2n$ end points (it contains at least n leaves and at least n petals) and $\hat{F}_n$ contains at least $n + p - 1$ end points of Y (it contains at least n leaves and $p - 1$ petals) (see (13) and (14)).
(25) At least two end points u and v of Y contained in $\hat{F}_n$ are contained also in U (see (7), (23) and (24)).
(26) The root $\hat{r}_n$ of $\hat{F}_n$ is contained in U.

(Proof. Both arcs uy and vy (contained in the flower $\hat{F}_n$) cross the boundary of U in Y at the singleton t (see (25) and (22)). This point is not a floweret of $\hat{F}_n$ (it is y, see (13)). We conclude that there is a ramification point w of $\hat{F}_n$ in U. The arc $\hat{r}_n y$ must contain w (see (13)), and cross the boundary of U in Y at the singleton t only (see (22)). We see that the root $\hat{r}_n$ of $\hat{F}_n$ is contained in U.)

(27) The point t is contained in $\hat{F}_n$ (because the arc $\hat{r}_n y$ contains t, see the proof of (26)).
(28) The component C of $Y \setminus \{\hat{r}_n\}$ containing y is a subset of $\hat{F}_n$ (see (27)).
(29) Both points y and y' are contained in $\hat{F}_n \setminus \{\hat{r}_n\}$ (the component C contains $Y \setminus U$, because $t \in C$, see (22), (27), (28) and Proposition C).
(30) Order of y' in Y is at most 3 (the only ramification point of $\hat{F}_n$ of order greater than 3 is the floweret y, see (13) and (29)).
(31) Items (30) and (19) give a contradiction.

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References

1. J. J. Charatonik, *Monotone mappings of universal dendrites*, Topology Appl. **38** (1991), 163–187.
2. J. J. Charatonik, W. J. Charatonik, *Dendrites*, Aportaciones Mat. Comun. **22** (1998), 227 – 253.
3. J. J. Charatonik, W. J. Charatonik, J. R. Prajs, *Mapping hierarchy for dendrites*, Dissertationes Math. (Rozprawy Mat.) **333** (1994), 1 – 52.
4. K. Kuratowski, *Topology, Vol. II*, Academic Press, New York, 1968.

5. S. B. Jr. Nadler, *Continuum theory: an introduction*, M. Dekker, Inc., New York, Basel and Hong Kong, 1992.
6. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publ. 28, Providence, 1942, reprinted with corrections 1971.