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Geometry and Convergence of some third-order methods

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Abstract

We study the global convergence of two third-order methods by using their geometric interpretation. A comparison of these methods with the classic ones is discussed also.

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Key Words and Phrases. Non-linear equations, third order iterative methods, global convergence

1 Introduction

We consider the basic problem, finding the zeros of a given function f, that is, finding the values α for which $f(\alpha) = 0$. Determining the zeros of a function has attracted the attention of pure and applied mathematicians for centuries. Solving equations numerically is a classical problem. General problems may be formulated in terms of finding zeros. The roots of a nonlinear equation $f(x) = 0$ cannot in general be expressed in closed form. Thus, in order to solve nonlinear equations, we have to use approximate methods.

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One of the most important techniques to study these equations is the use of iterative processes, starting from an initial approximation x_0, called pivot, successive approaches (until some predetermined convergence criterion is satisfied) x_i are computed , $i = 1, 2, \ldots$, with the help of certain iteration function $\Phi : E \to E$:

$$x_{i+1} := \Phi(x_i), \quad i = 0, 1, 2 \ldots \tag{1}$$

One of the most studied methods is Newton method:

$$x_{k+1} = \Phi(x_k) := x_k - \frac{f(x_k)}{f'(x_k)}. \tag{2}$$

The advance of computational techniques has allowed the development of some more complicated iterative methods in order to obtain greater order of convergence. When the evaluation of first and second derivatives is not very complicated, third order methods may be useful.

Definition 1 *A method with iteration function Φ is third order if $\Phi(\alpha) = \alpha$, $\Phi'(\alpha) = \Phi''(\alpha) = 0$ and $\Phi'''(\alpha) \neq 0$.*

On the other hand, unconstrained optimization deals with the problem of minimizing or maximizing a function in absence of any restrictions. Even though most practical optimization problems have side restrictions that must be satisfied, the study of techniques for unconstrained optimization is important for several reasons. In particular, many algorithms solve a constrained problem by converting it into a sequence of unconstrained problems see [3]. Usually Newton method is used, see [17]. But in some cases, the functions do not verify the hypothesis of the theorems, and the convergence is not clear. In order to model this type of problems we consider the simple equation $f(x) = \sqrt{|x|} = 0$. In this case $f(x)$ is not derivable and we can not apply the theory. Moreover, the function does not change its sign and we cannot apply typical first order methods. If we consider Newton type methods we will not have convergence, since in this case $x_{k+1} = -x_k, \forall k$. Thus, we consider third order methods. In table 4 and figure 4 (see section 4) we analize third order methods. We obtain linear converge. Let us remark that Chebyshev method (third order method) is exact for every initial data, but in general an order reduction takes place when we consider these type of equations.

In this paper, we present two third-order methods. As we shall see later on, these methods are based on an interpolation by an exponential and a logarithmic function, respectively. We derive sufficient conditions for their convergence. Finally, we compare these methods with the classical ones, both from an analytical and a numerical point of view.

2 Geometric and convegence of some iterative methods

The following well-known theorem, giving enough conditions for the convergence of Newton method, follows easily from its geometric interpretation.

Theorem 1 *Let f'' be continuous on an interval J, containing a root x^* of f; let $f' \neq 0$ and $f'' \geq 0$ or $f'' \leq 0$ on J. Then Newton method converges monotonically to x^* from any point $x_0 \in J$ such that $f(x_0)f''(x_0) \geq 0$.*

Melman (see [12]) proposes similar results for one of the most famous third order method, the Halley method

$$x_{k+1} = x_k - \frac{2f(x_k)f'(x_k)}{2f'^2(x_k) - f(x_k)f''(x_k)}. \tag{3}$$

Theorem 2 *Let f''' be continuous, $f' \neq 0$, and $((\eta f')^{-\frac{1}{2}})'' \geq 0$ on an interval J containing a root x^* of f with $\eta = sgn(f')$. Then Halley method converges monotonically to x^* from any point of the interval.*

Halley method uses a rational function $\frac{x+a}{bx+c}$, which interpolates the function f and its two first derivatives at the point x_k. And the zero of the rational function is the new approach.

Now, we introduce new third order methods (see definition 1), being based on same geometric arguments, previously mentioned. The idea is to consider new interpolation functions to obtain more alternatives.

2.1 Exponential method

It is considered the exponential function

$$g(x) = a + ke^{bx} \tag{4}$$

such that:

$$f(\overline{x}) = g(\overline{x}), \tag{5}$$

$$f'(\overline{x}) = g'(\overline{x}), \tag{6}$$

$$f''(\overline{x}) = g''(\overline{x}). \tag{7}$$

This third order interpolation leads to the iterative method (see [4])

$$x_{k+1} = x_k + \frac{f'(x_k)}{f''(x_k)} \log(1 - \frac{f(x_k)f''(x_k)}{f'(x_k)^2}). \tag{8}$$

As for the classical methods, we can obtain the following covergence theorem.

Theorem 3 *Let f''' be continuous on an interval J containing a root x^* of f, let $f' \neq 0$, and sgn $(f')\left(\log|f'|\right)'' \leq 0$ in J. Then the exponential method converges monotonically to x^* from any point of the interval.*

Proof

We suppose $f' > 0$ (for $f' < 0$ similar).

As $f' > 0$ then for hypothesis $\left(\log|f'(x)|\right)'' \leq 0$, therefore

$$kb \exp(bx) \geq f'(x) > 0.$$

For $x \geq \overline{x}$

$$\int_{\overline{x}}^{x} kb \exp(bt)\, dt \geq \int_{\overline{x}}^{x} f'(t)\, dt$$

so

$$a + k\exp(bx) - a - k\exp(b\overline{x}) \geq f(x) - f(\overline{x})$$

then

$$a + k\exp(bx) \geq f(x)$$

Therefore, if we begin from a point on the left of x^*, we will obtain a monotonic increasing sequence, bounded from above by x^*, then it converges at the limit $\gamma \leq x^*$. So, for the construction of the method and the continuity of f the convergence is obtained, $\gamma = x^*$. If we begin from a point on the right of root, we will obtain $a + k\exp(bx) \leq f(x)$ and the convergence will be monotonic from the right. □

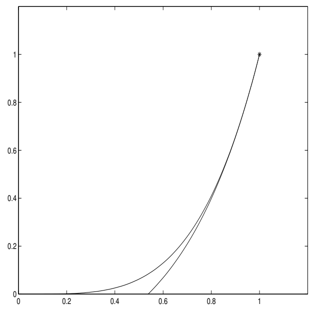

Figure 1: ' function, exponential interpolation, $x_0 = 1$'

2.2 Logarithmic method

We consider the logarithmic function

$$g(x) = a + k\log(x+b)$$

and the conditions (5), (6) and (7).

The iteration formula, based on logarithmic interpolation (see [4]) is given by

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}(\exp(\frac{f(x_k)f''(x_k)}{f'(x_k)^2}) - 1). \tag{9}$$

This leads to the following theorem.

Theorem 4 *Let f''' be continuous on an interval J containing a root x^* of f, let $f' \neq 0$, and $(\frac{1}{f'})(\frac{1}{f'})'' \geq 0$ (if and only if $2(f'')^2 \geq (f'''f')$) on J. Then the logarithmic method converges monotonically to x^* from any point of the interval .*

Proof

We suppose $f' > 0$ (for $f' < 0$ similar).

As $f' > 0$ then for hypothesis $\left(\frac{1}{f'}\right)'' \geq 0$, and therefore $\frac{k}{x+b} \geq f'(x) > 0$. For $x \geq \overline{x}$

$$\int_{\overline{x}}^{x} \frac{k}{t+b}dt \geq \int_{\overline{x}}^{x} f'(t)\,dt$$

so

$$a + k\log(x+b) - a - k\log(\overline{x}+b) \geq f(x) - f(\overline{x})$$

then

$$a + k\log(x+b) \geq f(x).$$

Therefore, if we begin from a point on the left of x^*, we will obtain a monotonic increasing sequence, bounded from above by x^*, then it converges to a limit $\gamma \leq x^*$. So, for the construction of the methods and the continuity of f the convergence is obtained, $\gamma = x^*$. If we begin from a point on the right of the root, we will obtain

$$a + k \log (x + b) \leq f(x)$$

and the convergence will be monotonic from the right. □

Notice that these methods can be written in the form

$$x_{k+1} = x_k + (1 + \theta_k + O(\theta_k^2)) \frac{f(x_k)}{f'(x_k)}$$

where $\theta_k := \frac{1}{2} \frac{f(x_k) f''(x_k)}{(f'(x_k))^2}$ see [1].

3 A comparison of sufficient conditions for global convergence

In the situation where the sufficient conditions of all four theorems hold, Newton method will always produce a smaller step than third-order methods. The reason for this is that when f is convex (concave), the tangent to the function at the approximation point always lie below (above) those three approximations (see [12] and figure 3).

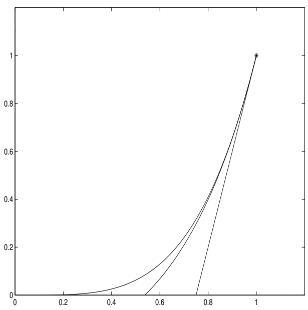

Figure 2: ‘ function, exponential interpolation, Newton interpolation (tangent), $x_0 = 1$ ’

In certain cases, it is possible to compare the third order methods. Considering, for instance, the case $f' > 0$ the conditions of the theorems (2), (3) and (4) are equivalent to:

$$\frac{3}{2}(f'')^2 \geq f'f''' \tag{10}$$

$$(f'')^2 \geq f'f''' \tag{11}$$

$$2(f'')^2 \geq f'f''' \tag{12}$$

respectively (see [2]).

These convergence conditions could also have been written in terms of the degree of logarithmic convexity of f' at x_0. The degree of logarithmic is a local functional defined by Garay and Hernández in [9]. If $g \in C^2(I)$, $x_0 \in I$, the degree of logarithmic convexity of g at x_0 is defined by

$$L_g(x_0) = \frac{g(x_0)g''(x_0)}{g'(x_0)^2}$$

Indeed, one can write the conditions as:

$$\frac{3}{2} \geq L_{f'}(t) \tag{13}$$

$$1 \geq L_{f'}(t) \tag{14}$$

$$2 \geq L_{f'}(t) \tag{15}$$

respectively.

Moreover, in [1], we can see the following interesting property:

Lemma 1 (Altman)

Let be f smooth enough such that $f(t) \geq 0$, $f' \neq 0$ in $t \in [0, t^*]$ and $f''' \geq 0$ in $t \in [0, t^*]$ and $f(t^*) = 0$. Then

$$L_f(t) \leq \frac{1}{2}. \tag{16}$$

The degree of logarithmic convexity is related to the speed of convergence and relation (16) is a typical hipothesis in convergence theorems, (for more details see [8],[9],[10]).

Remark 1 *It is possible to find (with similar ideas) a general global convergence theorem (we will assume $f' > 0$). If $g \in C^2$ such that conditions (5), (6) and (7) hold.*

Theorem 5 *Let f''' be continuous on an interval J containing a root x^* of f, let $f' > 0$, and $((f')^{-1})'' \cdot ((f')^{-1})' \geq 0$. Then the iterative method associated to g converges monotonically to x^* from any point of the interval.*

4 Numerical experiments and conclusions

In order to show the performance of the different iterative methods, we have tested them on the results obtained for the zeros of some equations.

In table 1 we study the equation $\sin(x) - x = 0$, for different initial data we compute the number of iterations (until the convergence criterion is satisfied), the exponential method needs less iterations. With the same equation, in table 2 we compute the error for different number of iterations, the best results are obtained with exponential method too.

In table 3 we consider $x^5 - 1 = 0$, in this case, $f(x)$ verifies the hypothesis of all the theorems. We obtain the same conclusions as before (the convergences are uniform).

Finally we consider in $[0, 1]$ the Cauchy problem:

$$\begin{aligned} y' &= -4 \cdot 10^4 \, y^3, \\ y(0) &= 1. \end{aligned} \tag{17}$$

We approximate the solution of the stiff problem (17) at $t = 1$ by applying implicit Euler method,

$$y_{n+1} = y_n + h \cdot f(t_{n+1}, y_{n+1}).$$

The different iterative processes are used for the non linear equation in the implicit method. We compute the number of iterations, in each step, until obtaining an error less than 10^{-4}. Exponential method needs less number of iterations ($j = jmax$) than the others, see table 5.

Remark 2 *Exponential method can decrease the error to* $3.00\ e - 04$ *with* $jmax = 15$ *(number of iterations implicit method=* 20*) and to* $9.29\ e - 06$ *with* $jmax = 6$ *(number of iterations implicit method=1000), but the accuracy is limited by the other methods.*

We have analyzed two new iterative methods. We have studied their convergence and we have tested their competitivity with respect the classical methods. Exponential method seemed to work very well in our preliminary numerical results. We have presented several reasons in order to use third order methods. The computational cost is not considered, since in 1-D all methods studied have similar properties. Nevertheless, 1-D problems are not trivial, since in some cases n-D problem are reduced to several 1-D problems based on local characteristic fields. In further papers we will establish these methods on Banach spaces, their convergence conditions and the computational evaluation.

Input Data	Exponential	Logarithmic	Halley
$x_0 = 0.001$	3	4	4
$x_0 = 0.01$	6	8	7
$x_0 = 0.1$	9	11	10
$x_0 = 0.3$	11	13	13
$x_0 = 0.5$	11	14	13
$x_0 = 0.9$	12	15	14
$x_0 = 1$	12	15	14
$x_0 = 2$	13	16	15

Table 1: Number of iterations untill to have an error $< 10^{-4}$, $f(x) = sin(x) - x$

Iterations	Exponential	Logarithmic	Halley
1	$4.50e-04$	$5.26e-04$	$4.99e-04$
2	$2.03e-04$	$2.76e-04$	$2.49e-04$
3	$9.15e-05$	$1.45e-04$	$1.24e-04$
4	$4.12e-05$	$7.66e-05$	$6.24e-05$
5	$1.85e-05$	$4.03e-05$	$3.12e-05$
6	$8.38e-06$	$2.12e-05$	$1.56e-05$
7	$3.77e-06$	$1.11e-05$	$4.81e-06$
8	$1.70e-06$	$5.87e-06$	$3.90e-06$
9	$7.67e-07$	$3.08e-06$	$1.95e-06$
10	$3.45e-07$	$1.62e-06$	$9.76e-07$

Table 2: Error, $x_0 = 0.001$, $f(x) = sin(x) - x$

Iterations	Exponential	Logarithmic	Halley
1	$5.51e-02$	$1.23e-01$	$1.00e-01$
2	$1.08e-04$	$4.59e-03$	$1.73e-03$
3	$8.50e-13$	$3.20e-07$	$1.04e-08$

Table 3: Error, $x_0 = 1.5$, $f(x) = x^5 - 1$

Iterations	Exponential	Logarithmic	Halley
1	$-3.86e-01$	$-2.64e-01$	$-3.33e-01$
2	$+1.49e-01$	$+6.98e-02$	$+1.11e-02$
3	$-5.76e-02$	$-1.84e-02$	$-3.70e-02$
4	$+2.22e-02$	$+4.87e-03$	$+1.23e-03$
5	$-8.60e-03$	$-1.28e-03$	$-4.11e-03$
6	$+3.32e-03$	$+3.40e-04$	$+1.37e-03$
7	$-1.28e-03$	$-8.99e-05$	$-4.57e-04$
10	$+7.39e-05$	$+1.65e-06$	$+1.69e-05$
20	$+5.47e-09$	$+3.94e-11$	$+2.86e-10$

Table 4: Iterations x_k, $x_0 = 1$, $f(x) = |x|^{\frac{1}{2}}$

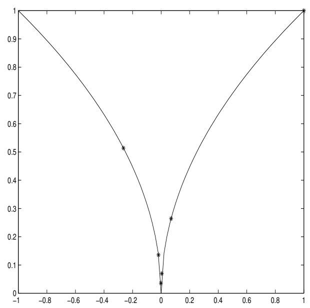

Figure 3: ‘ - ’ $\sqrt{|x|}$, ‘ * ’ log-iterations

Im.Met.	Expon.	Logarit.	Halley
20	$4.00e-04,\ j=12$	$4.00e-04,\ j=22$	$4.00e-04,\ j=25$
100	$7.89e-05,\ j=10$	$9.79e-05,\ j=20$	$9.79.e-04,\ j=25$
1000	$1.15e-05,\ j=5$	$1.51e-05,\ j=6$	$1.27e-05,\ j=10$

Table 5: Error and j =number iterations, stop when error$< 10^{-4}$

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