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WEAKLY STARLIKE LOGHARMONIC MAPPINGS

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Abstract. We consider the class of all weakly starlike p-valent logharmonic mappings. A complete characterization of these mappings is given.

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1. Introduction

Let $H(U)$ be the linear space of all analytic functions defined in the unit disc $U = \{z = x + iy;\ |z| < 1\}$ and let B be the set of all functions $a \in H(U)$ such that $|a(z)| < 1$ for all $z \in U$. We say a function f is logharmonic if f is a solution of the non-linear elliptic partial differential equation

$$\overline{f_{\bar{z}}} = (a\overline{f}/f)f_z;\ a \in B. \tag{1.1}$$

Observe that nonconstant logharmonic mappings are open and orienation preserving on U. If f has exactly p zeros $\{z_k\}_{k=1}^{p}$ in U (inclusive multiplicity), then f admits a global representation given by

$$f(z) = [\prod_{k=1}^{p}(z - z_k)|z - z_k|^{2\beta_k}]h(z)\overline{g(z)}, \tag{1.2}$$

where

i) $\beta_k = \overline{a(z_k)}\frac{(1+a(z_k))}{(1-|a(z_k)|^2)}$,

ii) h and g are analytic in U, such that $g(z_k) = 1$ and $h(z_k) \neq 0$.

If f is a univalent logharmonic mapping on U, then either $0 \notin f(U)$ and $\log\ f$ is univalent and harmonic on U, or, $f(z_0) = 0$, for a fixed $z_0 \in U$. In the latter

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case, the function $F(\zeta) = \log\ f(e^{\zeta})$ is univalent and harmonic on the half-plane $\{\zeta;\ \Re\,\zeta < 0\}$ and the representation (1.2) becomes

$$f(z) = (z - z_0)|z - z_0|^{2\beta} h(z)\overline{g(z)}.$$

If $f(U)$ is a starlike domain, then F is closely related with the theory of minimal surfaces over domains of the form $-\infty < u < u_0(v), u_0(v + 2\pi) = u_0(v)$. For details see e.g. [A1],A2], [AB], [AH1], [AH2], [AH3] , [AH4] and [AH5].

Definition. *let f be a nonconstant complex-valued function defined in U. We say that f is weakly starlike if*

$$\lim_{r\to 1} \inf[\min_{|z|=r} \frac{\partial \arg f(re^{it}, z_k)}{\partial t}] \geq 0. \tag{1.3}$$

Generalizations from univalent starlike analytic functions to weakly starlike analytic functions have been introduced by Hummel(see [H]). He has shown:

Theorem A. *ϕ is a weakly starlike analytic function with exactly p zeros in U if and only if there exists a normalized starlike univalent analytic ϕ such that*

$$\phi(z) = (\phi_0(z))^p \prod_{k=1}^{p} \psi_0(z, z_k)$$

where $z_1, z_2, \ldots.., z_p$ are the zeros of ϕ, and

$$\psi_0(z, z_k) = \frac{(z - z_k)(1 - \overline{z_k} z)}{z}.$$

Note that $\psi_0(z, z_k)$ is a univaelnt meromorphic fuction which satisfies (1.3) and maps U onto the exterior of the line segment whose end points are $(1 \pm |z_k|^2)^2$.

In this paper, we shall consider the class of weakly starlike p-valent logharmonic mappings. In Section 2 we shall include a characterization theorem for class of weakly starlike p-valent logharmonic mappings and also, a representation theorem for these mappings is given which is the analouge of Theorem A.

2. Representation theorems .

We shall need the following lemma:

Lemma 2.1. *Let f be a weakly starlike and logharmonic mapping with respect to a given $a \in B$. Then f is p-valent if and only if f has exactly p zeros (inclusive multiplicity).*

Proof. First, we suppose that f has p zeros (inclusive multiplicity). In order to show that f is p-valent, we shall adapt the method of Hummel (see [H]). Let $\triangle_r = \{\theta |\ \frac{\partial \arg\ f(re^{i\theta})}{\partial \theta} < 0\}$, where $0 < r < 1$. Given any $\epsilon > 0$, there exists a $\rho \in (0, 1)$ such that for all r with $\rho < r < 1$ we have

$$\int_{\triangle_r} \frac{\partial \arg\ f(re^{i\theta})}{\partial \theta} d\theta > -2\pi\epsilon. \tag{2.1}$$

Now we suppose that f is not p-valent then there is some $w_0(\neq 0)$ such that the value w_0 is taken on $p+1$ or more times by $f(z)$. Then there exists N points $z_1, ...z_N$, $N \geq p+1$, such that $g(z_i) = w_0$ for each i. Some of the points may be multiple. Since f is an open mapping there exists a disk W with center w_0, not containing the origin, and a real number r_0, $0 < r_0 < 1$, such that every point $w \in W$ is taken on N or more times by $f(z)$ in $\{z| \, |z| < r_0\}$. Let α be the angle subtended by W at the origin, then $0 < \alpha < \pi$. Let $L_\phi = \{w| \, \arg \, w = \phi\}$ be any ray from the origin which intersects W. If r is such that $r_0 \leq r \leq 1$, then $f(z)$ assumes every value in W at least N times for $|z| < r$.

Let C_r be the circle $\{|z| = r\}$ and let Γ_r be the image curve given by $f(re^{i\theta})$, $0 \leq \theta \leq 2\pi$. Now $f(z)$ has exactly p zeros in $|z| < r$ (we may assume that r_0 is large enough to insure this). Using the argument principle, we deduce that the winding number of the curve Γ_r (w.r.to the origin) is $n(\Gamma_r, 0) = p$.

On the other hand, if $w_1 \in W \bigcap L_\phi$ such that $w_1 \notin \Gamma_r$, the winding number of Γ_r (w.r.to the w_1) must be greater than N. It follows that Γ_r crosses the line segment joining 0 and w_1 in the direction of decreasing argument at least once. But since Γ_r is an analytic curve, there are only finitely many intervals $[\theta_i, \theta_i'] \subset [0, 2\pi]$ such that $\arg f(re^{it})$ is monotone decreasing in $[\theta_i, \theta_i']$. The union of these intervals makes up the closure of the set $\triangle_r$ above. Since the angle subtended by W is α, we have

$$\int_{\triangle_r} \frac{\partial \arg \, f(re^{i\theta})}{\partial\theta} d\theta < -\alpha$$

which contradicts (2.1).

Conversely, if f is p-valent, then f has at most p zeros. If f has m zeros, $m < p$, then f is m-valent.This completes the proof.

Let f be a logharmonic mapping with respect to a given $a \in B$ and let $z_k \in U$, $1 \leq k \leq p$, be the zeros of f. Using Lemma 2.2 in [AB], it follows that f admits the represenation

$$f(z) = [\prod_{k=1}^{p} (\frac{z - z_k}{1 - \overline{z_k}}) |\frac{z - z_k}{1 - \overline{z_k}}|^{2\beta_k}] h(z).\overline{g(z)} \tag{2.2}$$

where $\beta_k = \overline{a(z_k)} \frac{1 + a(z_k)}{1 - |a(z_k)|^2}$. The functions h and g are nonvanishing and analytic in U and $g(0) = 1$.

Now we associate to f the mapping $\phi(z) = [\prod_{k=1}^{p} (\frac{z - z_k}{1 - \overline{z_k}})] \frac{h(z)}{g(z)}$.

The following theorem gives a relation between f and ϕ:

Theorem 2.2. *a) Let f be a weakly starlike, p-valent and logharmonic mapping with respect to a given $a \in B$. Then ϕ is weakly starlike, p-valent and analytic in U.*

b) Let ϕ be a weakly starlike, p-valent and analytic function. For any given $a \in B$ there are h and g in $H(U)$ uniquely determined such that

i) $0 \notin h.g(U)$; $g(0) = 1$

ii) $\phi(z) = [\prod_{k=1}^{p} (\frac{z - z_k}{1 - \overline{z_k}})] h/g$

iii) f defined in (2.2) is a solution of (1.1) with respect to the given a and is weakly starlike p-valent mapping.

Proof. a) Let f defined in (2.2) be weakly starlike and logharmonic with respect to a given $a \in B$. Then ϕ has exactly p zeros and is weakly starlike analytic since

$$\arg\ f(re^{it}) = \arg\ \phi(re^{it}) + 2\sum_{k=1}^{p} \Im(\beta_k) \log|\frac{z-z_k}{1-\overline{z_k}}|. \tag{2.3}$$

By proposition 6 in [H], it follows that ϕ is p-valent.

b) Let $a \in B$ and let ϕ be a weakly starlike p-valent analytic function.

Put

$$g(z) = exp\ \int_0^z [\frac{a(s)}{(1-a(s)}\frac{\phi'(s)}{\phi(s)} + \sum_{k=1}^{p} \nu(s, z_k)]ds,$$

$$h(z) = \phi(z)g(z)/\prod_{k=1}^{p}[\frac{z-z_k}{1-\overline{z_k}}]$$

and

$$f(z) = f(z) = [\prod_{k=1}^{p}(\frac{z-z_k}{1-\overline{z_k}})|\frac{z-z_k}{1-\overline{z_k}}|^{2\beta_k}]h(z).\overline{g(z)}$$

$$= \phi(z)[\prod_{k=1}^{p}|\frac{z-z_k}{1-\overline{z_k}}|^{2\beta_k}]|g(z)|^2,$$

where $\nu(z, z_k) = \frac{(a(z\beta_k - \overline{\beta_k})(1-|z_k|^2)}{(1-a(z))(z-z_k)(1-\overline{z_k}z)}$ and $\beta_k = \overline{a(z_k)}\frac{1+a(z_k)}{1-|a(z_k)|^2}$. Then h and g are nonvanishing analytic functions defined on U, normalized by $g(0) = 1$, and f is a weakly starlike logharmonic mapping. Moreover, f has exactly p zeros and by Lemma 2.1, f is p-valent.

Next we introduce a class of fuctions which appear naturally in the represenation theorem.

Definition. *Let z_k be a point in U, and let $a \in B$ be given. We define*

$$\psi(z, z_k) = \frac{\psi_0(z, z_k)|\frac{z-z_k}{1-\overline{z_k}}|^{2\beta_k}}{|z|^{2\beta_0}} \exp\ 2\Re\,\sigma(z, z_k), \tag{2.4}$$

where

a) $\beta_k = \overline{a(z_k)}\frac{(1+a(z_k))}{(1-|a(z_k)|^2)}$,

b) $\beta_0 = \overline{a(0)}\frac{(1+a(0))}{(1-|a(0)|^2)}$,

c) $\psi_0(z, z_k) = \frac{(z-z_k)(1-\overline{z_k}z}{z}$,

d) $\sigma(z, z_k) = \int_0^z [\frac{(a(s)\psi_0'}{(1-a(s)\psi_0(s))} - \frac{a(s)\beta_0 - \overline{\beta_0}}{s(1-a(s))} + \nu(s, z_k)]$,

where

$\nu(s, z_k) = \frac{(a(z)\beta_k - \overline{\beta_k})(1-|z_k|^2)}{(1-a(z)(z-z_k)(1-\overline{z_k}z)}$.

As a direct consequence of the represenation (2.4), we have the following lemma:

Lemma 2.3. *For any given $z_k \in U$ and $a \in B$, the function $\psi(z, z_k)$ is continuous, real and positive on $|z| = 1$. Furthermore, $\psi(z, z_k)$ is a univalent meromorphic function which satisfies (1.1).*

Proof. Using (2.4), we have

$$\arg \psi(z, z_k) = \arg \psi_0(z, z_k) + 2\Im(\beta_k) \log |\frac{z - z_k}{1 - \overline{z_k} z}| - 2\Im(\beta_0) \log |z|$$

and since ψ_0 maps U conformally onto the exterior of a line segment joining the points $(1 \pm |z_k|^2)$, we deduce that ψ maps U also onto the exterior of a line segment I joining two points lying on the positive real axis. Moreover, we have

$$\lim_{r\to 1} inf[\min_{|z|=r} \frac{\partial \arg \psi(re^{it}, z_k)}{\partial t}] = \lim_{r\to 1} inf[\min_{|z|=r} \frac{\partial \arg \psi_0(re^{it}, z_k)}{\partial t}] = 0.$$

Thus, ψ satisfies (2.1) and direct calculations shows that ψ satisfies (1.1) with respect to the same a. It is left to show that ψ is univalent. Let C_r be the circle $|z| = r$ and let Γ_r be the image curve given by $\psi(re^{it}, z_k)$, $0 \le \theta \le 2\pi$. First let $w^\circ \in \mathbb{C} \setminus \mathbb{R}^+$. If we choose r sufficiently close to one, then the winding number of the curve Γ_r with respect to w° is $n(\Gamma_r, w^\circ) = 0$. But $n(\Gamma_r, w^\circ) = N_r(\psi - w^\circ) - P_r(\psi)$, where $N_r(\psi - w^\circ)$ is the number of zeros of $\psi - w^\circ$ in $|z| \le r$ and $P_r(\psi)$ is the number of poles of ψ in $|z| \le r$. Since $P_r(\psi) = 1$, we conclude that $N_r(\psi - w^\circ) = 1$. Next, if $w^\circ \in \mathbb{R}^+ \setminus I$, then there exists a sequence w_n in $\mathbb{C} \setminus \mathbb{R}^+$ such that w_n converges to w°. Now, there is an $r \in (0, 1)$ sufficiently close to 1 such that $n(\Gamma_r, w_n) = 0$ for all $n \in \mathbf{N}$. Since $\psi - w_n$ converges uniformly to $\psi - w^\circ$ on C_r, we get that $n(\gamma_r, w^\circ) = 0$.

We are now ready to state and prove the main result of this paper:

Theorem 2.4 (Representation Theorem). *Let f be a logharmonic function with respect to a given $a \in B$. Then a function f defined as in (2.2) is weakly starlike if and only if*

$$f(z) = (f_0(z))^p \prod_{k=1}^{p} \psi(z, z_k)$$

where $z_1, z_2, ..., z_p$ are in U and f_0 is a starlike univalent logharmonic mapping with respect to the same a.

Proof. Let $a \in B$ be given and let f be logharmonic with respect to a such that $f(z) = (f_0(z))^p \prod_{k=1}^{p} \psi(z, z_k)$, where $z_1, z_2, ..., z_p \in U$ and let f_0 be a starlike univalent logharmonic mapping. Then f has exactly p zeros and we have

$$\lim_{r\to 1} \inf[\min_{|z|=r} \frac{\partial \arg f(re^{it}, z_k)}{\partial t}] =$$

$$p \lim_{r\to 1} \inf[\min_{|z|=r} \frac{\partial \arg f_0(re^{it}, z_k)}{\partial t}] + \lim_{r\to 1} \inf[\min_{|z|=r} \frac{\partial \arg \psi(re^{it}, z_k)}{\partial t}] \ge 0.$$

Using Theorem 2.2, it follows that f is weakly starlike and p-valent and has the representation (2.2).

Conversely, let f be a logharmonic mapping with respect to a given $a \in B$. We suppose that f has the represenation (2.2) and is weakly starlike p-valent. By Theorem 2.2, we conclude that

$$\phi(z) = [\prod_{k=1}^{p} \frac{z - z_k}{1 - \overline{z_k} z}] \frac{h(z)}{g(z)}$$

is a weakly starlike p-valent analytic function in U. Theorem A implies that there exists a a starlike univalent analytic function ϕ_0 such that $\phi(z) = (\phi_0(z))^p \prod_{k=1}^{p} \psi_0(z, z_k)$ and therefore, $\phi_0(z) = z(\frac{h(z)}{g(z)\prod_{k=1}^{p}(1-\overline{z_k}z)^2})^{\frac{1}{p}}$. From Theorem 2.1 in [AH2] (with $\alpha = 0$), it follows that $f_0(z) = \phi_0(z)|z|^{2\beta_0}|g_1(z)|^2$ is a starlike univalent logharmonic, where $g_1(z) = \exp\ \int_0^z [\frac{a(s)\phi_0'(s)}{(1-a(s))\phi_0(s)} + \frac{a(s)\beta_0 - \overline{\beta_0}}{s(1-a(s))}] ds$ and $\beta_0 = \frac{\overline{a(0)}(1+a(0))}{(1-|a(0)|^2}$. Put

$$f_1(z) = (f_0(z))^p \prod_{k=1}^{p} \psi(z, z_k).$$

It follows that f_1 is a solution of (1.1) with respect to a and, furthermore, f_1 is a weakly starlike p-valent mapping. Straightforward calculations show that ϕ corresponds in Theorem 2.2 to the mapping f_1 and therefore, we have $f = f_1$.

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