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IRREDUCIBLE MAPPING AND THE LIGHTNESS OF OPEN MAPPINGS

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Abstract. We study continua with the property that each nonconstant open mapping is light. We define a large class of continua and show that each member of this class has this property. In this class we present an arcwise connected continuum which can be mapped under an irreducible mapping onto a continuum admitting a nonconstant non-light open mapping. Two closely related questions are answered.

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1. Introduction and notation.

For the convenience of the reader we repeat the relevant material. All spaces considered in the paper are assumed to be metric and all mappings are continuous. A *continuum* means a compact connected space. Given a space X and its subset S, we denote by $\operatorname{cl} S$ the closure of S.

Let X be a continuum and $A \subset X$. Then X is said to be *irreducible about* A provided no proper subcontinuum of X contains A. A continuum X is said to be *irreducible* provided that X is irreducible about $\{p, q\}$ for some $p, q \in X$ (then X is said to be *irreducible between* p *and* q).

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Let X be a continuum, and let A and B be nonempty compact subsets of X. We say that a subcontinuum C of X is *irreducible from A to B*, written $C = \text{irr}(A, B)$, provided that $C \cap A \neq \emptyset$, $C \cap B \neq \emptyset$, and no proper subcontinuum of C intersects both A and B. If A or B is a singleton set, we omit the set notation; thus, e.g. we write $C = \text{irr}(p, B)$, $C = \text{irr}(p, q)$.

Let X and Y be continua. A mapping $f : X \to f(X) = Y$ is said to be:
– *atomic* provided that for each subcontinuum $K \subset X$ with nondegenerate image $f(K)$ the equality $f^{-1}(f(K)) = K$ holds;
– *irreducible* provided that for each nondegenerate irreducible subcontinuum $I \subset Y$ the inverse image $f^{-1}(I)$ is irreducible;
– *monotone* provided that for each subcontinuum Q of Y the inverse image $f^{-1}(Q)$ is connected;
– *light* provided that for each point $y \in Y$ each component of the inverse image $f^{-1}(y)$ is a singleton;
– *interior at a point* $p \in X$ provided that for each open neighborhood U of p in X the point $f(p)$ is an interior point of the image $f(U)$ in Y;
– *open* provided that for each open subset U of X its image $f(U)$ is an open subset of Y.

Obviously a mapping is open if and only if it is interior at each point of its domain.

Given continua X and Y, we will denote by

$$\mathsf{Atomic}, \quad \mathsf{Irreducible}, \quad \text{and} \quad \mathsf{Monotone}$$

the family of all atomic, irreducible and monotone mappings $f : X \to f(X) = Y$, respectively. We recall from [15, Corollary 2.9, p. 36] (see also [12, 3.7, p. 13], [8, Theorem 1, p. 49]) the following

1.1. Statement. $\mathsf{Atomic} \subsetneq \mathsf{Irreducible} \subsetneq \mathsf{Monotone}$.

Let L denote the class of all continua X such that each nonconstant open mapping defined on X is light, and let $\mathfrak{M}$ be an arbitrary class of mappings between continua that contains the class of homeomorphisms. Consider classes $\mathfrak{M}$ for which the implication holds :

$$(\heartsuit) \qquad \text{if} \quad f \in \mathfrak{M} \quad \text{and} \quad X \in \mathsf{L}, \quad \text{then} \quad f(X) \in \mathsf{L}.$$

Recall the following assertions (see [13, Propositions 2.7-2.9, p. 783], [7, Example 2.10]).

(1.2) *The implication ($\heartsuit$) holds if $\mathfrak{M}$ is the class of open mappings.*
(1.3) *The implication ($\heartsuit$) does not hold if $\mathfrak{M}$ is the class of atomic mappings.*
(1.4) *The implication ($\heartsuit$) holds if $\mathfrak{M}$ is the class of atomic mappings and the continuum X is arcwise connected.*

Notice that each atomic mapping defined on an arcwise connected continuum is already a homeomorphism (see [12, 6.3, p. 51]). A natural question arises (see [16, Problem 35]) whether the implication ($\heartsuit$) is true if the continuum X is arcwise connected and $\mathfrak{M}$ is the class of irreducible mappings. Answering this question in the negative, we construct in Section 3 of the present paper a large family $\mathcal{S}_c$ of continua in L and in Example 3.3 an arcwise connected continuum in the family $\mathcal{S}_c$ as a counterexample.

Let us recall two (still open) problems (see [5, Problem 3, p. 214 and Problem 11, p. 217]).

1.5. Problem. What topological spaces X and Y have the property that each open mapping from X onto Y is light?

1.6. Problem. Characterize all continua X being in the class L.

Note that the family of continua in L that is defined in the present paper gives a partial answer to Problems 1.5 and 1.6.

2. Spirals.

A *ray* means a one-to-one image of the closed half-line $[0, +\infty)$, and the image of 0 is called the *end point of the ray.*

If S is a dense subspace of a compact space Z, then Z is called a *compactification* of S, and $Z \setminus S$ is called the *remainder* of S in Z (see e.g. [1, p. 34]). It is known that if S is a locally compact, noncompact, separable metric space, then each continuum is a remainder of S in some compactification of S, [1, Theorem, p. 35]. Taking as S a one-to-one image of the real half-line $[0, \infty)$ we conclude the following statement.

2.1. Statement. *Each nondegenerate continuum B is a remainder of a ray S in some compactification of S, and then $Y = B \cup S$ is a continuum having S as an arc-component with $B = \operatorname{cl} S \setminus S$.*

Let S^- and S^+ are two disjoint subcontinua of a continuum X. An open set $S \subset X$ is called a *spiral from S^- to S^+* if

- (S1) there is a one-to-one surjective mapping $g : (-\infty, \infty) \to S$ of the open interval $(-\infty, \infty)$ onto S;
- (S2) $S^+ \cup g([0, \infty))$ is a compactification of $g([0, \infty))$ having S^+ as the remainder of $g([0, \infty))$ in $S^+ \cup g([0, \infty))$;
- (S3) $S^- \cup g((-\infty, 0])$ is a compactification of $g((-\infty, 0])$ having S^- as the remainder of $g((-\infty, 0])$ in $S^- \cup g((-\infty, 0])$;
- (S4) $(S^- \cup S^+) \cap S = \emptyset$.

If conditions (S1)-(S4) are satisfied, then we say that the above mentioned mapping $g : (-\infty, \infty) \to S$ *describes the spiral S from S^- to S^+*, and the union

$$X = S^- \cup S \cup S^+$$

is named a *spiral-arc from S^- to S^+*. Obviously each arc is a spiral-arc (with its ends as degenerate continua S^- and S^+); and the $\sin(1/x)$-curve $\operatorname{cl}\{(x, \sin(1/x)) : x \in (0, 1]\}$ is a spiral-arc from a singleton $\{(1, \sin 1)\}$ to the closed segment $\{(0, y) : y \in [-1, 1]\}$.

If S is a spiral from a continuum A to a continuum B, we denote by S^- and S^+ the continua A and B, respectively.

We say that S is a *spiral* in X if S is a spiral from some subcontinuum (S^-) to some subcontinuum (S^+) in X.

For $S = g((-\infty, +\infty)) \subset X$ a spiral, the above notation means that

$$S^+ = \{x \in X, \text{ there exists } \{t_n\}_{n=1}^{\infty} \subset \mathbb{R} \text{ such that } \lim_{n\to\infty} t_n = +\infty \;\&\; \lim_{n\to\infty} g(t_n) = x\}$$
$$S^- = \{x \in X, \text{ there exists } \{t_n\}_{n=1}^{\infty} \subset \mathbb{R} \text{ such that } \lim_{n\to\infty} t_n = -\infty \;\&\; \lim_{n\to\infty} g(t_n) = x\}\,.$$

A spiral S is said to be *crooked near* a nondegenerate continuum K if for each point $a \in K$ there exists an arbitrarily small neighborhood U of a such that for each $b \in K \cap U$ there exists in $U \cap S$ a sequence of arcs J_n with end points α_n, β_n satisfying
(CS1) $\lim \alpha_n = \lim \beta_n \in \{a, b\}$;
(CS2) $\{a, b\} \subset \liminf J_n$.

A spiral S from a continuum S^- to a nondegenerate continuum S^+ is said to be *crooked (from S^- to S^+)* if S is crooked near S^+.

We will use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [11, §51, I, p. 254]). In particular, a point p of a subset S of a space X is called an *end point* of S provided that it is of order one in S, i.e., for each $\varepsilon > 0$ there is a neighborhood U of p such that $\operatorname{diam} U < \varepsilon$ and $\operatorname{card}(S \cap \operatorname{bd} U) = 1$.

We start with recalling a well known result, see [17, Chapter 8, (7.31), p. 147].

2.2. Statement. *The order of a point is never increased under an open mapping.*

2.3. Proposition. *If $S = g(\mathbb{R}) \subset X$ is a spiral and $f : X \to f(X) = Y$ is a nonconstant open mapping, then $f(S)$ is homeomorphic to one of the following four sets:*

(a) *the open interval* $(0, 1)$*;*
(b) *the half-open interval* $[0, 1)$*;*
(c) *the closed interval* $[0, 1]$*;*
(d) *the circle.*

Moreover, if (a) holds, then $f|S$ is injective and if (b) holds, then exists $c \in \mathbb{R}$ such that $f|g((-\infty, c])$ and $f|g([c, \infty))$ are injective.

Proof. The first part is obvious, because the open mapping does not increase the order of the point due to Statement 2.2. The first part of 'moreover' statement is also obvious, it is sufficient to prove the last one. To see this, let $h : f(S) \to [0, 1)$ be the above mentioned homeomorphism. Denote $p = h \circ f \circ g$. Choose $c \in \mathbb{R}$ such that $p(c) = 0$. Suppose on the contrary that $p|[c, \infty)$ is not injective. This means that there are $a, b \in \mathbb{R}$ with $c \le a < b$ such that $p(a) = p(b) = y$ for some $y \in [0, 1)$. Then $p([a, b])$ is an arc J in $[0, 1)$.

If $0 \in J$, then fix some $d \in (a, b]$ such that $p(d) = 0$; now $p([c, d))$ is an arc $0z$ for some $z \in (0, 1)$ and f is not interior at $f^{-1}(h^{-1}(z))$, a contradiction.

If $0 \notin J$, then fix an end point z distinct from y; now f is not interior at $f^{-1}(h^{-1}(z)) \in g([c, b])$, a contradiction.

We have shown that f is injective on $g([c, \infty))$. Similarly we can show that f is injective on $g((-\infty, c])$. Note further that the point c is uniquely determined. □

Let K be a subset of a continuum X and S be an open spiral in X. We say that a mapping f *cycles on S near K* if either

(i) $f|S \cap U \cap V$ is not injective for each open set U containing K and each open set V containing S^+

or

(ii) $f|S \cap U \cap V$ is not injective for each open set U containing K and each open set V containing S^-.

We say that a mapping f *cycles on* S if a mapping f cycles on S near S^+ or near S^-.

2.4. Proposition. *Let S be a spiral in a continuum X. If a nonconstant open mapping $f : X \to Y$ cycles on S near S^+ then $f(S^+) = f(S) = f(X)$ is homeomorphic either to the unit circle or to the unit interval.*

Proof. We know by Proposition 2.3 that $f(X)$ is homeomorphic either to the unit circle or to the unit interval. Fix first an open $U \supset S^+$.

Claim. In U there exist 3 distinct points with the same image under f.

(i) "circle case"

We find reals $s_1 < t_1 < s_2 < t_2 < s_3 < t_3$ such that $g([s_1, \infty)) \subset U$ and $f(g(s_i)) = f(g(t_i)) = y_i$ for $i \in \{1, 2, 3\}$. The image $f(g[s_i, t_i])$, for $i \in \{1, 2, 3\}$, is a closed connected subset of Y and equals Y (cannot be an arc J in Y because the mapping f must be interior at the preimage of the end point of J distinct from y_i). Then each $y \in Y$ has a preimage in $g[s_i, t_i]$, for $i \in \{1, 2, 3\}$. Hence Claim holds.

(ii) "interval case"

We find reals $s_i < t_i < s_{i+1}$ such that $g([s_1, \infty)) \subset U$ and $f(g(s_i)) = f(g(t_i)) = y_i$ for $i \in \{1, 2, 3, 4, 5\}$. Fix $i \in \{1, 2, 3, 4, 5\}$. The image $f(g[s_i, t_i])$ is an arc J_i in Y. At least one end point of J_i is distinct from y_i and belongs to $\{0, 1\}$ due to the openness of f at X. We have found $x_i \in (s_i, t_i)$ such that $f(g(x_i)) \in \{0, 1\}$. Among five such points $g(x_i)$ at least three have the same image under f. Claim is proved.

Let $a, b, c \in U$ be such that $f(a) = f(b) = f(c)$, $g^{-1}(a) < g^{-1}(b) < g^{-1}(c)$. Then $f(g([g^{-1}(a), g^{-1}(b)])) = f(g((g^{-1}(a), g^{-1}(b))))$ is both open and closed subset of Y, hence equals Y.

Finally, we conclude that $f(S^+) = Y$ due to the continuity of f. □

2.5. Proposition. *Let S be a spiral in a continuum X crooked near S^+. Let $f : X \to Y$ be a nonconstant open mapping. Then $f : X \to Y$ cannot cycle on S near S^+.*

Proof. Suppose on the contrary that f cycles on S near S^+. Using Proposition 2.4 and composing f with an open projection from a circle onto a unit interval (if necessary) we may assume that $f(X) = Y = [0, 1]$. By Proposition 2.4 we see that $f(S^+)$ (which equals $f(X)$) is not a singleton. Take $a \in S^+$ such that $0 \neq f(a) \neq 1$ such that arbitrarily close to a there are points in S^+ which are not mapped to $f(a)$. Then there is an open set W containing a such that $f(W) \subset (0, 1)$.

By the crookedness of S there is a neighborhood $U \subset W$ of a such that for any fixed point $b \in U \cap S^+$ with $f(b) \neq f(a)$ there exists in $U \cap S$ a sequence of arcs J_n with end points α_n, β_n satisfying conditions (CS1) - (CS2).

Let $\varepsilon = |f(b) - f(a)|/3$. By the continuity of f there are in U open neighborhoods V_a and V_b of a and of b respectively, such that $f(V_a) \subset \{y \in (0, 1) : |y - f(a)| < \varepsilon\}$ and $f(V_b) \subset \{y \in (0, 1) : |y - f(b)| < \varepsilon\}$.

In the sequence $\{J_n\}$ of arcs in S one can find an arc $J \subset C \cap U$ such that J meets both V_a and V_b and its end points α and β are contained in one of the mentioned sets; we may assume that they are contained in V_a. Note that $f(J)$ is an arc in $Y = [0,1]$ not containing the end points of Y. Moreover, $f(J)$ meets both $f(V_a)$ and $f(V_b)$. Hence, at least one of the end points of $f(J)$, denote it by y, is distinct from both $f(\alpha)$ and $f(\beta)$. But then f is not interior at that point $x \in J \setminus \{\alpha, \beta\}$ which is mapped onto y. We have obtained a contradiction with the openness of f. $\square$

We say that a subcontinuum K of a continuum X *is controlled by* a (finite or infinite) sequence of spirals $\{S_i\}_{i \in I}$ in X if K does not meet any S_i for $i \in I$ and in K there exists a point x with positive distance to

$$X \setminus \left(K \cup \bigcup_{i \in I} \operatorname{cl} S_i \right).$$

Denote the set of these points by $K * \{S_i\}_{i \in I}$. If K is a continuum controlled by a (finite or infinite) sequence of spirals $\{S_i\}_{i \in I}$ in X, then $K * \{S_i\}_{i \in I}$ contains a nondegenerate subcontinuum due to [14, Corollary 3.5, p. 74].

2.6. Proposition. *Let K be a nondegenerate subcontinuum of a continuum X. Let K be controlled by a finite sequence $\{S_i\}_{i=1}^n$ of spirals in X. Let $f : X \to Y$ be a nonconstant open mapping. If $f(K)$ is a singleton in Y, then for some $i \in \{1, \cdots, n\}$ the mapping f cycles on S_i .*

Proof. Let a nondegenerate subcontinuum K of a continuum X be controlled by a finite sequence $\{S_i\}_{i=1}^n$ of spirals in X. Let $f : X \to Y$ be a nonconstant open mapping, $f(K)$ be a singleton.

Suppose on the contrary that for $i \in \{1, 2, \ldots, n\}$ the mapping f does not cycle on S_i. We conclude a contradiction.

If K has a nonempty interior we obtain a contradiction with the openness of f. Hence we can assume that

(2.6.1) arbitrarily close to any point of K there are points of $\bigcup_{i=1}^n S_i$.

Notice first that if for some $i, j \in \{1, 2, \ldots, n\}$ the image $f(S_j^+)$ meets $f(S_i)$ in two distinct points, then f cycles on S_j near S_j^+ and the proof is finished.

Now we can assume that they meet at most at some singleton. If the inverse image of this singleton meets S_i arbitrarily close to either S_i^+ or S_i^- we conclude that f cycles on S_i and the proof is finished. The same arguments hold for S_j^-.

In all other cases we can replace the spiral S_i by two spirals $S_{i+1/3} \subset S_i$ from $c_i^+ \in S_i$ to S_i^+ and $S_{i-1/3} \subset S_i$ from $c_i^- \in S_i$ to S_i^- in such a way that $f(S_j^+) \cap f(S_{i \pm 1/3}) = \emptyset$ and $f(S_j^-) \cap f(S_{i \pm 1/3}) = \emptyset$ for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$. A similar replacement can be done to obtain f injective on each 'new' spiral.

Due to this discussion we can assume that K is controlled by a finite sequence $\{S_i\}_{i=1}^n$ of spirals in X with this properties :

(2.6.2) for $i \in \{1, 2, \ldots, n\}$ the mapping f is injective on S_i,
(2.6.3) $f(S_j^+) \cap f(S_i) = \emptyset$ for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$,
(2.6.4) $f(S_j^-) \cap f(S_i) = \emptyset$ for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, n\}$.

To obtain a contradiction suppose that $f(K) = \{y\}$ and take distinct points $x_1, x_2, \dots, x_{n+1}$ in K with a positive distance to

$$X \setminus \left(K \cup \bigcup_{i=1}^{n} \operatorname{cl} S_i \right)$$

and their disjoint open neighbourhoods $U_1, U_2, \dots, U_{n+1}$ such that $U_i \subset K \cup \bigcup_{j=1}^{n} \operatorname{cl} S_j$ for $i \in \{1, \dots, n+1\}$. Then for $i \in \{1, \dots, n+1\}$ the image $f(U_i)$ and also their intersection V is an open neighbourhood of y. Since f is a nonconstant open mapping and due to (2.6.1) we can take

$$y' \in \left(V \cap \bigcup_{i=1}^{n} f(S_i) \right) \setminus \{y\}.$$

For $i \in \{1, \dots, n+1\}$ there exists $z_i \in U_i$ such that $f(z_i) = y'$. Now z_i cannot be in K for $i \in \{1, \dots, n+1\}$ (because $f(K) = \{y\}$ and $y' \neq y$). Moreover, they cannot be in S_j^+ due to the construction. Indeed, the images of S_j^+ do not meet images of S_i for $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, n\}$ due to (2.6.3). The same is true for S_j^- due to (2.6.4).

We conclude that $z_i \in S_{j_i}$ for some $j_i \in \{1, 2, \dots, n\}$. There must be $j_i = j_k$ for some $i \neq k$ what is a contradiction with the injectivity of f on spirals in $\{S_i\}_{i=1}^{n}$ due to (2.6.2). $\square$

We show in Example 2.8 that the we cannot replace the finite sequence in Proposition 2.6 by an infinite one.

2.7. Lemma. *Let $\varepsilon > 0$ and $d \in [-1, 0]$ be given. Put $L = \{(x, 0) \in \mathbb{R}^2 : x \in [-1, 1]\}$, $K = \{(x, 0) \in \mathbb{R}^2 : x \in [-1, 0]\}$, $c = (\varepsilon, 0)$. For $n \in \mathbb{N}$ put $s_n = (0, \varepsilon/n)$ and $t_n = (-1, 3\varepsilon/4n)$. Put*

$$A = cs_1 \cup \bigcup_{n=1}^{\infty} s_n t_n \cup \bigcup_{n=1}^{\infty} t_n s_{n+1},$$

$B = \{(x, y) \in \mathbb{R}^2 : (x, -y) \in A\}$ and $X = A \cup B \cup L$.

Then there exists a nonconstant mapping $g : X \to [0, 1]$ which is interior at $(d, 0)$ such that $g|(X \setminus K)$ is open.

Proof. See Figure 1. Let $g|L$ be a mapping sending K to $0 \in [0, 1]$ and identity elsewhere. Let $\{A_n\}_{n=1}^{\infty}$ be any sequence of mutually disjoint arcs in A, $c \in A_1$, converging to $(d, 0)$ such that each $A_n \setminus \{c\}$ separates $\{c\}$ and A_{n+1} in A. Let $g|A$ be any homeomorphism of A onto the interval $(0, \varepsilon]$ already defined at $c \in A \cap L$. For $n \in \mathbb{N}$ let $\{B_n\}$ be any sequence of mutually disjoint arcs in $B \setminus \{c\}$ converging to $(d, 0)$ such that each B_n separates $\{c\}$ and B_{n+1} in B. Let $g|B$ be any homeomorphism of B onto the interval $(0, \varepsilon]$ already defined at $c \in B \cap L$ such that $g(B_n)$ meets both $g(A_n)$ and $g(A_{n+1})$. We see that g is interior at $(d, 0)$ and $g|(X \setminus K)$ is open. $\square$

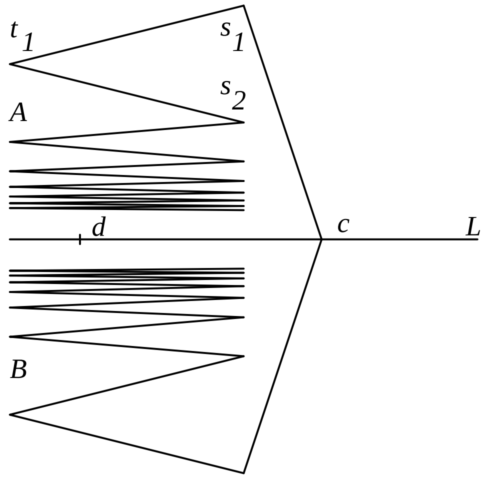

Figure 1 (Lemma 2.7)

Using Lemma 2.7 we show that the finite sequence of spirals in Proposition 2.6 cannot be changed to an infinite one.

2.8. Example. *There exist a continuum X and a nondegenerate subcontinuum K controlled by an infinite sequence $\{S_i\}_{i=1}^{\infty}$ of spirals in X and a nonconstant open mapping $f : X \to Y$ such that $f(K)$ is a singleton in Y and f is injective on all S_i for $i \in \mathbb{N}$.*

Proof. Put $L = \{(x, 0, 0) \in \mathbb{R}^3 : x \in [-1, 1]\}$ and let $\{(d_n, 0, 0)\}_{n=1}^{\infty}$ be a dense sequence in $K = \{(x, 0, 0) \in \mathbb{R}^3 : x \in [-1, 0]\}$. Fix $n \in \mathbb{N}$.

We use Lemma 2.7 with $\varepsilon = 1/n$ and $d = d_n$. We obtain $A_n = A$, $B_n = B$ and $g_n = g$. We put $S_n = \pi_n(A_n \cup B_n)$ where π_n is the orthogonal projection from the plane $P_0 = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$ onto the plane $P_n = \{(x, y, z) \in \mathbb{R}^3 : z = ny\}$.

Now let

$$X = L \cup \bigcup_{n=1}^{\infty} S_n.$$

We see that X is a continuum. Let $Y = [0, 1]$. We define a mapping $f : X \to Y$ in the following way.

Let $f|L$ be a mapping sending $(x, 0, 0) \in L$ to $\max(0, x) \in Y$. For $n \in \mathbb{N}$ define $f|S_n = g_n \circ \pi_n^{-1}$.

We see that in the continuum X there is a nondegenerate subcontinuum K controlled by an infinite sequence $\{S_i\}_{i=1}^{\infty}$ of spirals in X and that the nonconstant mapping $f : X \to Y$ satisfies

(2.8.1) f is open;
(2.8.2) $f(K)$ is a singleton in Y;
(2.8.3) f is injective on all S_i for $i \in \mathbb{N}$.

To see (2.8.1) we distinguish points in K, L and S_n. The openness of f follows from the definition of f in X and the density of points $(d_n, 0, 0)$ in K, because each g_n guarantees the interiority of f at $(d_n, 0, 0)$. Properties (2.8.2) and (2.8.3) are easy observations. This finishes the proof. □

2.9 Remark. In [7, Remark 3.5] a question is posed, whether the openness of a spiral S from a continuum to a nondegenerate continuum S^+ guarantees that the image of S^+ under any nonconstant open mapping f is nondegenerate. We have seen in Example 2.8 that the answer is negative. Indeed, there can be infinitely many open spirals which ensures the interiority of f at S^+.

In [7, Remark 3.5] a question is posed, whether the openness of a spiral S crooked from a subcontinuum S^- to a nondegenerate subcontinuum S^+ in a given continuum X guarantees, that if the image $f(X)$ is either an arc or a simple closed curve, then f is not open. We will see in the following example that the answer to this question is negative.

2.10 Example. *There exist a continuum X, an open mapping $f : X \to [-1, 1]$ and an open spiral S in X crooked near a nondegenerate subcontinuum S^+ of X.*

Proof. Put $L = [-1, 1]$. For $n \in \mathbb{N}$ let $r_{2n-1} = -1/5$ and $\{r_n\}_{n=1}^{\infty}$ be a sequence containing all rational numbers in the interval $[-1/5, 1/5]$. Put

$$A = \bigcup_{n=1}^{\infty} a_n a_{n+1},$$

where $a_n = (-1/n, r_n)$ for $n \in \mathbb{N}$. Put

$$B = \bigcup_{n=1}^{\infty} b_n b_{n+1},$$

where $b_n = (1/n, (-1)^n)$ for $n \in \mathbb{N}$. Now we set $X = A \cup B \cup L$. The continuum is pictured on Figure 2.

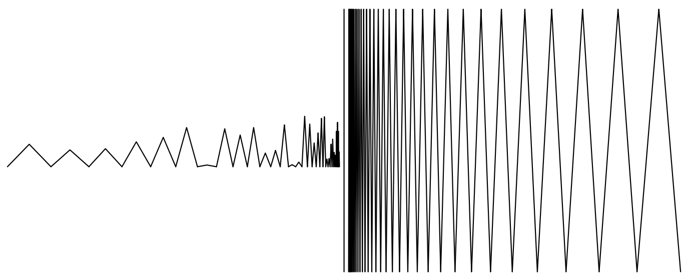

FIGURE 2 (EXAMPLE 2.10, CONTINUUM X)

We see that $S = A \setminus \{a_1\}$ is a spiral crooked near $S^+ = \{0\} \times [-1/5, 1/5]$. We define the mapping $f : X \to [-1, 1]$ in the following way

(2.10.1) $f|A$ is any homeomorphism of A onto $(0, 1]$ such that $f(a_1) = 1$;

(2.10.2) $f|(B \cup L) : B \cup L \to [-1, 1]$ is a modification of the open non-light mapping from the $\sin(1/x)$-curve onto $[-1, 1]$ sending $\{0\} \times [-1/3, 1/3]$ to $0 \in [-1, 1]$ (see [6, Remark 18, p. 242]).

Then we observe that f is an open mapping of the continuum X onto $[-1, 1]$. $\square$

3. Classes $\mathcal{S}$ and $\mathcal{S}_c$.

We have seen in Example 2.8 that we can guarantee the interiority of a mapping at n points of a subcontinuum with $2n$ spirals and that if there is an infinite number of spirals approximating a nondegenerate continuum, then there is an open mapping from this space which is not light. This motivates the definition of suitable classes of continua using only (locally) finitely many spirals in the following paragraph.

The class $\mathcal{S}$ denotes the class of all continua at which all subcontinua are controlled by a finite sequence of spirals.

The class $\mathcal{S}_c$ denotes the class of all continua in $\mathcal{S}$ which have a nondegenerate subcontinuum K such that K is controlled by a finite sequence $\{S_i\}$ of crooked spirals and $K \subset \cup S_i^+$.

Notice that continua pictured Figures 1-4 are in $\mathcal{S}$ and continuum pictured in Figure 3 is in $\mathcal{S}_c$. See also continua in [6, Figures 1-9] which are in $\mathcal{S}$ and continua pictured in [7, Figures 1-2] which are in $\mathcal{S}_c$.

3.1. Proposition. *Let $X \in \mathcal{S}$ and $f : X \to Y$ be a nonconstant open non-light mapping. Then*
(i) f cycles on some spiral in X;
(ii) $f(X)$ is homeomorphic either to the unit circle or to the unit interval.

Proof. We have (i) due to Proposition 2.6 and (ii) due to Proposition 2.4. $\square$

3.2. Theorem. $\mathcal{S}_c \subset \mathsf{L}$.

Proof. Let $X \in \mathcal{S}_c$ and $f : X \to Y$ be a nonconstant open non-light mapping. We conclude a contradiction.

According to Proposition 3.1., $f(X)$ is homeomorphic to the unit circle or to the unit interval. Composing f with the open projection from the unit circle onto the unit interval (if necessary) we may assume that $f(X)$ is homeomorphic to the unit interval. Let K be a subcontinuum of X controlled by a finite sequence $\{S_i\}_{i=1}^n$ of crooked spirals and $K \subset \bigcup_{i=1}^n S_i^+$.

For $i \in \{1, \cdots, n\}$ the mapping f cannot cycle on S_i near S_i^+ according to Proposition 2.5., and so we can assume that f is injective on S_i due to Proposition 2.3 (we can replace S_i by a spiral from some $s_i \in S_i$ to S_i^+). Hence $f(S^+)$ is a singleton because $f(X)$ is homeomorphic to the unit interval. We conclude that $f(K)$ is a singleton. By Proposition 2.6. we get that for some $i \in \{1, \cdots, n\}$ the mapping f cycles on the spiral S_i, what is the contradiction due to Proposition 2.5, because spirals in $\{S_j\}_{j=1}^n$ are crooked. $\square$

Notice that in [7, Theorem 2.6] only continua of the form $X = S^- \cup S \cup S^+$ are considered, where S is a crooked spiral. Theorem 3.2 extends these results to a very large family.

3.3. Example. *There exist an arcwise connected continuum X in* L and an irreducible mapping f such that $f(X)$ is not in L .

Proof. For t real we denote $a_t = (0, t)$ and $b_t = (8, t)$. Set

$$E = b_3 b_0 \cup b_0 a_0 \cup a_0 a_8 \cup a_8 b_8 \cup b_8 b_5.$$

For $n \in \mathbb{N}$ let $r_{2n-1} = 5$ and $\{r_n\}_{n=1}^{\infty}$ be a sequence containing all rational numbers in the interval $[5, 7]$. Put

$$C = \bigcup_{n=1}^{\infty} c_n c_{n+1},$$

where $c_n = (8/n, r_n)$ for $n \in \mathbb{N}$. Put

$$D = \bigcup_{n=1}^{\infty} d_n d_{n+1},$$

where $d_n = (8/n, 2-(-1)^n)$ for $n \in \mathbb{N}$. Now we set $X = C \cup D \cup E$. The continuum is pictured on Figure 3.

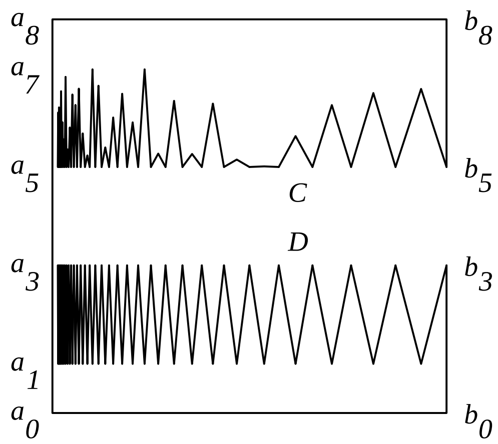

FIGURE 3 (EXAMPLE 3.3, CONTINUUM X)

We see that $X \in \mathcal{S}_c$. Now we construct an irreducible mapping $f : X \to Y$. We define f in the following way

(i) $f(a_5a_7) = \{a_5\}$;

(ii) for $(0, y) \in a_7a_8$ we define $f(0, y) = (0, 3y - 16)$;

(iii) for $(x, y) \in C$ we define $f(x, y) = (x, 5)$;

(iv) f is the identity elsewhere.

The mapping f is continuous and $Y = f(X)$ is a continuum. See Figure 4.

We show that f is an irreducible mapping and that and $Y \notin \mathsf{L}$.

We start with the proof that f is irreducible. Let $I \subset Y$ be a nondegenerate subcontiuum irreducible about $\{p, q\}$. Put $J = f^{-1}(I)$. If $\{p, q\} \cap \{a_5\} = \emptyset$, then J is irreducible about $\{f^{-1}(p), f^{-1}(q)\}$. If $p = a_5$ and J is an arc then it is irreducible about its end points. If J is not an arc then it contains either a spiral $C' \subset C$ from some $c \in C$ to a_5a_7 or a spiral $D' \subset D$ from some $d \in D$ to a_1a_3. In both cases J is irreducible about $\{f^{-1}(q), a_5\}$.

To show that $Y \notin \mathsf{L}$ we define an open non-light mapping $g : Y \to a_1a_3$ in the following way :

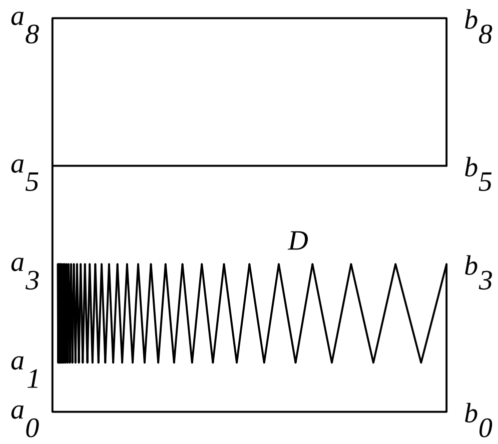

FIGURE 4 (EXAMPLE 3.3, CONTINUUM Y)

(α) We adapt the open non-light mapping f from the $\sin(1/x)$-curve (see [6, Remark 18, p. 242]) to get an open non-light surjective mapping $g : (D \cup a_1a_3) \to a_1a_3$, $g(b_3) = a_3$. This gives $g(a_{5/3}a_{7/3}) = \{a_2\}$.

(β) We define $g|(a_1a_0 \cup a_0b_0 \cup b_0b_3)$ to be any homeomorphism onto a_1a_3 (already defined in a_1 and b_3).

(γ) We define $g|(a_3a_8 \cup a_8b_8)$ to be any homeomorphism onto a_1a_3 (already defined in a_3).

(δ) We define $g|(a_5b_5 \cup b_5b_8)$ to be any homeomorphism onto $a_1g(a_5)$ (already defined in a_5 and b_8).

The continuity and the openness of g and the fact that g is not light follows from the construction of g. $\square$

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