Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 2, December 2002, pp. 77–82.
Submitted: February 22, 2002. Published: December 31 2002.

SALAGEAN-TYPE HARMONIC UNIVALENT FUNCTIONS

Jay M. Jahangiri, G. Murugusundaramoorthy, K. Vijaya

Abstract. A sufficient coefficient bound is determined for the Salagean-type class of complex-valued harmonic univalent functions consisting of functions of the form $f(z) = z + \sum_{m=2}^{\infty} a_m z^m + \sum_{m=1}^{\infty} \bar{b}_m \bar{z}^m$, $|z| < 1$. It is shown that this coefficient bound is also necessary if $f(z) = z - \sum_{m=2}^{\infty} a_m z^m + (-1)^n \sum_{m=1}^{\infty} b_m \bar{z}^m$ where $a_m \geq 0$, $b_m \geq 0$, and n is a whole number. Furthermore, extreme points, distortion theorem, convolution condition, and comnvex combinations for this latter class are also determined.

1991 A.M.S. (MOS) Subject Classification Codes. 30C45, 30C50

Key Words and Phrases. Harmonic, Univalent, Starlike, Convex

1. Introduction. A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain C if both u and v are real harmonic in C. In any simply connected domain $D \subset C$ we can write $f = h + \bar{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in D is that $|g'(z)| < |h'(z)|$ in D. See Clunie and Sheil-Small [2].

Denote by H the family of functions $f = h + \bar{g}$ which are harmonic univalent and orientation preserving in the open unit disk $U = \{z : |z| < 1\}$ so that $f = h + \bar{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Therefore, for $f = h + \bar{g} \in H$ we can express the analytic functions h and g by the following power series expansion

$$h(z) = z + \sum_{m=2}^{\infty} a_m z^m, \quad g(z) = \sum_{m=1}^{\infty} b_m z^m, \quad |b_1| < 1. \tag{1}$$

Note that the family H of orientation preserving, normalized harmonic univalent functions reduces to the class S of normalized analytic univalent functions if the co-analytic part of $f = h + \bar{g}$ is identically zero.

For $f = h + \bar{g}$ given by (1), we define the modified Salagean operator of f as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)} \tag{2}$$

where $D^n h(z) = z + \sum_{m=2}^{\infty} m^n a_m z^m$ and $D^n g(z) = \sum_{m=1}^{\infty} m^n b_m z^m$.

Kent State University, Burton, Ohio 44021-9500
E-mail: jay@geauga.kent.edu
Vellore Institute of Technology, Vellore - 632 014, TN, India
E-mail: gmsmoorthy@yahoo.com

For $0 \leq \alpha < 1$, we let $H(n, \alpha)$ denote the family of harmonic functions f of the form (1) such that

$$\Re\left(\frac{D^{n+1}f(z)}{D^n f(z)}\right) \geq \alpha, \quad 0 \leq \alpha < 1 \tag{3}$$

where $D^n f$ is defined by (2).

If the co-analytic part of $f = h + \bar{g}$ is identically zero, then the family $H(n, \alpha)$ turns out to be the class $S(n, \alpha)$ introduced by Salagean [5] for the analytic case.

We let the subclass $\overline{H}(n, \alpha)$ consist of harmonic functions $f_n = h + \overline{g_n}$ in $H(n, \alpha)$ so that h and g_n are of the form

$$h(z) = z - \sum_{m=2}^{\infty} a_m z^m, \quad g_n(z) = (-1)^n \sum_{m=1}^{\infty} b_m z^m, \quad a_m \geq 0,\ b_m \geq 0. \tag{4}$$

The class $H(n, \alpha)$ includes a variety of well-known subclasses of H. For example, $H(0, \alpha) \equiv HS(\alpha)$ is the class of orientation preserving, harmonic univalent functions f which are starlike of order α in U, that is, $\frac{\partial}{\partial\theta}\{arg(f(re^{i\theta}))\} \geq \alpha$, and $H(1, \alpha) \equiv HK(\alpha)$ is the class of orientation preserving, harmonic univalent functions f which are convex of order α in U, that is, $\frac{\partial}{\partial\theta}\{arg(\frac{\partial}{\partial\theta} f(re^{i\theta}))\} \geq \alpha$ where $z = re^{i\theta} \in U$.

For the harmonic functions f of the form (1) with $b_1 = 0$, Avci and Zlotkiewicz [1] showed that if $\sum_{m=2}^{\infty} m(|a_m| + |b_m|) \leq 1$ then $f \in HS(0)$ and if $\sum_{m=2}^{\infty} m^2(|a_m| + |b_m|) \leq 1$ then $f \in HK(0)$. Silverman [6] proved that the above two coefficient conditions are also necessary if $f = h + \bar{g}$ has negative coefficients. Later, Silverman and Silvia [7] improved the results of [1] and [6] to the case b_1 not necessarily zero. Jahangiri [3] and [4] generalized the results of [1], [6], and [7] by proving that $f \in HS(\alpha)$ if

$$\sum_{m=2}^{\infty} (m - \alpha)|a_m| + \sum_{m=1}^{\infty} (m + \alpha)|b_m| \leq 1 - \alpha \tag{5}$$

and $f \in HK(\alpha)$ if

$$\sum_{m=2}^{\infty} m(m - \alpha)|a_m| + \sum_{m=1}^{\infty} m(m + \alpha)|b_m| \leq 1 - \alpha. \tag{6}$$

In [3] and [4] is proved that the conditions (5) and (6) are also necessary if $f = h + \bar{g}$ has negative coefficients.

In this paper, the coefficient conditions given in [1, 3, 4, 6, 7] for the classes $HS(\alpha)$ and $HK(\alpha)$ are extended to the class $H(n, \alpha)$ of the forms (3) above. Furthermore, we determine extreme points, distortion theorem, convolution conditions and convex combinations for the functions in $\overline{H}(n, \alpha)$.

Main Results. In our first theorem, we introduce a sufficient coefficient bound for harmonic functions in $H(n, \alpha)$.

Theorem 1. Let $f = h + \bar{g}$ be given by (1). If

$$\sum_{m=1}^{\infty} m^n \left[(m - \alpha)|a_m| + (m + \alpha)|b_m|\right] \leq 2(1 - \alpha) \tag{7}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$, then f is orientation preserving, harmonic univalent in U, and $f \in H(n, \alpha)$.

Proof. If the inequality (7) holds for the coefficients of $f = h + \bar{g}$, then by (5), f is orientation preserving and harmonic univalent in U. It remains to show that $\Re\{(D^{n+1}f)/(D^n f)\} \geq \alpha$. According to (2) and (3) we have

$$\Re\left\{\frac{D^{n+1}f(z)}{D^n f(z)}\right\} = \Re\left\{\frac{D^{n+1}h(z) - (-1)^n \overline{D^{n+1}g(z)}}{D^n h(z) + (-1)^n \overline{D^n g(z)}}\right\} \geq \alpha,\ 0 \leq \alpha < 1.$$

Using the fact that $\Re\{\omega\}\geq\alpha$ if and only if $|1-\alpha+\omega|\geq|1+\alpha-\omega|$ it suffices to show that

$$|D^{n+1}f(z)+(1-\alpha)D^nf(z)|-|D^{n+1}f(z)-(1+\alpha)D^nf(z)|\geq 0. \tag{8}$$

Substituting for D^nf and $D^{n+1}f$ in (8) yields

$$\begin{aligned}
&|D^{n+1}f(z)+(1-\alpha)D^nf(z)|-|D^{n+1}f(z)-(1+\alpha)D^nf(z)|\\
&=|z+\textstyle\sum_{m=2}^{\infty}m^{n+1}a_mz^m-(-1)^n\sum_{m=1}^{\infty}m^{n+1}\bar{b}_m\bar{z}^m\\
&\qquad +(1-\alpha)z+(1-\alpha)\textstyle\sum_{m=2}^{\infty}m^na_mz^m+(-1)^n(1-\alpha)\sum_{m=1}^{\infty}m^n\bar{b}_m\bar{z}^m|\\
&\quad -|z+\textstyle\sum_{m=2}^{\infty}m^{n+1}a_mz^m-(-1)^n\sum_{m=1}^{\infty}m^{n+1}\bar{b}_m\bar{z}^m\\
&\qquad -(1+\alpha)z-(1+\alpha)\textstyle\sum_{m=2}^{\infty}m^na_mz^m-(-1)^n(1+\alpha)\sum_{m=1}^{\infty}m^n\bar{b}_m\bar{z}^m|\\
&=\left|(2-\alpha)z+\textstyle\sum_{m=2}^{\infty}m^n(m+1-\alpha)a_mz^m-(-1)^n\sum_{m=1}^{\infty}m^n(m-1+\alpha)\bar{b}_m\bar{z}^m\right|\\
&\quad -\left|-\alpha z+\textstyle\sum_{m=2}^{\infty}m^n(m-1-\alpha)a_mz^m-(-1)^n\sum_{m=1}^{\infty}m^n(m+1+\alpha)\bar{b}_m\bar{z}^m\right|\\
&\geq (2-\alpha)|z|-\textstyle\sum_{m=2}^{\infty}m^n(m+1-\alpha)|a_m||z|^m-\sum_{m=1}^{\infty}m^n(m-1+\alpha)|b_m||z|^m\\
&\qquad -\alpha|z|-\textstyle\sum_{m=2}^{\infty}m^n(m-1-\alpha)|a_m||z|^m-\sum_{m=1}^{\infty}m^n(m+1+\alpha)|b_m|z|^m\\
&= 2(1-\alpha)|z|\left\{1-\textstyle\sum_{m=2}^{\infty}\frac{m^n(m-\alpha)}{1-\alpha}|a_m||z|^{m-1}-\sum_{m=1}^{\infty}\frac{m^n(m+\alpha)}{1-\alpha}|b_m||z|^{m-1}\right\}\\
&\geq 2(1-\alpha)|z|\left\{1-\textstyle\sum_{m=2}^{\infty}\frac{m^n(m-\alpha)}{1-\alpha}|a_m|-\sum_{m=1}^{\infty}\frac{m^n(m+\alpha)}{1-\alpha}|b_m|\right\}.
\end{aligned}$$

This last expression is non-negative by (7), and so the proof is complete.

The starlike harmonic function

$$f(z)=z+\sum_{m=2}^{\infty}\frac{1-\alpha}{m^n(m-\alpha)}x_mz^m+\sum_{m=1}^{\infty}\frac{1-\alpha}{m^n(m+\alpha)}\bar{y}_m\bar{z}^m \tag{9}$$

where $\sum_{m=2}^{\infty}|x_m|+\sum_{m=1}^{\infty}|y_m|=1$ shows that the coefficient bound given by (7) is sharp.

The functions of the form (9) are in $H(n,\alpha)$ because

$$\sum_{m=1}^{\infty}\left(\frac{m^n(m-\alpha)}{1-\alpha}|a_m|+\frac{m^n(m+\alpha)}{1-\alpha}|b_m|\right)=1+\sum_{m=2}^{\infty}|x_m|+\sum_{m=1}^{\infty}|y_m|=2.$$

In the following theorem, it is shown that the condition (7) is also necessary for functions $f_n=h+\overline{g_n}$ where h and g_n are of the form (4).

Theorem 2. Let $f_n=h+\bar{g}_n$ be given by (4). Then $f_n\in\overline{H}(n,\alpha)$ if and only if

$$\sum_{m=1}^{\infty}m^n\left[(m-\alpha)a_m+(m+\alpha)b_m\right]\leq 2(1-\alpha). \tag{10}$$

Proof. Since $\overline{H}(n,\alpha)\subset H(n,\alpha)$, we only need to prove the "only if" part of the theorem. To this end, for functions f_n of the form (4), we notice that the condition $\Re[(D^{n+1}f_n(z))/(D^nf_n(z))]\geq\alpha$ is equivalent to

$$\Re\left\{\frac{(1-\alpha)z-\sum_{m=2}^{\infty}m^n(m-\alpha)a_mz^m-(-1)^{2n}\sum_{m=1}^{\infty}m^n(m+\alpha)b_m\bar{z}^m}{z-\sum_{m=2}^{\infty}m^na_mz^m+(-1)^{2n}\sum_{m=1}^{\infty}m^nb_m\bar{z}^m}\right\}\geq 0. \tag{11}$$

The above required condition (11) must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\frac{1-\alpha-\sum_{m=2}^{\infty}m^n(m-\alpha)a_mr^{m-1}-\sum_{m=1}^{\infty}m^n(m+\alpha)b_mr^{m-1}}{1-\sum_{m=2}^{\infty}m^na_mr^{m-1}+\sum_{m=1}^{\infty}m^nb_mr^{m-1}}\geq 0. \tag{12}$$

If the condition (10) does not hold, then the numerator in (12) is negative for r sufficiently close to 1. Hence there exist $z_0 = r_0$ in (0,1) for which the quotient in (12) is negative. This contradicts the required condition for $f_n \in \overline{H}(n, \alpha)$ and so the proof is complete.

Next we determine the extreme points of closed convex hulls of $\overline{H}(n, \alpha)$ denoted by *clco* $\overline{H}(n, \alpha)$.

Theorem 3. Let f_n be given by (4). Then $f_n \in \overline{H}(n, \alpha)$ if and only if

$$f_n(z) = \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_{n_m}(z)) \tag{13}$$

where $h_1(z) = z$, $h_m(z) = z - \frac{1-\alpha}{m^n(m-\alpha)} z^m$ $(m = 2, 3, ...)$, $g_{n_m}(z) = z + (-1)^n \frac{1-\alpha}{m^n(m+\alpha)} \bar{z}^m$ $(m = 1, 2, 3, ...,$ $\sum_{m=1}^{\infty}(X_m + Y_m) = 1$, $X_m \geq 0$, $Y_m \geq 0$. In particular, the extreme points of $\overline{H}(n, \alpha)$ are $\{h_m\}$ and $\{g_{n_m}\}$.

Proof. For functions f_n of the form (13) we have

$$\begin{aligned} f_n(z) &= \sum_{m=1}^{\infty} (X_m h_m(z) + Y_m g_{n_m}(z)) \\ &= \sum_{m=1}^{\infty} (x_m + Y_m) z - \sum_{m=2}^{\infty} \frac{1-\alpha}{m^n(m-\alpha)} X_m z^m + (-1)^n \sum_{m=1}^{\infty} \frac{1-\alpha}{m^n(m+\alpha)} Y_m \bar{z}^m. \end{aligned}$$

Then

$$\sum_{m=2}^{\infty} \frac{m^n(m-\alpha)}{1-\alpha} a_m + \sum_{m=1}^{\infty} \frac{m^n(m+\alpha)}{1-\alpha} b_m = \sum_{m=2}^{\infty} X_m + \sum_{m=1}^{\infty} Y_m = 1 - X_1 \leq 1$$

and so $f_n \in$ *clco* $\overline{H}(n, \alpha)$.

Conversely, suppose that $f_n \in$ *clco* $\overline{H}(n, \alpha)$. Setting

$$X_m = \frac{m^n(m-\alpha)}{1-\alpha} a_m; \ (m = 2, 3, ...), \quad Y_m = \frac{m^n(m+\alpha)}{1-\alpha} b_m; \ (m = 1, 2, 3, ...)$$

where $\sum_{m=1}^{\infty}(X_m + Y_m) = 1$ we obtain $f_n(z) = \sum_{m=1}^{\infty}(X_m h_m(z) + Y_m g_{n_m}(z))$ as required.

The following theorem gives the distortion bounds for functions in $\overline{H}(n, \alpha)$ which yields a covering result for this class.

Theorem 4. Let $f_n \in \overline{H}(n, \alpha)$. Then for $|z| = r < 1$ we have

$$(1 - b_1) r - \frac{1}{2^n} \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha} b_1 \right) r^2 \leq |f_n(z)| \leq (1 + b_1) r + \frac{1}{2^n} \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha} b_1 \right) r^2.$$

Proof. We only prove the right hand inequality. The proof for the left hand inequality is similar and will be omitted. Let $f_n \in \overline{H}(n, \alpha)$. Taking the absolute value of f_n we obtain

$$\begin{aligned} |f_n(z)| &\leq (1 + b_1) r + \sum_{m=2}^{\infty} (a_m + b_m) r^m \\ &\leq (1 + b_1) r + \sum_{m=2}^{\infty} (a_m + b_m) r^2 \\ &\leq (1 + b_1) r + \frac{1-\alpha}{2^n(2-\alpha)} \sum_{m=2}^{\infty} \left(\frac{2^n(2-\alpha)}{1-\alpha} a_m + \frac{2^n(2-\alpha)}{1-\alpha} b_m \right) r^2 \\ &\leq (1 + b_1) r + \frac{1-\alpha}{2^n(2-\alpha)} \left(1 - \frac{1+\alpha}{1-\alpha} b_1 \right) r^2 \\ &= (1 + b_1) r + \frac{1}{2^n} \left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha} b_1 \right) r^2. \end{aligned}$$

The following covering result follows from the left hand inequality in Theorem 4.

Corollary. Let f_n of the form (4) be so that $f_n \in \overline{H}(n,\alpha)$. Then

$$\left\{\omega : |\omega| < \frac{2^{n+1}-1-(2^n-1)\alpha}{2^n(2-\alpha)} - \frac{2^{n+1}-1-(2^n-1)\alpha}{2^n(2+\alpha)}b_1\right\} \subset f_n(U).$$

Remark. For n=1 and n=0 the above covering result coincides with those covering results given in [3] and [4], respectively.

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form $f_n(z) = z - \sum_{m=2}^{\infty} a_m z^m + (-1)^n \sum_{m=1}^{\infty} b_m \bar{z}^m$ and $F_n(z) = z - \sum_{m=2}^{\infty} A_m z^m + (-1)^n \sum_{m=1}^{\infty} B_m \bar{z}^m$ we define the convolution of f_n and F_n as

$$(f_n * F_n)(z) = f_n(z) * F_n(z) = z - \sum_{m=2}^{\infty} a_m A_m z^m + (-1)^n \sum_{m=1}^{\infty} b_m B_m \bar{z}^m. \tag{14}$$

Theorem 5. For $0 \le \beta \le \alpha < 1$, let $f_n \in \overline{H}(n,\alpha)$ and $F_n \in \overline{H}(n,\beta)$. Then the convolution $f_n * F_n \in \overline{H}(n,\alpha) \subset \overline{H}(n,\beta)$.

Proof. For f_n and F_n as in Theorem 5, write $f_n(z) = z - \sum_{m=2}^{\infty} a_m z^m + (-1)^n \sum_{m=1}^{\infty} b_m \bar{z}^m$ and $F_n(z) = z - \sum_{m=2}^{\infty} A_m z^m + (-1)^n \sum_{m=1}^{\infty} B_m \bar{z}^m$. Then the convolution $f_n * F_n$ is given by (14). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given in Theorem 2. For $F_n \in \overline{H}(n,\beta)$ we note that $|A_m| \le 1$ and $|B_m| \le 1$. Now, for the convolution function $f_n * F_n$ we obtain

$$\begin{aligned}\sum_{m=2}^{\infty} \frac{m^n(m-\beta)}{1-\beta} a_m A_m + \sum_{m=1}^{\infty} \frac{m^n(m+\beta)}{1-\beta} b_m B_m &\le \sum_{m=2}^{\infty} \frac{m^n(m-\beta)}{1-\beta} a_m + \sum_{m=1}^{\infty} \frac{m^n(m+\beta)}{1-\beta} b_m \\ &\le \sum_{m=2}^{\infty} \frac{m^n(m-\alpha)}{1-\alpha} a_m + \sum_{m=1}^{\infty} \frac{m^n(m+\alpha)}{1-\alpha} b_m \\ &\le 1\end{aligned}$$

since $0 \le \beta \le \alpha < 1$ and $f_n \in \overline{H}(n,\alpha)$. Therefore $f_n * F_n \in \overline{H}(n,\alpha) \subset \overline{H}(n,\beta)$.

Finally, we discuss the convex combinations of the class $\overline{H}(n,\alpha)$.

Theorem 6. The family $\overline{H}(n,\alpha)$ is closed under convex combination.

Proof. For $i = 1, 2, \ldots$, suppose that $f_{n_i} \in \overline{H}(n,\alpha)$ where

$$f_{n_i}(z) = z - \sum_{m=2}^{\infty} a_{i_m} z^m + (-1)^n \sum_{m=1}^{\infty} b_{i_m} \bar{z}^m.$$

Then, by Theorem 2,

$$\sum_{m=2}^{\infty} \frac{m^n(m-\alpha)}{1-\alpha} a_{i_m} + \sum_{m=1}^{\infty} \frac{m^n(m+\alpha)}{1-\alpha} b_{i_m} \le 1. \tag{15}$$

For $\sum_{m=1}^{\infty} t_i = 1,\ 0 \le t_i \le 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{m=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i_m}\right) z^m + (-1)^n \sum_{m=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{i_m}\right) \bar{z}^m.$$

Then, by (15),

$\sum_{m=2}^{\infty} \frac{m^n(m-\alpha)}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i a_{i_m}\right) + \sum_{m=2}^{\infty} \frac{m^n(m+\alpha)}{1-\alpha} \left(\sum_{i=1}^{\infty} t_i b_{i_m}\right)$

$= \sum_{i=1}^{\infty} t_i \left(\sum_{m=2}^{\infty} \frac{m^n(m-\alpha)}{1-\alpha} a_{i_m} + \sum_{m=2}^{\infty} \frac{m^n(m+\alpha)}{1-\alpha} b_{i_m}\right) \le \sum_{i=1}^{\infty} t_i = 1$

and therefore $\sum_{i=1}^{\infty} t_i f_{n_i} \in \overline{H}(n,\alpha)$.

References

1. Y. Avci and E. Zlotkiewicz, *On harmonic univalent mappings*, Ann. Univ. Marie Curie-Sklodowska Sect. A **44** (1991), 1-7.
2. J. Clunie and T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A.I Math. **9** (1984), 3-25.
3. J. M. Jahangiri, *Coefficient bounds and univalence criteria for harmonic functions with negative coefficients*, Ann. Univ. Marie Curie-Sklodowska Sect. A **52** (1998), 57-66.
4. J. M. Jahangiri, *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl. **235** (1999), 470-477.
5. G. S. Salagean, *Subclasses of univalent functions*, Complex Analysis - Fifth Romanian Finish Seminar, Bucharest, **1** (1981), 362-372.
6. H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl. **220** (1998), 283-289.
7. H. Silverman and E. M. Silvia, *Subclasses of harmonic univalent functions*, New Zealand J. Math. **28** (1999), 275-284.