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DYNAMIC ANALYSIS OF OLIGOPSONY UNDER ADAPTIVE EXPECTATIONS

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Abstract. This paper analyzes the global stability of equilibrium for oligopsony facing perfectly competitive product price under adaptive expectations, assuming each firm's production function to be quadratic and the factor price function to be linear. The general stability conditions are simplified for some special cases, and a formal similarity is noted between the stability for oligopoly and that for oligopsony.

AMS Subject Classification Codes. 90

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1. Introduction

In spite of many papers on the exitence and stability of Cournot oligopoly and other oligopolies under various forms of firms' expectations, few economists have analyzed the existence and stability of equilibrium in oligopsony and oligopoly with oligopsony (oligopoly–oligopsony). Okuguchi (1998) and Szidarovszky and Okuguchi (1997) have examined the existence of a unique equilibrium for Cournot oligopoly–oligopsony. Chiarella and Okuguchi (1997) have analyzed the existence and stability of Cournot duopoly–duopsony equilibrium under the assumption of two identical firms, one factor of production and a linear inverse demand function for the product. Earlier, Okuguchi (1970) had analyzed the stability of the equilibrium of oligopoly with adaptive expectations and without product differentiation, while Okuguchi and Szidarovszky (1999) have investigated the stability of the equilibrium of oligopoly with multi-product firms with adaptive expectations, and given a comprehensive summary of the earlier works on oligopoly–oligopsony.

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In this paper we will introduce adaptive expectations into oligopoly facing perfect competition in product market and with several factors of production, and analyze the global stability of the oligopsony equilibrium. We will assume quadratic production functions and linear factor price functions to make our analysis technically tractable. We will derive a general stability condition, which is then interpreted for some special cases including Cournot oligopsony. We will find a remarkable similarity between the stability condition for Cournot oligopoly and that for Cournot oligopsony.

2. Model

Similarly to the model examined in Okuguchi (1998), consider an n-firm oligopsony where only a single product is produced using m factors of production. We assume perfectly competitive product market and an oligopsonistic factor market. The profit of firm i is given as

$$\Pi_i = pf_i(\mathbf{l}_i) - \mathbf{l}_i^T w(\mathbf{L}) \quad (i = 1, 2, \dots, n) \tag{1}$$

where p is the competitive product price, f_i is the production of firm i, $\mathbf{l}_i$ is its m-element production factor, w is the m-element factor price vector and $\mathbf{L}$ is a m-element vector of total production factor amounts used by the industry: $\mathbf{L} = \sum_{i=1}^{n} \mathbf{l}_i$. For the sake of simplicity assume that

$$\text{(A)} \qquad f_i(\mathbf{l}_i) = \mathbf{l}_i^T \mathbf{C}_i \mathbf{l}_i + \mathbf{c}_i^T \mathbf{l}_i + \gamma_i \text{ and } w(\mathbf{L}) = \mathbf{A}\mathbf{L} + \mathbf{a}, \tag{2}$$

where $\mathbf{c}_i$ and $\mathbf{a}$ are m-element vectors, $\mathbf{C}_i$ and $\mathbf{A}$ are $m \times m$ matrices, and γ_i is a scalar.

Assume that at the initial time period $t = 0$, each firm selects an initial $\mathbf{l}_i(0)$, and at each later time period $t \geq 1$, each firm first predicts the production factor usage of the rest of the industry adaptively:

$$\mathbf{s}_i^E(t) = \mathbf{M}_i \sum_{j \neq i} \mathbf{l}_j(t-1) + (\mathbf{I} - \mathbf{M}_i)\mathbf{s}_i^E(t-1), \tag{3}$$

where $\mathbf{M}_i$ is an $m \times m$ given constant matrix, usually $\alpha_i \mathbf{I}$ with some $\alpha_i \in (0, 1]$. Then the expected profit of firm i at time period t can be expressed as

$$\Pi_i(t) = p\big(\mathbf{l}_i^T \mathbf{C}_i \mathbf{l}_i + \mathbf{c}_i^T \mathbf{l}_i + \gamma_i\big) - \mathbf{l}_i^T \big(\mathbf{A}\mathbf{l}_i + \mathbf{A}\mathbf{s}_i^E(t) + \mathbf{a}\big). \tag{4}$$

Excluding corner optimum, the expected profit maximizing production factors of firm i at time period t are the solutions of equation

$$p\big((\mathbf{C}_i + \mathbf{C}_i^T)\mathbf{l}_i + \mathbf{c}_i\big) - (\mathbf{A} + \mathbf{A}^T)\mathbf{l}_i - \mathbf{A}\mathbf{s}_i^T(t) - \mathbf{a} = \mathbf{0}. \tag{5}$$

Here we assume that the second-order optimality condition holds, that is,

(B) matrix $p(\mathbf{C}_i + \mathbf{C}_i^T) - (\mathbf{A} + \mathbf{A}^T)$ is negative definite for all i.

Combining equations (3) and (5), a discrete dynamic system is obtained with state variables $\mathbf{l}_i(t)$ and $\mathbf{s}_i^E(t)$ $(i = 1, 2, \ldots, n)$. The state transition equation for variables $\mathbf{l}_i$ is given as follows:

(6)

$$\mathbf{l}_i(t) = -\left(\mathbf{A} + \mathbf{A}^T - p(\mathbf{C}_i + \mathbf{C}_i^T)\right)^{-1}\mathbf{A}\left(\mathbf{M}_i \sum_{j \neq i} \mathbf{l}_j(t-1) + (\mathbf{I} - \mathbf{M}_i)\mathbf{s}_i^E(t-1)\right)$$
$$+ \ \left(\mathbf{A} + \mathbf{A}^T - p(\mathbf{C}_i + \mathbf{C}_i^T)\right)^{-1}(p\mathbf{c}_i - \mathbf{a}).$$

The state transition equations for variables $\mathbf{s}_i^E$ are given by (3). The asymptotical stability of this system will be analyzed in the next section of the paper.

3. Stability Analysis

Introduce the notation

$$\mathbf{B} = \left(\mathbf{A} + \mathbf{A}^T - p(\mathbf{C}_i + \mathbf{C}_i^T)\right)^{-1}\mathbf{A}, \tag{7}$$

then the coefficient matrix of the system can be written as

$$\mathbf{H} = \tag{8}$$

$$\left\{\begin{array}{cccccccc}
\mathbf{O} & -\mathbf{BM}_1 & \cdots & -\mathbf{BM}_1 & -\mathbf{B}(\mathbf{I}-\mathbf{M}_1) & & & \\
-\mathbf{BM}_2 & \mathbf{O} & \cdots & -\mathbf{BM}_2 & & -\mathbf{B}(\mathbf{I}-\mathbf{M}_2) & & \\
\vdots & \vdots & \vdots & & & & \ddots & \\
-\mathbf{BM}_n & -\mathbf{BM}_n & \cdots & \mathbf{O} & & & & -\mathbf{B}(\mathbf{I}-\mathbf{M}_n) \\
\mathbf{O} & \mathbf{M}_1 & \cdots & \mathbf{M}_1 & \mathbf{I}-\mathbf{M}_1 & & & \\
\mathbf{M}_2 & \mathbf{O} & \cdots & \mathbf{M}_2 & & \mathbf{I}-\mathbf{M}_2 & & \\
\vdots & \vdots & \vdots & & & & \ddots & \\
\mathbf{M}_n & \mathbf{M}_n & \cdots & \mathbf{O} & & & & \mathbf{I}-\mathbf{M}_n
\end{array}\right\}.$$

It is well known (see, for example, Szidarovszky and Bahill, 1998) that the equilibrium of the system is asymptotically stable if and only if all eigenvalues of matrix $\mathbf{H}$ are inside the unit circle. The eigenvalue equation can be written as

$$\left.\begin{array}{r}
-\mathbf{BM}_i \sum_{j \neq i} \mathbf{u}_j - \mathbf{B}(\mathbf{I}-\mathbf{M}_i)\mathbf{v}_i = \lambda\mathbf{u}_i \\
\mathbf{M}_i \sum_{j \neq i} \mathbf{u}_j - (\mathbf{I}-\mathbf{M}_i)\mathbf{v}_i = \lambda\mathbf{v}_i
\end{array}\right\} \quad (i = 1, 2, \ldots, n). \tag{9}$$

Adding the $\mathbf{B}$-multiple of the second equation to the first equation yields a relation

$$\lambda(\mathbf{u}_i + \mathbf{B}\mathbf{v}_i) = \mathbf{0}. \tag{10}$$

Since zero eigenvalues do not disturb asymptotical stability, we may assume that $\lambda \neq 0$. Then

$$\mathbf{u}_i = -\mathbf{B}\mathbf{v}_i, \tag{11}$$

and by substituting this relation into the second equation of (9) we have

$$-\mathbf{M}_i\mathbf{B}\sum_{j\neq i}\mathbf{v}_j + (\mathbf{I}-\mathbf{M}_i)\mathbf{v}_i = \lambda\mathbf{v}_i \quad (i=1,2,\dots,n). \tag{12}$$

That is, the nonzero eigenvalues of $\mathbf{H}$ coincide with the eigenvalues of matrix

$$\mathbf{H}^{(1)} = \begin{pmatrix} \mathbf{I}-\mathbf{M}_1 & -\mathbf{M}_1\mathbf{B} & \cdots & -\mathbf{M}_1\mathbf{B} \\ -\mathbf{M}_2\mathbf{B} & \mathbf{I}-\mathbf{M}_2 & \cdots & -\mathbf{M}_2\mathbf{B} \\ \vdots & \vdots & & \vdots \\ -\mathbf{M}_n\mathbf{B} & -\mathbf{M}_n\mathbf{B} & \cdots & \mathbf{I}-\mathbf{M}_n \end{pmatrix}. \tag{13}$$

Hence the equilibrium is asymptotically stable if and only if all eigenvalues of $\mathbf{H}^{(1)}$ are inside the unit circle.

We will next further reduce the size of the eigenvalue problem that has to be examined in order to establish the asymptotical stability of the equilibrium. With the new variable $\mathbf{v} = \sum_{j=1}^{n}\mathbf{v}_j$, we see that (12) is rewritten as

$$-\mathbf{M}_i\mathbf{B}\mathbf{v} + (\mathbf{I}-\mathbf{M}_i+\mathbf{M}_i\mathbf{B}-\lambda\mathbf{I})\mathbf{v}_i = 0. \tag{14}$$

Assuming that matrix $\mathbf{I}-\mathbf{M}_i+\mathbf{M}_i\mathbf{B}$ is non-singular,

$$\mathbf{v}_i = (\mathbf{I}-\mathbf{M}_i+\mathbf{M}_i\mathbf{B}-\lambda\mathbf{I})^{-1}\mathbf{M}_i\mathbf{B}\mathbf{v}, \tag{15}$$

and adding these equations for $i=1,2,\dots,n$, we have

$$\left(\sum_{i=1}^{n}(\mathbf{I}-\mathbf{M}_i+\mathbf{M}_i\mathbf{B}-\lambda\mathbf{I})^{-1}\mathbf{M}_i\mathbf{B}-\mathbf{I}\right)\mathbf{v} = \mathbf{0}. \tag{16}$$

Notice that $\mathbf{v}=\mathbf{0}$ implies that for all i, $\mathbf{v}_i=\mathbf{0}$ which is impossible, since eigenvectors must differ from zero. Therefore equation (16) has nonzero solutions if and only if

$$\det\left(\sum_{i=1}^{n}(\mathbf{I}-\mathbf{M}_i+\mathbf{M}_i\mathbf{B}-\lambda\mathbf{I})^{-1}\mathbf{M}_i\mathbf{B}-\mathbf{I}\right) = 0. \tag{17}$$

Hence, we have proved that under the assumption that matrices $\mathbf{I}-\mathbf{M}_i+\mathbf{M}_i\mathbf{B}$ are non-singular, the equilibrium is asymptotically stable if and only if all roots of equation (17) are inside the unit circle.

Simple sufficient conditions can be given for the asymptotical stability of the equilibrium by requiring that some norm of matrix $\mathbf{H}^{(1)}$ is less than one. Selecting the block-row norm, the condition is the following:

$$\|\mathbf{I}-\mathbf{M}_i\| + \sum_{j\neq i}\|\mathbf{M}_i\mathbf{B}\| < 1 \quad (i=1,2,\dots,n) \tag{18}$$

which holds if

$$\|\mathbf{I}-\mathbf{M}_i\| + (n-1)\|\mathbf{M}_i\|\cdot\|\mathbf{B}\| < 1 \tag{19}$$

which is equivalent to the assumption that

$$\|\mathbf{B}\| < \min_i \left\{ \frac{1 - \|\mathbf{I} - \mathbf{M}_i\|}{(n-1)\|\mathbf{M}_i\|} \right\}. \tag{20}$$

Selecting the block-column norm, the condition is the following:

$$\|\mathbf{I} - \mathbf{M}_i\| + \sum_{j \neq i} \|\mathbf{M}_j \mathbf{B}\| < 1 \quad (i = 1, 2, \dots, n) \tag{21}$$

which holds if

$$\|\mathbf{I} - \mathbf{M}_i\| + \|\mathbf{B}\| \cdot \sum_{j \neq i} \|\mathbf{M}_j\| < 1 \tag{22}$$

which is equivalent to the condition

$$\|\mathbf{B}\| < \min_i \left\{ \frac{1 - \|\mathbf{I} - \mathbf{M}_i\|}{\sum\limits_{j \neq i} \|\mathbf{M}_j\|} \right\}. \tag{23}$$

4. Special Cases

Assume first that $\mathbf{M}_i \equiv \mathbf{M}$, that is, the firms have identical expectations. In this case, matrix $\mathbf{H}^{(1)}$ can be rewritten as

$$\mathbf{H}^{(1)}_{spec1} = \begin{pmatrix} \mathbf{I}-\mathbf{M} & -\mathbf{MB} & \cdots & -\mathbf{MB} \\ -\mathbf{MB} & \mathbf{I}-\mathbf{M} & \cdots & -\mathbf{MB} \\ \vdots & \vdots & & \vdots \\ -\mathbf{MB} & -\mathbf{MB} & \cdots & \mathbf{I}-\mathbf{M} \end{pmatrix}, \tag{24}$$

and equation (17) simplifies to the following:

$$\det\big(n(\mathbf{I} - \mathbf{M} + \mathbf{MB} - \lambda\mathbf{I})^{-1}\mathbf{MB} - \mathbf{I}\big) = 0. \tag{25}$$

Notice that inequalities (20) and (23) reduce to the following:

$$\|\mathbf{B}\| < \frac{1 - \|\mathbf{I} - \mathbf{M}\|}{(n-1)\|\mathbf{M}\|}. \tag{26}$$

In the further special case of Cournot expectations, $\mathbf{M} = \mathbf{I}$. Then $\mathbf{H}^{(1)}$ becomes

$$\mathbf{H}^{(1)}_{spec2} = \begin{pmatrix} \mathbf{O} & -\mathbf{B} & \cdots & -\mathbf{B} \\ -\mathbf{B} & \mathbf{O} & \cdots & -\mathbf{B} \\ \vdots & \vdots & & \vdots \\ -\mathbf{B} & -\mathbf{B} & \cdots & \mathbf{O} \end{pmatrix}, \tag{27}$$

which is the Kronecker product of matrices

$$\mathbf{E} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} \tag{28}$$

and $-\mathbf{B}$. It is known (see, for example, Okuguchi and Szidarovskzy, 1999) that the eigenvalues of $\mathbf{E}$ are -1 and $(n-1)$, therefore the eigenvalues of $\mathbf{H}^{(1)}_{spec2}$ are λ_i and $-(n-1)\lambda_i$ where $\lambda_i (1 \le i \le m)$ are the eigenvalues of $\mathbf{B}$. Hence the equilibrium is asymptotically stable if and only if for $i = 1, 2, \ldots, m$,

$$|\lambda_i| < \frac{1}{n-1}\,. \tag{29}$$

Assume next that for all i, $\mathbf{C}_i = \mathbf{O}$ and $\mathbf{A}$ is an invertible symmetric matrix. Then $\mathbf{B} = \frac{1}{2}\mathbf{I}$, and therefore $\mathbf{H}^{(1)}$ has the special form

$$\mathbf{H}^{(1)}_{spec3} = \begin{pmatrix} \mathbf{I}-\mathbf{M}_1 & -\frac{1}{2}\mathbf{M}_1 & \cdots & -\frac{1}{2}\mathbf{M}_1 \\ -\frac{1}{2}\mathbf{M}_2 & \mathbf{I}-\mathbf{M}_2 & \cdots & -\frac{1}{2}\mathbf{M}_2 \\ \vdots & \vdots & & \vdots \\ -\frac{1}{2}\mathbf{M}_n & -\frac{1}{2}\mathbf{M}_n & \cdots & \mathbf{I}-\mathbf{M}_n \end{pmatrix}. \tag{30}$$

In order to apply equation (17), we have to assume that the matrix

$$\mathbf{I} - \mathbf{M}_i + \mathbf{M}_i\mathbf{B} = \mathbf{I} - \frac{1}{2}\mathbf{M}_i \tag{31}$$

is non-singular. In this case, equation (17) specializes as

$$\det\left(\sum_{i=1}^{n} \frac{1}{2}\left(\mathbf{I} - \frac{1}{2}\mathbf{M}_i - \lambda\mathbf{I}\right)^{-1}\mathbf{M}_i - \mathbf{I}\right) = 0. \tag{32}$$

In addition to the above conditions assume that for all i, $\mathbf{M}_i = \alpha_i\mathbf{I}$ with some $\alpha_i \in (0, 1]$. Then matrix (30) can be further reduced as follows:

$$\mathbf{H}^{(1)}_{spec4} = \begin{pmatrix} \mathbf{I}(1-\alpha_1) & -\frac{\alpha_1}{2}\mathbf{I} & \cdots & -\frac{\alpha_1}{2}\mathbf{I} \\ -\frac{\alpha_2}{2}\mathbf{I} & \mathbf{I}(1-\alpha_2) & \cdots & -\frac{\alpha_2}{2}\mathbf{I} \\ \vdots & \vdots & & \vdots \\ -\frac{\alpha_n}{2}\mathbf{I} & -\frac{\alpha_n}{2}\mathbf{I} & \cdots & \mathbf{I}(1-\alpha_n) \end{pmatrix}. \tag{33}$$

Since matrix (31) is $\mathbf{I}\left(1 - \frac{\alpha_i}{2}\right)$ and non-singular, we can use equation (32), which can be significantly simplied:

$$\sum_{i=1}^{n} \frac{\frac{\alpha_i}{2}}{\left(1 - \frac{\alpha_i}{2}\right) - \lambda} = 1. \tag{34}$$

We will next find sufficient and necessary conditions that the roots of this equation are inside the unit circle. Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_r \le 1$ denote the distinct α_i values with multipliticies $m_1, m_2, \ldots, m_r$. Then (34) can be rewritten as

$$g(\lambda) = \sum_{i=1}^{r} \frac{\frac{m_i\alpha_i}{2}}{\left(1 - \frac{\alpha_i}{2}\right) - \lambda} = 1. \tag{35}$$

Notice first that this is equivalent to a polynomial equation of degree r, therefore there are r real or complex roots. Furthermore, for all i,

$$\lim_{\lambda \to 1-\frac{\alpha_i}{2}+0} g(\lambda) = -\infty, \tag{36}$$

$$\lim_{\lambda \to 1-\frac{\alpha_1}{2}-0} g(\lambda) = \infty, \tag{37}$$

and

$$\lim_{\lambda \to \pm\infty} g(\lambda) = 0. \tag{38}$$

Since $g'(\lambda) > 0$, function g is locally strictly increasing everywhere. There are $r-1$ real roots between $1-\frac{\alpha_{i+1}}{2}$ and $1-\frac{\alpha_i}{2}$ $(i = 1, 2, \dots, r-1)$, and there is an additional real root below $1-\frac{\alpha_r}{2}$. Hence all roots are real, and they are between -1 and $+1$ if and only if $g(-1) < 1$. This condition can be rewritten as

$$\sum_{i=1}^{n} \frac{\alpha_i}{4-\alpha_i} < 1. \tag{39}$$

Assume first that $n = 2$. Then

$$\sum_{i=1}^{2} \frac{\alpha_i}{4-\alpha_i} \leq 2 \cdot \frac{1}{4-1} < 1, \tag{40}$$

therefore dupolies are always asymptotically stable under the above simplifying assumptions. If $n \geq 3$, then (39) is not satisfied necessarily. It holds if the α_i values are sufficiently small. For example, if $\alpha_i \equiv \alpha$, then (39) holds if and only if

$$\frac{n\alpha}{4-\alpha} < 1, \tag{41}$$

that is, when

$$\alpha < \frac{4}{n+1}. \tag{42}$$

Notice that this condition is equivalent to the asymptotical stability of dynamic oligopolies under similar simplifying conditions (see, Okuguchi and Szidarovszky, 1999, Theorem 4.2.4).

Assume finally that the factor market is also perfectly competitive. Then matrix $\mathbf{A}$ is zero, and therefore at each time period the profit (4) of firm i $(i = 1, 2, \dots, n)$ does not depend on the activity of the competitors:

$$\Pi_i(t) = p\big(\mathbf{l}_i^T \mathbf{C}_i \mathbf{l}_i + \mathbf{c}_i^T \mathbf{l}_i + \gamma_i\big) - \mathbf{l}_i^T \mathbf{a}. \tag{43}$$

Therefore at each time period the firms maximize their own payoffs independently of each other:

$$\mathbf{l}_i(t) = \big[p(\mathbf{C}_i + \mathbf{C}_i^T)\big]^{-1} (\mathbf{a} - p\mathbf{c}_i). \tag{44}$$

5. Conclusions

Special dynamic oligopsonies were examined with linear price and quadratic production functions. We proved that the equilibrium is asymptotically stable if and only if all eigenvalues of matrix (8) or (13) are inside the unit circle. We showed that under special conditions it is sufficient to examine the more simple determinant equation (17). Sufficient conditions were also derived (relations (20) and (23)) which are very easy to check in practical cases. Some special cases were discussed such as identical expectations, and in further special cases Cournot expectations, linear production functions, and special speeds of adjustment were analyzed. The general stability conditions were simplified in these special cases and the analogy between these conditions and those for oligopolies known from the literature were observed.

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