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# A GENERALIZATION OF BEATTY'S THEOREM

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**Abstract.** In 1926 Sam Beatty made the following discovery, which he posed as a problem in [1]. If $a$ is a positive irrational number, the sequences $m(1+a), m=1,2,\dots$ and $n(1+a^{-1}), n=1,2,\dots$ together contain exactly one number from each of the intervals $(k,k+1), k=1,2,3,\dots$. The problem was solved by Ostrowski and Aitken[4] and generalized to a larger class of sequences by Lambeck and Moser[3]. The authors are grateful for UNC Charlotte undergraduate James Rudzinski for suggesting the combinatorial game that lead to this paper.

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**Keywoards and Phrases:.** Beatty's theorem, combinatorial game, Ostrowski, Aitken

The purpose of this paper is to extend this theorem from sequences to continuous functions. We first establish some notation. Let $P = (a_0 = 0, a_1, a_2, \dots)$ be a strictly increasing unbounded sequence of real numbers. For any nonnegative number $x$, let $\lfloor x \rfloor_P$ be the largest member of $P$ that does not exceed $x$, and for positive $x$,

$$\rfloor x \lfloor_P = \begin{cases} \lfloor x \rfloor_P & \text{if } x \text{ is not in } P \\ a_{i-1} & \text{if } x \in P \text{ and } \lfloor x \rfloor_P = a_i. \end{cases}$$

Also, if $t > 0$, we define $N_t = \{0, t, 2t, 3t, 4t, \cdots\}$ and $N_t^+ = N_t \backslash \{0\}$.

**Main Theorem.** *Let $F$ and $G$ be real, continuous, strictly increasing, functions with domains $[0,\infty)$ satisfying $F(0) = G(0) = 0$ and $\lim_{x\to\infty}(F(x)+G(x)) = \infty$. For all $t > 0$, let $P_t = \{0, (F+G)^{-1}(t), (F+G)^{-1}(2t), (F+G)^{-1}(3t), \dots\}$, and $P_t^+ = P_t \backslash \{0\}$. Then the two sequences $A_t$ and $B_t$ defined by $A_t = \{\rfloor (F^{-1})(n) \lfloor_{P_t}:$*

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$n \in N_t^+\}$ *and* $B_t = \{\lfloor (G^{-1})(n) \rfloor_{P_t} : n \in N_t^+\}$ *partition* $P_t^+$ *for all* $t > 0$*. Also, the elements of* $A_t$ *are distinct and the elements of* $B_t$ *are distinct.*

We conjecture that the theorem remains true even when $F+G$ is bounded. Also, it is easy to see that an analogous version exists for functions $F : [a, b) \to [0, \infty)$ and $G : [a, b) \to [0, \infty)$, where $F$ and $G$ are continuous, strictly increasing, and satisfy $F(a) = G(a) = 0$ and $\lim_{x \to b}(F(x) + G(x)) = \infty$.

Note that the two functions $F$ and $G$ in the theorem are independent, unlike those in the Lambek and Moser paper where each one determines the other. This shows that using continuous functions provides a much richer theory. Note also that our conclusion is true for all $t > 0$. Thus we have an uncountable number of partitions. This forces rigid restrictions on the sequences $A_t$ and $B_t$.

The following is an example illustrating the theorem. We can choose $F$ and $G$ independently. However, we have chosen $F$ and $G$ so that we can compute $F^{-1}, G^{-1}, (F+G)^{-1}$ explicitly. Of course, if we cannot compute the inverse of one of our functions explicitly, then we just leave that inverse in implicit form just as we would for the function $sin^{-1}$. Let $F(x) = ax^2 + bx, G(x) = cx^2 + dx, a, b, c, d > 0$. Then $(F+G)(x) = (a+c)x^2 + (b+d)x, F^{-1}(n) = \frac{-b+\sqrt{b^2+4an}}{2a}, G^{-1}(n) = \frac{-d+\sqrt{d^2+4cn}}{2c}$ and $(F+G)^{-1}(n) = \frac{-(b+d)+\sqrt{(b+d)^2+4(a+c)n}}{2(a+c)}$. Then

$$P_t = \left\{ \frac{-(b+d) + \sqrt{(b+d)^2 + 4(a+c)n}}{2(a+c)} : n = 0, t, 2t, 3t, \ldots \right\},$$

$$A_t = \left\{ \left\lfloor \frac{-b + \sqrt{b^2 + 4an}}{2a} \right\rfloor_{P_t} : n = t, 2t, 3t, \ldots \right\},$$

and

$$B_t = \left\{ \left\lfloor \frac{-d + \sqrt{d^2 + 4cn}}{2c} \right\rfloor_{P_t} : n = t, 2t, 3t, \ldots \right\}.$$

The theorem guarantees that for all $t > 0$, $A_t$ and $B_t$ partition $P_t^+$ and the members of $A_t$ and $B_t$ are distinct. The functions $F(x) = \frac{x}{1+a}$ and $G(x) = \frac{ax}{1+a}$, where $a$ is a positive irrational number, can be used to prove Beatty's theorem. The reader might also like to use $F(x) = 3(x^2 + x), G(x) = x^3$ to give an example for himself.

We next prove the main theorem by first proving a more specialized result, for which we need more notation. Let $N = \{0, 1, 2, 3, \ldots\}, Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$, and for any $t > 0$, $Z_t = \{\ldots, -3t, -2t, -t, 0, t, 2t, 3t, \ldots\}$ and $N_t = \{0, t, 2t, 3t, \ldots\}$. Thus $N = N_1$ and $Z = Z_1$. Suppose $x = nt + \underline{\epsilon}$ where $n \in Z$ and $0 \leq \underline{\epsilon} < t$. Then $\lfloor x \rfloor_t = nt$. Suppose $= x = mt + \overline{\epsilon}$ where $m \in Z$ and $0 < \overline{\epsilon} \leq t$. Then $\lceil x \rceil_t = mt + t$. Thus $\lfloor x \rfloor_t$ and $\lceil x \rceil_t$ are the *floor* and *ceiling* functions with respect to the grid $Z_t$. Of course, $\lfloor x \rfloor_1 = \lfloor x \rfloor$ and $\lceil x \rceil_1 = \lceil x \rceil$ where $\lfloor x \rfloor$ and $\lceil x \rceil$ are the usual floor and ceiling functions. If $F : A \to B$ is a function on $A$ where $A$ is a countable linearly ordered set, then we consider $S = \{F(x) : x \in A\}$ to be both a set and a sequence. The reason for this is that we wish to discuss the set $S$ of functional values and we also need to discuss the fact that the members of the *sequence* are distinct.

Definition: Suppose $f$ and $g$ are any real functions with domain $[0, \infty)$. We say that $f$ and $g$ are complementary if $f$ and $g$ satisfy $f(0) = g(0) = 0$, $f$ and $g$ are strictly increasing on $[0, \infty)$, and $f(x) + g(x) = x$ for all $x \geq 0$.

It is easy to see that for any $\epsilon > 0$, $f(x+\epsilon)-f(x) < \epsilon$ and $0 < g(x+\epsilon)-g(x) < \epsilon$ for all non-negative $x$, so complimentary functions are continuous. Also, it is easy to see that $0 < f(x) < x$ and $0 < g(x) < x$ for all positive $x$.

**Theorem 1.** *Suppose $f : [0,\infty) \to R$, $g : [0,\infty) \to R$ are complementary. Then*

a. *$\forall t > 0$, the following two sets partition the set $N_t^+ = \{t, 2t, 3t, 4t, \cdots\}$ : $A = \{\lceil f^{-1}(n)\rceil_t : n \in N_t^+\}$ and $B = \{\lfloor g^{-1}(n) + t\rfloor_t : n \in N_t\}$. Also, the elements of the sequence $A$ are distinct and the elements of the sequence $B$ are distinct.*

b. *Also, for every $t > 0$, the following two sets partition $N_t^+$ : $C = \{\lceil f^{-1}(n) - t\rceil_t : n \in N_t^+\}$ and $D = \{\lfloor g^{-1}(n)\rfloor_t : n \in N_t^+\}$. Also, the elements of the sequence $C$ are distinct and the elements of the sequence $D$ are distinct.*
*We note that if one of $f$, $g$ is bounded above, then in both a. and b., one of the two sets is finite.*

We will now prove theorem 1 through a series of lemmas. It is convenient for notational reasons to assume that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Although we will assume this, a slightly more clumsy version holds without the assumption. In the following lemmas, we consider $t > 0$ to be arbitrary but fixed.

**Lemma 1.** *Suppose $h : N_t \to N_t$ satisfies $h(0) = 0$, $h(n+t) - h(n) \in \{0, t\}$ for all $n \in N_t$, and $\lim_{n\to\infty} h(n) = \lim_{n\to\infty}(n - h(n)) = \infty$.*

*Define $k : N_t \to N_t$ as follows: $\forall n \in N_t, k(n) = n - h(n)$. It is easy to see that $k(0) = 0, \lim_{n\to\infty} k(n) = \infty$ and $\forall n \in N_t, k(n+t) - k(n) \in \{0, t\}$. Also both $h$ and $k$ are non-decreasing on $N_t$.*

*Next, $\forall m \in N_t$, define $H(m)$ to be the smallest member of $N_t$ such that $h(H(m)) = m$ and $K(m)$ to be the smallest member of $N_t$ such that $k(K(m)) = m$. Then*

a. *$H(0) = K(0) = 0$.*
b. *$H$ and $K$ are strictly increasing.*
c. *the two sets $\{H(n) : n \in N_t^+\}$ and $\{K(n) : n \in N_t^+\}$ partition the set $N_t^+$.*

*Proof.* Parts a and b are easy to see. To prove c, first observe that $\forall n \in N_t^+$, either $h(n) - h(n-t) = t$ and $k(n) - k(n-t) = 0$ or $h(n) - h(n-t) = 0$ and $k(n) - k(n-t) = t$. This is because $h(n) + k(n) = n$. From this it follows that $\forall x \in N_t^+, x$ is in the range set of exactly one of $H$ and $K$. $\square$

<u>Definitions</u>: Let $f, g$ be the two complementary functions used in the hypothesis of the Theorem 1. For convenience we again assume that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.

Let us define for all positive numbers $t$, $h : N_t \to N_t$ by $h(n) = \lfloor f(n)\rfloor_t$ and define $k : N_t \to N_t$ by $k(n) = n - h(n)$. It follows that $k(n) = n - \lfloor f(n)\rfloor_t = n + \lceil -f(n)\rceil_t = \lceil n - f(n)\rceil_t = \lceil g(n)\rceil_t$. i.e., $k(n) = \lceil g(n)\rceil_t$. Note that $k$ is being defined from $h$ the same way as it was in lemma 1. These functions $h$ and $k$ are used in the remainder of our proof of the theorem.

**Lemma 2..** *The function $h$ defined above satisfies the hypothesis given in lemma 1, namely*

1. *$h(0) = 0$,*
2. *$\forall n \in N_t, h(n+t) - h(n) \in \{0, t\}$,*
3. *$\lim_{n\to\infty} h(n) = \lim_{n\to\infty}(n - h(n)) = \infty$.*

*Proof.* Since $h(n) = \lfloor f(n) \rfloor_t$, $n - h(n) = \lceil g(n) \rceil_t$, $f(0) = 0$ and $\lim_{n\to\infty} f(x) = \lim_{n\to\infty} g(x) = \infty$, conclusions 1 and 3 are easy to see. We now prove conclusion 2.

Now $0 < f(n+t) - f(n) < t$ since $\forall \epsilon > 0, \forall x \in [0,\infty), 0 < f(x+\epsilon) - f(x) < \epsilon$.

Therefore $\lfloor f(n+t) \rfloor_t - \lfloor f(n) \rfloor_t \in \{0, t\}$. It follows that $h(n+t) - h(n) \in \{0,t\}$ since $h(n+t) = \lfloor f(n+t) \rfloor_t$ and $h(n) = \lfloor f(n) \rfloor_t$. This completes the proof of lemma 2. $\square$

Since $k(n) = n - h(n)$, as in lemma 1, it follows that $k(0) = 0$, $\lim_{n\to\infty} k(n) = \infty$ and $\forall n \in N_t$, $k(n+t) - k(n) \in \{0,t\}$. Now $\forall m \in N_t$, $H(m), K(m)$ are defined in lemma 1 since $h$ and $k$ satisfy the conditions specified in lemma 1. That is, $\forall m \in N_t$, $H(m)$ is the smallest member of $N_t$ such that $h(H(m)) = \lfloor f(H(m)) \rfloor_t = m$ and $\forall m \in N_t$, $K(m)$ is the smallest member of $N_t$ such that $= k(K(m)) = \lceil g(K(m)) \rceil_t = m$. The following two lemmas will now add considerable power to lemma 1 when we use $h(n) = \lfloor f(n) \rfloor_t$ and $= k(n) = n - h(n) = \lceil g(n) \rceil_t$ in lemma 1.

Since $f : [0,\infty) \to R$ and $g : [0,\infty) \to R$ are complementary, we recall that $f(0) = g(0) = 0$, $f, g$ are strictly increasing and continuous on $[0,\infty)$ and also $\forall \epsilon > 0, \forall x \in [0,\infty), 0 < f(x+\epsilon) - f(x) < \epsilon, 0 < g(x+\epsilon) - g(x) < \epsilon$. As always, for convenience, we are assuming that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.

**Lemma 3.** $\forall n \in N_t, H(n) = \lceil f^{-1}(n) \rceil_t$.

*Proof.* Let us abbreviate $f^{-1}(n) = x$. Therefore $f(x) = n$.

Define $\overline{x} = \lceil x \rceil_t$. Of course, $= \overline{x} \in N_t$. Now by lemma 1, $H(0) = 0$. Also, $f^{-1}(0) = 0$. So $H(0) = \lceil f^{-1}(0) \rceil_t = \lceil 0 \rceil_t = 0$ is obvious. Therefore, we now assume that $n \in \{t, 2t, 3t, 4t, \cdots\}$. Then note that $f^{-1}(n) = x > t$ since $\forall x \in (0,\infty), 0 < f(x) < x$. Now since by lemma 2, $\forall \overline{n} \in= N_t, h(\overline{n}+t) - h(\overline{n}) \in \{0,t\}$, lemma 3 is proved once we prove the following:

a. $h(\overline{x}) = \lfloor f(\overline{x}) \rfloor_t = n$

b. $h(\overline{x} - t) = \lfloor f(\overline{x} - t) \rfloor_t = n - t$.

<u>Proof of a</u>: Now $\overline{x} - x = \overline{\epsilon}$ where $0 \le \overline{\epsilon} < t$. Now $\lfloor f(\overline{x}) \rfloor_t = \lfloor f(x + \overline{\epsilon}) \rfloor_t = \lfloor f(x+\overline{\epsilon}) - f(x) + f(x) \rfloor_t = \lfloor f(x+\overline{\epsilon}) - f(x) \rfloor_t + n$. Now $0 \le f(x+\overline{\epsilon}) - f(x) \le \overline{\epsilon} < t$ since $0 \le \overline{\epsilon}$. It follows that $\lfloor f(x+\overline{\epsilon}) - f(x) \rfloor_t = 0$. Therefore $\lfloor f(\overline{x}) \rfloor_t = \lfloor f(x+\overline{\epsilon}) - f(x) \rfloor_t + n = 0 + n = n$.

<u>Proof of b</u>: Define $\underline{x} = \overline{x} - t = \lceil x - t \rceil_t$. Let $x - \underline{x} = \underline{\epsilon}$, where $0 < \underline{\epsilon} \le t$. Now $\lfloor f(\overline{x} - t) \rfloor_t = \lfloor f(\underline{x}) \rfloor_t = \lfloor f(x - \underline{\epsilon}) \rfloor_t = \lfloor f(x - \underline{\epsilon}) - f(x) + f(x) \rfloor =_t= \lfloor f(x - \underline{\epsilon}) - f(x) \rfloor_t + n$. Since $0 < \underline{\epsilon}$, we see that $0 < f(x) - f(x - \underline{\epsilon}) < \underline{\epsilon} \le t$. Therefore $-t < f(x - \underline{\epsilon}) - f(x) < 0$. It follows that $\lfloor f(x - \underline{\epsilon}) - f(x) \rfloor_t = -t$. Hence $\lfloor f(\overline{x} - t) \rfloor_t = \lfloor f(x - \underline{\epsilon}) - f(x) \rfloor_t + n = n - t$. $\square$

**Lemma 4.** *$K(0) = 0$ and $\forall n \in N_t$, $K(n+t) = \lfloor g^{-1}(n) + t \rfloor_t$.*

*Proof.* Recall that $K$ was defined at the end of the proof of lemma 2. Note that $K(0) = 0$ from lemma 1. We now show that for all $n \in N_t$, $K(n+t) = \lfloor g^{-1}(n) + t \rfloor_t$. Let us first consider $n = 0$. We will show that $K(0+t) = \lfloor g^{-1}(0) + t \rfloor_t$. Now $g^{-1}(0) = 0$. Therefore, we need to show that $K(t) = \lfloor t \rfloor_t = t$. Now $0 < g(t) < t$ is true. Therefore, $k(t) = \lceil g(t) \rceil_t = t$. Since $k(0) = 0$, $= k(t) = t$, we see that $K(t) = t$ must be true by the definition of $K$. We will now suppose $n \in \{t, 2t, 3t, \cdots\}$. We must show that $K(n+t) = \lfloor g^{-1}(n) + t \rfloor_t$. Let us abbreviate $g^{-1}(n) = x$. Therefore, $g(x) = n$. Since $n \in \{t, 2t, \cdots\}$ and for all

$x \in (0,\infty)$, $0 < g(x) < x$, it follows that $x > t$. Define $\overline{x} = \lfloor x+t \rfloor_t$. Of course, $\overline{x} \in N_t$. Since $\forall \overline{n} \in N_t$, $k(\overline{n}+t) - k(\overline{n}) \in \{0,t\}$, the lemma is proved once we prove the following:

a. $k(\overline{x}) = \lceil g(\overline{x}) \rceil_t = n+t$.
b. $k(\overline{x}-t) = \lceil g(\overline{x}-t) \rceil_t = n$.

Proof of a: Define $\overline{x} - x = \overline{\epsilon}$, where $0 < \overline{\epsilon} \le t$. Now $\lceil g(\overline{x}) \rceil_t = \lceil g(x+\overline{\epsilon}) \rceil_t = \lceil g(x+\overline{\epsilon}) - g(x) + g(x) \rceil_t = \lceil g(x+\overline{\epsilon}) - g(x) \rceil_t + n$. Now $0 < g(x+\overline{\epsilon}) - g(x) < \overline{\epsilon} \le t$, since $0 < \overline{\epsilon}$. Therefore $\lceil g(x+\overline{\epsilon}) - g(x) \rceil_t = t$. Therefore $\lceil g(\overline{x}) \rceil_t = \lceil = g(x+\overline{\epsilon}) - g(x) \rceil_t + n = n+t$.

Proof of b: Remember $x > t$. Let $\underline{x} = \overline{x} - t = \lfloor x \rfloor_t$. We must show $\lceil = g(\underline{x}) \rceil_t = n$. Define $x - \underline{x} = \underline{\epsilon}$, where $0 \le \underline{\epsilon} < t$. Now $\lceil g(\underline{x}) \rceil_t = \lceil g(x-\underline{\epsilon}) \rceil_t = \lceil g(x-\underline{\epsilon}) - g(x) + g(x) \rceil_t = \lceil g(x-\underline{\epsilon}) - g(x) \rceil_t + n$. Now $0 \le g(x) - g(x-\underline{\epsilon}) \le \underline{\epsilon} < t$, since $0 \le \underline{\epsilon} < t$. Therefore $-t < g(x-\underline{\epsilon}) - g(x) \le 0$. Therefore $\lceil g(x-\underline{\epsilon}) - g(x) \rceil_t = 0$. Therefore $\lceil g(\underline{x}) \rceil_t = \lceil g(x-\underline{\epsilon}) - g(x) \rceil_t + n = 0 + n = n$. $\square$

We now have enough information to prove the theorem 1.

*Proof.* We know that $f$ and $g$ are complementary functions. Also, we know that $f$ and $g$ are continuous and bounded above by the identity function. We also recall that we defined $h : N_t \to N_t$ and $k : N_t \to N_t$ as follows: $\forall n \in N_t, h(n) = \lfloor f(n) \rfloor =_t, k(n) = n - h(n) = \lceil g(n) \rceil_t$. In lemma 2 we showed that this $h$ satisfies the hypothesis of lemma 1. Also this $k$ is defined from this $h$ in the same way as in lemma 1.

We also defined the functions $H(m)$ and $K(m)$ in lemma 1. We also know from lemma 3, 4 that $\forall n \in N_t, H(n) = \lceil f^{-1}(n) \rceil_t$ and $\forall n \in N_t, K(n+t) = \lfloor g^{-1}(n) + t \rfloor_t$.

¿From lemma 1, we know that $\forall t > 0$, the following two sets partition the set $N_t^+$ : $\{H(n) : n \in N_t^+\} = \{\lceil f^{-1}(n) \rceil_t : n \in N_t^+\}$ and $\{K(n) : n \in N_t^+\} = \{\lfloor g^{-1}(n) + t \rfloor_t : n \in N_t\}$.

Recall that $H$ and $K$ are strictly increasing functions whose image sets are disjoint. Therefore, the elements of these two sets, when viewed as a sequence, are all distinct.

Now since $g^{-1}(0) = 0$, we know that $\{\lfloor g^{-1}(n) + t \rfloor_t : n \in N_t\} = \{t\} \cup \{\lfloor g^{-1}(n) + t \rfloor_t : n \in N_t^+\}$. Therefore, $\{\lceil f^{-1}(n) \rceil_t : n \in N_t^+\}$ and $\{\lfloor g^{-1}(n) + t \rfloor_t : n \in N_t^+\}$ partition the set $\{2t, 3t, 4t, 5t, \cdots\}$. Therefore $\{\lceil f^{-1}(n) - t \rceil_t : n \in N_t^+\}$ and $\{\lfloor g^{-1}(n) \rfloor_t : n \in N_t^+\}$ partition $N_t^+$. Also, the elements of these two sets, when viewed as a sequence, are all distinct.

We now use theorem 1 to prove the Main Theorem.

Suppose $h : [0,\infty) \to [0,\infty)$ is continuous, strictly increasing, and satisfies both $h(0) = 0$ and $\lim_{x\to\infty} h(x) = \infty$. We define for all $t > 0$, the sets $P_t$ and $P_t^+$ as follows:

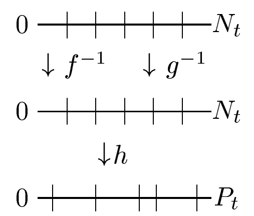
$$P_t = \{h(0) = 0, h(t), h(2t), \dots\} \text{and} P_t^+ = \{h(t), h(2t), \dots\}.$$

Next consider the following lemma.

**Lemma 5.** *Suppose $f$ and $g$ are complementary functions and $h$ satisfies the conditions listed above. Then the sets $A$ and $B$ defined by $A = \{\rfloor (fh^{-1})^{-1}(n) \lfloor_{P_t} : n \in N_t^+\}$ and $B = \{\lfloor (gh^{-1})^{-1}(n) \rfloor_{P_t} : n \in N_t^+\}$ partition $P_t^+$ for all $t > 0$. Also, the elements of $A$ (when viewed as a sequence) are distinct and the same is true for the elements of $B$.*

*Proof.* Note that $(fh^{-1})^{-1}(n) = hf^{-1}(n)$ and $(gh^{-1})^{-1}(n) = hg^{-1}(n)$. By part b of theorem 1, $\{\lceil f^{-1}(n) - t \rceil_t : n \in N_t^+\}$ and $\{\lfloor g^{-1}(n) \rfloor_t : n \in N_t^+\}$ partition $N_t^+$.

Also, the elements of each of the two sets, when viewed as sequences, are distinct. Since $h$ is an order preserving change of variable and since $\rfloor x \lfloor_{P_t}$ is defined for the partition of $P_t$ topologically analogous to the way $\lceil x - t \rceil_t$ is defined for the partition of $N_t$ and since $\lfloor x \rfloor_{P_t}$ is defined for the partition $P_t$ topologically analogous to the way $\lfloor x \rfloor_t$ is defined for the partition of $N_t$, the conclusion is easy to see. See figure 1. $\square$



$h$ is an order preserving change of variable.

We now prove the Main Theorem.

*Proof.* Since both $F$ and $G$ are strictly increasing, so is $F + G$. Also, $F + G$ is unbounded and $(F+G)(0) = 0$. It follows that $(F+G)^{-1}$ exists for all non-negative $x$. Also, $((F+G)(F+G)^{-1})(x) = (F(F+G)^{-1})(x) + (G(F+G)^{-1})(x) = x$ for all nonnegative $x$. For convenience of notation, denote $F(F+G)^{-1}$ by $f$, $G(F+G)^{-1}$ by $g$ and $(F+G)^{-1}$ by $h$. Note that $f, g$, and $h$ satisfy the hypothesis of the last lemma and that $fh^{-1} = F$ and $gh^{-1} = G$. The lemma completes the proof. $\square$

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