Electronic Journal: Southwest Journal of Pure and Applied Mathematics
Internet: http://rattler.cameron.edu/swjpam.html
ISSN 1083-0464
Issue 2, December 2001, pp. 43–52.
Submitted: April 2, 2001. Published: December 31 2001.

SOME FORMULAS FOR CONJUGATION

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Abstract. In this paper we prove a formula involving the canonical anti-automorphism χ of the mod-p Steenrod algebra and its dual.

1991 A.M.S. (MOS) Subject Classification Codes. 55S10, 55S05

Key Words and Phrases. Steenrod algebra, anti-automorphism, Milnor basis

1. INTRODUCTION AND MAIN RESULTS

It is well known that the mod-p Steenrod algebra $\mathcal{A}$, being a cocommutative Hopf algebra, has a canonical conjugation χ. This linear map was first introduced by Thom [8] and later Milnor [5] discovered most of what we know today about χ. In 1974 Davis [1] proved nice formulae involving canonical anti-automorphism.

We recall the structure of dual algebra $\mathcal{A}^*$ of $\mathcal{A}$ which was computed by Milnor [5]. The algebra $\mathcal{A}^*$ is the tensor product of a polynomial algebra on generators ξ_i $i \geq 1$ of degree $2(p^i-1)$ by an exterior algebra on generators τ_i $i \geq 0$ of degree $2p^i - 1$. So the monomial basis in $\mathcal{A}^*$ can be written in the form

$$\xi^I = \tau_0^{\epsilon_0}\xi_1^{r_1}\tau_1^{\epsilon_1}\cdots$$

Then dual to the the monomial basis ξ^I is a basis for the mod-p Steenrod algebra known as Milnor basis. So $Q_0^{\epsilon_0}Q_1^{\epsilon_1}\cdots P(r_1, r_2, \dots)$ will denote the dual of ξ^I.

The structure of the mod-p Steenrod algebra $\mathcal{A}$ is given as a graded $\mathbb{Z}_p$-module on the Milnor basis with product and coproduct defined in terms of that basis. If $R = (r_1, r_2, \dots, r_m)$ is a sequence of nonnegative integers, we will write $\mathcal{P}(R)$ for the corresponding Milnor basis element. The degree of $\mathcal{P}(R)$ is $2\,|R|$ where $|R| = \sum\limits_{i=1}^{m}\left(p^i-1\right)r_i$ and the excess of $\mathcal{P}(R)$ is $2e(R)$ where $e(R) = \sum\limits_{i=1}^{m} r_i$. Let B be the vector space of $\mathcal{A}$ with the basis the set of all $\mathcal{P}(R)$. For $\mathcal{P}(a) \in B$, write $\hat{\mathcal{P}}(a)$

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for $(-1)^a\chi(\mathcal{P}(a))$ where χ denotes the canonical conjugation. Given pair (k,n) of nonnegative integers, we define

$$X(k,n)=\mathcal{P}(p^k(p^{n+1}-1))\mathcal{P}(p^{k-1}(p^{n+1}-1))\cdots\mathcal{P}(p(p^{n+1}-1))\mathcal{P}((p^{n+1}-1)).$$

First main result which is a conjugation formula for a certain Steenrod opreations $X(k,n)$ is mod-p analogue of Theorem 3.1 in [6]

Theorem 1.1. *For all positive integers i and j, we have*

$$\widehat{X}(j,i)=X(i,j).$$

Our second main result is also a conjugation formula for the certain elements $\mathcal{W}_I(k)$ in the dual $\mathcal{A}^*$ of Steenrod algebra which will be defined in Section 3. This formula generalizes the Proposition 5.5 in [7].

Theorem 1.2. *For any integers $k\geq 1$ and $i_0>0$ we have*

$$\widehat{\mathcal{W}_I}(k)\equiv\xi_k^{\frac{p^{i_0}-1}{p-1}}\widehat{\mathcal{R}}_I(k)$$

modulo elements of length $>k$, where $\mathcal{R}_I(k)$is defined in Section 3.

2. PRELIMINARIES

We will introduce the useful notation: each natural number a has a unique p-adic expansion

$$a=\sum_{i=0}^{\infty}\alpha_i(a)p^i$$

with $0\leq\alpha_i(a)<p$. It is fact that

$$(1)\qquad \binom{a}{b}\equiv\prod_{i=0}^{\infty}\binom{\alpha_i(a)}{\alpha_i(b)}\mod p.$$

We have following definition;

Definition 2.1. We say that m and n have no carries if $\alpha_i(m)+\alpha_i(n)\leq p-1$ for all i. This is equivalent to the condition that the binomial coefficient $\binom{m+n}{m}$ is nonzero mod p . If m and n have no carries, we will write $m\asymp n$.

Using Davis' method [1] we can derive the formulae,

Proposition 2.2.

$$(2)\qquad \mathcal{P}(u)\cdot\widehat{\mathcal{P}}(v)=\sum_R\binom{|R|+e(R)}{pu}\mathcal{P}(R)$$

$$(3)\qquad \widehat{\mathcal{P}}(u)\cdot\mathcal{P}(v)=\sum_R\binom{e(R)}{v}\mathcal{P}(R)$$

where the sum is taken over all R for which $|R| = (p-1)(u+v)$.

Proof. Corollary 1a in [2] is exactly formula (2) when $t = 1$. So we will prove formula (3). Using Gallant's Proposition 1 that $\chi(\mathcal{P}(a))$ is $(-1)^a$ times the sum of all $\mathcal{P}(R)$ in the appropriate dimension, we see that the product $\widehat{\mathcal{P}}(u)\cdot\mathcal{P}(v)$ contains a term

$$\prod_i \binom{r_i}{s_i}\mathcal{P}(r_1, r_2, \dots)$$

for each Milnor matrix,

$$\begin{pmatrix} * & s_1 \\ r_1 - s_1 & s_2 \\ r_2 - s_2 & s_3 \\ \vdots & \vdots \\ r_{n-1} - s_{n-1} & s_n \\ r_n - s_n & s_{n+1} \\ \vdots & \vdots \end{pmatrix}$$

such that $v = \sum_i s_i$. Thus

$$\widehat{\mathcal{P}}(u)\cdot\mathcal{P}(v) = \sum_R \sum_S \prod_i \binom{r_i}{s_i}\mathcal{P}(R)$$

where S ranges over sequences $(s_1, s_2, \dots)$ having $\sum\limits_i s_i = v$. The equation $\sum\limits_S \prod\limits_i \binom{r_i}{s_i} = \binom{\sum_i r_i}{v}$ follows immediately from considering the coefficient of x^v in the expansion of

$$(1+x)^{r_1}\cdot(1+x)^{r_2}\cdots(1+x)^{r_n}.$$

Hence we have

$$\widehat{\mathcal{P}}(u)\cdot\mathcal{P}(v) = \sum_R \binom{\sum_i r_i}{v}\mathcal{P}(R) = \sum_R \binom{e(R)}{v}\mathcal{P}(R).$$

□

Using formulae above, we can prove the following proposition.

Proposition 2.3. *For nonnegative integers k, l, m, n and $k > l$, suppose that*

(1) $m + n = p^k - p^l$
(2) $m < p^{k-1}$
(3) $m \equiv 0 \mod p^l$.

Then

(i) *When $l = 0$, we have*

$$\mathcal{P}(m)\cdot\widehat{\mathcal{P}}(n) = \widehat{\mathcal{P}}(n-(p-1)m-1)\mathcal{P}(pm+1)$$

(ii) *When $l > 0$, we have*

$$\mathcal{P}(m)\cdot\widehat{\mathcal{P}}(n) = \widehat{\mathcal{P}}(n-(p-1)(m-p^l))\mathcal{P}(pm+(p-1)p^l) - \sum_{w=1}^{p-1}\binom{p-1}{w}\mathcal{P}(m+wp^{l-1})\cdot\widehat{\mathcal{P}}(n-wp^{l-1})$$

Proof.

(i) Let $l = 0$. Using Proposition 2.2, we have

$$\mathcal{P}(m) \cdot \widehat{\mathcal{P}}(n) = \sum_R \binom{|R| + e(R)}{pm} \mathcal{P}(R),$$

and

$$\widehat{\mathcal{P}}(n - (p-1)m - 1) \cdot \mathcal{P}(pm+1) = \sum_R \binom{e(R)}{pm+1} \mathcal{P}(R)$$

where $|R| = (p-1)(p^k - 1)$ and $1 \le e(R) \le p^k - 1$. In order to prove these sums are equivalent in mod-p, we need to show that their binomial coefficients are equivalent in mod-p i.e.

$$\binom{|R| + e(R)}{pm} \equiv \binom{e(R)}{pm+1} \mod p.$$

We know that $|R| = \sum_{i=1}^{\infty} r_i(p^i - 1)$ and $e(R) = \sum_{i=1}^{\infty} r_i$. Using these facts, we have

$$\begin{aligned}
p^k - 1 &= (p-1)(p^{k-1} + p^{k-2} + \cdots + p + 1) = \frac{|R|}{p-1} \\
&= \sum_{i=1}^{\infty} r_i(p^{i-1} + p^{i-2} + \cdots + p + 1) \\
&= r_1 + \sum_{i=2}^{\infty} r_i(p^{i-1} + p^{i-2} + \cdots + p + 1) \\
&= e(R) - \sum_{i=2}^{\infty} r_i + \sum_{i=2}^{\infty} r_i(p^{i-1} + p^{i-2} + \cdots + p + 1) \\
&= e(R) + \sum_{i=2}^{\infty} r_i p(p^{i-2} + p^{i-3} + \cdots + p + 1).
\end{aligned}$$

Since $\sum_{i=2}^{\infty} r_i p(p^{i-2} + p^{i-3} + \cdots + p + 1) \equiv 0 \mod p$, $e(R) \equiv p - 1 \mod p$. Using Equation 1, and the upper bounds of $e(R)$ and m, we have

$$\binom{|R| + e(R)}{pm} = \binom{(p-1)(p^k - 1) + e(R)}{pm} \equiv \binom{e(R)}{pm+1} \mod p.$$

This completes the proof of part (i).

(ii) Let $l > 0$. Again using Proposition 2.2, we have

$$\mathcal{P}(m) \cdot \widehat{\mathcal{P}}(n) = \sum_R \binom{|R| + e(R)}{pm} \mathcal{P}(R),$$

$$\mathcal{P}(m + wp^{l-1}) \cdot \widehat{\mathcal{P}}(n - wp^{l-1}) = \sum_R \binom{|R| + e(R)}{pm + wp^l} \mathcal{P}(R),$$

and

$$\widehat{\mathcal{P}}(n-(p-1)(m-p^l))\cdot\mathcal{P}(pm+(p-1)p^l)=\sum_R\binom{e(R)}{pm+(p-1)p^l}\mathcal{P}(R)$$

where $|R|=(p-1)(p^k-p^l)$ and $1\le e(R)\le p^k-p^l$. In order to prove the sums in part (ii) are equivalent, we need to show that

$$\binom{|R|+e(R)}{pm}+\sum_{w=1}^{p-1}\binom{p-1}{w}\binom{|R|+e(R)}{pm+wp^l}\equiv\binom{e(R)}{pm+(p-1)p^l}\mod p.$$

Case 1: $0\le\alpha_l(e(R))<p-1$. Then $\binom{e(R)}{pm+(p-1)p^l}$ are equivalent to zero in mod-p. So it is enough to show that

$$\binom{|R|+e(R)}{pm}+\sum_{w=1}^{p-1}\binom{p-1}{w}\binom{|R|+e(R)}{pm+wp^l}\equiv 0\mod p,$$

i.e.

$$1+\sum_{w=1}^{p-1}\binom{p-1}{w}\binom{1+\alpha_l(e(R))}{w}\equiv 0\mod p$$

for $0\le\alpha_l(e(R))<p-1$. The following equivalent holds

$$\sum_{w=0}^{p-1}\binom{p-1}{w}\binom{1+\alpha_l(e(R))}{w}\equiv\binom{p+\alpha_l(e(R))}{p-1}\mod p$$

by considering coefficient of x^{p-1} in the binomial expansion of

$$(x+1)^{p+\alpha_l(e(R))}=(x+1)^{p-1}(x+1)^{1+\alpha_l(e(R))}.$$

Since $0\le\alpha_l(e(R))<p-1$, $\binom{p+\alpha_l(e(R))}{p-1}$ is equivalent to zero in mod-p. Hence

$$\sum_{w=0}^{p-1}\binom{p-1}{w}\binom{1+\alpha_l(e(R))}{w}\equiv 0\mod p,$$

i.e.

$$1+\sum_{w=1}^{p-1}\binom{p-1}{w}\binom{1+\alpha_l(e(R))}{w}\equiv 0\mod p$$

So the result holds.

Case 2: $\alpha_l(e(R))=p-1$. Then

$$\sum_{w=1}^{p-1}\binom{p-1}{w}\binom{|R|+e(R)}{pm+wp^l}\equiv 0\mod p$$

and

$$\binom{|R|+e(R)}{pm}\equiv\binom{e(R)}{pm+(p-1)p^l}\mod p.$$

Therefore the result holds. $\square$

3. Conjugation in Dual of Steenrod algebra

Let $\mathcal{S}$ be the monoid of sequences of nonnegative integers almost all of which are zero; write $0_{\mathcal{S}}$ for the trivial element and $B(j)$ for the sequence whose i-th coordinate is δ_{ij}. If $S = (s_1, s_2, s_3, \dots)$ has $s_i = 0$ for $i > N$, we identify S with $S = (s_1, s_2, \dots, s_N)$. Suppose that $S = (s_1, s_2, \dots, s_N)$ with $s_N \neq 0$. The length of S is $len(S) = N$. We will write $\mathcal{S}^*$ for $\mathcal{S} \cup \{*\}$ where $*$ satisfies $x + * = *$ for all $x \in \mathcal{S}^*$.

Let $\mathcal{S}(k)$ be the symmetric group with identity Id^k acting on $\{0, 1, 2, 3, \dots, k-1\}$. For $\tau \in \mathcal{S}(k)$ and $i \geq 0$. Define

$$Z_i(k;\tau) = \begin{cases} \sum_{j=0}^{k-1} B(i + \tau(j) - j)p^j & i + \tau(j) - j \geq 0 \text{ for all } j \\ * & \text{else} \end{cases}$$

and

$$T_i(k;\tau) = \begin{cases} \prod_{j=0}^{k-1} \xi_{i+\tau(j)-j}^{p^j} & i + \tau(j) - j \geq 0 \text{ for all } j \\ 0 & \text{else.} \end{cases}$$

¿From these definitions we can observe that

(1) $Z_i(k; Id^k) = \frac{p^k-1}{p-1} B(i)$.

(2) $T_i(k; Id^k) = \xi_i^{\frac{p^k-1}{p-1}}$.

So define

$$\mathcal{W}_i(k) = \sum_{\tau \in \mathcal{S}(k)} T_i(k;\tau).$$

Then $\mathcal{W}_i(1) = T_i(1; Id^1) = \xi_i$. So we have following lemma;

Lemma 3.1. *For any integer $k \geq 1$, we have $\widehat{\xi}_k = \mathcal{W}_1(k)$.*

Proof:. We prove this by induction on k. Let $k = 1$. Then

$$\mathcal{W}_1(1) = \xi_1 = \widehat{\xi}_1.$$

Let $k = 2$.

$$\mathcal{W}_1(2) = T_1(2; Id^2) + T_1(2; (0,1)) = \xi_1^{p+1} + \xi_2^{p^0} \xi_0^{p^1} = \widehat{\xi}_2.$$

Suppose that the statement is true for $k' < k$. Note that if $T_1(k;\tau) \neq 0$, we must have $\tau(j) \geq j - 1$ for all j. So if $T_1(k;\tau) \neq 0$ and l satisfies $\tau(l) = k - 1$, we must have $\tau(k-1) = k-2$, $\tau(k-2) = k-3$, $\dots$, $\tau(l+1) = l$ so that τ has cycle-decomposition of the form $(k-1, k-2, \dots, l)\sigma$ for some $\sigma \in \mathcal{S}(l)$. In this case we have

$$T_1(k;\tau) = \xi_{k-1}^{p^l} T_1(k;\sigma).$$

So let $\mathcal{S}_l(k) = \{\tau \in \mathcal{S}(k) : \tau(l) = k-1\}$. Then

$$\mathcal{W}_1(k) = \sum_{l=0}^{k-1} \sum_{\tau \in \mathcal{S}_l(k)} T_1(k;\tau) = \sum_{l=0}^{k-1} \xi_{k-l}^{p^l} \sum_{\sigma \in \mathcal{S}(l)} T_1(k;\sigma)$$

$$\sum_{l=0}^{k-1} \xi_{k-l}^{p^l} \widehat{\xi}_l = \widehat{\xi}_k.$$

□

Definition 3.2. For $k \geq 1$, let $\mathcal{I}(k)$ be the set of nondecreasing sequences $(i_0, i_1, \ldots, i_{k-1})$ of positive integers. For $\tau \in \mathcal{S}(k)$ and $I \in \mathcal{I}(k)$, define

$$Z_I(k;\tau) = \begin{cases} \sum\limits_{j=0}^{k-1} B(i_{\tau(j)} + \tau(j) - j)p^j & i_{\tau(j)} + \tau(j) - j \geq 0 \text{ for all } j \\ * & \text{else} \end{cases}$$

$$T_I(k;\tau) = \begin{cases} \prod\limits_{j=0}^{k-1} \xi_{i_{\tau(j)}+\tau(j)-j}^{p^j} & i_{\tau(j)} + \tau(j) - j \geq 0 \text{ for all } j \\ 0 & \text{else.} \end{cases}$$

$$\mathcal{W}_I(k) = \sum_{\tau \in \mathcal{S}(k)} T_I(k;\tau).$$

We further define

$$P_I(k;\tau) = \begin{cases} \sum\limits_{j=0}^{k-1} B(i_{\tau(j)} + \tau(j) - j - i_0)p^j & i_{\tau(j)} + \tau(j) - j - i_0 \geq 0 \text{ for all } j \\ * & \text{else} \end{cases}$$

$$R_I(k;\tau) = \begin{cases} \prod\limits_{j=0}^{k-1} \xi_{i_{\tau(j)}+\tau(j)-j-i_0)}^{p^{i_0+j}} & i_{\tau(j)} + \tau(j) - j - i_0 \geq 0 \text{ for all } j \\ 0 & \text{else.} \end{cases}$$

$$\mathcal{R}_I(k) = \sum_{\tau \in \mathcal{S}(k)} R_I(k;\tau).$$

¿From this definition, we observe that

(1) If $I = (i, i, i, \ldots, i) \in I(k)$ is a constant sequence, then $Z_I(k;\tau) = Z_i(k;\tau)$. Moreover for $\tau \neq Id^k$, $P_I(k;\tau) = *$ and consequently $\mathcal{R}_I(k) = R_I(k; Id^k) = 1$.

(2) For general I, $Z_I(k; Id^k) = \sum\limits_{j=0}^{k-1} B(i_{\tau(j)})p^j$.

(3) If $I = (i_0, i_1, \ldots, i_{k-1}) \in \mathcal{I}(k)$ and $i_0 > 1$, write $I[-1]$ for the sequence $(i_0 - 1, i_1 - 1, \ldots, i_{k-1} - 1)$. Then $\mathcal{R}_I(k) = [\mathcal{R}_{I[-1]}(k)]^p$.

4. PROOFS OF MAIN RESULTS

Proof of Theorem 1.1. We are going to prove the theorem by induction on i under the assumption that $i \leq j$. For $i = 0$ and all j , $\widehat{X}(j, 0) = \mathcal{P}(p^{j+1} - 1) = X(0, j)$ by Davis' formula in [1]. Assume that for all $\hat{i} \leq i - 1$ and all j, and for $\hat{i} = i$ and $\hat{j} \leq j - 1$,

$$\widehat{X}(\hat{i}, \hat{j}) = X(\hat{j}, \hat{i}).$$

The inductive proof will draw on the following remark: under the above assumptions,

$$\widehat{\mathcal{P}}(cp^{l-1}) \cdot X(l-1, i) = 0 \tag{4}$$

where c is a unit in mod-p and $1 \leq l \leq i$. Indeed by induction on l, we have

$$\begin{aligned}
\widehat{\mathcal{P}}(cp^{l-1}) \cdot X(l-1,i) &= \widehat{[\widehat{X}(l-1,i) \cdot \mathcal{P}(cp^{l-1})]} \\
&= \widehat{[X(i,l-1) \cdot \mathcal{P}(cp^{l-1})]} \\
&= \widehat{[\mathcal{P}(p^i(p^l-1)) \cdots \mathcal{P}(p(p^l-1))\mathcal{P}((p^l-1)) \cdot \mathcal{P}(cp^{l-1})]}.
\end{aligned}$$

By Adem relations, $\mathcal{P}(p^l-1) \cdot \mathcal{P}(cp^{l-1}) = 0$. Therefore this verifies equation (4).

We claim that for $0 \leq l \leq j$

$$X(l,i-1) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-1)) = \widehat{\mathcal{P}}(p^i(p^{j+1}-p^{l+1})) \cdot X(l,i) \tag{5}$$

The case $l=0$ follows from Proposition 2.3 (i). Suppose that the statement is true for $l-1$. Then by induction and Proposition 2.3 (ii), we have

$$\begin{aligned}
X(l,i-1) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-1)) &= \mathcal{P}(p^l(p^i-1)) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-p^l)) \cdot X(l-1,i) \\
&= [\widehat{\mathcal{P}}(p^i(p^{j+1}-p^{l+1})) \cdot \mathcal{P}(p^l(p^{i+1}-1))] \cdot X(l-1,i) \\
&- [\sum_{w=1}^{p-1} \binom{p-1}{w} \mathcal{P}(p^l(p^i-1)+wp^{l-1}) \cdot \widehat{\mathcal{P}}(p^{i+j+1}-p^{l+i}-wp^{l-1})] \cdot X(l-1,i)
\end{aligned}$$

From Equation (4),

$$\widehat{\mathcal{P}}(p^{i+j+1}-p^{l+i}-wp^{l-1}) \cdot X(l-1,i) = 0$$

for every $w = 1,2,3,\ldots,p-1$.

Hence we have

$$X(l,i-1) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-1)) = \widehat{\mathcal{P}}(p^i(p^{j+1}-p^{l+1})) \cdot X(l,i).$$

This proves our claim. Finally, taking $l=j$, we find that

$$\begin{aligned}
\widehat{X}(i,j) &= \widehat{X}(i-1,j)) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-1)) \\
&= X(j,i-1) \cdot \widehat{\mathcal{P}}(p^i(p^{j+1}-1)) \\
&= \widehat{\mathcal{P}}(0) \cdot X(j,i).
\end{aligned}$$

This completes the proof. $\square$

Proof of Theorem 1.2. Fix k. We prove by induction on i_0. Let $i_0 = 1$.

$$\xi_k \widehat{\mathcal{R}}_I(k) = \sum_\tau \xi_k \widehat{\xi}^{p^k}_{i_{\tau(k-1)}+\tau(k-1)-k} \prod_{j \neq k-1} \widehat{\xi}^{p^{j+1}}_{i_{\tau(j)}+\tau(j)-(j+1)}. \tag{6}$$

On the other hand,

$$\widehat{\mathcal{W}}_I(k) = \sum_\rho \widehat{\xi}^{p^0}_{i_{\rho(0)}+\rho(0)} \prod_{j \neq 0} \widehat{\xi}^{p^j}_{i_{\rho(j)}+\rho(j)-j}.$$

For each $\rho \in \mathcal{S}(k)$, define ρ' by

$$\rho'(l) = \begin{cases} \rho(0) & l = k-1 \\ \rho(l+1) & 0 \le l \le k-2. \end{cases} \tag{6}$$

Then

$$\widehat{\mathcal{W}}_I(k) = \sum_{\rho'} \widehat{\xi}_{i_{\rho'(k-1)}+\rho'(k-1)} \prod_{l\neq k-1} \widehat{\xi}^{p^{l+1}}_{i_{\rho'(l)}+\rho'(l)-(l+1)}. \tag{8}$$

By Milnor formula,

$$\widehat{\xi}_{i_{\rho'(k-1)}+\rho'(k-1)} \equiv \sum_{n=1}^{k} \xi_n \widehat{\xi}^{p^n}_{i_{\rho'(k-1)}+\rho'(k-1)-n} \tag{9}$$

modulo terms of length $> k$. So we have

$$\widehat{\mathcal{W}}_I(k) \equiv \sum_{\rho'} (\sum_{n=1}^{k} \xi_n \widehat{\xi}^{p^n}_{i_{\rho'(k-1)}+\rho'(k-1)-n}) \prod_{l\neq k-1} \widehat{\xi}^{p^{l+1}}_{i_{\rho'(l)}+\rho'(l)-(l+1)}. \tag{10}$$

Claim that (6) and (10) agree. Indeed their difference is

$$\begin{aligned} &\sum_{\rho'} (\sum_{n=1}^{k-1} \xi_n \widehat{\xi}^{p^n}_{i_{\rho'(k-1)}+\rho'(k-1)-n}) \prod_{l\neq k-1} \widehat{\xi}^{p^{l+1}}_{i_{\rho'(l)}+\rho'(l)-(l+1))} \\ &= \sum_{n=1}^{k-1} \sum_{\rho'} \xi_n \widehat{\xi}^{p^n}_{i_{\rho'(k-1)}+\rho'(k-1)-n} \widehat{\xi}^{p^{(n-1)+1}}_{i_{\rho'(n-1)}+\rho'(n-1)-n} \prod_{l\neq n-1,k-1} \widehat{\xi}^{p^{l+1}}_{i_{\rho'(l)}+\rho'(l)-(l+1)}. \end{aligned} \tag{11}$$

It can be easily verified that summand in (11) associated to n and ρ' equals to the term associated to n and ρ'' where

$$\rho''(l) = \begin{cases} \rho'(l) & l \neq n-1, k-1 \\ \rho'(n-1) & l = k-1 \\ \rho(k-1) & l = n-1. \end{cases}$$

Each term in (11) appears p-times. So sums in (6) and (10) are equal. Hence $\xi_k \widehat{\mathcal{R}}_I(k)$ and $\widehat{\mathcal{W}}_I(k)$ are congruent modulo elements of length $> k$. So theorem holds for $i_0 = 1$.

The proof for general I is similar. Suppose that the statement is true for non-decreasing sequences $(i_0-1, i_1-1, \dots i_k-1) = I[-1]$. Let $i = i_0$. From Lemma 3.1 and induction hypothesis, we have

$$\begin{aligned} \xi_k^{\frac{p^i-1}{p-1}} \widehat{\mathcal{R}}_I(k) &= [\xi_k^{\frac{p^{i-1}-1}{p-1}} \widehat{\mathcal{R}}_{I[-1]}(k)]^p \xi_k \\ &\equiv (\widehat{\mathcal{W}}_{I[-1]}(k))^p \xi_k \\ &= \xi_k \sum_{\tau} \prod_{j=0}^{k-1} \widehat{\xi}^{p^{j+1}}_{(i_{\tau(j)}-1)+\tau(j)-j} \\ &= \sum_{\tau} \xi_k \widehat{\xi}^{p^k}_{(i_{\tau(k-1)}-1)+\tau(k-1)-(k-1)} \prod_{j\neq k-1} \widehat{\xi}^{p^{j+1}}_{(i_{\tau(j)}-1)+\tau(j)-j} \\ &= \sum_{\tau} \xi_k \widehat{\xi}^{p^k}_{i_{\tau(k-1)}+\tau(k-1)-k} \prod_{j\neq k-1} \widehat{\xi}^{p^{j+1}}_{i_{\tau(j)}+\tau(j)-(j+1)}. \end{aligned} \tag{12}$$

On the other hand, we have

$$\begin{aligned}
\widehat{\mathcal{W}}_I(k) &= \sum_\rho \prod_{j=0}^{k-1} \widehat{\xi}^{p^j}_{i_{\rho(j)}+\rho(j)-j} \\
&= \sum_\rho \widehat{\xi}^{p^0}_{i_{\rho(0)}+\rho(0)-0} \prod_{j\neq 0} \widehat{\xi}^{p^j}_{i_{\rho(j)}+\rho(j)-j}.
\end{aligned} \tag{13}$$

So one can define ρ' as in (7) and use (9) as in the case $i_0 = 1$ to establish the congruence of the sums in (12) and (13) modulo elements of length $> k$. This establishes the inductive step and hence completes the proof. □

Define

$$\widehat{\mathcal{W}}'_I(k) = \sum_{Id^k \neq \tau \in \mathcal{S}(k)} \widehat{T}_i(k;\tau).$$

Then we have following corollary;

Corollary 4.1. *For any integers $k \geq 1$ and $i_0 > 0$, we have*

$$\widehat{T}_I(k; Id^k) \equiv \widehat{\mathcal{W}}'_I(k) + \xi_k^{\frac{p^{i_0}-1}{p-1}} \widehat{\mathcal{R}}_I(k)$$

modulo elements of length $> k$.

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